TRANSIENT RESPONSE ANALYSIS

Test signals:

- Impulse
- Step
- Ramp
- Sin and/or cos

Transient Response: for $t$ between 0 and $T$

Steady-state Response: for $t \to \infty$

System Characteristics:

- Stability $\Rightarrow$ transient
- Relative stability $\Rightarrow$ transient
- Steady-state error $\Rightarrow$ steady-state
First order systems

\[ \frac{C(s)}{R(s)} = \frac{1}{Ts + 1} \]

Unit step response:

\[ C(s) = \frac{1}{Ts + 1} \cdot \frac{1}{s} = \frac{1}{s} - \frac{T}{sT + 1} \]

\[ c(t) = 1 - e^{-t/T} \quad t \geq 0 \]

\[ e(t) = r(t) - c(t) = e^{-t/T} \quad e(\infty) = 0 \]

\[ c(T) = 1 - e^{-1} = 0.632 \]

\[ \frac{dc(t)}{dt} \bigg|_{t=0} = \frac{1}{T} e^{-t/T} \quad \bigg|_{t=0} = \frac{1}{T} \]
Unit ramp response

\[
C(s) = \frac{1}{Ts+1} \cdot \frac{1}{s^2} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}
\]

\[
c(t) = t - T + Te^{-t/T} \quad t \geq 0
\]

\[
e(t) = r(t) - c(t) = T \left( 1 - e^{-t/T} \right) \quad t \geq 0
\]

\[
e(\infty) = T
\]

Unit-ramp response of the system
Impulse response:

\[ R(s) = 1 \quad r(t) = \delta(t) \]

\[ C(s) = \frac{1}{sT + 1} \]

\[ c(t) = \frac{e^{-t/T}}{T} \quad t \geq 0 \]

Unit-impulse response of the system

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ramp ( r(t) = t \quad t \geq 0 )</td>
<td>( c(t) = t - T + Te^{-t/T} \quad t \geq 0 )</td>
</tr>
<tr>
<td>Step ( r(t) = 1 \quad t \geq 0 )</td>
<td>( c(t) = 1 - e^{-t/T} \quad t \geq 0 )</td>
</tr>
<tr>
<td>Impulse ( r(t) = \delta(t) )</td>
<td>( c(t) = \frac{e^{-t/T}}{T} \quad t \geq 0 )</td>
</tr>
</tbody>
</table>
Observation:

Response to the derivative of an input equals to derivative of the response to the original signal.

\[ Y(s) = G(s) U(s) \quad U(s): \text{input} \]
\[ U_1(s) = s U(s) \quad Y_1(s) = s Y(s) \quad Y(s): \text{output} \]
\[ G(s) U_1(s) = G(s) s U(s) = s Y(s) = Y_1(s) \]

How can we recognize if a system is 1st order?

Plot \( \log |c(t) - c(\infty)| \)

If the plot is linear, then the system is 1st order

Explanation:

\[ c(t) = 1 - e^{-t/T} \quad c(\infty) = 1 \]

\[ \log |c(t) - c(\infty)| = \log |e^{-t/T}| = \frac{t}{T} \]
Second Order Systems

Block Diagram

Transfer function:

\[
\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Fs + K}
\]

\[
= \frac{K}{J} \left[ s + \frac{F}{2J} + \sqrt{\left( \frac{F}{2J} \right)^2 - \frac{K}{J}} \right] \left[ s + \frac{F}{2J} - \sqrt{\left( \frac{F}{2J} \right)^2 - \frac{K}{J}} \right]
\]
Substitute in the transfer function:

\[
\frac{K}{J} = \omega_n^2
\]

\[
\frac{F}{J} = 2 \zeta \omega_n = 2 \sigma
\]

\[
\zeta = \frac{F}{2 \sqrt{JK}}
\]

\(\zeta\): damping ratio  
\(\omega_n\): undamped natural frequency  
\(\sigma\): stability ratio

to obtain

\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

- **Underdamped** case: \(0 < \zeta < 1\)
  
  \(F^2 - 4JK < 0\) two complex conjugate poles

- **Critically damped** case: \(\zeta = 1\)
  
  \(F^2 - 4JK = 0\) two equal real poles

- **Overdamped** case: \(\zeta > 1\)
  
  \(F^2 - 4JK > 0\) two real poles
Under damped case \((0 < \zeta < 1)\):

