An Improved Lawson Local-Optimization Procedure and Its Application

Yue Fang

University of Victoria

yuefang@uvic.ca

April 23, 2018

Outline

1. **Introduction and Background**
   - Research Question and Purpose
   - Triangulation and Triangle Mesh Models
   - Triangulation-Cost Optimization
   - Edge Flip
   - Local Optimization Procedure(LOP)
   - Triangulation Cost Function
2. **N-flip Optimality**
3. **The Modified LOP(MLOP)**
   - Permissible Flip Sequence
   - Proposed Permissible Flip Sequence Policy
4. **Evaluation of Proposed Methods**
   - Mesh Optimization
   - Conclusion

Research Question and Purpose

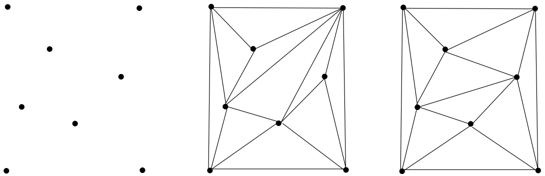
- study the problem of selecting the connectivity of a triangulation in order to minimize a given cost function
- great importance for generating triangle mesh models
- in early work, the local optimization procedure (LOP) and lookahead LOP (LLOP) can only yield triangulation that satisfy a weak optimality condition.
- our interests:
  - further improvements to LOP and LLOP
  - a more general framework, modified LOP (MLOP)
  - satisfy a stronger optimality condition
  - yield triangulation of significantly lower cost

Triangulation

Definition 1

(Triangulation). A triangulation of a finite set P of points in $\mathbb{R}^2$ is a set T of (non-degenerate) triangles satisfying the following conditions:

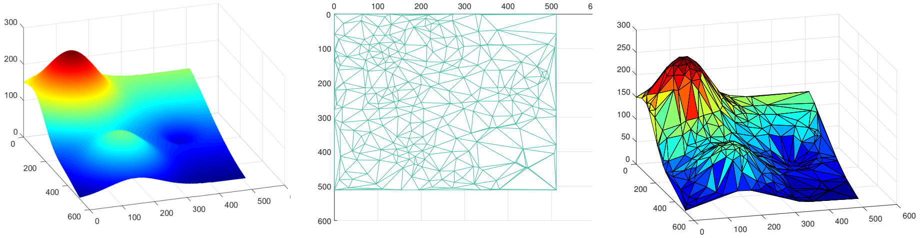
1. the union of all triangles in T is the convex hull of P;
2. the union of vertices of all triangles in T is P;
3. the interiors of any two triangles in T are disjoint; and
4. every edge of a triangle in T joins two and only two points from P.



(a) (b) (c)

Triangle-Mesh Models

- a (triangle) mesh model consists of:
  - a set $P = \{p_i\}$ of sample points;
  - a triangulation T of P; and
  - the function values $\{z_i = \phi(p_i)\}$ for $p_i \in P$.
- apply linear interpolation over each triangle face to construct the triangle mesh model.



(a) (b) (c)

Figure 1 : Mesh model of a bivariate function ϕ. (a)A bivariate function ϕ, (b) the triangulation and (c) the resulting triangle mesh model.

Triangulation-cost Optimization

Triangulation-cost Optimization Problem

Given a set P of points, a triangulation cost function[a] c, and a mapping S from a triangulation of P to a subset of all triangulations of P, find a triangulation T of P such that

$$c(T) \leq c(T') \quad \text{for all } T' \in S(T). \tag{1}$$

[a] a function maps triangulation to $\mathbb{R}$

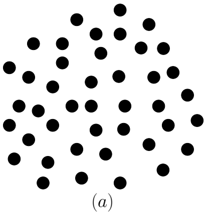
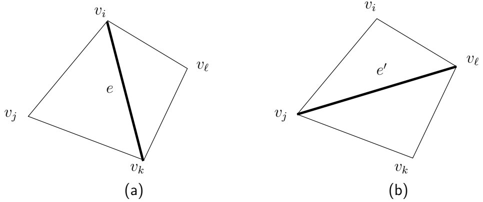


Figure 2 : (a) each black node represents a triangulations of P, (b) red and blue nodes represent T and T' respectively, where $c(T) \leq c(T')$

Edge Flip

- a very popular class of optimization schemes is the class based on edge flip.
- edge e is **flippable** if e has two incident faces and the union of these two faces is a strictly convex quadrilateral q.
- an **edge flip** is an operation that replace e by the diagonal e' of q.

