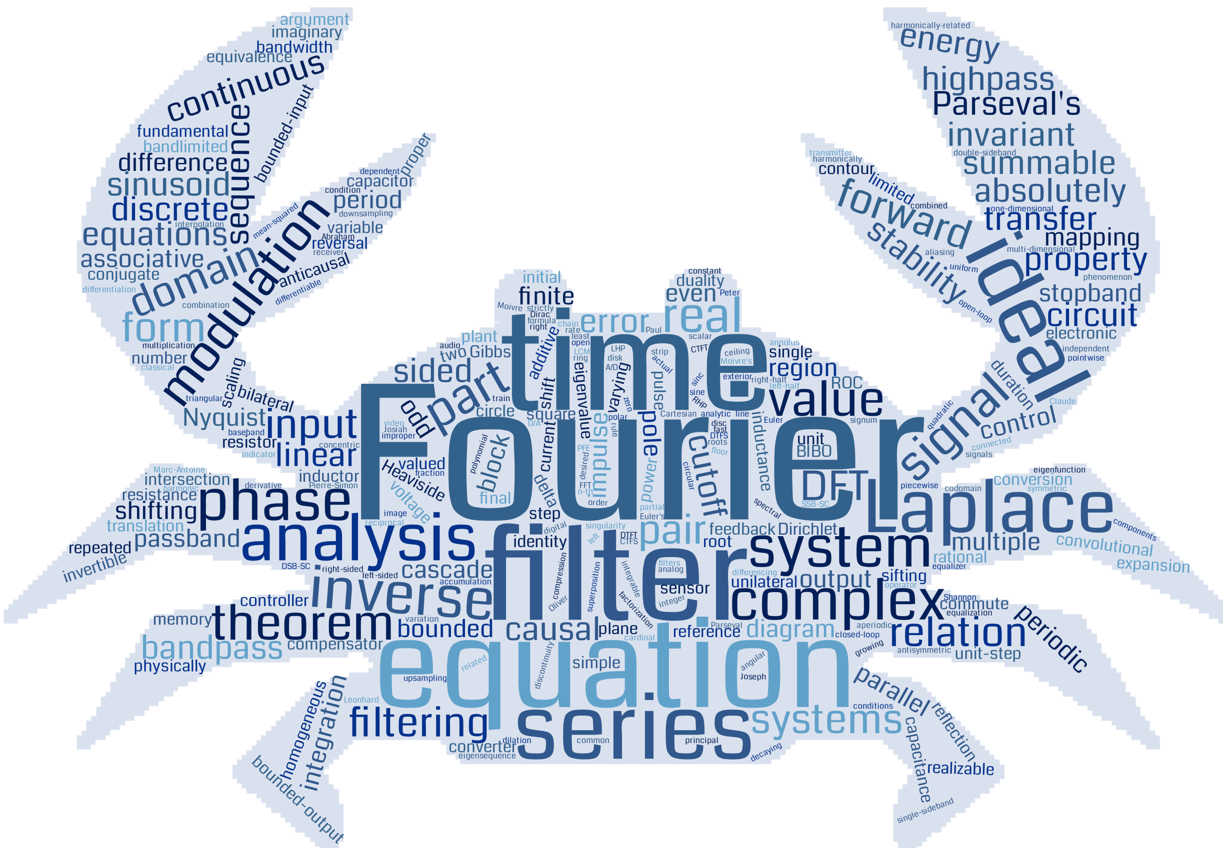
Signals and Systems

Edition 3.0

Michael D. Adams



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Edition 3.0

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Published by the University of Victoria, Victoria, British Columbia, Canada

Photography by Michael Adams

This book was typeset with LaTeX.

ISBN 978-1-55058-673-2 (print)
ISBN 978-1-55058-674-9 (PDF)

To my students, past, present, and future

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Preface

This book is primarily intended to be used as a text for undergraduate students in engineering (and related) disciplines. The book provides a basic introduction to continuous-time and discrete-time signals and systems. Since many engineering curricula use MATLAB as a teaching tool, the book also includes a detailed introduction to MATLAB as an appendix. The earlier editions of this book, which covered only the continuous-time case, evolved from a detailed set of lecture notes that the author prepared in order to teach two different undergraduate courses on continuous-time signals and systems at the University of Victoria (Victoria, BC, Canada). The first version of these lectures notes was developed while the author was teaching ELEC 260 (Signal Analysis) in the Summer 2003 term, and were subsequently augmented in order to accommodate the teaching of ELEC 255 (System Dynamics) in the Fall 2003 term. Over time, the lecture notes underwent many changes, eventually leading to the earlier versions of this book, which covered only the continuous-time case. More recently, coverage of the discrete-time case has been added, resulting in the book that you are now reading.

Acknowledgments

I would like to thank my colleague, Dr. Wu-Sheng Lu, for many interesting technical discussions that helped to clarify some of the finer points of the mathematics behind signals and systems. Also, I would like to thank my past students for their feedback regarding earlier revisions of this manuscript. They have helped me to eliminate numerous errors in this manuscript that would have otherwise gone undetected.

Michael Adams
Victoria, BC
2020-12-15

Guidance for Instructors

The theory of continuous-time (CT) and discrete-time (DT) signals and systems is taught in many engineering and related disciplines. The manner in which this material is split across courses (or terms in the case of multi-term courses) can vary dramatically from one institution or program to another. For this reason, this textbook has been organized in such a way as to accommodate the teaching of courses with a wide variety of structures. Some possibilities include (but are not limited to):

- sequential presentation of the CT and DT cases with the CT case covered first (e.g., two single-term courses, where the first covers the CT case and the second covers the DT case);
- sequential presentation of the CT and DT cases with the DT case covered first (e.g., two single-term courses, where the first covers the DT case and the second covers the CT case);
- integrated presentation of both the CT and DT cases (e.g., a one- or two-term course that covers both the CT and DT cases together).

Sequential Presentation of the CT and DT Cases

The sequential presentation of the CT and DT cases with the CT case treated first might, for example, cover material from the textbook as follows:

1. First Course/Term: The CT Case
   (a) Chapter 1 (Introduction) with an emphasis on the material most relevant to the CT case
   (b) if a review of complex analysis is desired: Appendix A (Complex Analysis)
   (c) Chapter 2 (Preliminaries) with an emphasis on the material most relevant to the CT case
   (d) Chapter 3 (Continuous-Time Signals and Systems)
   (e) Chapter 4 (Continuous-Time Linear Time-Invariant Systems)
   (f) Chapter 5 (Continuous-Time Fourier Series)
   (g) Chapter 6 (Continuous-Time Fourier Transform)
   (h) if an introduction to partial fraction expansions is needed: Appendix B (Partial Fraction Expansions)
   (i) Chapter 7 (Laplace Transform)
2. Second Course/Term: The DT Case
   (a) Chapter 1 (Introduction) with an emphasis on the material most relevant to the DT case
   (b) Chapter 2 (Preliminaries) with an emphasis on the material most relevant to the DT case
   (c) Chapter 8 (Discrete-Time Signals and Systems)
   (d) Chapter 9 (Discrete-Time Linear Time-Invariant Systems)
   (e) Chapter 10 (Discrete-Time Fourier Series)
   (f) Chapter 11 (Discrete-Time Fourier Transform)
   (g) Chapter 12 (z Transform)

To cover the DT case first, one should, for the most part, be able to simply swap the order of the two courses described above, since the textbook tries to minimize dependencies on whether the CT case is covered before the DT case. This said, however, there are a small number of dependencies. To resolve these dependencies, the following changes would need to be made. A small amount of material would need to be moved from the course covering the DT case to the course covering the CT case, namely:

- Section 11.17, which covers the relationship between the discrete-time Fourier transform (DTFT) and continuous-time Fourier series (CTFS); and
- Section 11.18, which covers the relationship between the DTFT and continuous-time Fourier transform (CTFT).

Also, a small amount of material would need to be moved from the course covering the CT case to the course covering the DT case, namely:

- Section 3.5.11, which introduces the delta function.

Integrated Presentation of the CT and DT Cases

The integrated presentation of the CT and DT cases might, for example, cover material from the textbook as follows:

1. First Term
   (a) Chapter 1 (Introduction)
   (b) if a review of complex analysis is desired: Appendix A (Complex Analysis)
   (c) Chapter 2 (Preliminaries)
   (d) Chapter 3 (Continuous-Time Signals and Systems)
   (e) Chapter 8 (Discrete-Time Signals and Systems)
   (f) Chapter 4 (Continuous-Time Linear Time-Invariant Systems)
   (g) Chapter 9 (Discrete-Time Linear Time-Invariant Systems)
   (h) Chapter 5 (Continuous-Time Fourier Series)
   (i) Chapter 10 (Discrete-Time Fourier Series)
2. Second Term
   (a) Chapter 6 (Continuous-Time Fourier Transform)
   (b) Chapter 11 (Discrete-Time Fourier Transform)
   (c) if an introduction to partial fraction expansions is needed: Appendix B (Partial Fraction Expansions)
   (d) Chapter 7 (Laplace Transform)
   (e) Chapter 12 (z Transform)

Video Lectures

Video lectures are available for some of the material covered in this textbook. These video lectures are likely to be helpful to instructors, either for planning their own courses or for using as additional reference material for their students. More information on these video lectures can be found in Appendix G.

About the Author

Michael Adams received the B.A.Sc. degree in computer engineering from the University of Waterloo, Waterloo, ON, Canada in 1993, the M.A.Sc. degree in electrical engineering from the University of Victoria, Victoria, BC, Canada in 1998, and the Ph.D. degree in electrical engineering from the University of British Columbia, Vancouver, BC, Canada in 2002. From 1993 to 1995, Michael was a member of technical staff at Bell-Northern Research in Ottawa, ON, Canada, where he developed real-time software for fiber-optic telecommunication systems. Since 2003, Michael has been on the faculty of the Department of Electrical and Computer Engineering at the University of Victoria, Victoria, BC, Canada, first as an Assistant Professor and currently as an Associate Professor.

Michael is the recipient of a Natural Sciences and Engineering Research Council (of Canada) Postgraduate Scholarship. He has served as a voting member of the Canadian Delegation to ISO/IEC JTC 1/SC 29 (i.e., Coding of Audio, Picture, Multimedia and Hypermedia Information), and been an active participant in the JPEG-2000 standardization effort, serving as co-editor of the JPEG-2000 Part-5 standard and principal author of one of the first JPEG-2000 implementations (i.e., JasPer). His research interests include software, signal processing, image/video/audio processing and coding, multiresolution signal processing (e.g., filter banks and wavelets), geometry processing, and data compression.

Other Works by the Author

Some other open-access textbooks and slide decks by the author of this book include:

1. M. D. Adams, *Lecture Slides for Signals and Systems*, Edition 3.0, University of Victoria, Victoria, BC, Canada, Dec. 2020, xvi + 625 slides, ISBN 978-1-55058-677-0 (print), ISBN 978-1-55058-678-7 (PDF). Available from Google Books, Google Play Books, University of Victoria Bookstore, and author's web site http://www.ece.uvic.ca/~mdadams/sigsysbook.

2. M. D. Adams, *Lecture Slides for Programming in C++ — The C++ Language, Libraries, Tools, and Other Topics (Version 2020-02-29)*, University of Victoria, Victoria, BC, Canada, Feb. 2020, xxii + 2543 slides, ISBN 978-1-55058-663-3 (print), ISBN 978-1-55058-664-0 (PDF). Available from Google Books, Google Play Books, University of Victoria Bookstore, and author's web site http://www.ece.uvic.ca/~mdadams/cppbook.

3. M. D. Adams, *Multiresolution Signal and Geometry Processing: Filter Banks, Wavelets, and Subdivision (Version 2013-09-26)*, University of Victoria, Victoria, BC, Canada, Sept. 2013, xxxviii + 538 pages, ISBN 978-1-55058-507-0 (print), ISBN 978-1-55058-508-7 (PDF). Available from Google Books, Google Play Books, University of Victoria Bookstore, and author's web site http://www.ece.uvic.ca/~mdadams/waveletbook.

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Chapter 1

Introduction

1.1 Signals and Systems

Mathematics has a very broad scope, encompassing many areas such as: linear algebra, calculus, probability and statistics, geometry, differential equations, and numerical methods. For engineers, however, an area of mathematics of particular importance is the one that pertains to signals and systems (which is, loosely speaking, the branch of mathematics known as functional analysis). It is this area of mathematics that is the focus of this book. Before we can treat this topic in any meaningful way, however, we must first explain precisely what signals and systems are. This is what we do next.

1.2 Signals

A **signal** is a function of one or more variables that conveys information about some (usually physical) phenomenon. Some examples of signals include:

- a human voice
- a voltage in an electronic circuit
- the temperature of a room controlled by a thermostat system
- the position, velocity, and acceleration of an aircraft
- the acceleration measured by an accelerometer in a cell phone
- the force measured by a force sensor in a robotic system
- the electromagnetic waves used to transmit information in wireless computer networks
- a digitized photograph
- a digitized music recording
- the evolution of a stock market index over time

1.2.1 Classification of Signals

Signals can be classified based on the number of independent variables with which they are associated. A signal that is a function of only one variable is said to be **one dimensional**. Similarly, a signal that is a function of two or more variables is said to be **multi-dimensional**. Human speech is an example of a one-dimensional signal. In this case, we have a signal associated with fluctuations in air pressure as a function of time. An example of a two-dimensional signal is a monochromatic image. In this case, we have a signal that corresponds to a measure of light intensity as a function of horizontal and vertical position.

A signal can also be classified on the basis of whether it is a function of continuous or discrete variables. A signal that is a function of continuous variables (e.g., a real variable) is said to be **continuous time**. Similarly, a signal that is a function of discrete variables (e.g., an integer variable) is said to be **discrete time**. Although the independent variable need not represent time, for matters of convenience, much of the terminology is chosen as if this were so.

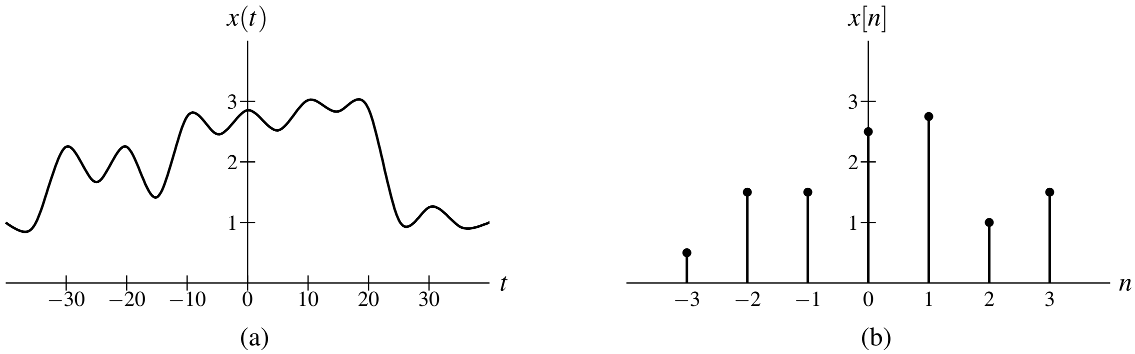


Figure 1.1: Graphical representations of (a) continuous-time and (b) discrete-time signals.

For example, a digital image (which consists of a rectangular array of pixels) would be referred to as a discrete-time signal, even though the independent variables (i.e., horizontal and vertical position) do not actually correspond to time.

If a signal is a function of discrete variables (i.e., discrete-time) and the value of the function itself is also discrete, the signal is said to be **digital**. Similarly, if a signal is a function of continuous variables, and the value of the function itself is also continuous, the signal is said to be **analog**.

Many phenomena in our physical world can be described in terms of continuous-time signals. Some examples of continuous-time signals include: voltage or current waveforms in an electronic circuit; electrocardiograms, speech, and music recordings; position, velocity, and acceleration of a moving body; forces and torques in a mechanical system; and flow rates of liquids or gases in a chemical process. Any signals processed by digital computers (or other digital devices) are discrete-time in nature. Some examples of discrete-time signals include digital video, digital photographs, and digital audio data.

A discrete-time signal may be inherently discrete or correspond to a sampled version of a continuous-time signal. An example of the former would be a signal corresponding to the Dow Jones Industrial Average stock market index (which is only defined on daily intervals), while an example of the latter would be the sampled version of a (continuous-time) speech signal.

