ON THE ULTRASPHERICAL FAMILY

OF

WINDOW FUNCTIONS

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OUTLINE

- **Part I: Fourier Series, Gibbs Phenomenon, and Window Functions**
- **Part II: The Ultraspherical Window and Spectral Characteristics**
- **Part III: Nonrecursive Digital Filter Design Using the Ultraspherical Window**

MOTIVATION

- **Windows are time-domain weighting functions that are used to reduce Gibbs' oscillations that are caused by the truncation of a Fourier series.**
- **They are employed in a variety of traditional applications including power spectral estimation, beamforming, and digital filter design.**
- **More recently, windows have been used in conjunction with electrocardiograms to facilitate the detection of irregular and abnormal heartbeat patterns in patients.**
- **Medical imaging systems, such as the ultrasound, have also shown enhanced performance when windows are used to improve the contrast resolution of the system.**
- **Windows have also been employed to aid in the classification of cosmic data and to improve the reliability of weather prediction models.**

FOURIER SERIES

- **A periodic function $x(t)$ defined over the interval $[-\tau_0/2, \tau_0/2]$ can be represented by the Fourier series**

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos \omega_0 t + b_k \sin k\omega_0 t)$$

where

$$a_k = \frac{1}{\pi}\int_{-\tau_0/2}^{\tau_0/2} x(t)\cos k\omega_0 t \; dt \quad \text{and} \quad b_k = \frac{1}{\pi}\int_{-\tau_0/2}^{\tau_0/2} x(t)\sin k\omega_0 t \; dt$$

- **Alternatively, the Fourier series of $x(t)$ can be expressed as**

$$x(t) = \sum_{k=-\infty}^{\infty} A_k e^{jk\omega_0 t} \quad \text{where} \quad A_k = \frac{1}{2\pi}\int_{-\tau_0/2}^{\tau_0/2} x(t)e^{-jk\omega_0 t}dt$$

- **The two representations are interrelated in terms of the following formula**

$$A_k = \begin{cases} \frac{1}{2}(a_k + jb_k) & \text{for } k < 0 \\ \frac{1}{2}a_0 & \text{for } k = 0 \\ \frac{1}{2}(a_k - jb_k) & \text{for } k > 0 \end{cases}$$

EXAMPLE

Consider the function

$$x(t) = \begin{cases} 0 & \text{for } -\pi \le t < -\pi/2 \\ 1 & \text{for } -\pi/2 < t < \pi/2 \\ 0 & \text{for } \pi/2 < t \le \pi \end{cases}$$

Since $x(t)$ is an even function, we have

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\cos kt\ dt = \frac{2}{\pi}\int_{0}^{\pi/2} \cos kt\ dt \\ &= \frac{2}{\pi k}\sin\frac{k\pi}{2} \\ a_0 &= \frac{2}{\pi}\int_{0}^{\pi/2} dt = 1 \end{aligned}$$

and $b_k = 0$. Hence,

$$\begin{aligned} x(t) &= \frac{1}{2} + \frac{2}{\pi}\left[\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - ...\right] \\ &= \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\cos(2k+1)t}{(2k+1)} \end{aligned}$$

Jean Baptiste Joseph Fourier [1]

(1768-1830)

Jean Baptiste Joseph Fourier *Cont'd*

- **Fourier was a French mathematician who had Lagrange and Laplace as teachers.**

  His interests included mechanics and heat transfer.

- **At 19 he began studying to become a priest but changed his mind before too long.**

- **He got himself involved with the French Revolution and on two occasions he was imprisoned.**

- **In due course, he joined Napoleon's army in its invasion of Egypt as a scientific advisor.**

- **Fourier returned form Egypt in 1801 to find himself appointed by Napoleon as Prefect (Chief Administrative Officer) stationed in Grenoble.**

  His achievements as Prefect included draining the swamps of Bourgoin and supervising the construction of a new highway from Grenoble to Turin!

- **Fourier most likely did not like this assignment but who could refuse Napoleon — his power was absolute in 1801!**

Jean Baptiste Joseph Fourier *Cont'd*

- It was during his time at Grenoble that he developed his ideas on the Fourier series.

  He presented his work in a treatise (memoir) titled *On the Propagation of Heat in Solid Bodies* in 1807.

- He claimed that an arbitrary function defined within a finite interval can always be expressed as a sum of sinusoids.

- The treatise was reviewed by Lagrange, Laplace, and two others but it was rejected for the following reasons:

  - Not original enough.
  - A certain Biot claimed that Fourier did not refer to Biot's work on the derivation of certain heat transfer equations.

  NOTE: Biot's derivation is now known to be in error!

- In 1811, Fourier submitted his treatise along with some other work in a competition on the propagation of heat in solid bodies.

  The competition had only one other candidate and it was judged by Lagrange, Laplace, Legendre, and two others.

  Fourier won the prize but the review included the following:

  ... the manner in which the author arrives at these equations is not exempt of difficulties and that his analysis to integrate them still leaves something to be desired on the score of generality and even rigour ...

- After much controversy, the Academie of Sciences published Fourier's prize winning essay titled *Theorie Analytique de la Chaleur*.

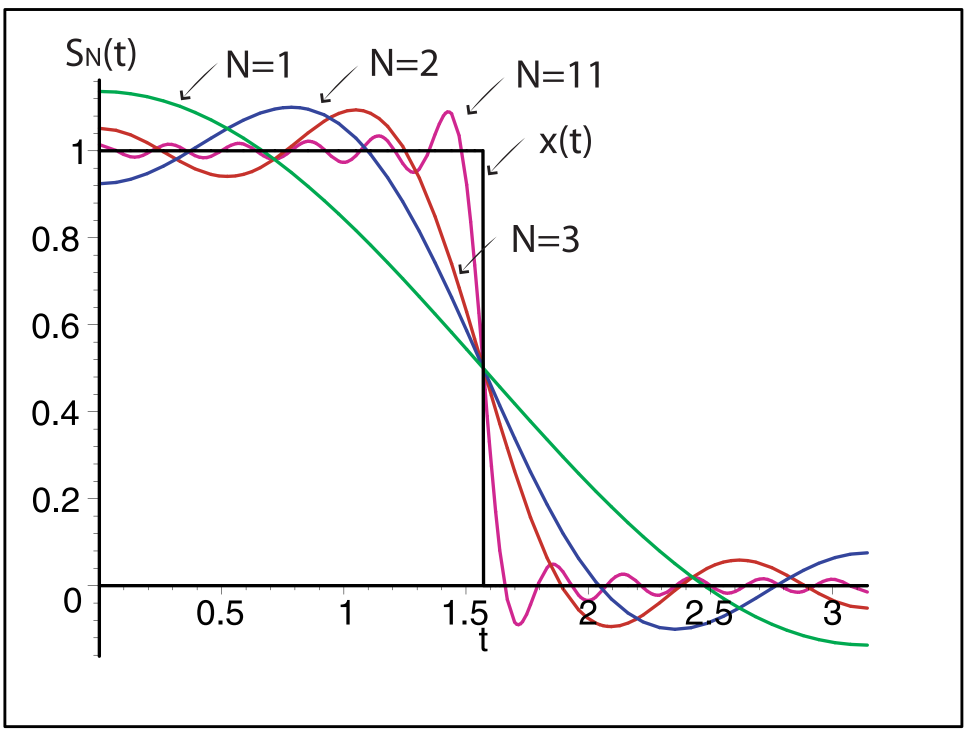
  And the rest is history.