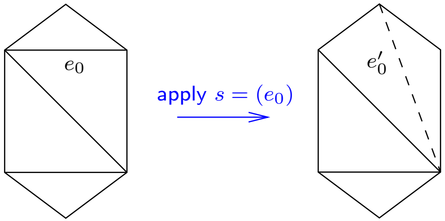


- a **flip sequence** is a sequence of edge flips. $s = (e_0, e_1, ..., e_{n-1})$

Local Optimization Procedure(LOP)

In order to address the triangulation-cost optimization problem

- the LOP chooses $S(T) = S_{\text{LOP}}(T)$.
- $S_{\text{LOP}}(T)$ is the set of all triangulation of P that are reachable from T by a single edge flip.
- no single edge flip applied to T can yield a new triangulation with strictly lower cost than T.



Triangulation Cost Function

Triangulation cost function is defined as an accumulation of cost for individual triangulation elements, such as edges or faces.

- an **edge-based cost function** c has the form:

$$c(T) = \sum_{e \in \mathcal{E}(T)} \mathrm{edgeCost}(e), \qquad (2)$$

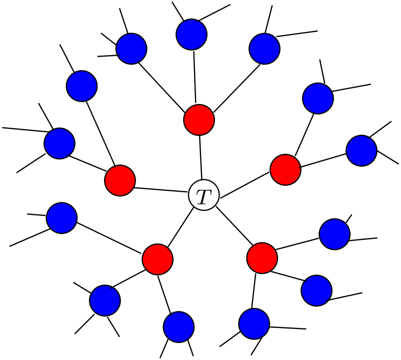
- an **face-based cost function** c has the form:

$$c(T) = \sum_{f \in \mathcal{F}(T)} \mathrm{faceCost}(f), \qquad (3)$$

- cost functions considered in our work:
  1. **angle between normals (ABN)**
  2. **absolute mean curvature (AMC)**
  3. **deviations from linear polynomials (DLP)**
  4. **distances from planes (DP)**
  5. **jump in normal derivatives (JND)**
  6. **squared error (SE)**
  7. **Yu-Morse-Sederberg (YMS)**

N-flip Optimality

- each node corresponds to a distinct triangulation.
- two nodes are connected by an edge if the two triangulations are reachable from one another by a single edge flip.



- T is 1-flip optimal if no red node triangulation has a lower cost.
- T is 2-flip optimal if no red and blue node triangulation has a lower cost.

The Modified LOP(MLOP)

- permissible flip-sequence policy
  - specify $S(T)$ indirectly in terms of flip sequences.
  - a rule that specifies the set of flip sequences starting with a flippable edge e_0.
  - this set is denoted as $\mathrm{permFlipSeqs}_T(e_0)$.
- an edge e_0 is **optimial** if all the sequence of $\mathrm{permFlipSeqs}_T(e_0)$ can not reduce the cost of triangulation.
- an edge e_0 whose optimality is uncertain is said to be **suspect**

The Modified LOP(MLOP) framework

given a triangulation and a triangulation cost function as input, the MLOP produces an optimal triangulation as output.

1. initially, mark each flippable edge in the triangulation as suspect edge
2. test if there exists a permissible flip sequence starting with a suspect edges can reduce the triangulation cost
3. if such a flip sequence exists, apply it, and then determine which edges become suspect.
4. iterate performing the step 2 and 3 in triangulation until no suspect edges remain.

Permissible Flip Sequence

- permissible flip-sequence policy: determine the particular set of flip sequences be considered.
  - four input parameters control the premissible flip sequence generation process.
  - `maxLevel`: determine the size of the region which edge flip can take place.
  - `skip`: the distance between the two adjacent edges in sequence is NOT necessary to be one if enable.
  - `inward`: allow to flip the edge, which has just been flipped, if enable
  - `maxLength`: restrict the maximum length of the flip sequences.

