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TLM-based solutions of the Klein–Gordon equation (Part II)‡

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SUMMARY

In Part I, two TLM-based solutions were presented for the Klein–Gordon Equation in its basic form, with the TLM pulses representing the primary variable. In Part II, two further approaches are presented in which the TLM pulses now represent derivatives of the primary variable, with respect to either space or time. As in Part I, the two solution schemes were verified symbolically and numerically. They illustrate further ways to extend the power of TLM beyond its traditional application areas. Some of these areas are discussed briefly. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: TLM; transmission line matrix; dispensive wave equation; dispersion; Klein–Gordon equation; numerical methods; forced wave equation

1. INTRODUCTION

In Part I, two TLM-based numerical solution methods were presented for the Klein–Gordon equation

$$u_{tt} = c^2\, u_{xx} - hu \tag{1}$$

The derivation of the equation and its physical significance were also presented. As in Part I, to facilitate discussion and nomenclature, it will be assumed here that the physical application is that of an elastically anchored vibrating string. However, the methods and results will naturally be valid for any application obeying this equation.

In Part I, the TLM pulses represented the variable u in equation (1) directly. Thus for the elastically anchored string, their sum, at any time and at any point in space, represented the instantaneous value of the local displacement of the string. In this part of the paper the TLM pulses instead represent (or are proportional to) the spatial or temporal derivative of the primary variable, that is, they represent the local gradient or velocity of the string. Again this approach gives rise to two numerical solution methods.

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2. FORCE/VELOCITY PROBLEM FORMULATION

It is well known and easily shown, that the local *gradient* or *slope* of a vibrating string obeys exactly the same wave equation as the displacement (as, indeed, does the string velocity). It is therefore reasonable to choose the local gradient of the string as the fundamental wave variable. A slightly better idea, because it has more physical meaning, is to use the gradient multiplied by the tension T, or $T\delta u/\delta x$, corresponding, within the standard approximations, to the normal component of the tension, in Newtons. The variation over time and space of this tension component is to be described by a rightward-going wave $r_i(x - ct) = -T\delta u/\delta x|_+$ superposed on a leftward going wave $l_i(x + ct) = +T\delta u/\delta x|_-$. The minus sign in the rightward-going wave is because a negative slope with respect to x causes a positive change in the displacement. The total normal component of the tension, $T\delta u/\delta x$, is then $(-r_i + l_i)$ N.

It is worth clarifying that this normal component of tension at a point, while it is a "force", is not the total normal force *on a string element* at the point. The latter is the spatial derivative of the total normal tension, or $T\delta^2 u/\delta x^2$.

Perhaps surprisingly, the instantaneous string velocity associated with a single wave propagating in one direction is directly proportional to the instantaneous normal tension component, the constant of proportionality being the mechanical impedance, $Z = \rho c$. If the waves are defined as above, positive values of r_i and l_i are each associated with a positive velocity, and the velocity at a point will be $(r_i + l_i)/Z$ m/s.

Note in passing that other formulations are possible. For example, the TLM pulses r_i and l_i could both be made equal to component $-T\delta u/\delta x$ terms, independently of direction, and then the velocity would be $(r_i - l_i)/Z$ m/s. This formulation makes the normal component of the tension analogous to voltage and the velocity analogous to the current in a transmission line. Alternatively, one could make the TLM pulses represent the total normal force on a string element at a point, or $T\delta^2 u/\delta x^2$. The acceleration at a point would then be proportional to the difference between the counter-propagating pulses at the point. A double integration with respect to space or time would then give the string displacement. Neither of these options, however, will be considered here.

For the Klein–Gordon problem, there is at least one significant advantage in having the TLM pulses represent force/velocity rather than displacement (as in Part I). In Part I, when it came to adding in a "new" or "extra" force effect (that is, a force not present in the standard wave problem), difficulties arose precisely because one cannot directly add a force variable to a displacement variable. If, however, TLM pulses represent normal force components, the extra (elastic) force can simply be added directly to the TLM pulses at each time interval. (Bear in mind that what is "added" is a negative force in this particular case). This is a very convenient feature.

As before, half the force should be "added" to the left-going pulse stream, half to the right-going one. If the pulses are scaled by $1/Z$ to represent velocities, the same argument applies: one is adjusting the local velocity in each Δt, that is, causing an *acceleration* (temporal rate of change of velocity) proportional to the local elastic force, as required.