1.2.2 Notation and Graphical Representation of Signals

In the case of discrete-time signals, we sometimes refer to the signal as a **sequence**. The nth element of a sequence x is denoted as either $x(n)$ or x_n. Figure 1.1 shows how continuous-time and discrete-time signals are represented graphically.

1.2.3 Examples of Signals

A number of examples of signals have been suggested previously. Here, we provide some graphical representations of signals for illustrative purposes. Figure 1.2 depicts a digitized speech signal. Figure 1.3 shows an example of a monochromatic image. In this case, the signal represents light intensity as a function of two variables (i.e., horizontal and vertical position).

1.3 Systems

A **system** is an entity that processes one or more input signals in order to produce one or more output signals, as shown in Figure 1.4. Such an entity is represented mathematically by a system of one or more equations.

In a communication system, the input might represent the message to be sent, and the output might represent the received message. In a robotics system, the input might represent the desired position of the end effector (e.g., gripper), while the output could represent the actual position.

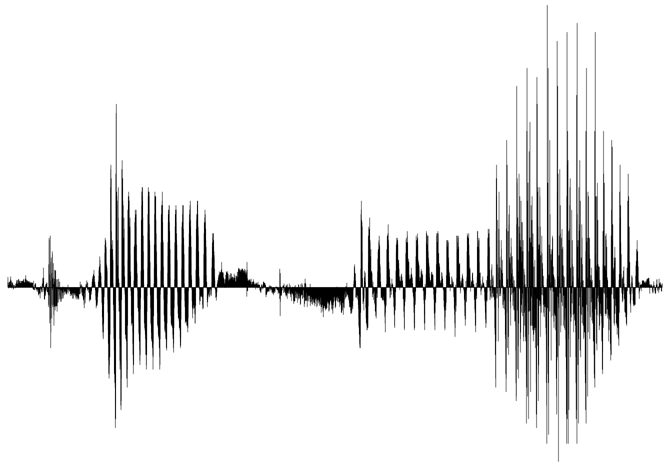


Figure 1.2: Segment of digitized human speech.

Figure 1.3: A monochromatic image.

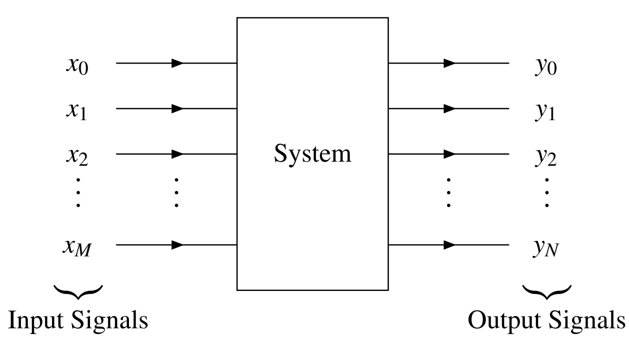


Figure 1.4: System with one or more inputs and one or more outputs.

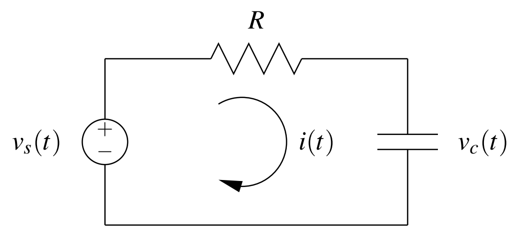


Figure 1.5: A simple RC network.

1.3.1 Classification of Systems

A system can be classified based on the number of inputs and outputs it has. A system with only one input is described as **single input**, while a system with multiple inputs is described as **multi-input**. Similarly, a system with only one output is said to be **single output**, while a system with multiple outputs is said to be **multi-output**. Two commonly occurring types of systems are single-input single-output (SISO) and multi-input multi-output (MIMO).

A system can also be classified based on the types of signals with which it interacts. A system that deals with continuous-time signals is called a **continuous-time system**. Similarly, a system that deals with discrete-time signals is said to be a **discrete-time system**. A system that handles both continuous- and discrete-time signals, is sometimes referred to as a **hybrid system** (or sampled-data system). Similarly, systems that deal with digital signals are referred to as **digital**, while systems that handle analog signals are referred to as **analog**. If a system interacts with one-dimensional signals, the system is referred to as **one-dimensional**. Likewise, if a system handles multi-dimensional signals, the system is said to be **multi-dimensional**.

1.3.2 Examples of Systems

Systems can manipulate signals in many different ways and serve many useful purposes. Sometimes systems serve to extract information from their input signals. For example, in the case of speech signals, systems can be used in order to perform speaker identification or voice recognition. A system might analyze electrocardiogram signals in order to detect heart abnormalities. Amplification and noise reduction are other functionalities that systems could offer.

One very basic system is the resistor-capacitor (RC) network shown in Figure 1.5. Here, the input would be the source voltage v_s and the output would be the capacitor voltage v_c.

Consider the signal-processing systems shown in Figure 1.6. The system in Figure 1.6(a) uses a discrete-time system (such as a digital computer) to process a continuous-time signal. The system in Figure 1.6(b) uses a continuous-time system (such as an analog computer) to process a discrete-time signal. The first of these types of systems is ubiquitous in the world today.

Consider the communication system shown in Figure 1.7. This system takes a message at one location and reproduces this message at another location. In this case, the system input is the message to be sent, and the output is the estimate of the original message. Usually, we want the message reproduced at the receiver to be as close as possible to the original message sent by the transmitter.

A system of the general form shown in Figure 1.8 frequently appears in control applications. Often, in such applications, we would like an output to track some reference input as closely as possible. Consider, for example, a robotics application. The reference input might represent the desired position of the end effector, while the output represents the actual position.

1.4 Why Study Signals and Systems?

As can be seen from the earlier examples, there are many practical applications in which we need to develop systems that manipulate signals. In order to do this, we need a formal mathematical framework for the study of such systems. Such a framework can be used to guide the design of new systems as well as to analyze the behavior of already existing systems. Over time, the complexity of systems designed by engineers has continued to grow. Today, most systems

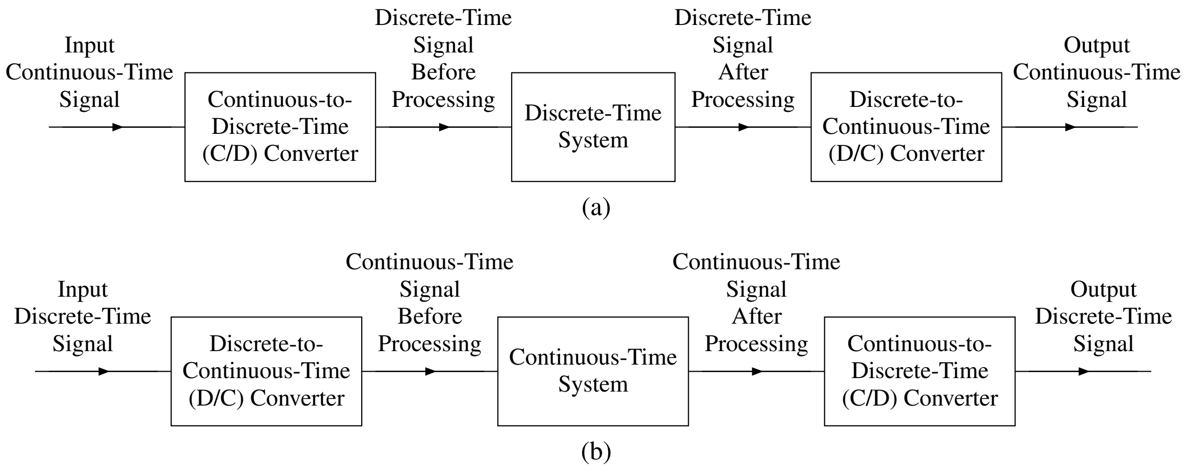


Figure 1.6: Signal processing systems. (a) Processing a continuous-time signal with a discrete-time system. (b) Processing a discrete-time signal with a continuous-time system.

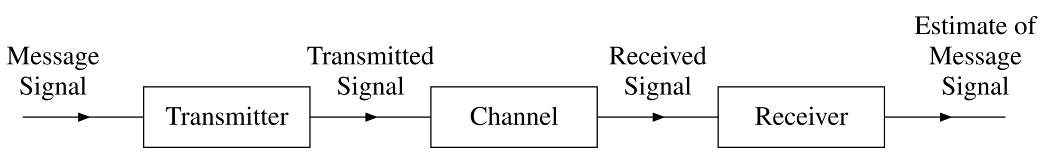


Figure 1.7: Communication system.

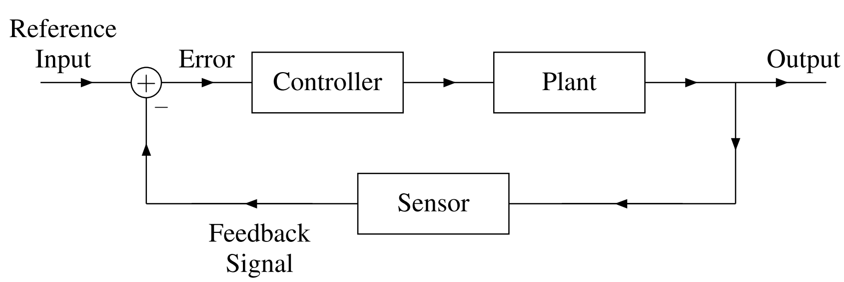


Figure 1.8: Feedback control system.

of practical interest are highly complex. For this reason, a formal mathematical framework to guide the design of systems is absolutely critical. Without such a framework, there is little hope that any system that we design would operate as desired, meeting all of the required specifications.

1.5 Overview of This Book

This book presents the mathematical tools used for studying both signals and systems. Although most systems considered herein are SISO, the mathematics extends in a very straightforward manner to systems that have multiple inputs and/or multiple outputs. Only the one-dimensional case is considered herein, however, as the multi-dimensional case is considerably more complicated and beyond the scope of this book.

The remainder of this book is organized as follows. Chapter 2 presents some mathematical preliminaries that are essential for both the continuous-time and discrete-time cases. Then, Chapters 3, 4, 5, 6, and 7 cover material that relates primarily to the continuous-time case. Then, Chapters 8, 9, 10, 11, and 12 cover material that relates primarily to the discrete-time case.

The material on continuous-time signals and systems consists of the following. Chapter 3 examines signals and systems in more depth than covered in earlier chapters. A particular class of systems known as (continuous-time) linear time-invariant (LTI) systems is introduced. Chapter 4 then studies continuous-time LTI systems in depth. Finally, Chapters 5, 6, and 7 introduce continuous-time Fourier series, the continuous-time Fourier transform, and the Laplace transform, respectively, which are important mathematical tools for studying continuous-time signals and systems.

The material on discrete-time signals and systems consists of the following. Chapter 8 examines discrete-time signals and systems in more depth than earlier chapters. A particular class of systems known as (discrete-time) linear time-invariant (LTI) systems is introduced. Chapter 9 then studies discrete-time LTI systems in depth. Finally, Chapters 10, 11, and 12 introduce discrete-time Fourier series, the discrete-time Fourier transform, and the z transform, respectively, which are important mathematical tools for studying discrete-time signals and systems.

This book also includes several appendices, which contain supplemental material related to the topics covered in the main chapters of the book. Appendix A provides a review of complex analysis. Appendix B introduces partial fraction expansions. Appendix C presents time-domain methods for solving for differential equations. Appendix D presents a detailed introduction to MATLAB. Appendix E provides some additional exercises. Appendix F offers a list of some useful mathematical formulas. Finally, Appendix G presents some information about video lectures available for this book.

Chapter 2

Preliminaries

2.1 Overview

Before we can proceed to study signals and systems in more depth, we must first establish some basic mathematical preliminaries. In what follows, we introduce some background related to sets, mappings, functions, sequences, operators, and transforms. Also, we present some basic properties of functions and sequences.

2.2 Sets

As a matter of terminology, a **rational number** is a number of the form x/y, where x and y are integers and $y \neq 0$. That is, a rational number is a ratio of integers (that does not result in division by zero). For example, $\frac{1}{2}$, $5 = \frac{5}{1}$, and $-\frac{2}{3}$ are rational numbers. In contrast, π is not a rational number, as it cannot be represented exactly as a ratio of integers.

The sets of integers, rational numbers, real numbers, and complex numbers are denoted as $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. It is important to note that the sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, do not include infinity. The omission of infinity from these sets is necessary in order to allow for a consistent system with respect to all of the usual arithmetic operations (i.e., addition, subtraction, multiplication, and division). For readers not familiar with complex numbers, detailed coverage of complex numbers can be found in Appendix A.

Often, it is convenient to be able to concisely denote a set consisting of a consecutive range of integers. Consequently, we introduce some notation for this purpose. For two integers a and b, we define the following notation for sets of consecutive integers:

$$\begin{aligned}
[a\,..\,b] &= \{x \in \mathbb{Z} : a \leq x \leq b\}, \\
[a\,..\,b) &= \{x \in \mathbb{Z} : a \leq x < b\}, \\
(a\,..\,b] &= \{x \in \mathbb{Z} : a < x \leq b\}, \quad \text{and} \\
(a\,..\,b) &= \{x \in \mathbb{Z} : a < x < b\}.
\end{aligned}$$

In this notation, a and b indicate the endpoints of the range for the set, and the type of brackets used (i.e., parenthesis versus square bracket) indicates whether each endpoint is included in the set. A square bracket indicates that the corresponding endpoint is included in the set, while a parenthesis (i.e., round bracket) indicates that the corresponding endpoint is not included. For example, we have that:

- $[0\,..\,4]$ denotes the set of integers $\{0,1,2,3,4\}$,
- $[0\,..\,4)$ denotes the set of integers $\{0,1,2,3\}$,
- $(0\,..\,4)$ denotes the set of integers $\{1,2,3\}$, and
- $(0\,..\,4]$ denotes the set of integers $\{1,2,3,4\}$.

The variants of this notation that exclude one or both of the endpoints of the range are typically useful when excluding an endpoint would result in a simpler expression. For example, if one wanted to denote the set $\{0,1,2,\ldots,N-1\}$, we could write this as either $[0\,..\,N)$ or $[0\,..\,N-1]$, but the first of these would likely be preferred since it is more compact.

Often, it is convenient to be able to concisely denote an interval (i.e., range) on the real line. Consequently, we introduce some notation for this purpose. For two real numbers a and b, we define the following notation for intervals:

$$\begin{aligned}
[a,b] &= \{x \in \mathbb{R} : a \leq x \leq b\}, \\
(a,b) &= \{x \in \mathbb{R} : a < x < b\}, \\
[a,b) &= \{x \in \mathbb{R} : a \leq x < b\}, \quad \text{and} \\
(a,b] &= \{x \in \mathbb{R} : a < x \leq b\}.
\end{aligned}$$

In this notation, a and b indicate the endpoints of the interval for the set, and the type of brackets used (i.e., parenthesis versus square bracket) indicates whether each endpoint is included in the set. A square bracket indicates that the corresponding endpoint is included in the set, while a parenthesis (i.e., round bracket) indicates that the corresponding endpoint is not included. For example, we have that:

- $[0,100]$ denotes the set of all real numbers from 0 to 100, including both 0 and 100;
- $(-\pi,\pi]$ denotes the set of all real numbers from $-\pi$ to π, excluding $-\pi$ but including π;
- $[-\pi,\pi)$ denotes the set of all real numbers from $-\pi$ to π, including $-\pi$ but excluding π; and
- $(0,1)$ denotes the set of all real numbers from 0 to 1, excluding both 0 and 1.

2.3 Mappings

A **mapping** is a relationship involving two sets that associates each element in one set, called the **domain**, with an element from the other set, called the **codomain**. The notation $f : A \to B$ denotes a mapping f whose domain is the set A and whose codomain is the set B. An example of a very simple mapping is given below.

Example 2.1. Let A and B be the sets given by

$$A = \{1,2,3,4\} \quad \text{and} \quad B = \{0,1,2,3\}.$$

Let $f : A \to B$ such that

$$f(x) = \begin{cases} 0 & x \in \{1,2\} \\ 1 & x = 4 \\ 2 & x = 3. \end{cases}$$

The mapping f maps each element x in the domain A to the element $f(x)$ in the codomain B. The mapping f is expressed in pictorial form in Figure 2.1. ■

The mapping in the above example is quite simple. So, we now consider a few more interesting examples of mappings. The sine function (i.e., sin) is an example of a mapping. For each real number t, the sine function maps t to the sine of t (i.e., $\sin t$). In the case of the sine function, both the domain and codomain are the real numbers. As another example, consider the function trunc that rounds a real number t to an integer by discarding the fractional part of t (i.e., trunc rounds towards zero). The trunc function is a mapping where the domain and codomain are the real numbers and integers, respectively.