Proposed Policies

Table 1 : Parameter selections for defining various permissible flip-sequence policies

Policy	Parameter			
	maxLevel	inward	skip	maxLength
$\mathrm{PFS}_{\mathrm{IO}}(L)$	L	**true**	**false**	$\infty^{\dagger}$
$\mathrm{PFS}_{\mathrm{IOS}}(L)$	L	**true**	**true**	$\infty^{\dagger}$
$\mathrm{PFS}_{\mathrm{MLT}}(L)$	L	**false**	**true**	2
$\mathrm{PFS}_{\mathrm{LOP}}$	0	**false**	**false**	1
$\mathrm{PFS}_{\mathrm{LLOP}}$	1	**false**	**false**	2

†Any value greater than or equal to $L+1$ is equivalent to ∞.

Proposed Methods for Using the MLOP

1. $\text{MLOP}_{\text{A}}(L)$: apply the MLOP with $\text{PFS}_{\text{MLT}}(L)$
2. $\text{MLOP}_{\text{B}}(L, M)$: apply the MLOP in two stages. Apply the MLOP with $\text{PFS}_{\text{IO}}(M)$, and then apply MLOP with $\text{PFS}_{\text{MLT}}(L)$. Reasons for two stages optimization:
   - avoid poor local minimum
   - allow the second stage optimization to be seeded with an initial triangulation that is in a more desirable region of the solution.
3. $\text{MLOP}_{\text{C}}(L)$: apply the MLOP with $\text{PFS}_{\text{IOS}}(L)$

Optimality Properties of the LOP, LLOP, MLOP

- LOP only guarantees 1-flip optimality.
- LLOP can only guarantees 2-flip optimality if $\mathrm{inflDist}(c) \leq 1$ (i.e. SE cost function)
- $\mathrm{MLOP_A}(L)$, $\mathrm{MLOP_B}(L, M)$ and $\mathrm{MLOP_C}(L)$ methods are guaranteed 2-flip optimality as long as L is sufficiently large.
- $L = 2$ is large enough to ensure 2-flip optimality for the cost function considered.

Evaluation of Proposed Methods—Mesh Optimization

Table 2 : Triangulation costs obtained using the various optimization methods.

Cost Func.	Mesh	Triangulation Cost				
		LOP	LLOP	$\text{MLOP}_{\text{A}}(2)$	$\text{MLOP}_{\text{B}}(2,2)$	$\text{MLOP}_{\text{C}}(2)$
ABN	lena@1-ED@0.02	11916.60	10271.55	10089.16	9539.82	**9417.04**
ABN	n36-w113@0.005	7349.00	7002.41	6972.81	6905.48	**6897.76**
AMC	lena@1-ED@0.02	106401.89	101248.28	100792.96	99810.94	**99481.09**
AMC	n36-w113@0.005	6918326.24	6868876.57	6857626.68	6853149.95	**6851738.72**
DLP	lena@1-ED@0.02	973861.70	749538.76	751934.44	693624.80	**693765.19**
DLP	n36-w113@0.005	7673511.72	7065988.61	7008287.94	6890063.14	**6838935.10**
DP	lena@1-ED@0.02	99480.75	77684.29	78242.12	**67863.62**	70426.29
DP	n36-w113@0.005	5909169.34	5312215.54	5262803.93	5122322.64	**5082415.04**
JND	lena@1-ED@0.02	203874.69	190001.90	188589.17	187041.87	**186462.70**
JND	n36-w113@0.005	9831.93	9519.02	9479.73	9431.13	**9422.08**
SE	lena@1-ED@0.02	22684438	19285947	19134957	**18297542**	18333058
SE	n36-w113@0.005	899889559	874421792	873570760	**869878114**	869931291
YMS	lena@1-ED@0.02	3056206.81	2792159.74	2781951.65	2727975.11	**2724628.16**
YMS	n36-w113@0.005	2049.38	1791.59	1748.12	1685.08	**1680.91**