\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta \omega_n + j \omega_d)(s + \zeta \omega_n - j \omega_d)}
\]

\[
\omega_d = \omega_n \sqrt{1 - \zeta^2}
\]

\(\omega_n\): undamped natural frequency
\(\omega_d\): damped natural frequency
\(\zeta\): damping ratio
Unit step response:

\[ R(s) = \frac{1}{s} \]

\[ C(s) = \frac{1}{s} - \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_d^2} - \frac{\zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_d^2} \]

\[ c(t) = 1 - e^{-\zeta \omega_n t} \left( \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right) \quad t \geq 0 \]

\[ r \]

\[ c(t) = 1 - \frac{1}{\beta} e^{-\zeta \omega_n t} \sin \left( \omega_n \beta t + \theta \right) \quad t \geq 0 \]

\[ \beta = \sqrt{1 - \zeta^2} \quad \theta = \tan^{-1} \left( \frac{\beta}{\zeta} \right) \]

\[ e(t) = r(t) - c(t) = e^{-\zeta \omega_n t} \left( \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right) \quad t \geq 0 \]

Unit step response curves of a second order system
Undamped case \((\zeta = 0)\):

Unit step response:

\[ c(t) = 1 - \cos \omega_n t \quad t \geq 0 \]

Critically damped case \((\zeta = 1)\):

Unit step Response: \( R(s) = 1/s \)

\[ \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n + \omega_n^2} = \frac{\omega_n^2}{(s + \omega_n)^2} \]

\[ C(s) = \frac{1}{s(s + \omega_n)^2} \]

\[ c(t) = 1 - e^{-\omega_n t} (1 + \omega_n t) \quad t \geq 0 \]
Overdamped case ($\zeta > 1$):

Unit step Response: \[ R(s) = \frac{1}{s} \]

\[
C(s) = \frac{\omega_n^2}{s + \zeta \omega_n + \omega_n \sqrt{\zeta^2 - 1}} \left( \frac{s + \zeta \omega_n - \omega_n \sqrt{\zeta^2 - 1}}{s + \zeta \omega_n + \omega_n \sqrt{\zeta^2 - 1}} \right) \frac{1}{s}
\]

\[
c(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left( \frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2} \right) \quad t \geq 0
\]

with \[ s_1 = \left( \zeta + \sqrt{\zeta^2 - 1} \right) \omega_n \]
\[ s_2 = \left( \zeta - \sqrt{\zeta^2 - 1} \right) \omega_n \]

if \( |s_2| \ll |s_1| \), the transfer function can be approximated by

\[
\frac{C(s)}{R(s)} = \frac{s_2}{s + s_2}
\]

and for \( R(s) = \frac{1}{s} \)

\[ c(t) = 1 - e^{-s_2 t} \quad t \geq 0 \]

with \[ s_2 = \left( \zeta - \sqrt{\zeta^2 - 1} \right) \omega_n \]
Unit step response curves of a critically damped system.
Transient Response Specifications

Unit step response of a 2\textsuperscript{nd} order underdamped system:

- **$t_d$** delay time: time to reach 50\% of $c(\infty)$ for the first time.
- **$t_r$** rise time: time to rise from 0 to 100\% of $c(\infty)$.
- **$t_p$** peak time: time required to reach the first peak.
- **$M_p$** maximum overshoot: \[ \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\% \]
- **$t_s$** settling time: time to reach and stay within a 2\% (or 5\%) tolerance of the final value $c(\infty)$.

$$0.4 < \zeta < 0.8$$

Gives a good step response for an underdamped system
Rise time $t_r$: time from 0 to 100% of $c(\infty)$

\[ c(t_r) = 1 \Rightarrow 1 - e^{-\zeta \omega_d t_r} (\cos \omega_d t_r + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t_r) = 1 \]

\[ \cos \omega_d t_r + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t_r = 0 \]

\[ \tan \omega_d t_r = -\frac{\sqrt{1 - \zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma} \]

\[ t_r = \frac{1}{\omega_d} \tan^{-1}\left(\frac{\omega_d}{\sigma}\right) \]

Peak time $t_p$: time to reach the first peak of $c(t)$

\[ \frac{dc(t)}{dt} \bigg|_{t=t_p} = 0 \Rightarrow (\sin \omega_d t_p) \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t_p} = 0 \]

\[ \sin \omega_d t_p = 0 \]

\[ t_p = \frac{\pi}{\omega_d} \]
**Maximum overshoot** $M_p$:

$$t = t_p = \frac{\pi}{\omega_d}$$

$$M_p = c(t_p) = 1 - e^{-\xi\omega_0(t_p)}(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\pi)$$

$$\frac{\xi\omega_0}{\omega_d} = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{\frac{-\sigma\pi}{\omega_d}}$$