Although very many types of mappings exist, the types of most relevance to our study of signals and systems are: functions, sequences, system operators, and transforms. In the sections that follow, we will introduce (or further elaborate upon) each of these types of mappings.

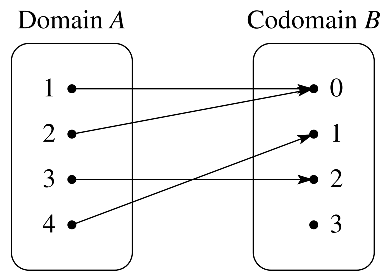


Figure 2.1: The mapping f.

2.4 Functions

A **function** is a mapping where the domain is a set that is continuous in nature, such as the real numbers, complex numbers, or a subset of either of these. In practice, the codomain is typically either the real numbers or complex numbers. Functions are also commonly referred to as **continuous-time (CT) signals**. Herein, we focus mostly on functions of a single independent variable (i.e., one-dimensional functions). The trigonometric sine (i.e., sin) function is an example of a function. It associates each real number t with the real number $\sin t$. In this case, the domain and codomain are the real numbers.

For a function f, the expression "$f(t)$" denotes the value of the function f evaluated at t. Note that "$f(t)$" does not denote the function f itself. This distinction is an extremely important one to make—a point that we shall revisit shortly. Normally, the parameters to a function are placed in (round, square, or curly) brackets. For standard mathematical functions with only a single parameter (e.g., sin and cos), brackets generally tend to be omitted when not needed for grouping. That is, we typically write "$\sin t$" instead of "$\sin(t)$". In the case of a non-standard function like f, we would always write "$f(t)$", not "ft".

Many notational schemes can be used to specify a function mathematically. In engineering, however, it is arguably most common to specify a function in terms of a defining equation. For example, we can specify a function f that maps the real number t to t^2 (i.e., a squaring function) using the equation $f(t) = t^2$. Here, f is the function, and t is a dummy variable that is used only for the purposes of writing the equation $f(t) = t^2$ that defines the function f. Since t is only a dummy variable, the equations $f(t) = t^2$ and $f(x) = x^2$ define exactly the same function. Note that "f" and "$f(t) = t^2$" are, strictly speaking, different things. That is, "f" is the function itself whereas "$f(t) = t^2$" is not the function f but rather an equation that defines the function f. So, strictly speaking, to define a function f, we should use wording like: "let f be the function defined by the equation $f(t) = \ldots$". Often, in practice, we abuse notation somewhat and simply write "the function $f(t) = \ldots$" or simply "$f(t) = \ldots$". Fortunately, this type notational abuse does not usually lead to problems in most cases.

Since notational conventions play a crucial role, it is worthwhile to take some time in order to clearly introduce such conventions. In what follows, we present several examples to illustrate various aspects of the notation associated with functions.

Example 2.2. Let f and g each denote a real-valued function of a real variable. Let t denote an arbitrary real number. The expression $f+g$ denotes a *function*, namely, the function formed by adding the functions f and g. The expression $f(t)+g(t)$ denotes a *number*, namely, the sum of: 1) the value of the function f evaluated at t; and 2) the value of the function g evaluated at t. The expression $(f+g)(t)$ denotes the result obtained by: 1) first computing a new function h that is the sum of the functions f and g; and 2) then evaluating h at t. Note that the meanings of the expressions $(f+g)(t)$ and $f(t)+g(t)$ are subtly different. In the first case, the addition operation is being applied to two functions, while in the second case, the addition operation is being applied to two numbers. Although the meanings of these two expressions are subtly different, they are always equal. In other words, it is always true that

$$(f+g)(t) = f(t)+g(t) \quad \text{for all } t.$$

This is due to the fact that preceding equation is precisely how the addition of functions is defined. In other words, we add functions by adding their values at corresponding points (i.e., the addition of functions is defined in a pointwise manner). ■

Example 2.3. For two functions x_1 and x_2, the expression x_1+x_2 denotes the function that is the sum of the functions x_1 and x_2. The expression $(x_1+x_2)(t)$ denotes the function x_1+x_2 evaluated at t. Since the addition of functions can be defined pointwise (i.e., we can add two functions by adding their values at corresponding pairs of points), the following relationship always holds:

$$(x_1+x_2)(t) = x_1(t)+x_2(t) \quad \text{for all } t.$$

Similarly, since subtraction, multiplication, and division can also defined pointwise, the following relationships also hold:

$$(x_1-x_2)(t) = x_1(t)-x_2(t) \quad \text{for all } t,$$
$$(x_1x_2)(t) = x_1(t)x_2(t) \quad \text{for all } t, \quad \text{and}$$
$$(x_1/x_2)(t) = x_1(t)/x_2(t) \quad \text{for all } t.$$

It is important to note, however, that not all mathematical operations involving functions can be defined in a pointwise manner. That is, some operations fundamentally require that their operands be functions. The convolution operation (for functions), which will be considered later, is one such example. If some operator, which we denote for illustrative purposes as "$\diamond$", is defined in such a way that it can only be applied to functions, then the expression $(x_1 \diamond x_2)(t)$ is mathematically valid, but the expression $x_1(t) \diamond x_2(t)$ is not. The latter expression is not valid since the $\diamond$ operator requires two functions as operands, but the provided operands $x_1(t)$ and $x_2(t)$ are numbers (namely, the values of the functions x_1 and x_2 each evaluated at t). Due to issues like this, one must be careful in the use of mathematical notation related to functions. Otherwise, it is easy to fall into the trap of writing expressions that are ambiguous, contradictory, or nonsensical. ■

2.5 Sequences

A **sequence** is a mapping where the domain is a set that is discrete in nature, such as the integers, or a subset thereof. In practice, the codomain is typically either the real numbers or complex numbers. Sequences are also commonly referred to as **discrete-time (DT) signals**. Herein, we focus mostly on sequences with a single independent variable (i.e., one-dimensional sequences). The sequence $1^2, 2^2, 3^2, 4^2, \ldots$ of perfect squares is an example of a sequence. It associates each (strictly) positive integer n with the integer n^2. In this case, the domain and codomain are the positive integers.

Similar comments as above also apply to expressions involving sequences. For a sequence f, the expression "$f(n)$" denotes the value of the sequence f evaluated at n, whereas the expression "f" denotes the sequence f itself. It is often critically important to make a clear distinction between a sequence and its value. A sequence is typically specified by a defining equation. For example, a sequence f that maps the integer n to n^2 (i.e., a sequence of squares) can be specified using the equation $f(n) = n^2$.

Since notational conventions play a crucial role, it is worthwhile to take some time in order to clearly introduce such conventions. In what follows, we present several examples to illustrate various aspects of the notation associated with sequences.

Example 2.4. Let f and g each denote a real-valued sequence with an integer index. Let n denote an arbitrary integer. The expression fg denotes a *sequence*, namely, the sequence formed by multiplying the sequences f and g. The expression $f(n)g(n)$ denotes a *number*, namely, the product of: 1) the value of the sequence f evaluated at n; and 2) the value of the sequence g evaluated at n. The expression $(fg)(n)$ denotes the result obtained by: 1) first computing a new sequence h that is the product of the sequences f and g; and 2) then evaluating h at n. Note that the meanings of the expressions $(fg)(n)$ and $f(n)g(n)$ are subtly different. In the first case, the multiplication operation is being applied to two sequences, while in the second case, the multiplication operation is being applied to two numbers. Although the meanings of these two expressions are subtly different, they are always equal. In other words, it is always true that

$$(fg)(n) = f(n)g(n) \quad \text{for all } n.$$

This is due to the fact that preceding equation is precisely how the multiplication of sequences is defined. In other words, we multiply sequences by multiplying their values at corresponding points (i.e., the multiplication of sequences is defined in a pointwise manner). ■

Example 2.5. For two sequences x_1 and x_2, we have two ways in which we can express the equality of these sequences. First, we can simply write that $x_1 = x_2$. Second, we can write that $x_1(n) = x_2(n)$ for all n. The first approach is probably preferable since it is less verbose. ■

Example 2.6. For two sequences x_1 and x_2, the expression $x_1 + x_2$ denotes the sequence that is the sum of the sequences x_1 and x_2. The expression $(x_1 + x_2)(n)$ denotes the sequence $x_1 + x_2$ evaluated at n. Since the addition of sequences can be defined pointwise (i.e., we can add two sequences by adding their values at corresponding pairs of points), the following relationship always holds:

$$(x_1 + x_2)(n) = x_1(n) + x_2(n) \quad \text{for all } n.$$

Similarly, since subtraction, multiplication, and division can also defined pointwise, the following relationships also hold:

$$\begin{aligned} (x_1 - x_2)(n) &= x_1(n) - x_2(n) \quad \text{for all } n, \\ (x_1x_2)(n) &= x_1(n)x_2(n) \quad \text{for all } n, \quad \text{and} \\ (x_1/x_2)(n) &= x_1(n)/x_2(n) \quad \text{for all } n. \end{aligned}$$

It is important to note, however, that not all mathematical operations involving sequences can be defined in a pointwise manner. That is, some operations fundamentally require that their operands be sequences. The convolution operation (for sequences), which will be considered later, is one such example. If some operator, which we denote for illustrative purposes as "$\diamond$", is defined in such a way that it can only be applied to sequences, then the expression $(x_1 \diamond x_2)(n)$ is mathematically valid, but the expression $x_1(n) \diamond x_2(n)$ is not. The latter expression is not valid since the $\diamond$ operator requires two sequences as operands, but the provided operands $x_1(n)$ and $x_2(n)$ are numbers (namely, the values of the sequences x_1 and x_2 each evaluated at n). Due to issues like this, one must be careful in the use of mathematical notation related to sequences. Otherwise, it is easy to fall into the trap of writing expressions that are ambiguous, contradictory, or nonsensical. ■

2.6 Remarks on Abuse of Notation

Unfortunately, for a function f, it is common practice for engineers to abuse mathematical notation and write "$f(t)$" to refer to the function f itself. A similar issue also exists for the case of sequences. The abuse of mathematical notation, however, can often lead to trouble. In some contexts, it is critically important to make a clear distinction between a function (or sequence) and its value, and failing to do so can lead to many problems, such as writing mathematical expressions that are ambiguous, contradictory, or nonsensical. For this reason, it is strongly recommended that one try to maintain a clear distinction between a function (or sequence) and its value.

With the above said, notational abuse in trivial cases is not likely to cause problems. For example, if we simply write "the function $f(t)$" instead of more correctly writing the "the function f", this is unlikely to cause confusion. Where notational abuse is much more likely to become problematic is when the expressions that are being referred to as functions contain mathematical operations or more than one variable, such as expressions like: $f(t-1)$, $f(7t)$, $f(at-b)$, $tf(t)$, and $f(t-\tau)$. In cases like these, abuse of notation makes the intended meaning much less clear, opening the possibility of misinterpretation. For example, in the case of "$f(t-\tau)$", more than one reasonable interpretation exists if one allows notation to be abused. In particular, "$f(t-\tau)$" might mean:

1. a number that is equal to the function f evaluated at $t-\tau$;
2. an anonymous (i.e., unnamed) function that is equal to the function g, where $g(x) = f(x-\tau)$ (i.e., t and τ are interpreted as a variable and constant, respectively);
3. an anonymous function that is equal to the function g, where $g(x) = f(t-x)$ (i.e., t and τ are interpreted as a constant and variable, respectively);

Table 2.1: Examples of dot notation for functions and sequences. Examples for (a) functions and (b) sequences.

(a)

Named Function f	Corresponding Unnamed Function
$f(t) = g(t)$	$g(\cdot)$
$f(t) = t^2$	$(\cdot)^2$
$f(t) = \sqrt[3]{t}$	$\sqrt[3]{\cdot}$
$f(t) = e^t$	$e^{(\cdot)}$
$f(t) = \lvert t\rvert$	$\lvert\cdot\rvert$
$f(t) = t^2 + 3t + 1$	$(\cdot)^2 + 3 \cdot +1$
$f(t) = g(at - b)$	$g(a \cdot -b)$
$f(t) = g(t-1)$	$g(\cdot - 1)$
$f(t) = g(3t)$	$g(3\cdot)$

(b)

Named Sequence f	Corresponding Unnamed Sequence
$f(n) = g(n)$	$g(\cdot)$
$f(n) = n^2$	$(\cdot)^2$
$f(n) = \sin(\frac{2\pi}{3}n)$	$\sin[\frac{2\pi}{3}(\cdot)]$
$f(n) = g(3n)$	$g(3\cdot)$
$f(n) = g(n-1)$	$g(\cdot - 1)$
$f(n) = ng(n)$	$(\cdot)g(\cdot)$
$f(n) = 2n^2 + n + 5$	$2(\cdot)^2 + (\cdot) + 5$

4. an anonymous function that is equal to the constant function g, where $g(x) = f(t - \tau)$ (i.e., t and τ are both interpreted as constants).

In circumstances like this one, notational problems can be easily avoided by simply specifying the desired function in terms of an equation. In other words, we can give a name to the function being specified and then define the function in terms of an equation using the given name. For example, instead of saying "the function $f(t-\tau)$", we can say "the function $g(t) = f(t-\tau)$". This latter notation makes clear that τ is a constant, for example. As another example, instead of saying "the function $f(t-1)$", we can say "the function $g(t) = f(t-1)$".

Due to problems like those above, great care must be exercised when using anonymous functions in order to avoid ambiguous notation. Since ambiguous notation is a frequent source of problems, the author would suggest that anonymous functions are best avoided in most circumstances.

2.7 Dot Notation for Functions and Sequences

Sometimes a situation may arise where one would like to distinguish a function from the value of a function, but without resorting to giving the function a name or other more verbose notational approaches. A similar comment also applies for the case of sequences. In this regard, the dot notation for functions and sequences is quite useful. If we wish to indicate that an expression corresponds to a function (as opposed to the value of a function), we can denote this using the interpunct symbol (i.e., "·"). In each place where the variable for the function would normally appear, we simply replace it with an interpunct symbol (i.e., "·"). For example, $\sqrt{\cdot}$ denotes the square root function, whereas $\sqrt{t}$ denotes the value of the square root function evaluated at t. Some additional examples of the dot notation for functions can be found in Table 2.1(a). A similar convention can also be applied to sequences. Some examples of the dot notation for sequences can be found in Table 2.1(b). Since some readers may find this dot notation to be somewhat strange, this book minimizes the use of this notation. It is, however, used in a few limited number of places in order to achieve clarity without the need for being overly verbose. Although the dot notation may appear strange at first, it is a very commonly used notation by mathematicians. Sadly, it is not used as much by engineers, in spite of its great utility.

2.8 System Operators

A system operator is a mapping used to represent a system. In what follows, we will focus exclusively on the case of single-input single-output systems, since this case is our primary focus herein. A (single-input single-output) **system operator** maps a function or sequence representing the input of a system to a function or sequence representing the output of the system. For example, the system $\mathcal{H}$ that maps a function to a function and is given by

$$\mathcal{H}x(t) = 2x(t)$$

multiplies its input function x by a factor of 2 in order to produce its output function. The system $\mathcal{H}$ that maps a sequence to a sequence and is given by

$$\mathcal{H}x(n) = x(n) + 1$$

adds one to its input sequence x in order to produce its output sequence.

For a system operator $\mathcal{H}$ and function x, the expression $\mathcal{H}(x)$ denotes the output produced by the system $\mathcal{H}$ when the input is the function x. Since only a single symbol x follows the operator $\mathcal{H}$, there is only one way to group the operations in this expression. Therefore, the parentheses can be omitted without any risk of changing the meaning of the expression. In other words, we can equivalently write $\mathcal{H}(x)$ as $\mathcal{H}x$. Since $\mathcal{H}x$ is a function, we can evaluate this function at t, which corresponds to the expression $(\mathcal{H}x)(t)$. We can omit the first set of parentheses in this expression without changing its meaning. In other words, the expressions $(\mathcal{H}x)(t)$ and $\mathcal{H}x(t)$ have identical meanings. This is due to the fact that there is only one possible way to group the operations in $\mathcal{H}x(t)$ that is mathematically valid. For example, the grouping $\mathcal{H}[x(t)]$ is not mathematically valid since $\mathcal{H}$ must be provided a function as an operand, but the provided operand $x(t)$ is a number (namely, the value of the function x evaluated at t).