Evaluation of Proposed Methods—Mesh Optimization

Table 3 : Average rankings taken across all test cases

Cost Func.	Average Rank (with Standard Deviation†)				
	LOP	LLOP	$\mathrm{MLOP_A}(2)$	$\mathrm{MLOP_B}(2,2)$	$\mathrm{MLOP_C}(2)$
ABN	5.00 (0.00)	3.86 (0.34)	3.14 (0.34)	1.79 (0.41)	**1.21** (0.41)
AMC	5.00 (0.00)	3.97 (0.18)	3.03 (0.18)	1.72 (0.45)	**1.28** (0.45)
DLP	5.00 (0.00)	3.93 (0.25)	3.07 (0.25)	1.62 (0.49)	**1.38** (0.49)
DP	5.00 (0.00)	3.76 (0.43)	3.21 (0.48)	1.62 (0.55)	**1.41** (0.49)
JND	5.00 (0.00)	3.97 (0.18)	3.03 (0.18)	1.83 (0.38)	**1.17** (0.38)
SE	5.00 (0.00)	3.72 (0.45)	3.24 (0.50)	1.55 (0.50)	**1.48** (0.56)
YMS	5.00 (0.00)	3.45 (0.72)	3.07 (0.83)	2.14 (0.90)	**1.34** (0.54)
Overall	5.00 (0.00)	3.81 (0.44)	3.11 (0.46)	1.75 (0.58)	**1.33** (0.49)

† The standard deviation is given in parentheses.

Evaluation of Proposed Methods—Mesh Optimization

Table 4 : Comparison of reduction in triangulation cost obtained with various methods relative to the LOP.

Cost Func.	Median Reduction in Triangulation Cost Relative to LOP (%)			
	LLOP	$MLOP_A(2)$	$MLOP_B(2,2)$	$MLOP_C(2)$
ABN	13.80	14.76	19.72	**19.77**
AMC	4.30	4.64	5.85	**5.96**
DLP	13.53	14.57	**18.21**	17.89
DP	20.70	21.35	**28.34**	28.25
JND	6.14	6.63	7.42	**7.59**
SE	5.63	5.72	**7.11**	7.07
YMS	22.08	24.10	27.58	**30.50**

Evaluation of Proposed Methods—Mesh Optimization

Table 5 : Triangulation costs obtained using $\mathrm{MLOP_B}(2, M)$ for various choices of M

Cost Func.	Mesh	Triangulation Cost $M=2$	$M=3$	$M=4$	$M=5$
ABN	lena@1-ED@0.02	9539.82	9235.90	8950.96	**8881.31**
ABN	n36-w113@0.005	6905.48	6877.19	6862.76	**6840.55**
AMC	lena@1-ED@0.02	99810.94	98830.81	98579.41	**98540.33**
AMC	n36-w113@0.005	6853149.95	6852002.26	6851130.46	**6848777.86**
DLP	lena@1-ED@0.02	693624.80	689832.86	678211.34	**631582.68**
DLP	n36-w113@0.005	6890063.14	6817818.32	**6718116.16**	6721997.02
DP	lena@1-ED@0.02	67863.62	66484.86	64318.27	**62851.59**
DP	n36-w113@0.005	5122322.64	5001998.30	4962355.40	**4933473.46**
JND	lena@1-ED@0.02	187041.87	185816.54	185728.60	**185482.90**
JND	n36-w113@0.005	9431.13	9408.14	9392.62	**9376.06**
SE	lena@1-ED@0.02	18297542	17832396	17735184	**17431940**
SE	n36-w113@0.005	869878114	868600878	867165140	**866664396**
YMS	lena@1-ED@0.02	2727975.11	2691297.40	**2657434.01**	2661270.97
YMS	n36-w113@0.005	1685.08	1651.44	1634.73	**1623.68**

Evaluation of Proposed Methods—Mesh Optimization

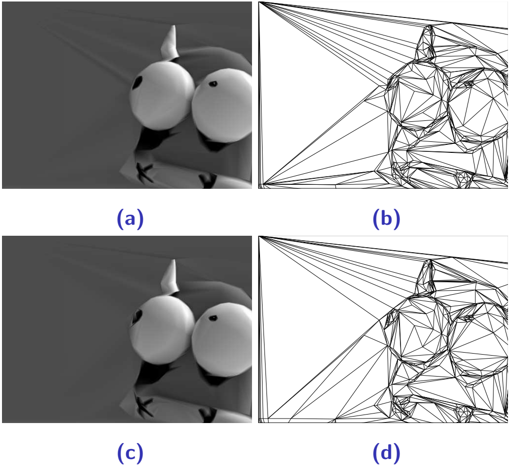
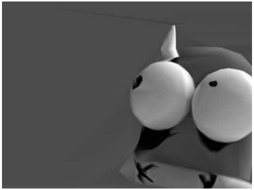
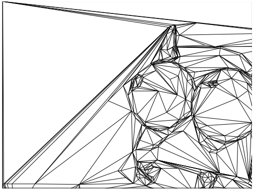
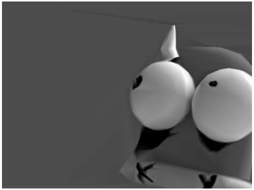
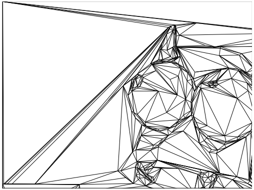