**Settling time** $t_s$:

$$c(t) = 1 - \frac{e^{-\xi\omega_0 t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Approximate $t_s$ using envelope curves:

$$\text{env}(t) = 1 \pm \frac{e^{-\xi\omega_0 t}}{\sqrt{1-\zeta^2}}$$

Pair of envelope curves for the unit-step response curve

2% band: $t_s = \frac{4}{\sigma} = \frac{4}{\xi\omega_n}$

5% band $t_s = \frac{3}{\sigma} = \frac{3}{\xi\omega_n}$
Settling time $t_s$ versus $\zeta$ curves \( \{ T = 1/(\zeta \omega_n) \} \)
Impulse response of second-order systems

\[ C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad R(s) = 1 \]

underdamped case \((0 < \zeta < 1):\)

\[ c(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \omega_n \sqrt{1 - \zeta^2} t \quad t \geq 0 \]

the first peak occurs at \(t = t_0\)

\[ t_0 = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta} \]

and the maximum peak is

\[ c(t_0) = \omega_n \exp \left( -\frac{\zeta}{\sqrt{1 - \zeta^2}} \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta} \right) \]
critically damped case ($\zeta = 1$):

$$c(t) = \omega_n^2 te^{-\omega_n t} \quad t \geq 0$$

overdamped case ($\zeta > 1$):

$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-s_1 t} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-s_2 t} \quad t \geq 0$$

where

$$s_1 = (\zeta - \sqrt{\zeta^2 - 1}) \omega_n$$
$$s_2 = (\zeta + \sqrt{\zeta^2 - 1}) \omega_n$$

Unit-impulse response for 2nd order systems
**Remark:** Impulse Response = d/dt (Step Response)

Relationship between $t_p$, $M_p$ and the unit-impulse response curve of a system

**Unit ramp response of a second order system**

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} \cdot \frac{1}{s^2} \quad R(s) = \frac{1}{s^2}$$

for an underdamped system \((0 < \zeta < 1)\)

$$c(t) = t - \frac{2\zeta}{\omega_n} + e^{-\zeta \omega_n t} \left(\frac{2\zeta}{\omega_n} \cos \omega_d t + \frac{2\zeta^2 - 1}{\omega_n \sqrt{1 - \zeta^2}} \sin \omega_d t\right) \quad t \geq 0$$

and the error:

$$e(t) = r(t) - c(t) = t - c(t)$$

at steady-state:

$$e(\infty) = \lim_{t \to \infty} e(t) = \frac{2\zeta}{\omega_n}$$
Examples:

a. Proportional Control

\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}
\]

with

\[
\frac{K}{J} = \omega_n^2
\]
\[
\frac{F}{J} = 2\zeta\omega_n = 2\sigma
\]
\[
\zeta = \frac{F}{2\sqrt{JK}}
\]

Choose \( K \) to obtain ‘good’ performance for the closed-loop system

For good transient response:

\[ 0.4 < \zeta < 0.8 \quad \rightarrow \quad \text{acceptable overshoot} \]
\[ \omega_n \text{ sufficiently large} \quad \rightarrow \quad \text{good settling time} \]

For small steady-state error in ramp response:

\[
e(\infty) = \lim_{t \to \infty} e(t) = \frac{2\zeta}{\omega_n} = \frac{2F}{2\sqrt{JK\zeta}} \cdot \sqrt{\frac{\zeta}{K}} = \frac{F}{K} \quad \rightarrow \quad \text{large } K
\]

Large \( K \) reduces \( e(\infty) \) but also leads to small \( \zeta \) and large \( M_p \)

\[ \rightarrow \text{compromise necessary} \]
b. Proportional plus derivative control:

\[
C(s) = \frac{K_p + K_ds}{J s^2 + (F + K_d)s + K_p}
\]

with

\[
\zeta = \frac{F + K_d}{2 \sqrt{K_p J}} \quad \omega_n = \sqrt{\frac{K_p}{J}}
\]