GIBBS' PHENOMENON

- When a Fourier series is truncated, it will exhibit certain oscillations known as ***Gibbs' oscillations***.
- From Example on Foil 5, the partial sum is given by

$$S_N(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{N} (-1)^{k-1} \frac{\cos(2k-1)t}{(2k-1)}.$$

- Gibbs' oscillations are most pronounced near discontinuities and due to the slow convergence of the Fourier series.
- The amplitude of Gibbs oscillations tends to be independent of the number of terms retained in the Fourier series.



EARLY SMOOTHING

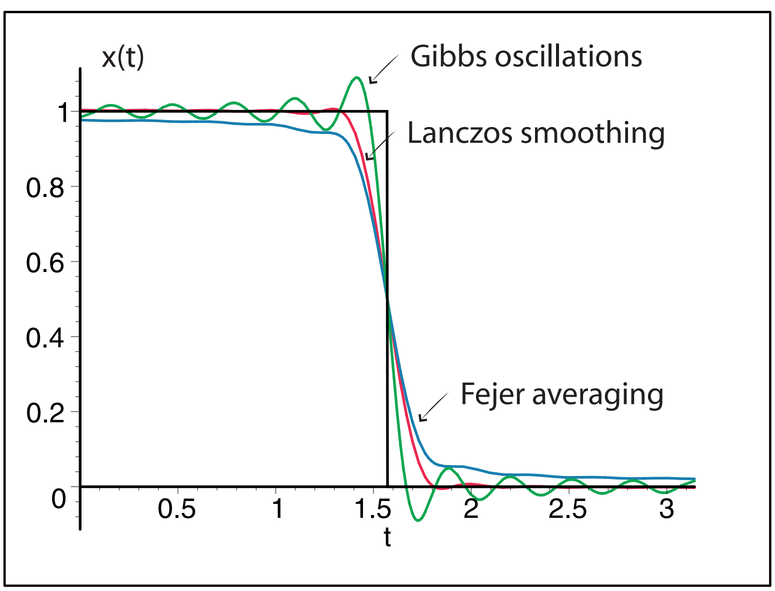
- **First, Lipot Fejer suggested averaging the nth partial sum of the truncated Fourier series [2].**
- **Next, Cornelius Lanczos observed that the ripple of the truncated Fourier series had the period of either the first term neglected or the last term kept [3].**

  He argued that smoothing the partial sum over this period would reduce the ripple.
- **These methods can be applied by using the multiplicative factors as follows:**

$$A(N,k) = \begin{cases} \frac{N-k}{N} & \textbf{Fejer Averaging} \\ \dfrac{\sin \pi k/N}{\pi k/N} & \textbf{Lanczos Smoothing} \end{cases}$$

to the truncated Fourier series such that

$$S_N(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} A(N,k)[a_k \cos k\omega_0 t + b_k \sin k\omega_0 t]$$



WINDOW FUNCTIONS

- **A more comprehensive view of the truncation and smoothing operations can be observed through the use of window functions.**
- **The truncated Fourier series can be obtained by assigning**

$$A_n = 0 \quad \text{for } |n| > N$$

in the Fourier series

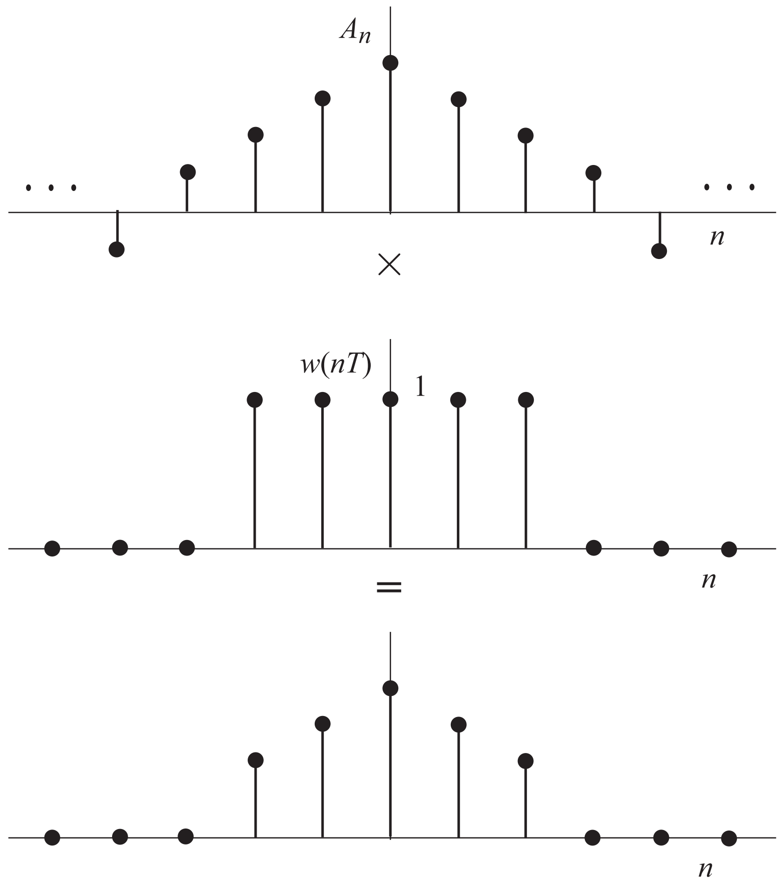
$$x(t) = \sum_{n=-\infty}^{\infty} A_n e^{jn\omega_0 t}$$

- **This operation is accomplished by using the multiplicative factor**

$$w_R(nT) = \begin{cases} 1 & \text{for } |n| \leq N \\ 0 & \text{otherwise} \end{cases}$$

which is said to be the *rectangular window function*.

WINDOW FUNCTIONS *Cont'd*



SPECTRAL CHARACTERISTICS

- **Windows are often quantified in terms of their spectral characteristics.**
- **The spectral representation for a window** $w(nT)$ **of length** $N = 2M + 1$ **defined over the range** $-M \le n \le M$ **is given by the** z **transform of** $w(nT)$ **evaluated on the unit-circle of the** z **plane, i.e.,**

$$W(e^{j\omega T}) = \sum_{n=-M}^{M} w(nT)e^{-j\omega nT}$$

- **The frequency spectrum is given by**

$$W(e^{j\omega T}) = e^{-j\omega MT} W_0(e^{j\omega T})$$

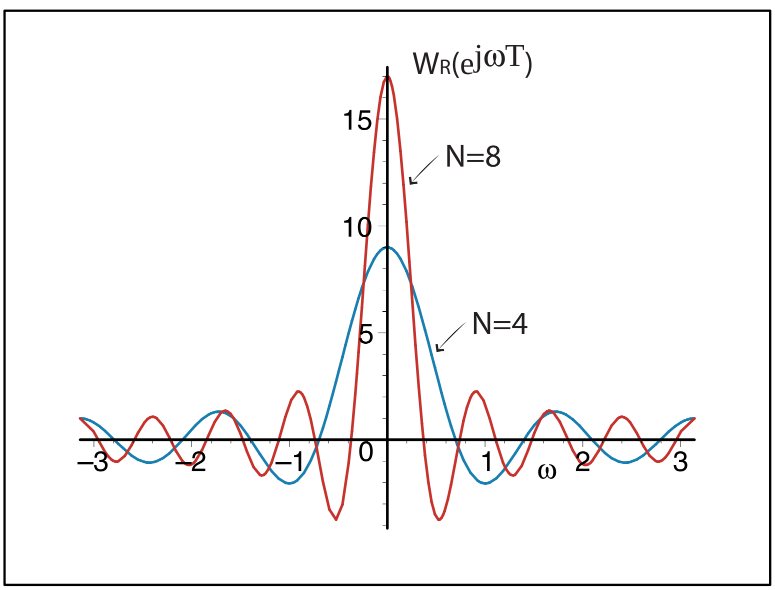
  where $W_0(e^{j\omega T})$ **is called the *amplitude function*.**