Figure 3 : Reconstructed images based on the mesh optimized by (a)(b) LOP method(34.70dB) (c)(d) LLOP method(35.36dB)

Evaluation of Proposed Methods—Mesh Optimization

(a) **(b)**

(c) **(d)**

Figure 4 : Reconstructed images based on the mesh optimized by (a)(b) MLOPB(2,2) method(35.48dB) (c)(d) MLOPB(2,6) method(35.58dB)

Conclusion

- An improved version of the LOP, called the MLOP, is proposed. The MLOP framework has a number of degrees of freedom.
- $\mathrm{MLOP_B}(L, M)$ and $\mathrm{MLOP_C}(L)$ is guaranteed to yield 2-flip optimal triangulations for arbitrary cost functions if L is sufficiently large.
- $\mathrm{MLOP_B}(2, 2)$ and $\mathrm{MLOP_C}(2)$ can produce triangulations of significantly lower cost than the LOP and LLOP.
- $\mathrm{MLOP_B}(2, 2)$ is the best choice for the cost function DP, DLP, or SE while $\mathrm{MLOP_C}(2)$ is the best choice for the other cost function.
- $\mathrm{MLOP_B}(2, i)$ and $\mathrm{MLOP_C}(i)$ can achieve even better results by increasing i at the expense of increased computational cost.

THANK YOU

Convex Set

Definition 2

(Convex set). A set P of points in $\mathbb{R}^2$ is convex if and only if for every pair of points $a, b \in P$, every point on the line segment which joins points a and b is completely contained in P.

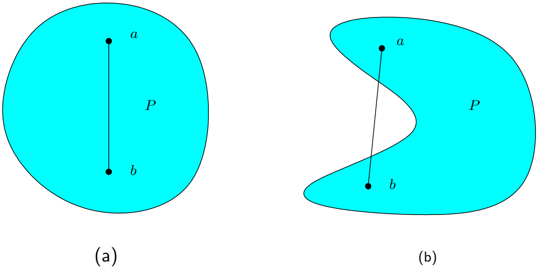


Figure 5 : Examples of (a) convex and (b) non-convex sets

Convex Hull

Definition 3

(Convex hull). The convex hull of a set P of points in $\mathbb{R}^2$ is the intersection of all convex sets that contain P.

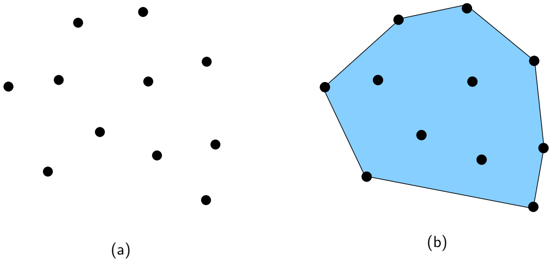
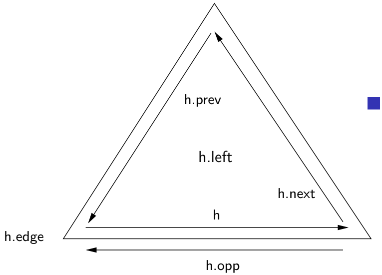


Figure 6 : Convex hull example. (a) A set P of points. (b) The convex hull of P.

Halfedge Structure

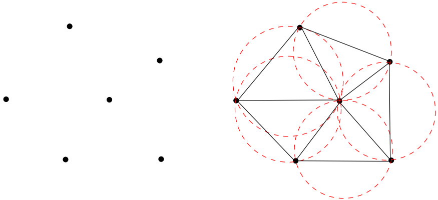


- Each edge is represented as a pair of directed edges(halfedges)

Delaunay triangulation

Definition 4

A Delaunay triangulation of a set P of points in $\mathbb{R}^2$ is a triangulation T such that no point in P is inside the circumcircle of any triangle in T.