The error for a ramp response is:

\[
E(s) = \frac{s^2 J + s F}{s^2 J + s (F + K_d) + K_p} \cdot R(s)
\]

and at steady-state:

\[
e(\infty) = \lim_{s \to 0} s E(s) = \frac{F}{K_p}
\]

using

\[
z = \frac{K_p}{K_d}
\]

\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{z} \cdot \frac{s + z}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

Choose \(K_p, K_d\) to obtain ‘good’ performance of the closed-loop system

For small \textit{steady-state error in ramp response} \(\rightarrow K_p\) large

For good \textit{transient response} \(\rightarrow K_d\) so that \(0.4 < \zeta < 0.8\)
c. Servo mechanism with velocity feedback

Transfer function

\[
\frac{\Theta(s)}{R(s)} = \frac{K}{Js^2 + (F + KK_h)s + K}
\]

where

\[
\zeta = \frac{F + KK_h}{2\sqrt{KJ}}
\]

\[
\omega_n = \sqrt{\frac{K}{J}} \quad \text{(not affected by velocity feedback)}
\]

\[
e(\infty) = \frac{F}{K} \quad \text{for a ramp}
\]

Choose K, K_h to obtain ‘good’ performance for the closed-loop system

For small \textit{steady-state error in ramp response} \rightarrow K \text{ large}

For good \textit{transient response} \rightarrow K_h \text{ so that } 0.4 < \zeta < 0.8

\textbf{Remark:} The damping ratio \( \zeta \) can be increased without affecting the natural frequency \( \omega_n \) in this case.
Effect of a zero in the step response of a 2\textsuperscript{nd} order system

\[ \frac{C(s)}{R(s)} = \frac{\omega_n^2}{z} \cdot \frac{s + z}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \zeta = 0.5 \]

Unit-step response curves of 2\textsuperscript{nd} order systems
Unit step Response of 3rd order systems

\[ \frac{C(s)}{R(s)} = \frac{\omega_n^2 p}{s^2 + 2\zeta \omega_n s + \omega_n^2 (s + p)} \quad 0 < \zeta < 1 \quad R(s) = 1/s \]

\[ c(t) = 1 - \frac{e^{-pt}}{\beta \xi^2 (\beta - 2) + 1} - \frac{e^{-\xi \omega_n t}}{\beta \xi^2 (\beta - 2) + 1} \cdot \left\{ \beta \xi^2 (\beta - 2) \cos(\sqrt{1-\zeta^2} \omega_n t) + \frac{\beta \xi \xi^2 (\beta - 2) + 1}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2} \omega_n t) \right\} \]

where

\[ \beta = \frac{p}{\zeta \omega_n} \]

Unit-step response curves of the third-order system, \( \zeta = 0.5 \)

The effect of the pole at \( s = -p \) is:

- Reducing the maximum overshoot
- Increasing settling time
Transient response of higher-order systems

\[
\frac{C(s)}{R(s)} = \frac{b_0 s^m + \ldots + b_{m-1} s + b_m}{s^n + \ldots + d_{n-1} s + a_n} = \frac{K(s + z_1)(s + z_m)}{(s + p_1)(s + p_n)} \quad n > m
\]

Unit step response

\[
C(s) = \frac{K \sum_{i=1}^{m} (s + z_i)}{\sum_{j=1}^{q} (s + p_j) \sum_{k=1}^{r} (s^2 + 2\zeta_k \omega_k s + \omega_k^2)} \cdot \frac{1}{s}
\]

\(0 < \zeta_k < 1 \quad k=1,\ldots,r \quad \text{and} \quad q + 2r = n\)

\[
C(s) = a + \sum_{j=1}^{q} \frac{a_j}{s + p_j} + \sum_{k=1}^{r} b_k \frac{(s + \zeta_k \omega_k) + c_k \omega_k \sqrt{1 - \zeta_k^2}}{s^2 + 2\zeta_k \omega_k + \omega_k^2}
\]

\[
c(t) = a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} b_k e^{-\zeta_k \omega_k t} \cos \left( \omega_k \sqrt{1 - \zeta_k^2 t} \right)
\]

\[+ \sum_{k=1}^{r} c_k e^{-\zeta_k \omega_k t} \sin \left( \omega_k \sqrt{1 - \zeta_k^2 t} \right) \quad t \geq 0
\]

Dominant poles: the poles closest to the imaginary axis.
STABILITY ANALYSIS

\[
G(s) = \frac{B(s)}{A(s)} = \frac{\sum_{i=0}^{m} b_i s^{m-i}}{\sum_{i=0}^{n} a_i s^{n-i}}
\]

Conditions for Stability:

A. **Necessary** condition for stability:

   All coefficients of A(s) have the same sign.