- $A(\omega) = |W_0(e^{j\omega T})|$ **and** $\theta(\omega) = -\omega MT$ **are called the *amplitude and phase spectrums*, respectively.**
- $|W_0(e^{j\omega T})|/W_0(0)$ **is a normalized version of the amplitude spectrum.**

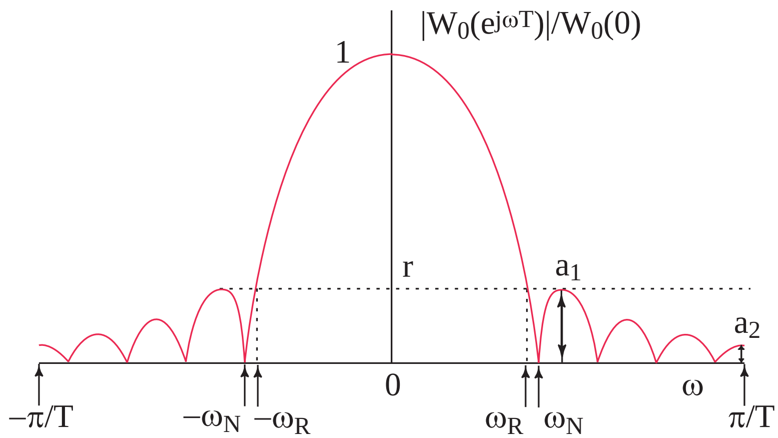
EXAMPLE

The frequency spectrum of a rectangular window of length $N = 2M + 1$ is given by

$$\begin{aligned} W_R(e^{j\omega T}) &= \sum_{n=-M}^{M} e^{-j\omega nT} \\ &= e^{j\omega MT} + e^{j\omega(M-1)T} + \cdots + e^{-j\omega(M-1)T} + e^{-j\omega MT} \\ &= \frac{e^{j\omega MT} - e^{-j\omega(M+1)T}}{1 - e^{-j\omega T}} \\ &= \frac{e^{j\omega(2M+1)T/2} - e^{-j\omega(2M+1)T/2}}{e^{j\omega T/2} - e^{-j\omega T/2}} \\ &= \frac{\sin(\omega NT/2)}{\sin \omega T/2} \end{aligned}$$



SPECTRAL CHARACTERISTICS *Cont'd*



- **The null-to-null width B_n and the main-lobe width B_r are defined by $B_n = 2\omega_n$ and $B_r = 2\omega_r$.**
- **The ripple ratio r is defined as**

$$r = \frac{\textbf{maximum side-lobe amplitude}}{\textbf{main-lobe amplitude}}.$$

- **The side-lobe roll-off ratio s which is defined as**

$$s = \frac{a_1}{a_2}$$

For the side-lobe roll-off ratio to have meaning, the envelope of the side-lobe pattern should be monotonically increasing or decreasing.

CONVOLUTION

- **The z transform of two discrete-time signals is equal to the complex convolution of the z transforms of the two signals.**
- **Evaluating the complex convolution on the unit circle of the z plane yields**

$$X_w(e^{j\omega T}) = \frac{T}{2\pi} \int_0^{2\pi/T} X(e^{j\varpi T}) W(e^{j(\omega-\varpi)T})\ d\varpi$$

which is the convolution of the frequency spectrums of the two windows.

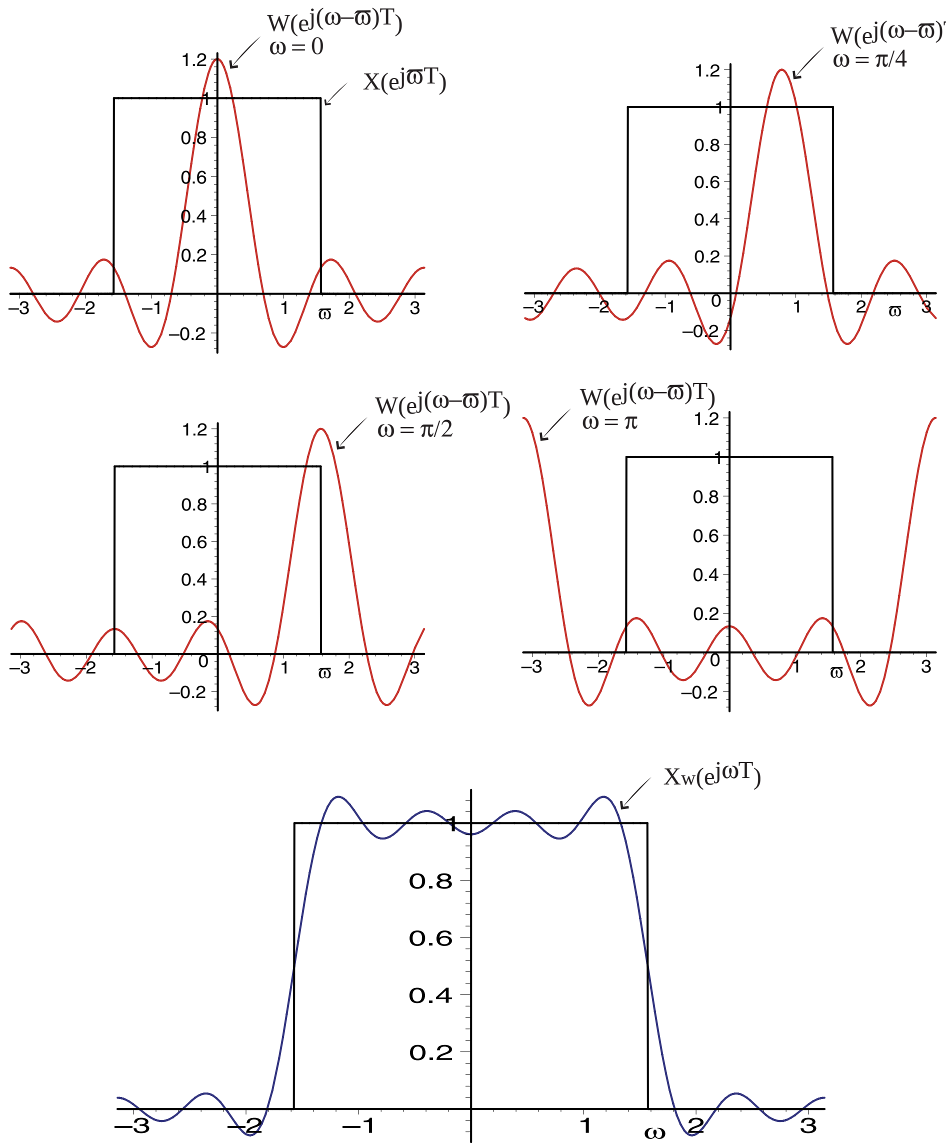
- **The effects of a window on a signal can be illustrated by considering a signal $x(t)$ with the frequency spectrum**

$$X(e^{j\omega T}) = \begin{cases} 1 & \text{for } -\pi/2 \le \omega \le \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

and some arbitrary window with a spectrum $W(e^{j\omega T})$.