A slightly unfortunate consequence of this force/velocity formulation is that the displacement is no longer available directly from the TLM scheme. In general, this may not be significant. Frequently, as in many acoustics problems, the derivative term is of greater interest than the displacement itself. In the particular case of the Klein–Gordon equation however, this displacement is always needed, to allow the calculation of the elastic acceleration, *hu*. But there is an easy solution, at a very modest price. The displacement can be determined either by integration of the velocity with respect to time or by integration of the gradient with respect to space. These give Methods 3 and 4 respectively (numbered to follow Methods 1 and 2 in Part I).

3.1. Solution Method 3. TLM pulses represent normal force, $\pm T\,\delta u/\delta x$. Elastic force by temporal integration of velocity, added directly to pulses

The obvious way to keep track of the string displacement, and thereby the elastic force, is to set up another array to store this displacement and to update it at each time interval by integrating the velocity, $(r_i + l_i)/Z$, with respect to time at each point along the string. The elastic force can then be calculated from this displacement, and half of it added to each component TLM pulse, before allowing them to propagate.

Again, this system works perfectly well. It has been verified both numerically and symbolically, as in Methods 1 and 2 in Part I. The symbolic version and the resulting table, however, are complex and difficult to decipher, mainly due to the "indirect" calculation of the displacement variable from the TLM variables and the presence of many higher order terms. For brevity it has not been included here. The numerical verification was again based on comparison with Fourier solutions [1], with results indistinguishable from those shown graphically in Figure 1, Part I.

The "Matlab" algorithm is as follows:

f_right = f_right $-\frac{1}{2}*h*\,\rho*\Delta l*$ (displacement)
f_left = f_left $-\frac{1}{2}*h*\,\rho*\Delta l*$ (displacement)
displacement = displacement + $(\Delta t^2/\rho*\Delta l)*$(f_right + f_left)
[Propagate force TLM pulses f_right and f_left as normal]

"f_right" and "f_left" are arrays of TLM pulses representing normal tension component waves, $\pm T\,\delta u/\delta x$, with the total normal tension $T\,\delta u/\delta x = (-$ f_right + f_left); "displacement" array stores the string displacement derived by time integration of velocity.

3.2. Solution Method 4. TLM pulses represent normal force, $\pm T\delta u/\delta x$. Elastic force by spatial integration of slope, added directly to pulses

Alternatively, as the TLM pulses represent $\pm T\,\delta u/\delta x$, the displacement u can be determined at each time step by a spatial integration (that is, a cumulative sum) of $(-r_i + l_i)/T$ times the TLM pulses along the string. The most convenient starting point for the integration (thereby determining the constant of integration) is at the boundaries where the displacement values are known.

This Method 4 works well. It has the minor, theoretical advantage over Method 3 that, strictly speaking, no extra array is required to be stored, so there is some saving on memory, although at the cost of more calculations per time step. Note, however, that if the displacement with time is required in any case, this extra array will have to be calculated and stored for its own sake, so this minor advantage is lost.

The algorithm is

f_right = f_right $-\frac{1}{2}*h*\,\rho*\Delta l*$ (displacement)
f_left = f_left $-\frac{1}{2}*h*\,\rho*\Delta l*$ (displacement)
displacement = $u_0 + (\Delta t^2/\rho*\,\Delta l)*$ cumsum (f_left − f_right)
[Propagate force TLM pulses f_right and f_left as normal]

with

> "f_right" and "f_left": arrays of TLM pulses with ($-$ f_right $+$ f_left) representing $T\delta u/\delta x$; "displacement" array stores the string displacement from $u = u_{x=0} + \int_{x_0}^{x} (\partial u/\partial x)\,\mathrm{d}x$. Typically, the displacement at the boundary, $u_0 = u_{x=0}$, is zero.

This method was also verified numerically and was shown to be completely consistent with the other methods, and with the Fourier solution, to within an arbitrarily small error as the time and space discretisation became finer.