Again, since notational conventions play a crucial role, it is worthwhile to take some time in order to clearly introduce such conventions. In what follows, we present several examples to illustrate various aspects of the notation associated with system operators.

Example 2.7. For a system operator $\mathcal{H}$, a function x, a real variable t, and a real constant t_0, the expression $\mathcal{H}x(t-t_0)$ denotes the result obtained by taking the function y produced as the output of the system $\mathcal{H}$ when the input is the function x and then evaluating y at $t-t_0$. ■

Example 2.8. For a system operator $\mathcal{H}$, function x', and real number t, the expression $\mathcal{H}x'(t)$ denotes result of taking the function y produced as the output of the system $\mathcal{H}$ when the input is the function x' and then evaluating y at t. ■

Example 2.9. For a system operator $\mathcal{H}$, function x, and a complex constant a, the expression $\mathcal{H}(ax)$ denotes the output from the system $\mathcal{H}$ when the input is the function ax (i.e., a times x). ■

Example 2.10. For a system operator $\mathcal{H}$ and the functions x_1 and x_2, the expression $\mathcal{H}(x_1+x_2)$ denotes the output produced by the system $\mathcal{H}$ when the input is the function x_1+x_2. Note that, in this case, we cannot omit the parentheses without changing the meaning of the expression. That is, $\mathcal{H}x_1+x_2$ means $(\mathcal{H}x_1)+x_2$, which denotes the sum of the function x_2 and the output produced by the system $\mathcal{H}$ when the input is the function x_1. ■

Example 2.11. For a system operator $\mathcal{H}$ and the functions x_1 and x_2, and the complex constants a_1 and a_2, the expression $\mathcal{H}(a_1x_1+a_2x_2)$ denotes the output produced by the system $\mathcal{H}$ when the input is the function $a_1x_1 + a_2x_2$. ■

Example 2.12. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the operators representing two systems and let x denote a function. Consider the expression $\mathcal{H}_2\mathcal{H}_1x$. The implied grouping of operations in this expression is $\mathcal{H}_2(\mathcal{H}_1x)$. So, this expression denotes the output produced by the system $\mathcal{H}_2$ when its input is the function y, where y is the output produced by the system $\mathcal{H}_1$ when its input is the function x. ■

2.9 Transforms

Later, we will be introduced to several types of mappings known as transforms. Transforms have a mathematical structure similar to system operators. That is, transforms map functions/sequences to functions/sequences. Due to

Table 2.2: Examples of transforms

Name	Domain	Codomain
CT Fourier Series	T-periodic functions (with domain $\mathbb{R}$)	sequences (with domain $\mathbb{Z}$)
CT Fourier Transform	functions (with domain $\mathbb{R}$)	functions (with domain $\mathbb{R}$)
Laplace Transform	functions (with domain $\mathbb{R}$)	functions (with domain $\mathbb{C}$)
DT Fourier Series	N-periodic sequences (with domain $\mathbb{Z}$)	N-periodic sequences (with domain $\mathbb{Z}$)
DT Fourier Transform	sequences (with domain $\mathbb{Z}$)	2π-periodic functions (with domain $\mathbb{R}$)
Z Transform	sequences (with domain $\mathbb{Z}$)	functions (with domain $\mathbb{C}$)

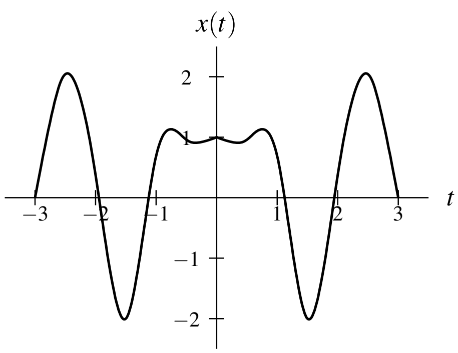


Figure 2.2: Example of an even function.

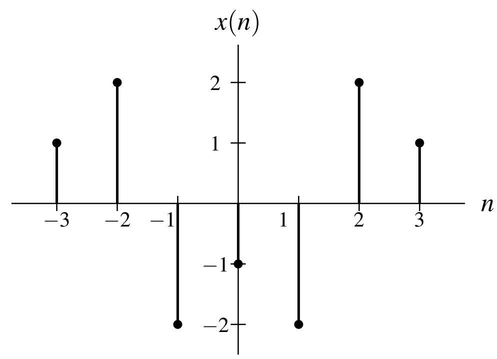


Figure 2.3: Example of an even sequence.

this similar structure, many of the earlier comments about system operators also apply to the case of transforms. Some examples of transforms of interest in the study of signals and systems are listed in Table 2.2. For example, the Fourier transform (introduced later) is denoted as $\mathcal{F}$ and the result of applying the Fourier transform operator to the function/sequence x is denoted as $\mathcal{F}x$.

2.10 Basic Signal Properties

Signals can possess a number of interesting properties. In what follows, we introduce a few very basic properties that are frequently useful.

2.10.1 Symmetry of Functions and Sequences

A function x is said to be **even** if it satisfies

$$x(t) = x(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).} \tag{2.1}$$

Similarly, a sequence x is said to be **even** if it satisfies

$$x(n) = x(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).} \tag{2.2}$$

Geometrically, an even function or sequence is symmetric with respect to the vertical axis. Examples of an even function and sequence are given in Figures 2.2 and 2.3. Some other examples of even functions include the cosine, absolute value, and square functions. Some other examples of even sequences include the unit-impulse and rectangular sequences (to be introduced later).

A function x is said to be **odd** if it satisfies

$$x(t) = -x(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).} \tag{2.3}$$

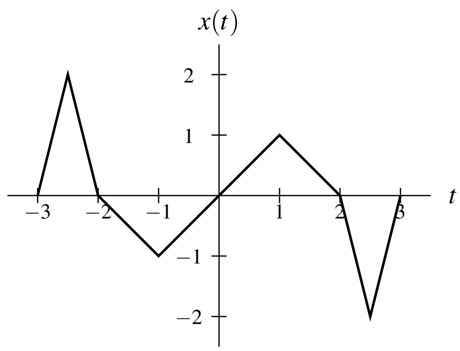


Figure 2.4: Example of an odd function.

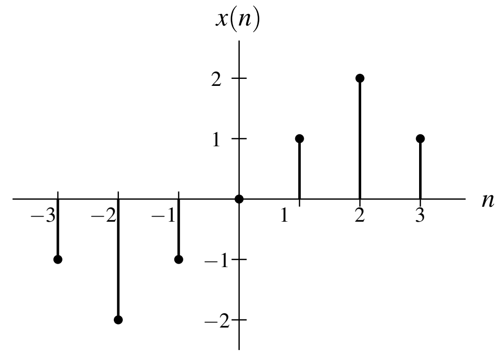


Figure 2.5: Example of an odd sequence.

Similarly, a sequence x is said to be **odd** if it satisfies

$$x(n) = -x(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).} \tag{2.4}$$

Geometrically, an odd function or sequence is symmetric with respect to the origin. Examples of an odd function and odd sequence are shown in Figures 2.4 and 2.5. One can easily show that an odd function or sequence x must be such that $x(0) = 0$, assuming that the domain of x includes 0 (i.e., $x(0)$ is defined). Some other examples of odd functions include the sine, signum, and cube (i.e., $x(t) = t^3$) functions.

A function x is said to be **conjugate symmetric** if it satisfies

$$x(t) = x^*(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).} \tag{2.5}$$

Similarly, a sequence x is said to be **conjugate symmetric** if it satisfies

$$x(n) = x^*(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).} \tag{2.6}$$

An example of a conjugate symmetric function is a complex sinusoid $x(t) = \cos(\omega t) + j\sin(\omega t)$, where ω is a real constant. The real part of a conjugate symmetric function or sequence is even. The imaginary part of a conjugate symmetric function or sequence is odd.

2.10.2 Periodicity of Functions and Sequences

A function x is said to be **periodic** with **period** T (or simply T**-periodic**) if, for some strictly positive real constant T,

$$x(t) = x(t+T) \quad \text{for all } t \text{ (where } t \text{ is a real number).} \tag{2.7}$$

In other words, the graph of a T-periodic function repeats in value every T units along the horizontal axis. A T-periodic function x is said to have the **frequency** $\frac{1}{T}$ and **angular frequency** $\frac{2\pi}{T}$. Similarly, a sequence x is said to be **periodic** with **period** N (or simply N**-periodic**) if, for some strictly positive integer N,

$$x(n) = x(n+N) \quad \text{for all } n \text{ (where } n \text{ is an integer).} \tag{2.8}$$

An N-periodic sequence x is said to have **frequency** $\frac{1}{N}$ and **angular frequency** $\frac{2\pi}{N}$. A function or sequence that is not periodic is said to be **aperiodic**. Examples of a periodic function and sequence are shown in Figures 2.6 and 2.7. Some other examples of periodic functions include the cosine and sine functions.

The period of a periodic function or sequence is not uniquely determined. In particular, a function or sequence that is periodic with period T is also periodic with period kT, for every strictly positive integer k. In most cases, we are interested in the smallest (positive) value of T or N for which (2.7) or (2.8) is satisfied, respectively. We refer to this value as the **fundamental period**. Moreover, the frequency corresponding to the fundamental period is called the **fundamental frequency**.

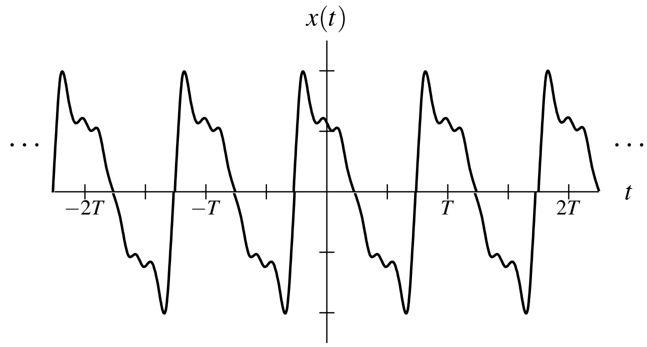


Figure 2.6: Example of a periodic function (with a fundamental period of T).

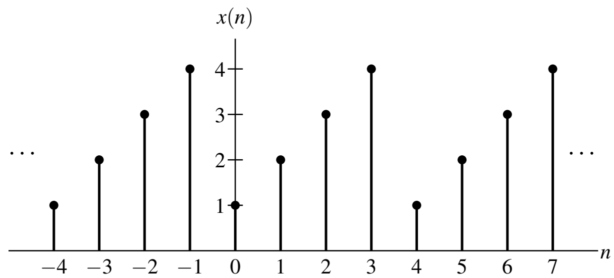


Figure 2.7: Example of a periodic sequence (with a fundamental period of 4).

Consider, for example, the periodic function x shown in Figure 2.6. The function x is periodic with periods T, $2T, 3T$, and so on. The fundamental period of x, however, is uniquely determined and has the value T.

When a function or sequence is periodic, it cycles through a particular set of values in each period. Normally, we are interested simply in the rate of oscillation (i.e., the number of cycles completed per unit time). In some contexts, however, it can be meaningful to associate the oscillations with a direction (or order). For example, a function or sequence might cycle through the set of values in each period in either the forward or reverse direction. In contexts where oscillations can be associated with a direction and this direction is of key importance, it is often convenient to attach a sign (i.e., positive or negative) to the frequency to indicate the direction in which periodic cycles occur. This leads to the notion of signed frequency. That is, a **signed frequency** is simply a quantity whose magnitude corresponds to the rate of oscillation (i.e., number of cycles per unit time) and whose sign (i.e., positive or negative) indicates the direction of oscillation.

To give a more concrete example, consider a wheel rotating at some fixed rate. Not only might we be interested in how many times the wheel rotates per second (i.e., what we would normally refer to as frequency), we might also be interested in whether the wheel is turning in the counterclockwise or clockwise direction. In this case, we could employ the notion of signed frequency. That is, we could define the frequency as a signed quantity, where the magnitude of the signed frequency indicates the oscillation rate (i.e., the number of rotations per second), while the sign (i.e., positive or negative) of the signed frequency indicates the direction of rotation. Suppose that we adopt the convention that the counterclockwise and clockwise directions correspond to positive and negative signed frequencies, respectively. Then, a signed frequency of 10 would correspond to 10 rotations of the wheel per second in the counterclockwise direction, while a signed frequency of -10 would correspond to 10 rotations per second in the clockwise direction.

Often, in the interest of brevity, we simply refer to “signed frequency” as “frequency”. This does not normally cause confusion, since it is usually clear from the context when signed frequency is being employed.

2.11 Exercises

2.11.1 Exercises Without Answer Key

2.1 Let each of $\mathcal{G}$ and $\mathcal{H}$ denote a system operator that maps a function to a function; let x and y denote functions; and assume that all other variables denote numbers. Fully parenthesize each of the expressions below in order to show the implied grouping of all operations.
(a) $\mathcal{H}x(t) = t^2 + 1$;
(b) $\mathcal{G}\mathcal{H}y(t)$;
(c) $\mathcal{H}x + y$; and
(d) $x\mathcal{H}\mathcal{G}y$.

2.2 Let $\mathcal{H}$ denote a system operator that maps a function to a function; let x and y denote functions; and let all other variables denote numbers. Using strictly-correct mathematical notation, write an expression for each quantity specified below. Only use brackets for grouping when strictly required. Use $\mathcal{D}$ to denote the derivative operator.
(a) the output of the system $\mathcal{H}$ when its input is y;
(b) the output of the system $\mathcal{H}$ evaluated at $2t-1$ when the input to the system is x;
(c) the output of the system $\mathcal{H}$ evaluated at t when the input to the system is ax;
(d) the output of the system $\mathcal{H}$ evaluated at $5t$ when the input to the system is $x+y$;
(e) the derivative of the output of the system $\mathcal{H}$ when its input is ax;
(f) the output of the system $\mathcal{H}$ when its input is the derivative of ax;
(g) the sum of: 1) the output of the system $\mathcal{H}$ when its input is x; and 2) the output of the system $\mathcal{H}$ when its input is y;
(h) the output of the system $\mathcal{H}$ when its input is $x+y$; and
(i) the derivative of x evaluated at $5t-3$.

2.3 Let $\mathcal{D}$ denote the derivative operator; let $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ denote the system operators $\mathcal{H}_1x(t) = tx(t)$, $\mathcal{H}_2x(t) = x(t-1)$, and $\mathcal{H}_3x(t) = x(3t)$; and let x denote the function $x(t) = t-1$. Evaluate each of the expressions given below.
(a) $\mathcal{H}_2\mathcal{H}_1x(t)$;
(b) $\mathcal{H}_1\mathcal{H}_2x(t)$;
(c) $\mathcal{H}_1x(t-1)$;
(d) $\mathcal{D}\mathcal{H}_3x(t)$;
(e) $\mathcal{H}_3\mathcal{D}x(t)$; and
(f) $\mathcal{D}\{3x\}(t)$.

2.11.2 Exercises With Answer Key

Currently, there are no exercises available with an answer key.

Part I

Continuous-Time Signals and Systems

Chapter 3

Continuous-Time Signals and Systems

3.1 Overview

In this chapter, we will examine continuous-time signals and systems in more detail.

3.2 Transformations of the Independent Variable

An important concept in the study of signals and systems is the transformation of a signal. Here, we introduce several elementary signal transformations. Each of these transformations involves a simple modification of the independent variable.

3.2.1 Time Shifting (Translation)

The first type of signal transformation that we shall consider is known as time shifting. **Time shifting** (also known as **translation**) maps a function x to the function y given by

$$y(t) = x(t-b), \tag{3.1}$$

where b is a real constant. In other words, the function y is formed by replacing t by $t-b$ in the expression for $x(t)$. Geometrically, the transformation (3.1) shifts the function x (to the left or right) along the time axis to yield y. If $b > 0$, y is shifted to the right relative to x (i.e., delayed in time). If $b < 0$, y is shifted to the left relative to x (i.e., advanced in time).

The effects of time shifting are illustrated in Figure 3.1. By applying a time-shifting transformation to the function x shown in Figure 3.1(a), each of the functions in Figures 3.1(b) and (c) can be obtained.