CONVOLUTION *Cont'd*

$$X_w(e^{j\omega T}) = \frac{T}{2\pi}\int_0^{2\pi/T} X(e^{j\varpi T})W(e^{j(\omega-\varpi)T})\ d\varpi$$



CONVOLUTION *Cont'd*

- **The side ripples in the spectrum of the window cause ripples in the spectrum of the signal whose amplitude is increases with the ripple ratio.**
- **The main-lobe width causes transition bands in $X_w(e^{j\omega T})$ whose width is directly proportional to the main-lobe width.**

STANDARD WINDOWS

Many window functions have been proposed over the years.

- **triangular**
- **Blackman**
- **von Hann**
- **Hamming**
- **Kaiser**
- **Dolph-Chebyshev**
- **Saramäki**
- **Ultraspherical**

STANDARD WINDOWS *Cont'd*

Standard windows can be *fixed or adjustable*.

- **Fixed windows have one parameter, namely, the window length which controls the main-lobe width.**
- **Adjustable windows have two or more independent parameters, namely, the window length, as in fixed windows, and one or more additional parameters that can control other window characteristics.**
- **The and Saramäki windows [4][5] have two parameters and achieve close approximations to discrete prolate functions that have maximum energy concentration in the main lobe relative to that in the side lobes .**
- **The Dolph-Chebyshev window [6] has two parameters and produces the minimum main-lobe width for a specified maximum side-lobe level.**

THE ULTRASPHERICAL WINDOW

- **The ultraspherical window has three parameters, namely, the window length, as in fixed windows, and two additional parameters [7].**
- **The window can control the width of the main lobe and the relative amplitude of the side lobes, as in the Kaiser, Saramäki, and Dolph-Chebyshev windows and, in addition, arbitrary side-lobe patterns can be achieved.**

THE ULTRASPHERICAL WINDOW *Cont'd*

The coefficients of the ultraspherical window of length $N = 2M + 1$ are calculated as [8]

$$w(nT) = \begin{cases} \widehat{w}(nT)/\widehat{w}(AT) & \textbf{for } |n| \leq M \\ 0 & \textbf{otherwise} \end{cases} \tag{A}$$

with $A = 0$ and 0.5 for odd and even N, respectively,

$$\begin{aligned} \widehat{w}(nT) &= \frac{\mu x_\mu^{2M}}{M + |n|} \binom{\mu + M + |n| - 1}{M + |n| - 1} \\ &\quad \cdot \sum_{m=0}^{M-|n|} \binom{\mu + M - |n| - 1}{M - |n| - m} \binom{M + |n|}{m} B^m \end{aligned}$$

where μ, x_μ, and N are independent parameters, $B = 1 - x_\mu^{-2}$, and the binomial coefficients $\binom{\alpha}{p}$ can be calculated using the following formulas:

$$\binom{\alpha}{0} = 1; \qquad \binom{\alpha}{p} = \frac{\alpha(\alpha - 1) \cdots (\alpha - p + 1)}{p!} \quad \textbf{for } p \geq 1$$

THE ULTRASPHERICAL WINDOW *Cont'd*

- **The amplitude function for the ultraspherical window is given by**

$$W_0(e^{j\omega T}) = C^{\mu}_{N-1}\left[x_\mu \cos(\omega T/2)\right]$$

where $C^{\mu}_{n}(x)$ is the ultraspherical polynomial which can be calculated using the recurrence relationship

$$C^{\mu}_{r}(x) = \frac{1}{r}\left[2x(r+\mu-1)C^{\mu}_{r-1}(x) - (r+2\mu-2)C^{\mu}_{r-2}(x)\right]$$

for $r = 2, 3, \ldots, n$, where $C^{\mu}_{0}(x) = 1$ and $C^{\mu}_{1}(x) = 2\mu x$.

- **The Dolph-Chebyshev window is the special case for $\mu = 0$, which results in**

$$W_0(e^{j\omega T}) = T_{N-1}\left[x_\mu \cos(\omega T/2)\right]$$

where

$$T_n(x) = \cos(n \cos^{-1} x)$$

is the Chebyshev polynomial of the first kind.

- **The Saramäki window is the special case for $\mu = 1$, which results in**

$$W_0(e^{j\omega T}) = U_{N-1}\left[x_\mu \cos(\omega T/2)\right]$$

where

$$U_n(x) = \frac{\sin[(n+1)\cos^{-1} x]}{\sin(\cos^{-1} x)}$$

is the Chebyshev polynomial of the second kind.

PRESCRIBED SPECTRAL CHARACTERISTICS

- **With the appropriate selection of parameters μ, x_μ, and N, ultraspherical windows can be designed to achieve prescribed**
  - **side-lobe roll-off ratio,**
  - **ripple ratio, and**
  - **one of the two width characteristics simultaneously [9].**
- **Parameter μ alters the side-lobe roll-off ratio, x_μ alters the ripple ratio, and N alters the main-lobe width.**
- **The next few transparencies explain how each specification can be achieved.**

PRESCRIBED SIDE-LOBE ROLL-OFF RATIO

- **To achieve a prescribed side-lobe roll-off ratio s, one selects the parameter μ appropriately for a fixed N by solving**

$$\underset{\mu_L \leq \mu \leq \mu_H}{\text{minimize}}\, F = \left(s - \left| \frac{C^{\mu}_{N-1}\left(x^{(\mu+1)}_{N-2}\right)}{C^{\mu}_{N-1}(0)} \right| \right)^2$$

- **The upper and lower bounds are**

$$\begin{array}{ll} \mu_L = 0 \text{ and } \mu_U = 10 & \text{for } s > 1 \\ \mu_L = -0.9999 \text{ and } \mu_U = 0 & \text{for } 0 < s < 1 \end{array}$$

- **The parameter $x^{(\mu+1)}_{N-2}$, which is the largest zero of $C^{\mu+1}_{N+2}(x)$, is found using Algorithm 1 (see next transparency) with the inputs $\lambda = \mu + 1$, $n = N - 2$, and $\varepsilon = 10^{-6}$.**

LARGEST ZERO OF $C_n^\lambda(x)$

The largest zero of $C_n^\lambda(x)$, denoted as $x_n^{(\lambda)}$, can be found using the following algorithm.