4. SOME WIDER IMPLICATIONS

4.1. Wave-like problems with more general forcing functions

Although outside the scope of this paper, it is worth noting here that the force/velocity formulation of the wave problem, used in Methods 3 and 4, certainly facilitates the solution of a more general class of problem where an arbitrary forcing (or accelerating-causing) function $f(x, t, u, u_t, u_x, \dots)$ acts on any wave-like system obeying an equation of the general form

$$u_{tt} - c^2\, u_{xx} = f(x, t, u, u_t, u_x, \dots) \tag{8}$$

This is because the forcing variable is more easily added to the TLM pulses, as has been seen above in the special case of the Klein–Gordon Equation.

In general the displacement u may not be required (unlike in the case of the Klein–Gordon Equation), neither to evaluate the forcing function that typically will be of the form $f(x, t)$, nor as part of the solution, where for example only force and velocity may be of interest. This is frequently the case in acoustics, for example. The 'forcing" function f in Equation (8) may be due to the physical effects arising within the system or coming from outside the system. An example of the latter is when two systems were coupled, each one appearing as an external driver for the other.

Equation (8) is sometimes referred to as the "forced wave equation" or the "non-homogeneous wave equation" (see below regarding this second name), and is encountered in many areas of Physics. Important examples include:

(a) a vibrating string (one-dimension) or membrane (two-dimensional wave structure) immersed in a viscous medium, where the (passive) forcing function is directly proportional to the string/membrane velocity, which is immediately obtainable from the TLM pulses;
(b) acoustic pressure acting on a string or membrane, as occurs for example is microphones, where f appears to the vibrating system as an (active) external force;
(c) electromagnetic forces acting on taut wires, meshes, or membranes, where again f is external and active;
(d) other special equations, such as the Sine–Gordon equation of solid-state electronics:

$$u_{tt} - u_{xx} + \sin u = 0 \tag{9}$$

(e) a conducting line with magnetic and/or electric coupling to another conductor.

The telegrapher's equation has the same form as the equations found in cases in (a) above.

The description “inhomogeneous” requires clarification. Not all “forced” wave equations are “inhomogeneous”. It depends on the nature of the forcing term on the right-hand side of Equation (8). In particular, both the Klein–Gordon equation and the telegrapher’s equation can be considered as forced wave equations, expressible in the form of Equation (8), yet both are mathematically “homogeneous” in that they can also be expressed as $L(u) = 0$, where L is a linear operator. In any case, whether homogeneous or not, the proposed TLM methods will work for all equations expressible in the form of Equation (8).

4.2. Mathematical comments on the new TLM algorithms

There is a degree of duality about the two methods presented in both Parts I and II. Thus, integration with respect to time in one method becomes integration with respect to space in the other, related by a constant of proportionality corresponding to the wave speed. Such interrelationships are a feature of wave-like phenomena. This is not surprising in the light of the relationship $u_t = -c\,u_x$ (Equation (5), Part I), an equation which can also be taken as the prototypical wave equation [2].

Although not derived in this way, the new TLM methods can be considered, in retrospect, as closely related to D’Alembert’s general solution of the wave equation for arbitrary initial conditions and of Duhamel’s principle for the forced wave equation [2–4]. Both of these express analytical solutions as the sum of a free wave solution and other terms involving the integration of derivative terms with respect to space and/or time. The TLM schemes can be considered as numerical equivalents of these analytical techniques, but presentation of the details will not be attempted here.

5. DISCUSSION

In Part II, two further methods have been presented for solving the Klein–Gordon equation using TLM. By making the TLM pulses represent force/velocity terms rather than displacement, the effect of the elastic forcing term can be added directly to the TLM pulses. This avoids the subtle complications that arose in Part I which demanded special care. Being able to add forcing terms to forcing terms directly simplifies and clarifies the problem. The only cost is minor. The displacement now has to be determined indirectly, but this is easily achieved.

The two methods were verified as in Part I, both numerically, by comparison with Fourier series analytical solutions, and symbolically, by comparision with finite difference methods.

By viewing the Klein–Gordon equation as a special case of a wider class of problems described by the forced wave equation, the methods presented open up new application areas for TLM. Work is already under way to progress matters much further, allowing the TLM solution method to extend to a range of fundamental partial differential equations of solid and fluid mechanics.

The present paper considered only one-dimensional problems, but all these techniques seem easily extendable to at least two dimensions. Indeed Method 3 has already been successfully applied to the acoustic coupling of a stretched membrane (a 2-D vibrating system) and an acoustic field [5].

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