B. **Necessary and sufficient** condition for stability:

   \[ A(s) \neq 0 \quad \text{for} \quad \text{Re}[s] \geq 0 \]

   or, equivalently

   All poles of G(s) in the left-half-plane (LHP)

Relative stability:

The system is stable and further, all the poles of the system are located in a sub-area of the left-half-plane (LHP).
Necessary condition for stability:

\[ A(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n \]
\[ = a_0 (s + p_1)(s + p_2)\ldots(s + p_n) \]
\[ = a_0 s^n + a_0 (p_1 + p_2 + \ldots p_n) s^{n-1} \]
\[ + a_0 (p_1 p_2 + \ldots + p_{n-1} p_n) s^{n-2} \]
\[ \vdots \]
\[ + a_0 (p_1 p_2 \ldots p_n) \]

-p_1 to -p_n are the poles of the system.

If the system is stable \( \rightarrow \) all poles have negative real parts
\( \rightarrow \) the coefficients of a stable polynomial have the same sign.

Examples:

\[ A(s) = s^3 + s^2 + s + 1 \]
\( \text{can be stable or unstable} \)

\[ A(s) = s^3 - s^2 + s + 1 \]
\( \text{is unstable} \)

Stability testing

Test whether all poles of \( G(s) \) (roots of \( A(s) \)) have negative real parts.

Find all roots of \( A(s) \) \( \rightarrow \) too many computations

Easier Stability test?
# Routh-Hurwitz Stability Test

\[ A(s) = \alpha_0 s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n \]

\[
\begin{array}{cccc}
  s^n & \alpha_0 & \alpha_2 & \alpha_4 & \ldots \\
  s^{n-1} & \alpha_1 & \alpha_3 & \alpha_5 & \ldots \\
  s^{n-1} & b_1 & b_2 & b_3 \\
  & c_1 & c_2 \\
  \ldots \\
  s^2 & e_1 & e_2 \\
  s^1 & f_1 \\
  s^0 & g_1 \\
\end{array}
\]

\[
\begin{align*}
  b_1 &= \frac{1}{-a_1} \left| \begin{array}{cc} a_0 & a_2 \\ a_1 & a_3 \end{array} \right| = \frac{a_1 a_2 - a_0 a_3}{a_1} \\
  b_2 &= \frac{1}{-a_1} \left| \begin{array}{cc} a_0 & a_4 \\ a_1 & a_5 \end{array} \right| = \frac{a_1 a_4 - a_0 a_5}{a_1} \\
  c_1 &= \frac{1}{-b_1} \left| \begin{array}{cc} a_1 & a_3 \\ b_1 & b_2 \end{array} \right| = \frac{a_3 b_1 - a_1 b_2}{b_1} \\
\end{align*}
\]

etc

## Properties of the Ruth-Hurwitz table:

1. Polynomial \( A(s) \) is stable (i.e. all roots of \( A(s) \) have negative real parts) if there is **no sign change in the first column**.

2. The **number of sign changes in the first column** is equal to the number of roots of \( A(s) \) with positive real parts.
Examples:

\[ A(s) = a_0 s^2 + \alpha_1 s + \alpha_2 \]

\[ \begin{array}{ccc}
  s^2 & a_0 & a_2 \\
  s^1 & a_1 \\
  s^0 & a_2 \\
\end{array} \]

\[ \alpha_0 > 0, \quad \alpha_1 > 0, \quad \alpha_2 > 0 \text{ or } \]
\[ \alpha_0 < 0, \quad \alpha_1 < 0, \quad \alpha_2 < 0 \]

For 2\textsuperscript{nd} order systems, the condition that all coefficients of \( A(s) \) have the same sign is necessary and sufficient for stability.

\[ A(s) = \alpha_0 s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 \]

\[ \begin{array}{ccc}
  s^3 & a_0 & a_2 \\
  s^2 & a_1 & a_3 \\
  s^1 & a_1 a_2 - a_0 a_3 \\
  s^0 & a_3 \\
\end{array} \]

\[ \alpha_0 > 0, \quad \alpha_1 > 0, \quad \alpha_3 > 0, \quad \alpha_1 \alpha_2 - \alpha_0 \alpha_3 > 0 \]

(or all first column entries are negative)
Special cases:

1. The properties of the table do not change when all the coefficients of a row are multiplied by the same positive number.