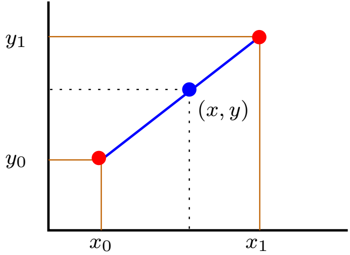


(a) (b)

- maximize the minimum interior angle of the triangles in the triangulation

Linear Interpolation

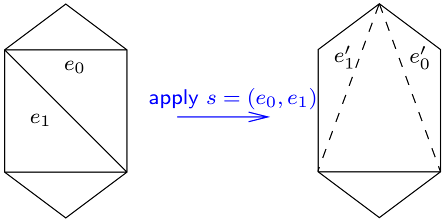
- linear interpolation is a method of curve fitting using linear polynomials to construct new data points within the range of a discrete set of known data points.



Given the two red points, the blue line is the linear interpolant between the points, and the value y at x may be found by linear interpolation.

Lookahead LOP

- a variant of the LOP
- the LLOP chooses $S(T) = S_{\text{LLOP}}(T)$, where $S_{\text{LLOP}}(T) = S_{\text{LOP}}(T) \cup \Gamma(T)$.
- $\Gamma(T)$ is the set of all triangulations that are reachable from T by a valid length-2 flip sequence of the form (e_0, e_1), where (e_0, e_1) share a common face.



N-flip Optimality

Definition 5

A triangulation T is said to be n-**flip optimal** with respect to the triangulation cost function c if the following condition holds:

$$c(T) \leq c(T') \quad \text{for all } T' \in S_{\text{NFO}}(T, n), \tag{4}$$

where $S_{\text{NFO}}(T, n)$ is the set of all triangulations that are reachable from T by a (valid) flip sequence of length $\ell \leq n$.

- LOP: $\mathrm{permFlipSeqs}_T(e_0)$ chosen as $\mathsf{PFS}_{\mathsf{LOP}}$, where

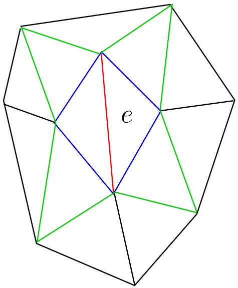
$$\mathsf{PFS}_{\mathsf{LOP}}(e_0) = \{(e_0)\} \tag{5}$$

- LLOP: $\mathrm{permFlipSeqs}_T(e_0)$ chosen as $\mathsf{PFS}_{\mathsf{LLOP}}$, where

$$\mathsf{PFS}_{\mathsf{LLOP}}(e_0) = \mathsf{PFS}_{\mathsf{LOP}}(e_0) \cup \Gamma(e_0) \tag{6}$$

$\Gamma(e_0)$: set of flip sequence (e_0, e_1) where e_1 share a common face with e_0.

Layers and Distance between edges



$$\text{layers}_T(e,i) = \begin{cases} e & i = 0 \\ \cup_{e \in \text{layers}_T(i-1)} \text{qe}_T(e) & i \geq 1, \end{cases}$$

where $\text{qe}_T(e)$ is defined as a set of edges belonging to faces incident on e.

- $\text{layers}_T(e,0)$ is the set of red edge
- $\text{layers}_T(e,1)$ is the set of red and blue edges
- $\text{layers}_T(e,2)$ is the set of red, blue, and green edges

$d_T(e,e_1)$: the smallest nonnegative integer k for which $e_1 \in \text{layers}_T(e,k)$.

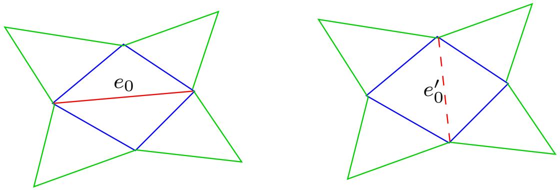
- $d_T(e,e_1) = 0$ if e_1 is the red edge
- $d_T(e,e_1) = 1$ if e_1 is the blue edge
- $d_T(e,e_1) = 2$ if e_1 is the green edge

Influence distance

the **influence distance** of a cost function c, denoted $\mathrm{inflDist}(c)$.