3.2.2 Time Reversal (Reflection)

The next type of signal transformation that we consider is called time reversal. **Time reversal** (also known as **reflection**) maps a function x to the function y given by

$$y(t) = x(-t). \tag{3.2}$$

In other words, the function y is formed by replacing t with $-t$ in the expression for $x(t)$. Geometrically, the transformation (3.2) reflects the function x about the origin to yield y.

To illustrate the effects of time reversal, an example is provided in Figure 3.2. Applying a time-reversal transformation to the function x in Figure 3.2(a) yields the function in Figure 3.2(b).

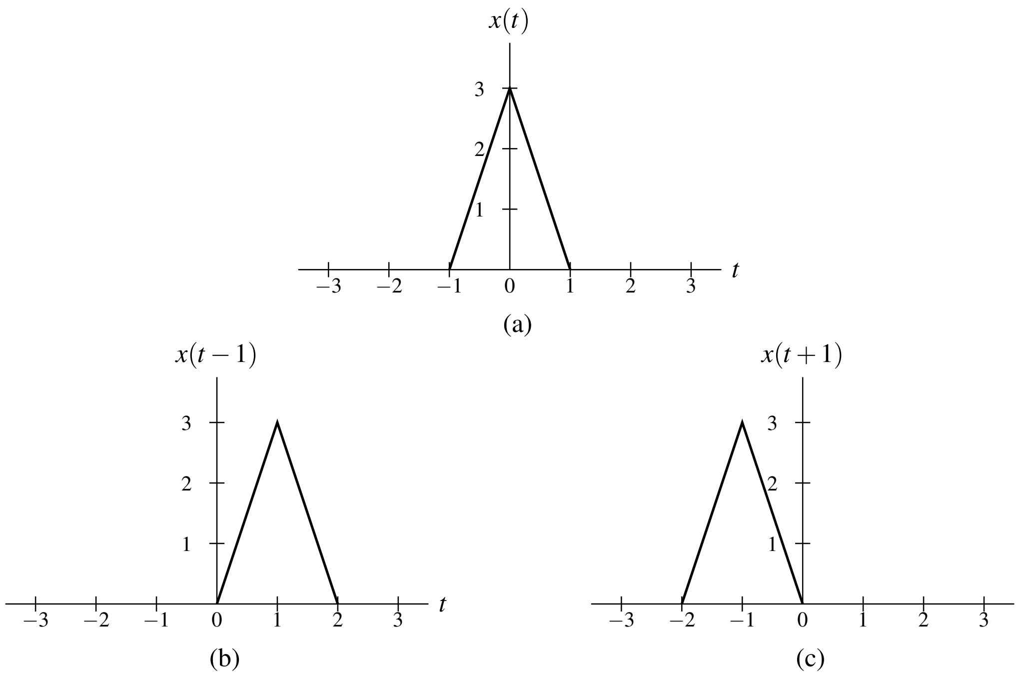


Figure 3.1: Example of time shifting. (a) The function x; and the result of applying a time-shifting transformation to x with a shift of (b) 1 and (c) -1.

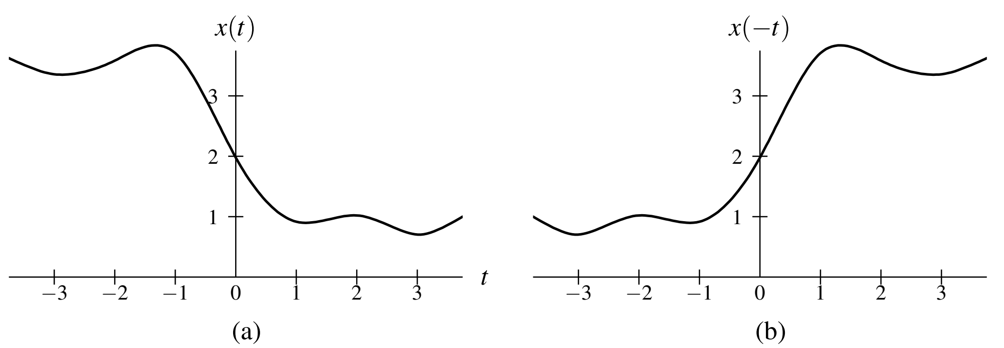


Figure 3.2: Example of time reversal. (a) The function x; and (b) the result of applying a time-reversal transformation to x.

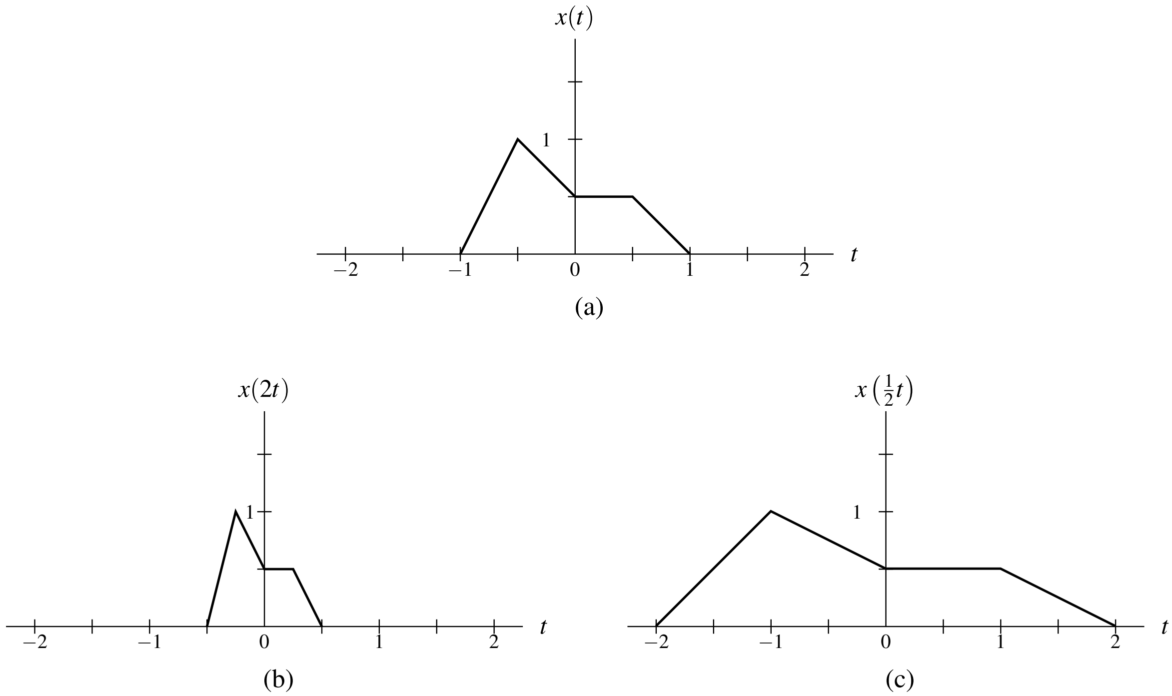


Figure 3.3: Example of time compression/expansion. (a) The function x; and the result of applying a time compression/expansion transformation to x with a scaling factor of (b) 2 and (c) $\frac{1}{2}$.

3.2.3 Time Compression/Expansion (Dilation)

The next transformation to be considered is called time compression/expansion. **Time compression/expansion** (also known as **dilation**) maps a function x to the function y given by

$$y(t) = x(at), \tag{3.3}$$

where a is a strictly positive real constant. In other words, the function y is formed by replacing t by at in the expression for $x(t)$. The constant a is referred to as the scaling factor. The transformation in (3.3) is associated with a compression/expansion along the time axis. If $a > 1$, y is compressed along the horizontal axis by a factor of a, relative to x. If $a < 1$, y is expanded (i.e., stretched) along the horizontal axis by a factor of $\frac{1}{a}$, relative to x.

To illustrate the effects of time compression/expansion, an example is provided in Figure 3.3. By applying a time compression/expansion transformation to the function x in Figure 3.3(a), each of the functions shown in Figures 3.3(b) and (c) can be obtained.

3.2.4 Time Scaling (Dilation/Reflection)

Another type of signal transformation is called time scaling. **Time scaling** maps a function x to the function y given by

$$y(t) = x(at), \tag{3.4}$$

where a is a nonzero real constant. In other words, the function y is formed by replacing t with at in the expression for $x(t)$. The quantity a is referred to as the scaling factor. Geometrically, the transformation (3.4) is associated with a compression/expansion along the time axis and/or reflection about the origin. If $|a| < 1$, the function is expanded (i.e., stretched) along the time axis. If $|a| > 1$, the function is instead compressed. If $|a| = 1$, the function is neither expanded nor compressed. Lastly, if $a < 0$, the function is reflected about the origin. Observe that time scaling includes both time compression/expansion and time reversal as special cases.

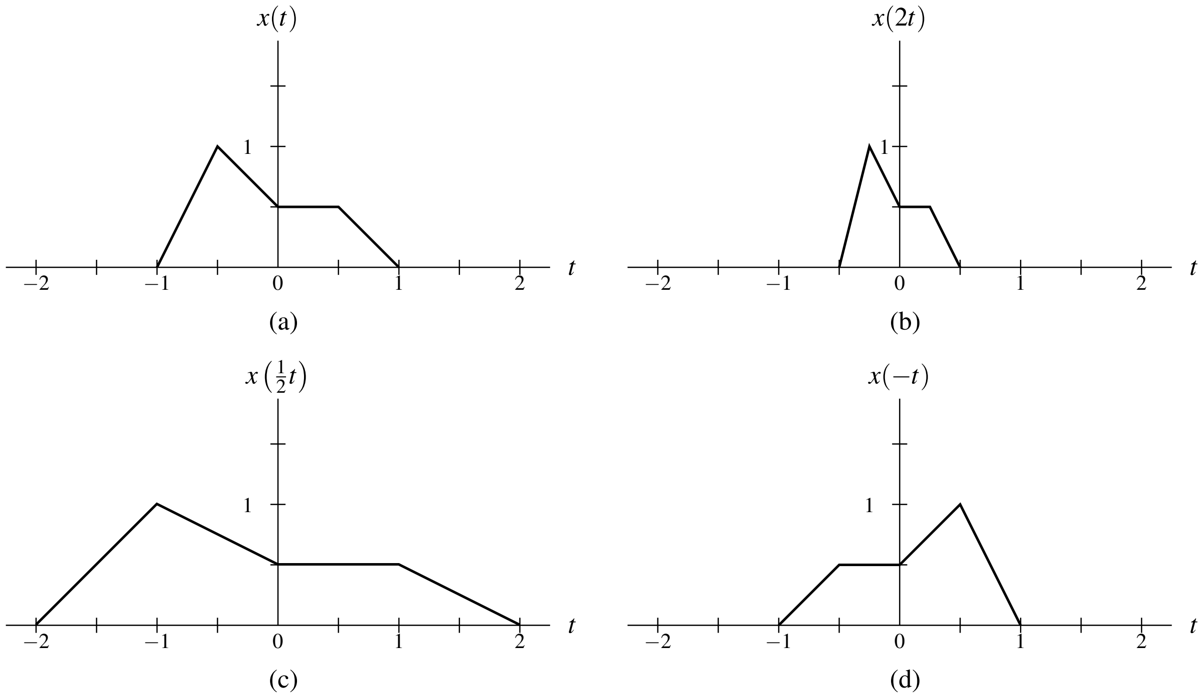


Figure 3.4: Example of time scaling. (a) The function x; and the result of applying a time-scaling transformation to x with a scaling factor of (b) 2, (c) $\frac{1}{2}$, and (d) -1.

To illustrate the behavior of time scaling, we provide an example in Figure 3.4. Each of the functions shown in Figures 3.4(b), (c), and (d) can be obtained by applying a time-scaling transformation to the function x given in Figure 3.4(a).

3.2.5 Combining Time Shifting and Time Scaling

In the preceding sections, we introduced the time shifting and time scaling transformations. Moreover, we observed that time scaling includes time compression/expansion and time reversal as special cases. Some independent-variable transformations commute, while others do not. The issue of commutativity is important, for example, when trying to simplify or manipulate expressions involving combined transformations. Time-scaling, time-reversal, and time-compression/expansion operations commute. Time-shifting (with a nonzero shift) and each of time-scaling, time-reversal, and time-compression/expansion operations do not commute.

Now, we introduce a more general transformation that combines the effects of time shifting and time scaling. This new transformation maps a function x to the function y given by

$$y(t) = x(at - b), \tag{3.5}$$

where a and b are real constants and $a \neq 0$. In other words, the function y is formed by replacing t with $at - b$ in the expression for $x(t)$. One can show that the transformation (3.5) is equivalent to first time shifting x by b, and then time scaling the resulting function by a. Geometrically, this transformation preserves the shape of x except for a possible expansion/compression along the time axis and/or a reflection about the origin. If $|a| < 1$, the function is stretched along the time axis. If $|a| > 1$, the function is instead compressed. If $a < 0$, the function is reflected about the origin.

The above transformation has two distinct but equivalent interpretations. That is, it is equivalent to each of the following:

1. first, time shifting x by b, and then time scaling the result by a.
2. first, time scaling x by a, and then time shifting the result by $\frac{b}{a}$.

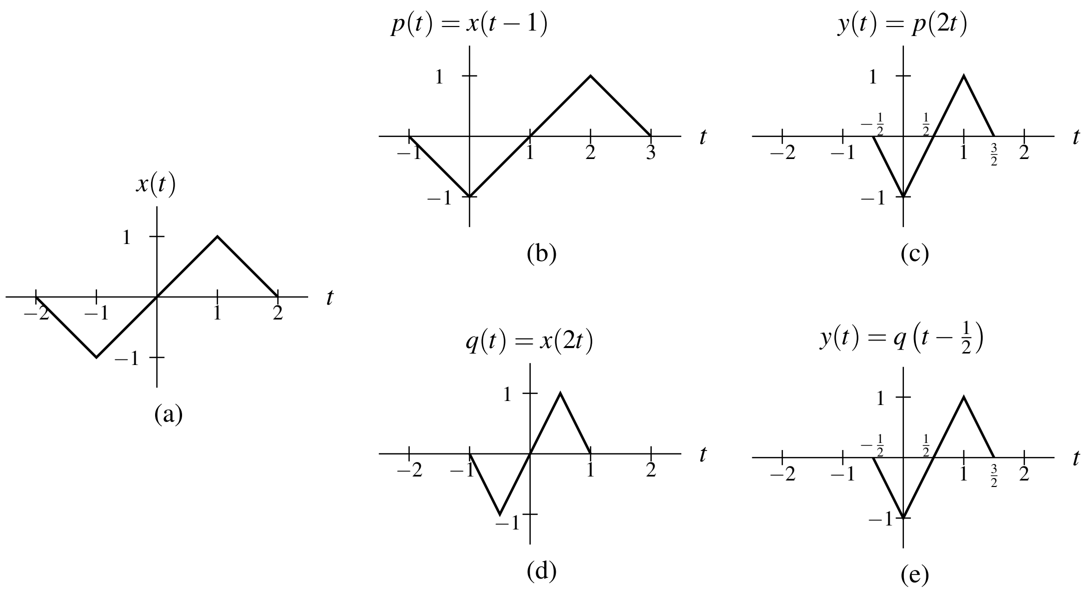


Figure 3.5: Two different interpretations of a combined time-shifting and time-scaling transformation. (a) Original function. Results obtained by shifting followed by scaling: (b) intermediate result and (c) final result. Results obtained by scaling followed by shifting: (d) intermediate result and (e) final result.

Observe that the shifting amount differs in these two interpretations (i.e., b versus $\frac{b}{a}$). This is due to the fact that time shifting and time scaling do not commute. The proof that the above two interpretations are valid is left as an exercise for the reader in Exercise 3.3.

Example 3.1. To illustrate the two equivalent interpretations of this combined transformation, we consider a simple example. Consider the function x shown in Figure 3.5(a). Let us now determine the transformed function $y(t) = x(at-b)$, where $a=2$ and $b=1$.

Solution. First, we consider the shift-then-scale method. In this case, we first shift the function x by b (i.e., 1). This yields the function in Figure 3.5(b). Then, we scale this new function by a (i.e., 2) in order to obtain y as shown in Figure 3.5(c). Second, we consider the scale-then-shift method. In this case, we first scale the function x by a (i.e., 2). This yields the function in Figure 3.5(d). Then, we shift this new function by $\frac{b}{a}$ (i.e., $\frac{1}{2}$) in order to obtain y as shown in Figure 3.5(e). ■

3.2.6 Two Perspectives on Independent-Variable Transformations

A transformation of the independent variable can be viewed in terms of:

1. the effect that the transformation has on the *function*; or
2. the effect that the transformation has on the *horizontal axis*.