2. If the first-column term becomes zero, replace 0 by $\varepsilon$ and continue.
   - If the signs above and below $\varepsilon$ are the same, then there is a pair of (complex) imaginary roots.
   - If there is a sign change, then there are roots with positive real parts.

Examples:

$$A(s) = s^3 + 2s^2 + s + 2$$

<table>
<thead>
<tr>
<th>$s^3$</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^2$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$s^1$</td>
<td>0 $\to$ $\varepsilon$</td>
<td>$\to$ pair of imaginary roots ($s = \pm j$)</td>
</tr>
<tr>
<td>$s^0$</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

$$A(s) = s^3 - 3s + 2 = (s-1)^2(s+2)$$

<table>
<thead>
<tr>
<th>$s^3$</th>
<th>1</th>
<th>$-3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^2$</td>
<td>0 $\approx$ $\varepsilon$</td>
<td>2</td>
</tr>
<tr>
<td>$s^1$</td>
<td>$-3 \frac{2}{\varepsilon}$</td>
<td>$\to$ two roots with positive real parts</td>
</tr>
<tr>
<td>$s^0$</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
3. If all coefficients in a line become 0, then \( A(s) \) has roots of equal magnitude radially opposed on the real or imaginary axis. Such roots can be obtained from the roots of the auxiliary polynomial.

**Example:**

\[
A(s) = s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50
\]

\[
\begin{array}{ccc}
  s^5 & 1 & 24 & -25 \\
  s^4 & 2 & 48 & -50 \\
  s^3 & 0 & 0 & \rightarrow \text{ auxiliary polynomial } p(s) \\
\end{array}
\]

\[
p(s) = 2s^4 + 48s^2 - 50
\]

\[
\frac{dp(s)}{ds} = 8s^3 + 96s
\]

\[
\begin{array}{ccc}
  s^3 & 8 & 96 \\
  s^2 & 24 & -50 \\
  s^1 & 112.7 & 0 \\
  s^0 & -50 & \\
\end{array}
\]

- \( A(s) \) has two radially opposed root pairs \((+1,-1)\) and \((+5j,-5j)\) which can be obtained from the roots of \( p(s) \).

- One sign change indicates \( A(s) \) has one root with positive real part.

Note:

\[
A(s) = (s+1)(s-1)(s+5j)(s-5j)(s+2)
\]

\[
p(s) = 2(s^2 - 1) (s^2 + 25)
\]
Relative stability

Question: Have all the roots of $A(s)$ a distance of at least $\sigma$ from the imaginary axis?

Substitute $s$ with $s = z - \frac{1}{G(s)}$ and apply the Routh-Hurwitz test to $A(z)$.

Closed-loop System Stability Analysis

Question: For what value of $K$ is the closed-loop system stable?

Apply the Routh-Hurwitz test to the denominator polynomial of the closed-loop transfer function $\frac{KG(s)}{1 + KG(s)}$. 

Steady-State Error Analysis

Evaluate the steady-state performance of the closed-loop system using the steady-state error \( e_{ss} \)

\[
e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)
\]

\[
E(s) = \frac{1}{1 + G(s)H(s)} \cdot R(s)
\]

for the following input signals:  
- Unit step input  
- Unit ramp input  
- Unit parabolic input

**Assumption:** the closed-loop system is stable

**Question:** How can we obtain the steady-state error \( e_{ss} \) of the closed-loop system from the open-loop transfer function \( G(s)H(s) \)?
**Classification of systems:**

For an open-loop transfer function

\[ G(s)H(s) = \frac{K(T_1s + 1)(T_2s + 1)\cdots}{s^N(T_1s + 1)(T_2s + 1)\cdots} \]

**Type of system:** Number of poles at the origin, i.e., \( N \)

**Static Error Constants:** \( K_p, K_v, K_a \)

Open-loop transfer function: \( G(s)H(s) \)

Closed-loop transfer function: \( G_{\text{tot}}(s) = \frac{G(s)}{1 + G(s)H(s)} \)

**Static Position Error Constant:** \( K_p \)

Unit step input to the closed-loop system shown in fig, p. B33.

\[ R(s) = \frac{1}{s} \]

\[ e_{ss} = \lim_{s \to 0} sE(s) = \frac{1}{1 + G(0)H(0)} \]