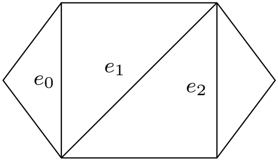
- measures the size of influence region of a single edge flip
- the maximum distance between the flipped edge e_0 and the edge e_1 whose optimality might change.

$$\mathrm{inflDist}(c) = \begin{cases} 1 & \text{if } c \text{ is SE} \\ 2 & \text{if } c \text{ is ABN, AMC, DLP, DP, JND, YMS.} \end{cases} \quad (7)$$



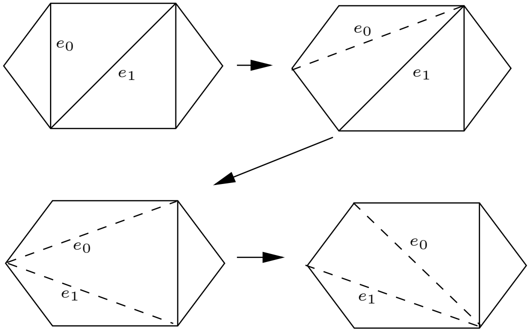
Permissible Flip Sequence I

- the particular set of flip sequences considered.
- four input parameters control the permissible flip sequence generation process.
- `maxLevel`: determine the size of the region which edge flip can take place.
  - `maxLevel=0`: $(e_0) \in \mathrm{permFlipSeqs}_T(e_0)$
  - `maxLevel=1`: $(e_0, e_1) \in \mathrm{permFlipSeqs}_T(e_0)$
  - `maxLevel=2`: $(e_0, e_1, e_2) \in \mathrm{permFlipSeqs}_T(e_0)$
  - ......
- `skip`: the distance between the two adjacent edges in sequence is NOT necessary to be one if enable.
  - `maxLevel=2 && skip=true`: $(e_0, e_2) \in \mathrm{permFlipSeqs}_T(e_0)$



Permissible Flip Sequence II

- `inward`: allow to flip the edge, which has just been flipped, if enable
  - `maxLevel=2 && inward=true`: $(e_0, e_1, e_0) \in \text{permFlipSeqs}_T(e_0)$



- `maxLength`: restrict the maximum length of the flip sequences.

Venn diagram to illustrate PFS policies

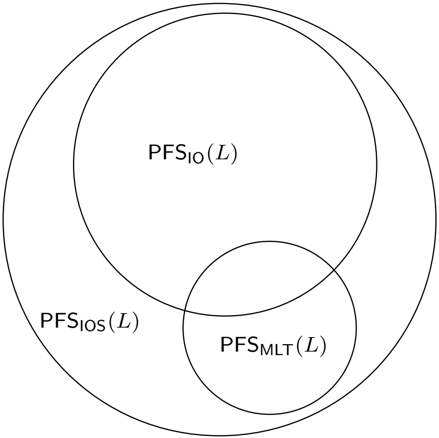


Figure 7 : Venn diagram to illustrate the permissible flip sequences considered by various PFS policies

Determination of Suspect Edges

Determination of Suspect Edges

$$\text{suspects}(T, s) = \text{edges}\left(\text{propagate}_{T'}(\text{flipAffReg}(T, s), L + \text{inflDist}(c) - 1)\right)$$

- L is the value of the `maxLevel` parameter for the PFS policy.

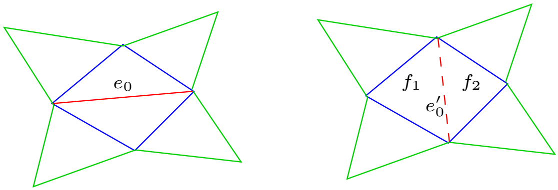


Figure 8 : Apply an edge flip sequence $s = \{e_0\}$ with $L = 0$

Determination of Suspect Edges

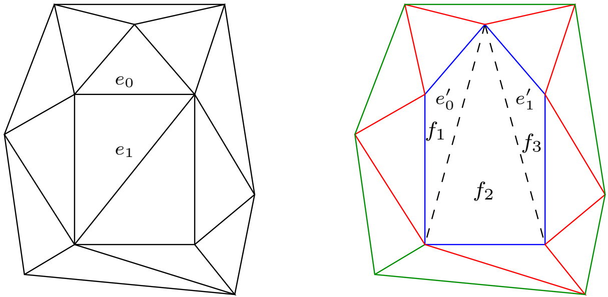


Figure 9 : Apply an edge flip sequence $s = \{e_0, e_1\}$ with $L = 1$

- $\text{flipAffReg}(T, s)$ is the set of faces f_1, f_2, f_3.
- $\text{suspects}(T, s)$ contains blue and red edges if $\text{inflDist}(c) = 1$.
- $\text{suspects}(T, s)$ contains red, blue and green edges if $\text{inflDist}(c) = 2$.