Algorithm 1

- ***Step 1***
  Input λ, n, **and** ε.
  If $\lambda = 0$, **then output** $x^* = \cos(\pi/2n)$ **and stop.**
  Set $k = 1$, **and compute**

$$y_1 = \frac{\sqrt{n^2 + 2n\lambda - 2\lambda - 1}}{n + \lambda}$$

- ***Step 2***
  Compute

$$y_{k+1} = y_k - \frac{C_n^\lambda(y_k)}{2\lambda C_{n-1}^{\lambda+1}(y_k)}$$

- ***Step 3***
  If $|y_{k+1} - y_k| \leq \varepsilon$, **then output** $x^* = y_{k+1}$ **and stop.**
  Set $k = k + 1$, **and repeat from Step 2.**

NOTES:

- **A termination tolerance $\varepsilon = 10^{-6}$ causes the algorithm to converge in 5 or 6 iterations.**
- **The algorithm uses the Newton-Raphson method as a line search because of its simplicity and efficiency but many other methods can also be used.**

PRESCRIBED NULL-TO-NULL WIDTH

- **To achieve a prescribed null-to-null half width of ω_n rad/s, one selects the parameter x_μ appropriately for a fixed μ and N using**

$$x_\mu = \frac{x_{N-1}^{(\mu)}}{\cos(\omega_n/2)}$$

- **The zero $x_{N-1}^{(\mu)}$ is found using Algorithm 1 with the inputs $\lambda = \mu$, $n = N - 1$, and $\varepsilon = 10^{-6}$.**

PRESCRIBED MAIN-LOBE WIDTH

- **To achieve a prescribed main-lobe half width of ω_r rad/s, one selects the parameter x_μ appropriately for a fixed μ and N using**

$$x_\mu = \frac{x_a}{\cos(\omega_r/2)}$$

- **Parameter x_a is found through a two-step process:**
  - **Find the zero $x_{N-2}^{(\mu+1)}$ using Algorithm 1 with the inputs $\lambda = \mu + 1$, $n = N - 2$, and $\varepsilon = 10^{-6}$ and then calculate the parameter $a = \left| C_{N-1}^{\mu}\left(x_{N-2}^{(\mu+1)}\right)\right|$.**
  - **Find x_a using a modified version of Algorithm 1 where the second equation is replaced with**

$$y_{k+1} = y_k - \frac{C_n^\mu(y_k) - a}{2\mu C_{n-1}^{\mu+1}(y_k)}$$

which uses the inputs $\lambda = \mu$, $n = N - 1$, and $\varepsilon = 10^{-6}$.

NOTE: **Instead of finding the largest zero of $f(x) = C_n^\mu(x)$, the modified algorithm finds the largest zero of $f(x) = C_n^\mu(x) - a$, which is parameter x_a.**

PRESCRIBED RIPPLE RATIO

- **To achieve a prescribed ripple ratio r, one selects the parameter x_μ appropriately for a fixed μ and N are fixed using a two-step process:**
  - **Find the zero $x_{N-2}^{(\mu+1)}$ using Algorithm 1 with the inputs $\lambda = \mu + 1$, $n = N - 2$, and $\varepsilon = 10^{-6}$ and then calculate the parameter $a = \left| C_{N-1}^{\mu}\left(x_{N-2}^{(\mu+1)}\right)\right|$.**
  - **Find x_μ using a modified version of Algorithm 1 where the second equation is replaced with**

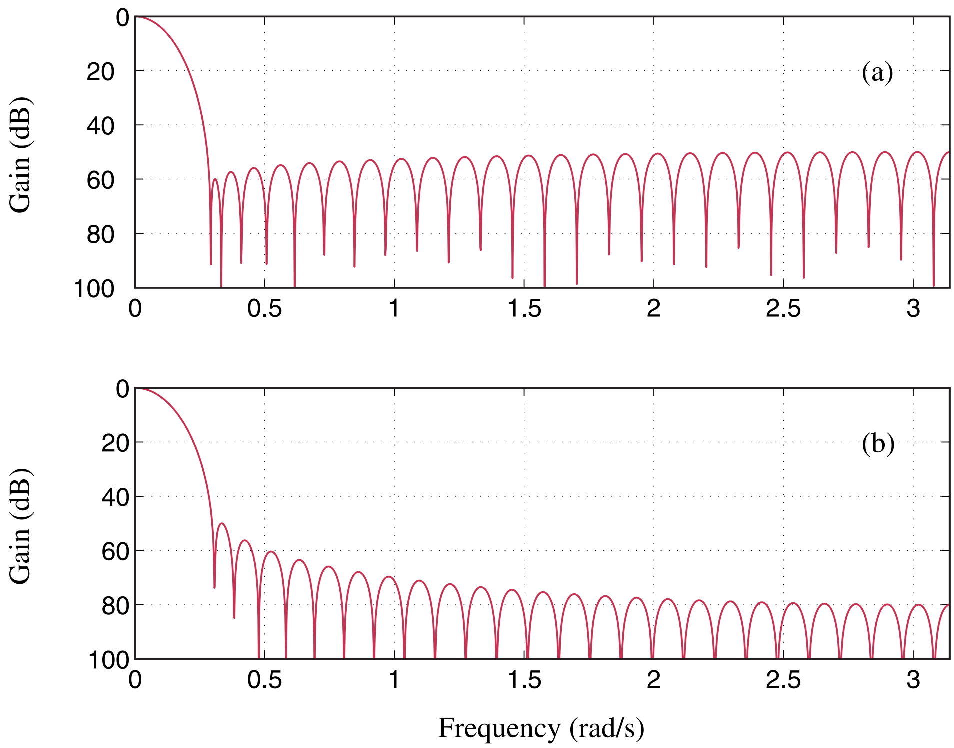
$$y_{k+1} = y_k - \frac{C_n^\mu(y_k) - a/r}{2\mu C_{n-1}^{\mu+1}(y_k)}$$

which uses the inputs $\lambda = \mu$, $n = N - 1$, and $\varepsilon = 10^{-6}$.

NOTE: **Instead of finding the largest zero of $f(x) = C_n^\mu(x)$, the modified algorithm finds the largest zero of $f(x) = C_n^\mu(x) - a/r$ which is the parameter x_μ.**

PRESCRIBED SPECIFICATIONS — EXAMPLE

For $N = 51$, generate the ultraspherical windows that will yield $R = 50$ dB for (a) $S = -10$ dB and (b) $S = 30$ dB.



Both designs meet the prescribed specifications and produced main-lobe widths of (a) $\omega_r = 0.2783$ rad/s and (b) $\omega_r = 0.2975$ rad/s.

Using the methods described resulted in (a) $\mu = -0.3914$ and (b) $\mu = 1.5151$ and (a) $x_\mu = 1.0107$ and (b) $x_\mu = 1.0091$.