**Define:** \( K_p = \lim_{s \to 0} G(s)H(s) = G(0)H(0) \)

Type 0 system \( K_p = K \)

Type 1 and higher \( K_p = \infty \)

\[ e_{ss} = \frac{1}{1 + K_p} \]

\[ e_{ss} = 0 \]
**Static Velocity Error Constant:** $K_v$

Unit ramp input to the closed-loop system shown if fig, p. B33.

$$R(s) = \frac{1}{s^2} \quad e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)H(s)} \cdot \frac{1}{s^2} = \lim_{s \to 0} \frac{1}{sG(s)H(s)}$$

**Define:**

$$K_v = \lim_{s \to 0} sG(s)H(s)$$

Type 0 system $K_v = 0 \quad e_{ss} = \infty$

Type 1 system $K_v = K \quad e_{ss} = 1/K_v$

Type 2 and higher $K_v = \infty \quad e_{ss} = 0$

**Static Acceleration Error Constant:** $K_a$

Unit parabolic input to the closed-loop system shown in fig, p. B33

$$R(s) = \frac{1}{s^3} \quad e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)H(s)} \cdot \frac{1}{s^3} = \lim_{s \to 0} \frac{1}{s^2G(s)H(s)}$$

**Define:**

$$K_a = \lim_{s \to 0} s^2 H(s)G(s)$$

Type 0 system $K_a = 0 \quad e_{ss} = \infty$

Type 1 system $K_a = 0 \quad e_{ss} = \infty$

Type 2 system $K_a = K \quad e_{ss} = 1/ K_a$

Type 3 and higher $K_a = \infty \quad e_{ss} = 0$
Summary:

Consider a closed-loop system:

![Closed-loop system diagram]

with an open-loop transfer function:

\[ G(s)H(s) = \frac{K(T_a s + 1) \cdot (T_b s + 1) \cdots}{s^N (T_1 s + 1) \cdot (T_2 s + 1) \cdots} \]

and static error constants defined as:

\[ K_p = \lim_{s \to 0} G(s)H(s) = G(0)H(0) \]
\[ K_v = \lim_{s \to 0} sG(s)H(s) \]
\[ K_a = \lim_{s \to 0} s^2 H(s)G(s) \]

The steady-state error \( e_{ss} \) is given by:

<table>
<thead>
<tr>
<th>Type</th>
<th>Unit step ( r(t) = 1 )</th>
<th>Unit ramp ( r(t) = t )</th>
<th>Unit parabolic ( r(t) = t^2/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 0</td>
<td>( e_{ss} = \frac{1}{1 + K_p} (= \frac{1}{1 + K}) )</td>
<td>( e_{ss} = \infty )</td>
<td>( e_{ss} = \infty )</td>
</tr>
<tr>
<td>Type 1</td>
<td>( e_{ss} = 0 )</td>
<td>( e_{ss} = \frac{1}{K_v} (= \frac{1}{K}) )</td>
<td>( e_{ss} = \infty )</td>
</tr>
<tr>
<td>Type 2</td>
<td>( e_{ss} = 0 )</td>
<td>( e_{ss} = 0 )</td>
<td>( e_{ss} = \frac{1}{K_a} (= \frac{1}{K}) )</td>
</tr>
</tbody>
</table>
Correlation between the Integral of error in step response and Steady-state error in ramp response

\[ E(s) = L[e(t)] = \int_{0}^{\infty} e^{-st} e(t)dt \]

\[ \lim_{s \to 0} E(s) = \lim_{s \to 0} \int_{0}^{\infty} e^{-st} e(t)dt = \int_{0}^{\infty} e(t)dt \]

substitute \( E(s) = \frac{R(s)}{1 + G(s)} \) in the above eq.

\[ \lim_{s \to 0} \frac{R(s)}{1 + G(s)} = \int_{0}^{\infty} e(t)dt \]

\[ \lim_{s \to 0} \frac{1}{1 + G(s)} \cdot \frac{1}{s} = \lim_{s \to 0} \frac{1}{s} \cdot G(s) = \frac{1}{K_v} \]

\( \frac{1}{K_v} = \text{Steady-state error in unit-ramp input} = e_{ssr} \)

\[ e_{ssr} = \int_{0}^{\infty} e(t)dt \]
\[ \int_{0}^{\infty} e(t) dt \]

Diagram showing:
- \( c(t) \) as the cumulative function.
- \( r(t) \) as a linear function.
- \( e_{ssr} \) as the error signal.