This distinction is important because such a transformation has *opposite* effects on the function and horizontal axis. For example, the (time-shifting) transformation that replaces t by $t-b$ (where b is a real number) in the expression for $x(t)$ can be viewed as a transformation that

1. shifts the function x *right* by b units; or
2. shifts the horizontal axis *left* by b units.

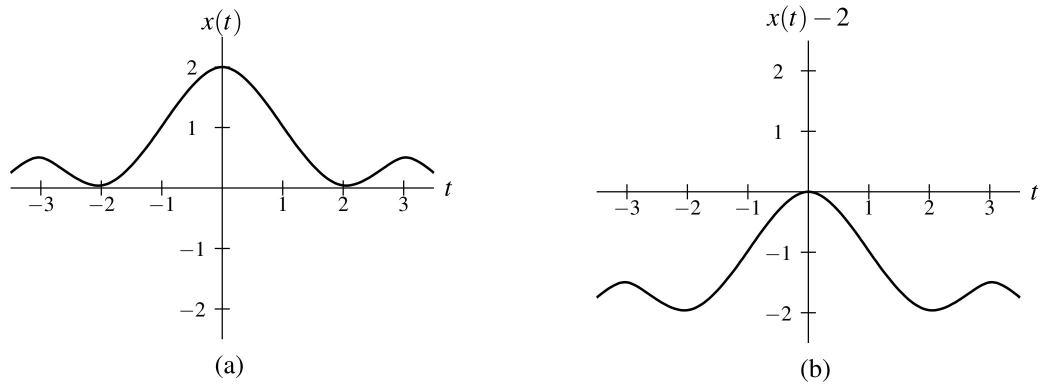


Figure 3.6: Example of amplitude shifting. (a) The function x; and the result obtained by applying an amplitude-shifting transformation to x with a shifting value of -2.

In our treatment of independent-variable transformations, we are only interested in the effect that a transformation has on the *function*. If one is not careful to consider that we are interested in the function perspective (as opposed to the axis perspective), many aspects of independent-variable transformations will not make sense.

3.3 Transformations of the Dependent Variable

In the preceding sections, we examined several transformations of the independent variable. Now, we consider some transformations of the dependent variable.

3.3.1 Amplitude Shifting

The first transformation that we consider is referred to as amplitude shifting. **Amplitude shifting** maps a function x to the function y given by

$$y(t) = x(t) + b,$$

where b is a scalar constant. Geometrically, the function y is displaced vertically relative to x. If $b > 0$, y is shifted upwards by $|b|$ relative to x. If $b < 0$, y is shifted downwards by $|b|$ relative to x.

The effects of amplitude shifting are illustrated by Figure 3.6. The function shown in Figure 3.6(b) can be obtained by applying an amplitude-shifting transformation to the function x given in Figure 3.6(a).

3.3.2 Amplitude Scaling

The next transformation that we consider is referred to as amplitude scaling. **Amplitude scaling** maps a function x to the function y given by

$$y(t) = ax(t),$$

where a is a scalar constant. Geometrically, the function y is expanded/compressed in amplitude and/or reflected about the horizontal axis, relative to x.

To illustrate the effects of amplitude scaling, an example is given in Figure 3.7. Each of the functions shown in Figures 3.7(b), (c), and (d) can be obtained by applying an amplitude-scaling transformation to the function x given in Figure 3.7(a).

3.3.3 Combining Amplitude Shifting and Scaling

In the previous sections, we considered the amplitude-shifting and amplitude-scaling transformations. We can define a new transformation that combines the effects of amplitude shifting and amplitude scaling. This transformation maps

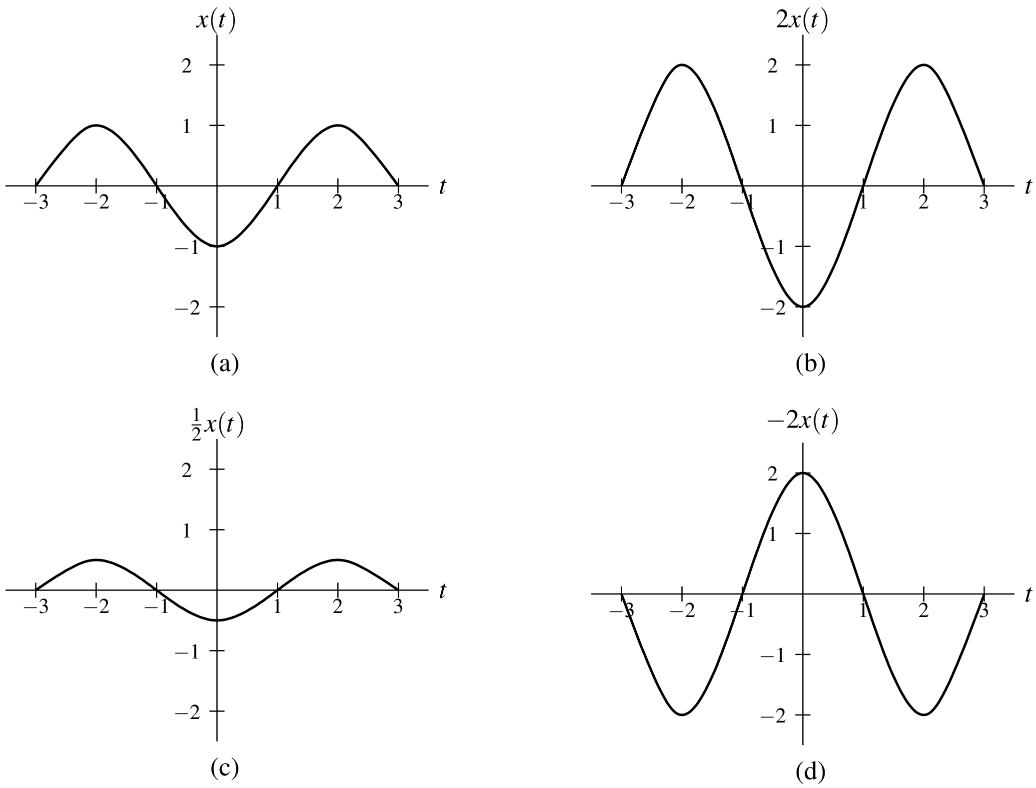


Figure 3.7: Example of amplitude scaling. (a) The function x; and the result of applying an amplitude-scaling transformation to x with a scaling factor of (b) 2, (c) $\frac{1}{2}$, and (d) -2.

the function x to the function y given by

$$y(t) = ax(t) + b, \tag{3.6}$$

where a and b are scalar constants. One can show that this transformation is equivalent to first scaling x by a and then shifting the resulting function by b. Moreover, since (3.6) can be rewritten as $y(t) = a[x(t) + \frac{b}{a}]$, this transformation is also equivalent to first amplitude shifting x by $\frac{b}{a}$ and then amplitude scaling the resulting function by a.

3.4 Properties of Functions

Functions can possess a number of interesting properties. In what follows, we consider the properties of symmetry and periodicity (introduced earlier) in more detail. Also, we present several other function properties. The properties considered are frequently useful in the analysis of signals and systems.

3.4.1 Remarks on Symmetry

At this point, we make some additional comments about even and odd functions (introduced earlier). Since functions are often summed or multiplied, one might wonder what happens to the even/odd symmetry properties of functions under these operations. In what follows, we introduce a few results in this regard.

Sums involving even and odd functions have the following properties:

- The sum of two even functions is even.
- The sum of two odd functions is odd.
- The sum of an even function and odd function is neither even nor odd, provided that the functions are not identically zero.

Products involving even and odd functions have the following properties:

- The product of two even functions is even.
- The product of two odd functions is even.
- The product of an even function and an odd function is odd.

(The proofs of the above properties involving sums and products of even and odd functions is left as an exercise for the reader in Exercise 3.9.)

As it turns out, any arbitrary function can be expressed as the sum of an even and odd function, as elaborated upon by the theorem below.

Theorem 3.1 (Decomposition of function into even and odd parts)**.** *Any arbitrary function x can be uniquely represented as the sum of the form*

$$x(t) = x_{\mathrm{e}}(t) + x_{\mathrm{o}}(t), \tag{3.7}$$

where x_{e} and x_{o} are even and odd, respectively, and given by

$$x_{\mathrm{e}}(t) = \tfrac{1}{2}\left[x(t) + x(-t)\right] \quad \textit{and} \tag{3.8}$$

$$x_{\mathrm{o}}(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right]. \tag{3.9}$$

As a matter of terminology, x_{e} is called the ***even part*** *of x and is denoted* $\mathrm{Even}\{x\}$*, and x_{o} is called the* ***odd part*** *of x and is denoted* $\mathrm{Odd}\{x\}$.

Proof. From (3.8) and (3.9), we can easily confirm that $x_{\mathrm{e}} + x_{\mathrm{o}} = x$ as follows:

$$\begin{aligned} x_{\mathrm{e}}(t) + x_{\mathrm{o}}(t) &= \tfrac{1}{2}[x(t) + x(-t)] + \tfrac{1}{2}[x(t) - x(-t)] \\ &= \tfrac{1}{2}x(t) + \tfrac{1}{2}x(-t) + \tfrac{1}{2}x(t) - \tfrac{1}{2}x(-t) \\ &= x(t). \end{aligned}$$

Furthermore, we can easily verify that x_e is even and x_o is odd. From the definition of x_e in (3.8), we have

$$\begin{aligned} x_e(-t) &= \tfrac{1}{2}[x(-t) + x(-[-t])] \\ &= \tfrac{1}{2}[x(t) + x(-t)] \\ &= x_e(t). \end{aligned}$$

Thus, x_e is even. From the definition of x_o in (3.9), we have

$$\begin{aligned} x_o(-t) &= \tfrac{1}{2}[x(-t) - x(-[-t])] \\ &= \tfrac{1}{2}[-x(t) + x(-t)] \\ &= -x_o(t). \end{aligned}$$

Thus, x_o is odd.

Lastly, we show that the decomposition of x into the sum of an even function and odd function is unique. Suppose that x can be written as the sum of an even function and odd function in two ways as

$$x(t) = f_e(t) + f_o(t) \quad \text{and} \tag{3.10a}$$

$$x(t) = g_e(t) + g_o(t), \tag{3.10b}$$

where f_e and g_e are even and f_o and g_o are odd. Equating these two expressions for x, we have

$$f_e(t) + f_o(t) = g_e(t) + g_o(t).$$

Rearranging this equation, we have

$$f_e(t) - g_e(t) = g_o(t) - f_o(t).$$

Now, we consider the preceding equation more carefully. Since the sum of even functions is even and the sum of odd functions is odd, we have that the left- and right-hand sides of the preceding equation correspond to even and odd functions, respectively. Thus, we have that the even function $f_e(t) - g_e(t)$ is equal to the odd function $g_o(t) - f_o(t)$. The only function, however, that is both even and odd is the zero function. (A proof of this fact is left as an exercise for the reader in Exercise 3.15.) Therefore, we have that

$$f_e(t) - g_e(t) = g_o(t) - f_o(t) = 0.$$

In other words, we have that

$$f_e(t) = g_e(t) \quad \text{and} \quad f_o(t) = g_o(t).$$

This implies that the two decompositions of x given by (3.10a) and (3.10b) must be the same decomposition (i.e., they cannot be distinct). Thus, the decomposition of x into the sum of an even function and odd function is unique. ■

3.4.2 Remarks on Periodicity

Since we often add functions, it is helpful to know if the sum of periodic functions is also periodic. We will consider this issue next, but before doing so we first must introduce the notion of a least common multiple.

The **least common multiple (LCM)** of two strictly positive real numbers a_1 and a_2, denoted $\operatorname{lcm}(a_1, a_2)$, is the smallest positive real number that is an integer multiple of each a_1 and a_2. For example, $\operatorname{lcm}(6\pi, 10\pi) = 30\pi$, since 30π is the smallest positive real number that can be evenly divided by both 6π and 10π. More generally, the **least common multiple** of a set of positive real numbers $\{a_1, a_2, \ldots, a_N\}$, denoted $\operatorname{lcm}\{a_1, a_2, \ldots, a_N\}$, is the smallest positive real number that is an integer multiple of each a_k for $k = 1, 2, \ldots, N$.

Having introduced the notation of a least common multiple, we can now consider whether the sum of two periodic functions is periodic. In this regard, the theorem below is enlightening.

Theorem 3.2 (Sum of periodic functions). *Let x_1 and x_2 be (continuous) periodic functions with periods T_1 and T_2, respectively. Then, the function $x = x_1 + x_2$ is a periodic if and only if the ratio T_1/T_2 is a rational number (i.e., the quotient of two integers). Suppose now that x is periodic. Let $T_1/T_2 = q/r$ where q and r are integers and coprime (i.e., have no common factors). Then, x is periodic with period $T = \operatorname{lcm}(T_1, T_2) = rT_1 = qT_2$.*

Proof. We provide only a partial proof. Assuming that x is periodic, we show that it must be periodic with period T. Since x is periodic, $T = \operatorname{lcm}(T_1, T_2)$ must exist. Since T is an integer multiple of both T_1 and T_2, we can write $T = k_1 T_1$ and $T = k_2 T_2$ for some positive integers k_1 and k_2. Thus, we have

$$\begin{aligned} x(t+T) &= x_1(t+T) + x_2(t+T) \\ &= x_1(t+k_1T_1) + x_2(t+k_2T_2) \\ &= x_1(t) + x_2(t) \\ &= x(t). \end{aligned}$$

Thus, x is periodic with period T. ■

In passing, we note that the above result can be extended to the more general case of the sum of N periodic functions. The sum of N periodic functions $x_1, x_2, \ldots, x_N$ with periods $T_1, T_2, \ldots, T_N$, respectively, is periodic if and only if the ratios of the periods are rational numbers (i.e., T_1/T_k is rational for $k = 2, 3, \ldots, N$). If the sum is periodic, then the fundamental period is simply $\operatorname{lcm}\{T_1, T_2, \ldots, T_N\}$.

Example 3.2. Let $x_1(t) = \sin(\pi t)$ and $x_2(t) = \sin t$. Determine whether the function $y = x_1 + x_2$ is periodic.

Solution. Denote the fundamental periods of x_1 and x_2 as T_1 and T_2, respectively. We then have

$$T_1 = \tfrac{2\pi}{\pi} = 2 \quad \text{and} \quad T_2 = \tfrac{2\pi}{1} = 2\pi.$$

Here, we used the fact that the fundamental period of $\sin(\alpha t)$ is $\frac{2\pi}{|\alpha|}$. Thus, we have

$$\tfrac{T_1}{T_2} = \tfrac{2}{2\pi} = \tfrac{1}{\pi}.$$

Since π is an irrational number, $\frac{T_1}{T_2}$ is not rational. Therefore, y is not periodic. ■

Example 3.3. Let $x_1(t) = \cos(2\pi t + \frac{\pi}{4})$ and $x_2(t) = \sin(7\pi t)$. Determine if the function $y = x_1 + x_2$ is periodic, and if it is, find its fundamental period.

Solution. Let T_1 and T_2 denote the fundamental periods of x_1 and x_2, respectively. Thus, we have

$$T_1 = \tfrac{2\pi}{2\pi} = 1 \quad \text{and} \quad T_2 = \tfrac{2\pi}{7\pi} = \tfrac{2}{7}.$$

Taking the ratio of T_1 to T_2, we have

$$\tfrac{T_1}{T_2} = \tfrac{7}{2}.$$

Since $\frac{T_1}{T_2}$ is a rational number, y is periodic. Let T denote the fundamental period of y. Since 7 and 2 are coprime,

$$T = 2T_1 = 7T_2 = 2.$$ ■

Example 3.4. Let $x_1(t) = \cos(6\pi t)$ and $x_2(t) = \sin(30\pi t)$. Determine if the function $y = x_1 + x_2$ is periodic, and if it is, find its fundamental period.