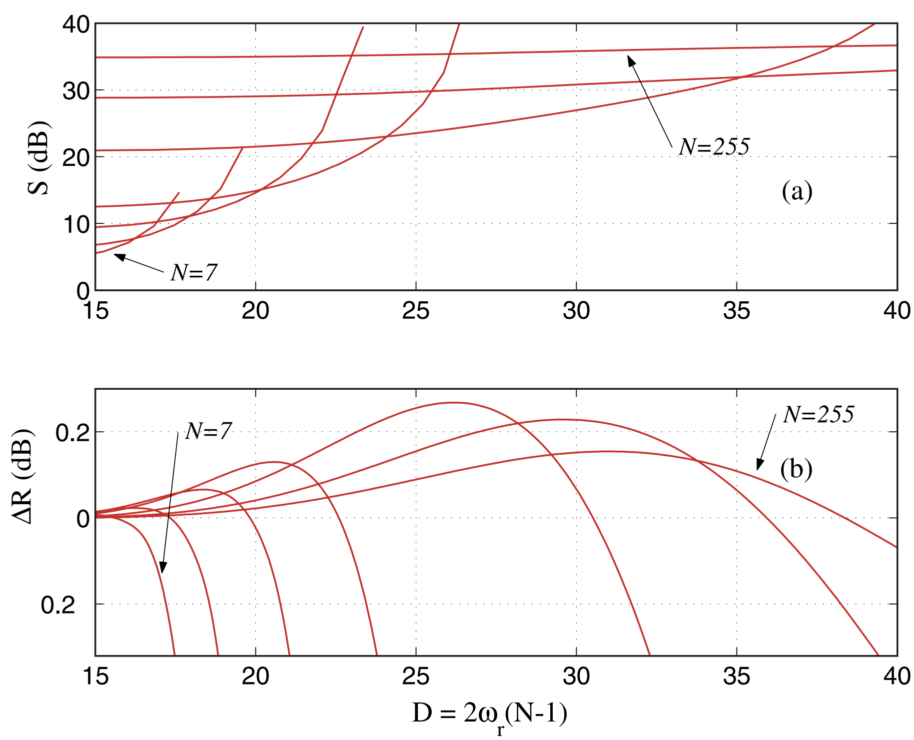
COMPARISON OF ULTRASPHERICAL WITH OTHER WINDOWS

- **Ultraspherical windows of the same length were designed to achieve the side-lobe roll-off ratio and main-lobe width produced by the Kaiser window, for values of the Kaiser-window parameter α in the range $[1, 10]$.**

- **The resulting ripple ratios for the two window families were measured and compared using**

$$\Delta R = R_U - R_K$$

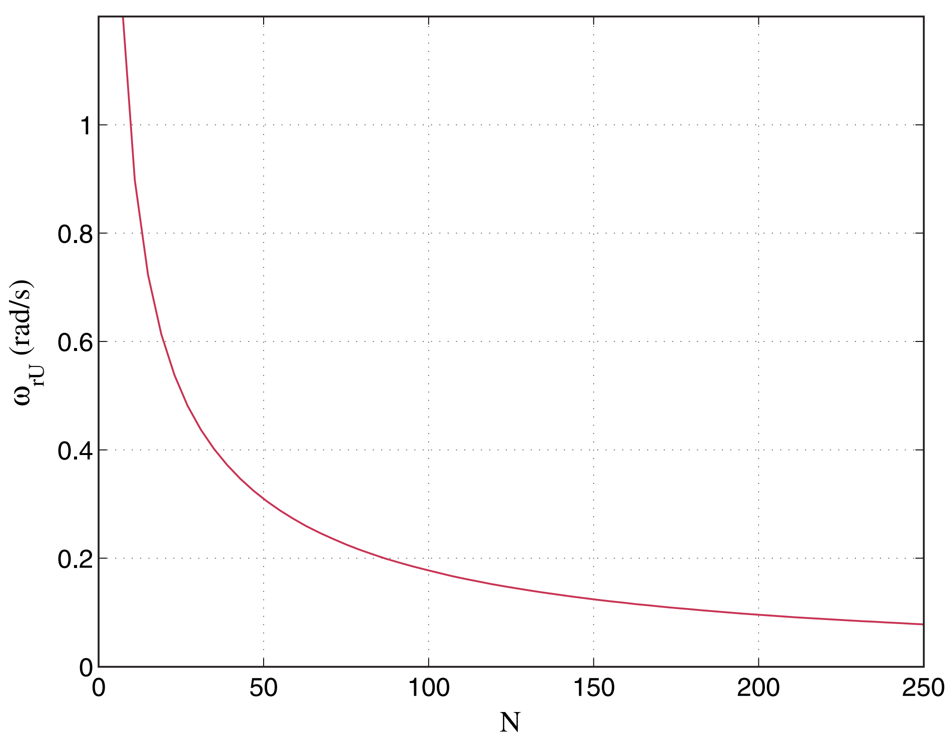
where R_U and R_K are the ripple ratios of the ultraspherical and Kaiser windows, respectively, in dB.



COMPARISON *Cont'd*

- **Thus for a given window length, there is a corresponding main-lobe half width, say, ω_{rU}, for which the ultraspherical window gives a better ripple ratio than the Kaiser window.**
- **For main-lobe half widths that are larger than ω_{rU}, the Kaiser window gives a larger ripple ratio.**

  In effect, if the point $[N,\ \omega_r]$ is located below the curve, the ultraspherical window is preferred and if it is located above the curve, the Kaiser window is preferred.



DESIGN OF NONRECURSIVE LOWPASS FILTERS

The design of nonrecursive filters involves four general steps as follows:

1. **An idealized frequency response is assumed and through the use of the Fourier series, an idealized infinite-duration noncausal design is obtained.**
2. **A suitable window is selected and the parameters of the window are chosen to achieve the desired filter specifications.**
3. **The window function is constructed and applied.**
4. **The resulting finite-duration noncausal filter is converted into a causal filter.**

DESIGN OF NONRECURSIVE FILTERS *Cont'd*

Infinite-duration impulse response

The infinite-duration impulse response of the noncausal lowpass filter is obtained by applying the Fourier series to the idealized frequency response

$$H(e^{j\omega T}) = \begin{cases} 1 & \textbf{for } |\omega| \le \omega_c \\ 0 & \textbf{for } \omega_c < |\omega| \le \omega_s/2 \end{cases} \tag{B}$$

Straightforward analysis gives

$$h_{id}(nT) = \begin{cases} \omega_c/\pi & \text{for } n = 0 \\ \dfrac{\sin \omega_c nT}{n\pi} & \text{for } n \ne 0 \end{cases} \tag{C}$$

DESIGN OF NONRECURSIVE FILTERS *Cont'd*

Finite-duration impulse response

A finite-duration impulse response is obtained by applying a window $w(nT)$, say, of length $N = 2M + 1$ as

$$h_0(nT) = w(nT)h_{id}(nT) \tag{D}$$

DESIGN OF NONRECURSIVE FILTERS *Cont'd*

Choice of Window Parameters [10]

- **The window parameters, i.e., μ and x_μ for the ultraspherical window, must be chosen such that the filter specifications are satisfied with the lowest possible filter length N.**
- **Given a set of specifications, the optimum values of μ and x_μ can be determined through simple trial-and-error techniques but such an approach is laborious and time-consuming.**
- **As it turned out, we were able to develop a fairly general, although empirical, method that can be used to determine the window parameters for arbitrary filter specifications.**

  The steps involved are detailed in the next two or three transparencies.

Choice of Window Parameters

- **A nonrecursive (noncausal) lowpass filter is typically required to satisfy the equations**

$$\begin{aligned} 1-\delta_p \le H(e^{j\omega T}) &\le 1+\delta_p \quad \text{for } \omega \in [0, \omega_p] \\ -\delta_a \le H(e^{j\omega T}) &\le \delta_a \qquad\quad \text{for } \omega \in [\omega_a, \omega_s/2] \end{aligned}$$

  where δ_p and δ_a are the passband and stopband ripples and ω_p and ω_a are the passband and stopband edge frequencies, respectively.

- **In nonrecursive filters designed with the window method, the passband ripple turns out to be approximately equal to the stopband ripple.**

  Therefore, one can design a filter that has a passband ripple δ_p or a filter that has a stopband ripple δ_a.