Evaluation of Proposed Methods—Mesh Generation

Table 6 : Individual results for mesh-generation with LOP, LLOP and $MLOP_B(2,2)$ as the postprocessing

Function	Samp. Dens (%)	PSNR (dB)		
		LOP	LLOP	$MLOP_B(2,2)$
lena	0.5	25.90	26.49	**26.62**
lena	1.0	28.36	29.38	**29.57**
lena	2.0	31.61	32.13	**32.25**
ct	0.5	37.82	38.24	**38.33**
ct	1.0	42.49	42.84	**42.93**
ct	2.0	47.12	47.42	**47.47**
n36-w113	0.5	66.52	66.99	**67.11**
n36-w113	1.0	69.69	70.12	**70.22**
n36-w113	2.0	72.81	73.25	**73.35**

Evaluation of Proposed Methods—Mesh Generation

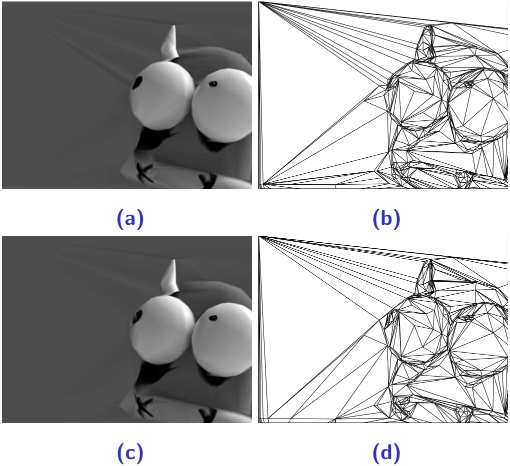


Figure 10 : Reconstructed images based on the mesh optimized by (a)(b) LOP, PSNR=34.70 dB (c)(d) LLOP, PSNR=35.36 dB

Evaluation of Proposed Methods—Mesh Generation

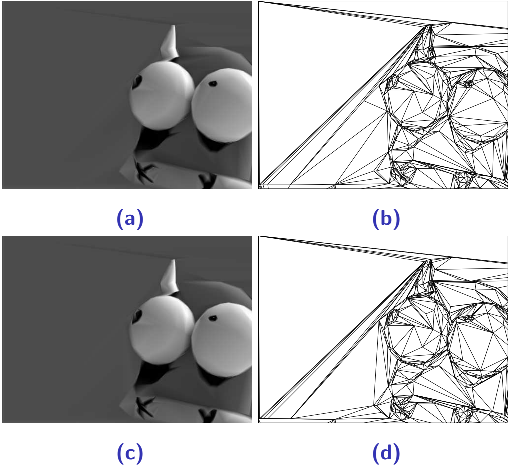


Figure 11 : Reconstructed images based on the mesh optimized by $\text{MLOP}_\text{B}(2, i)$ (a)(b) $i = 2$, PSNR=35.48 dB (c)(d) $i = 6$, PSNR=35.58 dB

Modified LOP Algorithm I

- Modified LOP Algorithm

Set the current triangulation T to the input triangulation.

For each edge e in T, mark e as suspect if it is flippable; otherwise, mark e as not suspect.

while suspect edges remain in T **do**

Select a suspect edge e_0 in T and mark e_0 as not suspect.

Clear the list ℓ of good flip sequences.

if e_0 is flippable **then**

Let p denote the set of all permissible flip sequences starting with e_0, as determined by the permissible flip-sequence policy (i.e., $p = \mathrm{permFlipSeqs}_T(e_0)$).

for each flip sequence s in p **do**

if applying s to T would strictly reduce the triangulation cost **then**

Add s to ℓ.

end if

Modified LOP Algorithm II

end for
if ℓ is not empty **then**
Select a sequence s from ℓ, using the good flip-sequence selection policy (i.e., $s = \text{selGood}(\ell)$).
Update T by applying s to T.
Determine the set σ of all edges in T that become suspect as a result of s being applied to T (i.e., $\sigma = \text{suspects}(T, s)$).
Mark each edge s in σ as suspect (if not already so marked).
end if
end if
end while
Output T as the final triangulation.