Solution. Let T_1 and T_2 denote the fundamental periods of x_1 and x_2, respectively. We have

$$T_1 = \tfrac{2\pi}{6\pi} = \tfrac{1}{3} \quad \text{and} \quad T_2 = \tfrac{2\pi}{30\pi} = \tfrac{1}{15}.$$

Thus, we have

$$\tfrac{T_1}{T_2} = (\tfrac{1}{3})/(\tfrac{1}{15}) = \tfrac{15}{3} = \tfrac{5}{1}.$$

Since $\frac{T_1}{T_2}$ is a rational number, y is periodic. Let T denote the fundamental period of y. Since 5 and 1 are coprime, we have

$$T = 1T_1 = 5T_2 = \tfrac{1}{3}. \qquad \blacksquare$$

3.4.3 Support of Functions

We can classify functions based on the interval over which their function value is nonzero. This is sometimes referred to as the support of a function. In what follows, we introduce some terminology related to the support of functions.

A function x is said to be **left sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t > t_0.$$

In other words, the value of the function is always zero to the right of some point. A function x is said to be **right sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t < t_0.$$

In other words, the value of the function is always zero to the left of some point. A function that is both left sided and right sided is said to be **time limited** or **finite duration**. A function that is neither left sided nor right sided is said to be **two sided**. Note that every function is exactly one of: left sided but not right sided, right sided but not left sided, finite duration, or two sided. Examples of left-sided (but not right-sided), right-sided (but not left-sided), finite-duration, and two-sided functions are shown in Figure 3.8.

A function x is said to be **causal** if

$$x(t) = 0 \quad \text{for all } t < 0.$$

A causal function is a special case of a right-sided function. Similarly, a function x is said to be **anticausal** if

$$x(t) = 0 \quad \text{for all } t > 0.$$

An anticausal function is a special case of a left-sided function. Note that the qualifiers "causal" and "anticausal", when applied to functions, have nothing to do with cause and effect. In this sense, this choice of terminology is arguably not the best.

3.4.4 Bounded Functions

A function x is said to be **bounded** if there exists some (finite) nonnegative real constant A such that

$$|x(t)| \leq A \quad \text{for all } t$$

(i.e., $x(t)$ is finite for all t). For example, the sine and cosine functions are bounded, since

$$|\sin t| \leq 1 \text{ for all } t \quad \text{and} \quad |\cos t| \leq 1 \text{ for all } t.$$

In contrast, the tangent function and any nonconstant polynomial function p (e.g., $p(t) = t^2$) are unbounded, since

$$\lim_{t\to\pi/2} |\tan t| = \infty \quad \text{and} \quad \lim_{|t|\to\infty} |p(t)| = \infty.$$

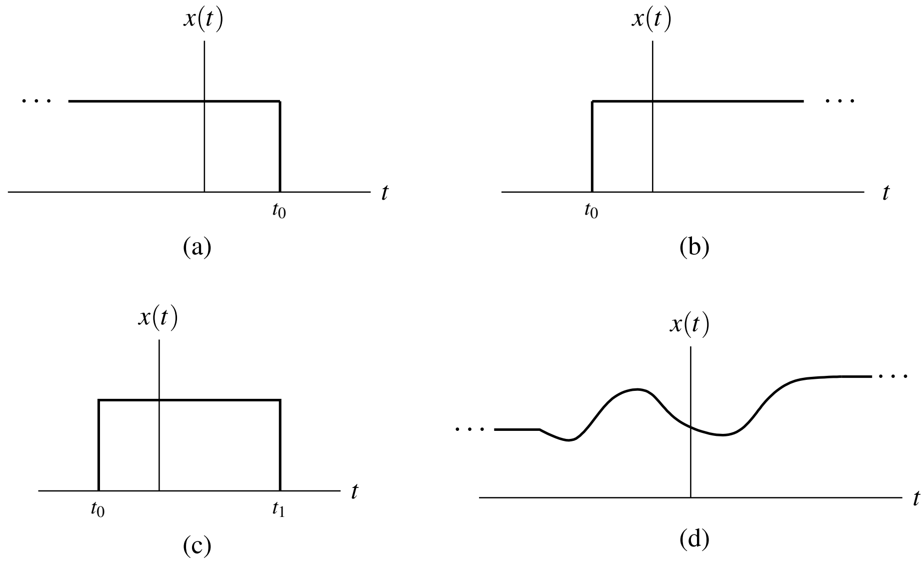


Figure 3.8: Examples of functions with various sidedness properties. A function that is (a) left sided but not right sided, (b) right sided but not left sided, (c) finite duration, and (d) two sided.

3.4.5 Signal Energy and Power

The **energy** E contained in the function x is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

As a matter of terminology, a function x with finite energy is said to be an **energy signal**. The **average power** P contained in the function x is given by

$$P = \lim_{T \to \infty} \tfrac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt.$$

As a matter of terminology, a function x with (nonzero) finite average power is said to be a **power signal**.

3.4.6 Examples

Below, we consider several examples that use signal properties in various ways.

Example 3.5. Let x be a function with the following properties:

$$v(t) = x(t-3) \text{ is causal; and}$$
$$x \text{ is odd.}$$

Determine for what values of t the quantity $x(t)$ must be zero.

Solution. Since v is causal, we know that $v(t) = 0$ for all $t < 0$. Since $v(t) = x(t-3)$, this implies that

$$x(t) = 0 \quad \text{for } t < -3. \tag{3.11}$$

Since x is odd, we know that

$$x(t) = -x(-t) \quad \text{for all } t. \tag{3.12}$$

From (3.11) and (3.12), we have $-x(-t) = 0$ for $t < -3$ which implies that

$$x(t) = 0 \;\text{ for } t > 3. \tag{3.13}$$

Substituting $t = 0$ into (3.12) yields $x(0) = -x(0)$ which implies that

$$x(0) = 0. \tag{3.14}$$

Combining (3.11), (3.13), and (3.14), we conclude that $x(t)$ must be zero for

$$t < -3, \quad t > 3, \quad \text{or} \quad t = 0.$$

■

Example 3.6. Consider a function x with the following properties:

- $x(t) = t + 5$ for $-5 \le t \le -3$;
- $v_1(t) = x(t-5)$ is causal; and
- $v_2(t) = x(t-3)$ is even.

Find $x(t)$ for all t.

Solution. Since $v_1(t) = x(t-5)$ is causal, we have that

$$\begin{aligned}
& v_1(t) = 0 \;\text{ for } t < 0 \\
\Rightarrow\quad & x(t-5) = 0 \;\text{ for } t < 0 \\
\Rightarrow\quad & x([t+5]-5) = 0 \;\text{ for } (t+5) < 0 \\
\Rightarrow\quad & x(t) = 0 \;\text{ for } t < -5.
\end{aligned}$$

From this and the fact that $x(t) = t+5$ for $-5 \le t \le -3$, we have

$$x(t) = \begin{cases} t+5 & -5 \le t \le -3 \\ 0 & t < -5. \end{cases} \tag{3.15}$$

So, we only need to determine $x(t)$ for $t > -3$. Since $v_2(t) = x(t-3)$ is even, we have

$$\begin{aligned}
& v_2(t) = v_2(-t) \\
\Rightarrow\quad & x(t-3) = x(-t-3) \\
\Rightarrow\quad & x([t+3]-3) = x(-[t+3]-3) \\
\Rightarrow\quad & x(t) = x(-t-6).
\end{aligned}$$

Using this with (3.15), we obtain

$$\begin{aligned}
x(t) &= x(-t-6) \\
&= \begin{cases} (-t-6)+5 & -5 \le -t-6 \le -3 \\ 0 & -t-6 < -5 \end{cases} \\
&= \begin{cases} -t-1 & 1 \le -t \le 3 \\ 0 & -t < 1 \end{cases} \\
&= \begin{cases} -t-1 & -3 \le t \le -1 \\ 0 & t > -1. \end{cases}
\end{aligned}$$

Therefore, we conclude that

$$x(t) = \begin{cases} 0 & t < -5 \\ t+5 & -5 \le t < -3 \\ -t-1 & -3 \le t < -1 \\ 0 & t \ge -1. \end{cases}$$

A plot of x is given in Figure 3.9.

■

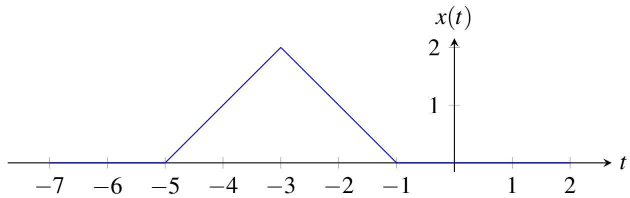


Figure 3.9: The function x from Example 3.6.

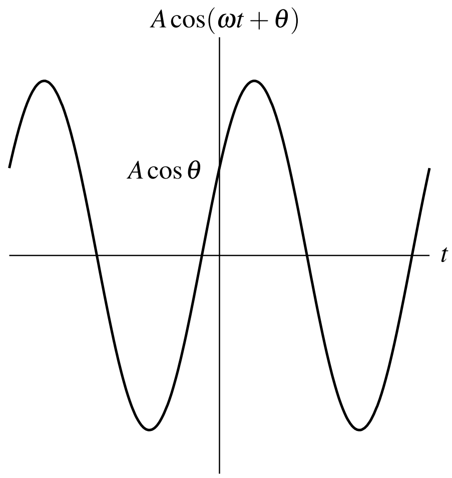


Figure 3.10: Real sinusoidal function.

3.5 Elementary Functions

A number of elementary signals are particularly useful in the study of signals and systems. In what follows, we introduce some of the more beneficial ones for our purposes.

3.5.1 Real Sinusoidal Functions

One important class of functions is the real sinusoids. A **real sinusoidal function** x has the general form

$$x(t) = A\cos(\omega t+\theta),$$

where A, ω, and θ are real constants. Such a function is periodic with fundamental period $T = \frac{2\pi}{|\omega|}$, and has a plot resembling that shown in Figure 3.10.

3.5.2 Complex Exponential Functions

Another important class of functions is the complex exponentials. A **complex exponential function** x has the general form

$$x(t) = Ae^{\lambda t}, \tag{3.16}$$

where A and λ are complex constants. Complex exponentials are of fundamental importance to systems theory, and also provide a convenient means for representing a number of other classes of functions. A complex exponential can exhibit one of a number of distinctive modes of behavior, depending on the values of its parameters A and λ. In what follows, we examine some special cases of complex exponentials, in addition to the general case.

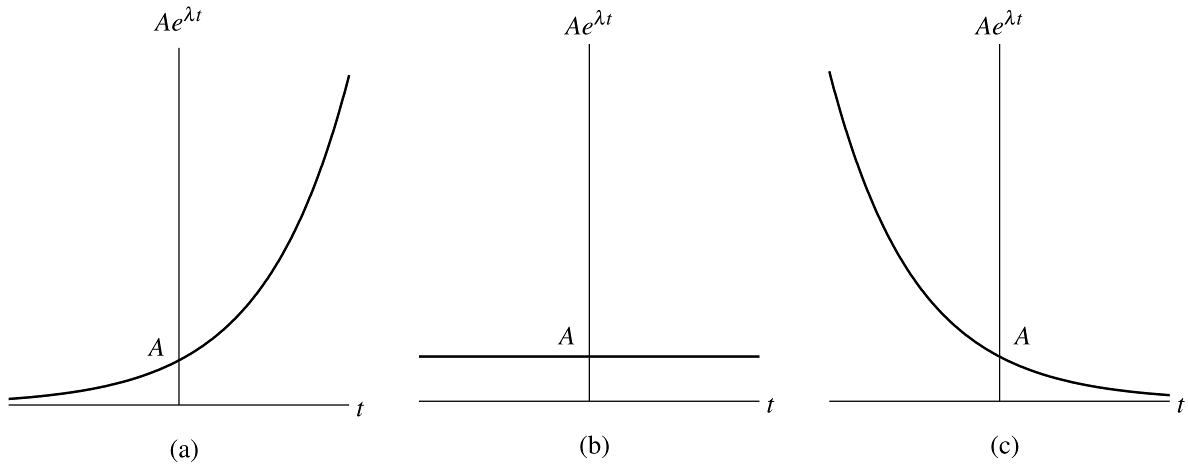


Figure 3.11: Real exponential function for (a) $\lambda > 0$, (b) $\lambda = 0$, and (c) $\lambda < 0$.

3.5.2.1 Real Exponential Functions

The first special case of the complex exponentials to be considered is the real exponentials. In the case of a **real exponential function**, we restrict A and λ in (3.16) to be real. A real exponential can exhibit one of three distinct modes of behavior, depending on the value of λ, as illustrated in Figure 3.11. If $\lambda > 0$, $x(t)$ increases exponentially as t increases (i.e., a growing exponential). If $\lambda < 0$, $x(t)$ decreases exponentially as t increases (i.e., a decaying or damped exponential). If $\lambda = 0$, $x(t)$ simply equals the constant A for all t.

3.5.2.2 Complex Sinusoidal Functions

The second special case of the complex exponentials that we shall consider is the complex sinusoids. In the case of a **complex sinusoidal function**, the parameters in (3.16)) are such that A is complex and λ is purely imaginary (i.e., $\operatorname{Re}(\lambda) = 0$). For convenience, let us re-express A in polar form and λ in Cartesian form as follows:

$$A = |A|\, e^{j\theta} \quad \text{and} \quad \lambda = j\omega,$$

where θ and ω are real constants. Using Euler's relation (A.3), we can rewrite (3.16) as

$$\begin{aligned} x(t) &= Ae^{\lambda t} \\ &= |A|\, e^{j\theta} e^{j\omega t} \\ &= |A|\, e^{j(\omega t+\theta)} \\ &= |A|\cos(\omega t+\theta) + j\,|A|\sin(\omega t+\theta). \end{aligned}$$

From the above equation, we can see that each of the real and imaginary parts of x is periodic with fundamental period $\frac{2\pi}{|\omega|}$. Furthermore, from this, we can deduce that x is also periodic with fundamental period $\frac{2\pi}{|\omega|}$. To illustrate the form of a complex sinusoid, we plot its real and imaginary parts in Figure 3.12. The real and imaginary parts are the same except for a phase difference.

The complex sinusoidal function $x(t) = e^{j\omega t}$ is plotted for the cases of ω equal to 2π and -2π in Figures 3.13(a) and (b), respectively. From these figures, one can see that the curve traced by $x(t)$ as t increases is a helix. If the curve is viewed by looking straight down the t axis in the direction of $-\infty$, the helix appears to turn in a counterclockwise direction if $\omega > 0$ (i.e., a right-handed helix) and a clockwise direction if $\omega < 0$ (i.e., a left-handed helix). Note that, although both of these complex sinusoids oscillate at the same rate, they are not translated (i.e., time-shifted) and/or reflected (i.e., time-reversed) versions of one another. Left- and right-handed helices are fundamentally different shapes. One cannot be made into the other by translation (i.e., time shifting) and/or reflection (i.e., time reversal).

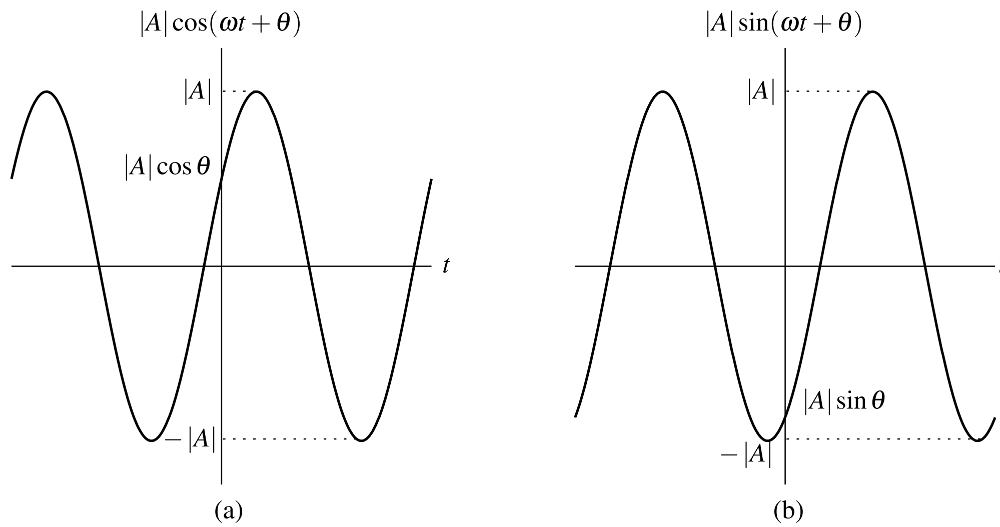


Figure 3.12: Complex sinusoidal function. (a) Real and (b) imaginary parts.