- **If the specifications call for an arbitrary passband ripple A_p and a minimum stopband attenuation A_a, both specified in dB, then it can be easily shown that**

$$\delta_p = \frac{10^{0.05A_p}-1}{10^{0.05A_p}+1} \quad \text{and} \quad \delta_a \;=\; 10^{-0.05A_a} \tag{E}$$

  By designing a filter on the basis of

$$\delta = \min(\delta_p,\ \delta_a) \tag{F}$$

  then if $\delta = \delta_p$ a filter will be obtained that has a passband ripple which is equal A_p dB and a minimum stopband attenuation which is greater than A_a dB; and if $\delta = \delta_a$ a filter will be obtained that has a minimum stopband attenuation which is equal to A_a dB and a passband ripple which is less than A_p dB.

Choice of Window Parameters — μ

- **Through extensive experimentation, we found out that parameters μ and x_μ control the passband and stopband ripples and, consequently, the actual stopband attenuation, namely,**

$$A_a = -20\log_{10}(\delta) \tag{G}$$

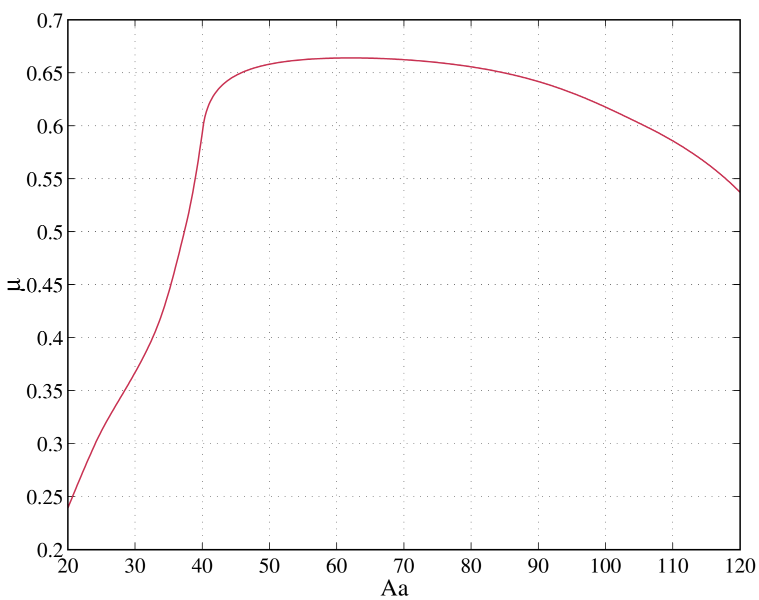
On the other hand, the filter length N controls the transition width of the filter, namely,

$$\Delta\omega = \omega_a - \omega_p$$

Conversely, the required μ and N are critically dependant on the actual stopband attenuation A_a and the transition width $\Delta\omega$, respectively.

- **It turns out that the required value of N is largely dependent on μ and is relatively independent of x_μ.**

The value of μ that minimizes the filter length obeys the law in the graph shown below:



Choice of Window Parameters — μ

Through curve fitting, an empirical formula was derived for the optimal μ as

$$\mu = aA_a^2 + bA_a + c \quad \text{for } A_L \leq A_a \leq A_H \tag{H}$$

where coefficients a, b, and c and bounds A_L and A_H are given in the table shown:

Parameters for formula for μ

A_L	A_H	a	b	c
20	30	− 3.570E-4	3.051E-2	−2.285E-1
30	40	1.461E-3	− 8.053E-2	1.471E+0
40	42	−7.910E-3	6.663E-1	− 1.340E+1
42	50	− 3.543E-4	3.569E-2	− 2.415E-1
50	65	− 4.272E-5	5.258E-3	5.023E-1
65	90	− 3.239E-5	4.165E-3	5.296E-1
90	120	− 5.576E-5	8.353E-3	3.407E-1

Choice of Window Parameters — N

The optimum filter length N can be determined as the lowest odd value of N that satisfies the inequality

$$N \geq \frac{D}{\Delta\omega/\omega_s} + 1 \tag{I}$$

where D is given by the empirical formula

$$D = 4.517 \times 10^{-5} A_a^2 + 6.227 \times 10^{-2} A_a - 4.839 \times 10^{-1} \tag{J}$$

Choice of Window Parameters — x_μ

Parameter x_μ is chosen as

$$x_\mu = \frac{x_{N-1}^{(\mu)}}{\cos(\beta\pi/N)} \tag{K}$$

where parameter β is given by the empirical formula

$$\beta = \begin{cases} 4.024 \times 10^{-5} A_a^2 + 2.423 \times 10^{-2} A_a & \\ \quad +3.574 \times 10^{-1} & \text{for } A_a \le 60 \\ 7.303 \times 10^{-5} A_a^2 + 2.079 \times 10^{-2} A_a & \\ \quad +4.447 \times 10^{-1} & \text{for } A_a > 60 \end{cases} \tag{L}$$

and $x_{N-1}^{(\mu)}$ is the largest zero of $C_{N-1}^{\mu}(x)$, which can be obtained by using Algorithm 1 with $\lambda = \mu$, $n = N - 1$, and $\varepsilon = 10^{-6}$ as input.

DESIGN ALGORITHM *Cont'd*

A lowpass nonrecursive filter satisfying the specifications

Passband edge:	ω_p
Stopband edge:	ω_a
Passband ripple:	A_p
Stopband ripple:	A_a
Sampling frequency:	ω_s

can be designed through the design algorithm to be described in the next few transparencies.

DESIGN ALGORITHM

Algorithm 2

Step 1: **Design an infinite-duration nonrecursive filter with the idealized frequency response**

$$H(e^{j\omega T}) = \begin{cases} 1 & \textbf{for } |\omega| \le \omega_c \\ 0 & \textbf{for } \omega_c < |\omega| \le \omega_s/2 \end{cases}$$

with $\omega_c = (\omega_p + \omega_a)/2$**.**

The impulse response is given by

$$h_{id}(nT) = \begin{cases} \omega_c/\pi & \text{for } n = 0 \\ \frac{\sin \omega_c nT}{n\pi} & \text{for } n \neq 0 \end{cases}$$

(see Eqs. (B)–(C)).

Step 2: **Find the required 'design' ripple**

$$\delta = \min(\delta_p,\ \delta_a)$$

where

$$\delta_p = \frac{10^{0.05A_p} - 1}{10^{0.05A_p} + 1} \quad \text{and} \quad \delta_a \ = \ 10^{-0.05A_a}$$

and update A_a **as**

$$A_a = -20\log_{10}(\delta)$$

(see Eqs. (E)–(G)).

DESIGN ALGORITHM *Cont'd*

Step 3: **Obtain the window parameter μ as**

$$\mu = aA_a^2 + bA_a + c \quad \textbf{for } A_L \leq A_a \leq A_H$$

(see Eq. (H)).

Step 4: **Calculate the minimum filter length as**

$$N \geq \frac{D}{\Delta\omega/\omega_s} + 1$$

where D is given by the empirical formula

$$D = 4.517 \times 10^{-5} A_a^2 + 6.227 \times 10^{-2} A_a - 4.839 \times 10^{-1}$$

(see Eqs. (I)–(J)).

Step 5: **Calculate window parameter x_μ as**

$$x_\mu = \frac{x_{N-1}^{(\mu)}}{\cos(\beta\pi/N)}$$

where parameter β is given by the empirical formula

$$\beta = \begin{cases} 4.024 \times 10^{-5} A_a^2 + 2.423 \times 10^{-2} A_a & \\ \quad +3.574 \times 10^{-1} & \text{for } A_a \leq 60 \\ 7.303 \times 10^{-5} A_a^2 + 2.079 \times 10^{-2} A_a & \\ \quad +4.447 \times 10^{-1} & \text{for } A_a > 60 \end{cases}$$

and $x_{N-1}^{(\mu)}$ is the largest zero of $C_{N-1}^{\mu}(x)$, which can be obtained by using Algorithm 1 with $\lambda = \mu$, $n = N - 1$, and $\varepsilon = 10^{-6}$ as input. (see Eqs. (K)–(L)).

DESIGN ALGORITHM *Cont'd*

Step 6: **With μ, N, and x_μ known, the coefficients of the ultraspherical window can be calculated from Eq. (A).**

Step 7: **Obtain a finite-duration impulse response as**

$$h_0(nT) = w(nT)h_{id}(nT)$$

(see Eq. (D)).

Step 8: **Obtain a causal design by delaying the impulse response by M samples, i.e.,**

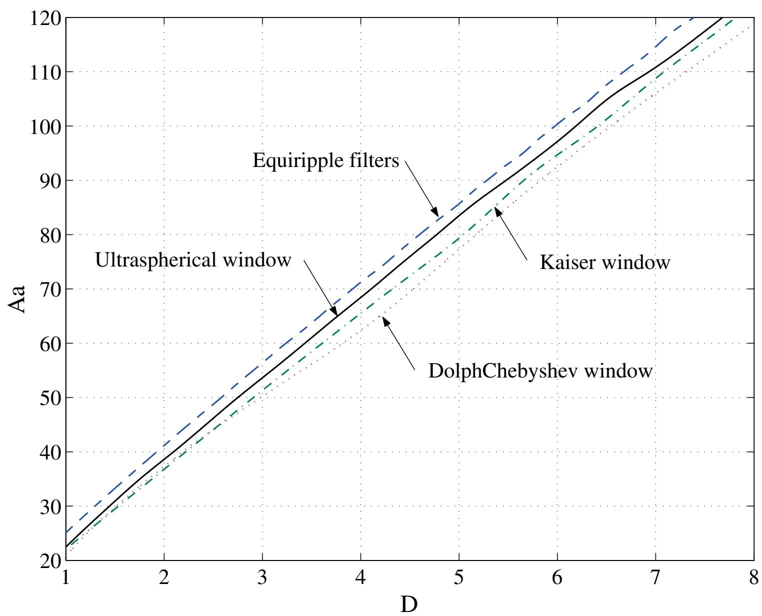
$$h(nT) = h_0\left[(n-M)\,T\right] \quad \text{for } 0 \leq n \leq N-1$$

Step 9: **Check your design to ensure that the filter satisfies the prescribed specifications.**

NOTE: **The design method can be easily extended to other types of filters, e.g., highpass, bandpass, and bandstop filters by following the procedure of Antoniou [11].**

COMPARISON WITH OTHER WINDOWS

- **The performance of adjustable windows for filter design can be measured by comparing the attenuation and performance factor $D = \Delta\omega(N-1)/\omega_s$.**
- **For $N = 127$, $\omega_c = 0.4\pi$, and $\omega_s = 2\pi$ rad/s the following results were obtained.**

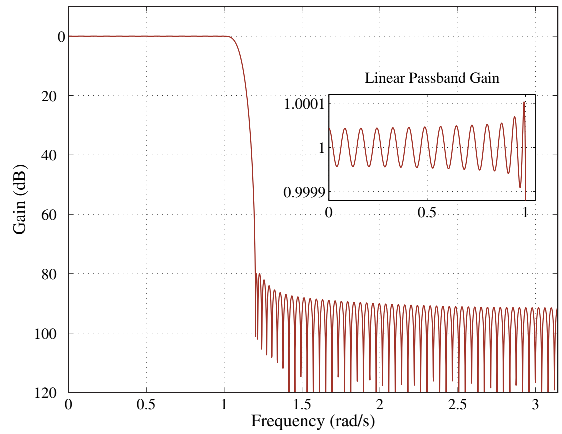


NOTE: **This relative ranking is relatively independent of N, i.e., it holds true for low as well as medium values of N.**

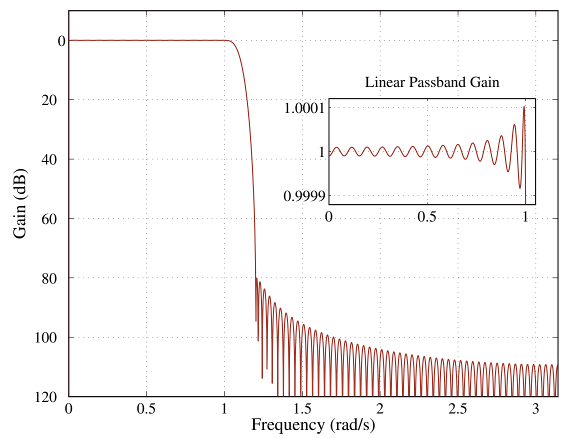
EXAMPLE

Design a lowpass filter with $\omega_p = 1$, $\omega_a = 1.2$ rad/s, and $A_a = 80$ dB using the ultraspherical, Kaiser, and Dolph-Chebyshev windows.

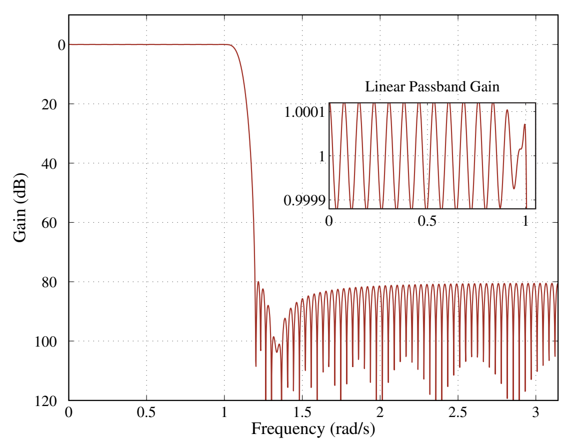
The filter lengths required to achieve the specifications were $N = 153$ for the ultraspherical window, $N = 159$ for the Kaiser window, and $N = 165$ for the Dolph-Chebyshev window.



(a) Ultraspherical Window



(b) Kaiser Window



(c) Dolph-Chebyshev Window

CONCLUSIONS

- **The ultraspherical window is a three-parameter window that can control the width of the main lobe, relative amplitude of the side lobes, and the side-lobe pattern.**
- **Conventional two-parameter windows cannot control the side-lobe pattern in the same fashion.**
- **The ultraspherical window includes both the Dolph-Chebyshev and Saramäki windows as special cases.**
- **When applied to digital filter design, the ultraspherical window has proven to yield lower order filters (improved cost) relative to other windows.**
- **Alternatively, for a fixed filter length, the ultraspherical window gives reduced passband ripple and increased attenuation (better performance) relative to other windows.**
- **The Remez method yields more efficient filters but a huge amount of computation is required which makes the Remez unsuitable for real or quasi-real time applications.**

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