As demonstrated above in Figure 3.13, the complex sinusoid $x(t) = e^{j\omega t}$ behaves fundamentally differently for $\omega > 0$ and $\omega < 0$. For this reason, signed frequency (as introduced earlier in Section 2.10.2) is often used when working with complex sinusoids. That is, we often refer to ω as the frequency of the complex sinusoid. This is an example of the use of signed frequency (as ω is clearly a signed quantity). The positive direction of rotation (i.e., $\omega > 0$) corresponds to a right-handed helix, while the negative direction of rotation (i.e., $\omega < 0$) corresponds to a left-handed helix.

3.5.2.3 General Complex Exponential Functions

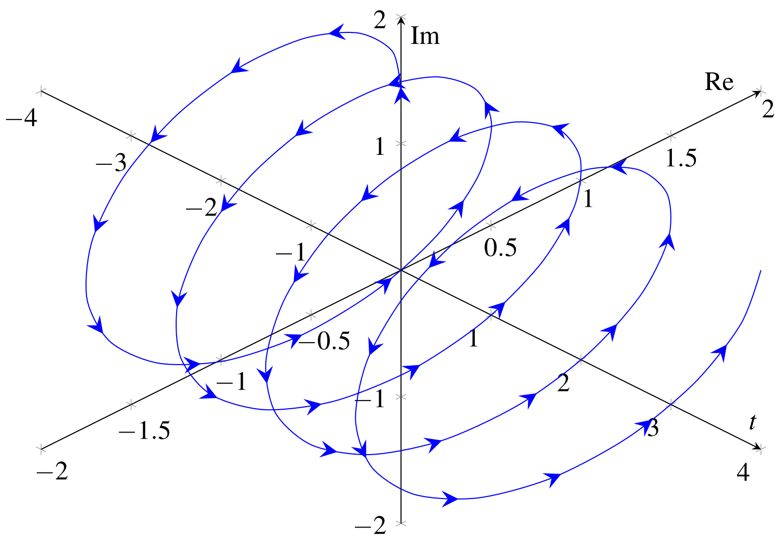
Lastly, we consider general complex exponential functions. That is, we consider the general case of (3.16) where A and λ are both complex. For convenience, let us re-express A in polar form and λ in Cartesian form as

$$A = |A|\,e^{j\theta} \quad \text{and} \quad \lambda = \sigma + j\omega,$$

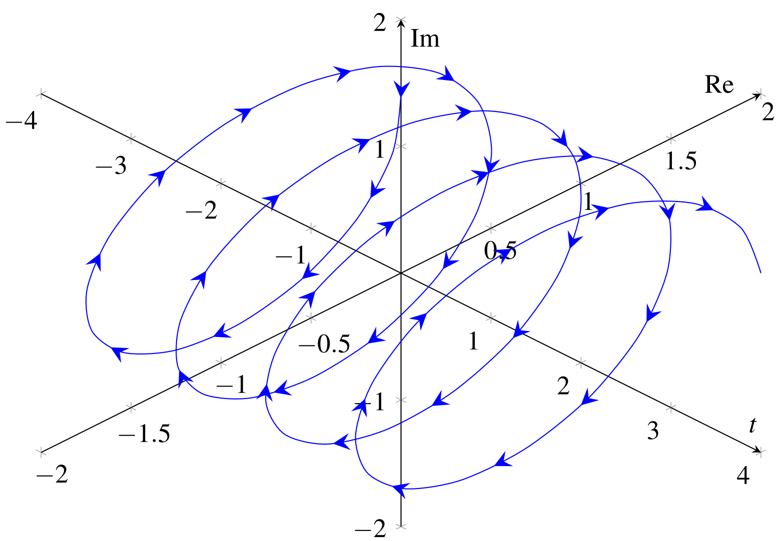
where θ, σ, and ω are real constants. Substituting these expressions for A and λ into (3.16), we obtain

$$\begin{aligned} x(t) &= Ae^{\lambda t} \\ &= |A|\,e^{j\theta}e^{(\sigma+j\omega)t} \\ &= |A|\,e^{\sigma t}e^{j(\omega t+\theta)} \\ &= |A|\,e^{\sigma t}\cos(\omega t+\theta) + j\,|A|\,e^{\sigma t}\sin(\omega t+\theta). \end{aligned}$$

We can see that $\operatorname{Re}[x(t)]$ and $\operatorname{Im}[x(t)]$ have a similar form. Each is the product of a real exponential and real sinusoid. One of three distinct modes of behavior is exhibited by x, depending on the value of σ. If $\sigma = 0$, $\operatorname{Re}(x)$ and $\operatorname{Im}(x)$ are real sinusoids. If $\sigma > 0$, $\operatorname{Re}(x)$ and $\operatorname{Im}(x)$ are each the product of a real sinusoid and a growing real exponential. If $\sigma < 0$, $\operatorname{Re}(x)$ and $\operatorname{Im}(x)$ are each the product of a real sinusoid and a decaying real exponential. These three cases are illustrated for $\operatorname{Re}(x)$ in Figure 3.14.



(a)



(b)

Figure 3.13: The complex sinusoidal function $x(t) = e^{j\omega t}$ for (a) $\omega = 2\pi$ and (b) $\omega = -2\pi$.

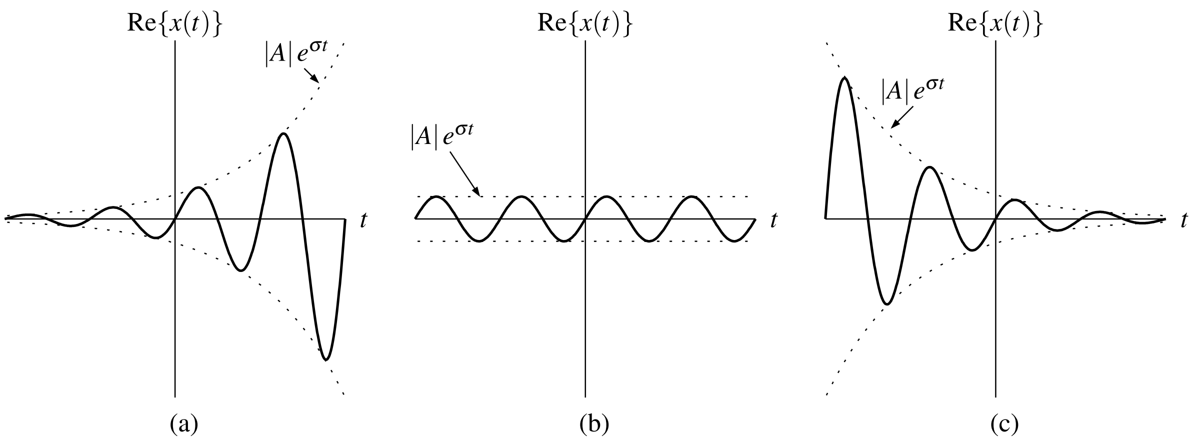


Figure 3.14: Real part of a general complex exponential function for (a) $\sigma > 0$, (b) $\sigma = 0$, and (c) $\sigma < 0$.

3.5.3 Relationship Between Complex Exponential and Real Sinusoidal Functions

A real sinusoid can be expressed as the sum of two complex sinusoids using the identity

$$A\cos(\omega t+\theta) = \frac{A}{2}\left(e^{j(\omega t+\theta)}+e^{-j(\omega t+\theta)}\right) \quad \text{and} \tag{3.17}$$

$$A\sin(\omega t+\theta) = \frac{A}{2j}\left(e^{j(\omega t+\theta)}-e^{-j(\omega t+\theta)}\right). \tag{3.18}$$

This result follows from Euler's relation and is simply a restatement of (A.7).

3.5.4 Unit-Step Function

Another elementary function often used in systems theory is the unit-step function. The **unit-step function** (also known as the **Heaviside function**) is denoted as u and defined as

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

A plot of this function is given in Figure 3.15.

Clearly, u is discontinuous at the origin. At this point of discontinuity, we have chosen to define u such that its value is 1 (i.e., $u(0) = 1$). As it turns out, however, this choice is somewhat arbitrary. That is, for most practical purposes, due to technical reasons beyond the scope of this book, what is important is that $u(0)$ is finite, not its specific value. For this reason, some authors choose to leave the value of $u(0)$ as an unspecified (but finite) constant, while others choose to assign a specific value to $u(0)$. In cases where a specific value is chosen for $u(0)$, the values most commonly used are 0, $\frac{1}{2}$, and 1. Obviously, in the case of this book, the author has chosen to use $u(0) = 1$.

3.5.5 Signum Function

Another function closely related to the unit-step function is the so called signum function. The **signum function**, denoted sgn, is defined as

$$\operatorname{sgn} t = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0. \end{cases} \tag{3.19}$$

A plot of this function is given in Figure 3.16. From (3.19), one can see that the signum function simply computes the sign of a number.

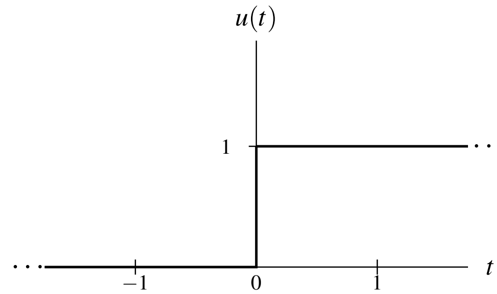


Figure 3.15: Unit-step function.

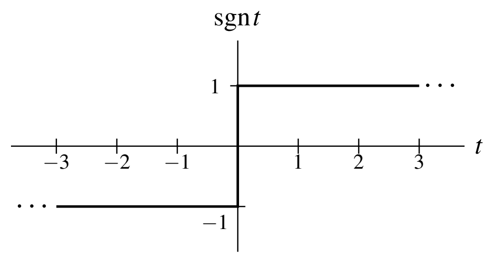


Figure 3.16: Signum function.

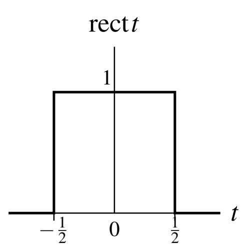


Figure 3.17: Rectangular function.

3.5.6 Rectangular Function

Another useful function is the rectangular function. The **rectangular function** (also known as the **unit-rectangular pulse function**) is denoted as rect and is defined as

$$\operatorname{rect} t = \begin{cases} 1 & -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

A plot of this function is shown in Figure 3.17.

Example 3.7 (Extracting part of a function with a rectangular pulse)**.** Use the rectangular function to extract one period of the function x shown in Figure 3.18(a).

Solution. Let us choose to extract the period of $x(t)$ for $-\frac{T}{2} < t \leq \frac{T}{2}$. In order to extract this period, we want to multiply x by a function that is one over this interval and zero elsewhere. Such a function is simply $v(t) = \operatorname{rect}\left(\frac{1}{T}t\right)$ as shown in Figure 3.18(b). Multiplying v and x results in the function shown in Figure 3.18(c). ■

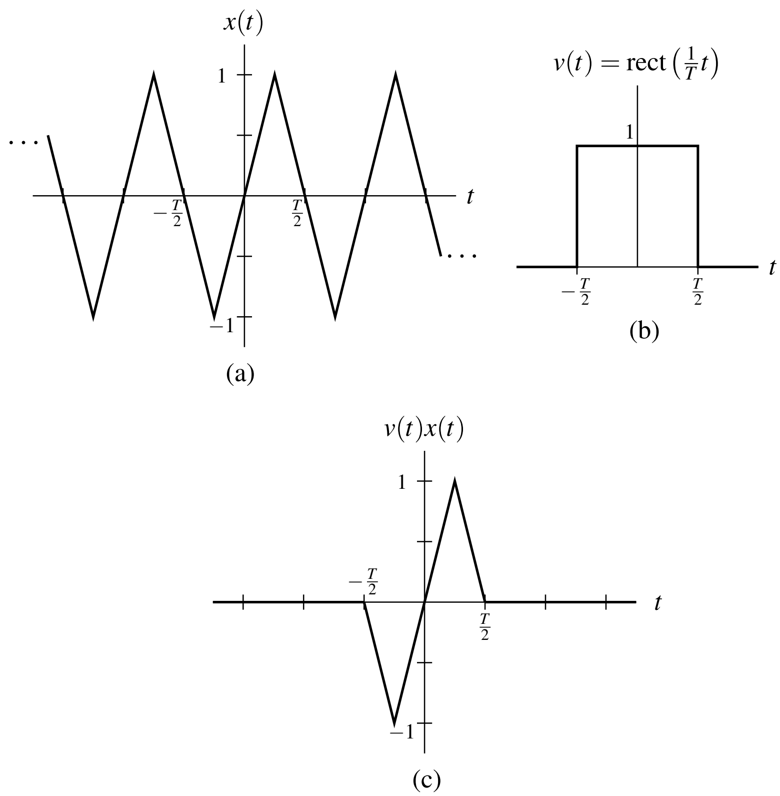


Figure 3.18: Using the rectangular function to extract one period of a periodic function x. (a) The function x. (b) A time-scaled rectangular function v. (c) The product of x and v.

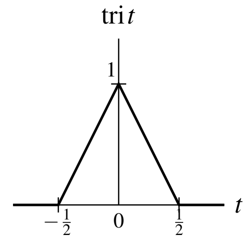


Figure 3.19: Triangular function.

3.5.7 Indicator Function

Functions and sequences that are one over some subset of their domain and zero elsewhere appear very frequently in engineering (e.g., the unit-step function, rectangular function, the unit-step sequence appearing later in Chapter 8, and the delta sequence appearing later in Chapter 8). Indicator function notation provides a concise way to denote such functions and sequences. The **indicator function** of a subset S of a set A, denoted χ_S, is defined as

$$\chi_S(t) = \begin{cases} 1 & \text{if } t \in S \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.8. A rectangular pulse (defined on $\mathbb{R}$) having an amplitude of 1, a leading edge at a, and falling edge at b is $\chi_{[a,b]}$. The unit-step function (defined on $\mathbb{R}$) is $\chi_{[0,\infty)}$. The unit-rectangular pulse (defined on $\mathbb{R}$) is $\chi_{[-1/2,1/2)}$. ■

3.5.8 Triangular Function

Another useful elementary function is the **triangular function** (also known as the **unit triangular pulse function**), which is denoted as tri and defined as

$$\operatorname{tri} t = \begin{cases} 1 - 2|t| & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

A plot of this function is given in Figure 3.19.

3.5.9 Cardinal Sine Function

In the study of signals and systems, a function of the form $x(t) = \frac{\sin t}{t}$ frequently appears. Therefore, as a matter of convenience, this particular function is given a special name, the cardinal sine function. More formally, the **cardinal sine function** (also known as the **sinc function**) is denoted sinc and defined as

$$\operatorname{sinc} t = \frac{\sin t}{t}. \tag{3.20}$$

The name “sinc” is simply a contraction of the function’s full Latin name “sinus cardinalis” (cardinal sine). By using l’Hopital’s rule, one can confirm that $\operatorname{sinc} t$ is well defined for $t = 0$. That is, $\operatorname{sinc} 0 = 1$.

It is worthwhile to note that a definition of the sinc function different from the one above is also sometimes used in the literature. In particular, the sinc function is sometimes defined as

$$x(t) = \frac{\sin(\pi t)}{\pi t}. \tag{3.21}$$

In order to avoid any possible confusion, we will refer to the function x defined by (3.21) as the **normalized sinc function**. As some examples of the practices followed by others, the definition of the sinc function in (3.20) is used by [4, 5, 9], while the definition in (3.21) is employed by [3, 8, 10] as well as the MATLAB software.

3.5.10 Rounding-Related Functions

Rounding can sometimes be an operation of interest. In what follows, we introduce several functions that are related to rounding.

The **floor function**, denoted $\lfloor\cdot\rfloor$, is a function that maps a real number x to the largest integer not more than x. In other words, the floor function rounds a real number to the nearest integer in the direction of negative infinity. For example,

$$\left\lfloor -\tfrac{1}{2} \right\rfloor = -1, \quad \left\lfloor \tfrac{1}{2} \right\rfloor = 0, \quad \text{and} \quad \lfloor 1 \rfloor = 1.$$

The **ceiling function**, denoted $\lceil\cdot\rceil$, is a function that maps a real number x to the smallest integer not less than x. In other words, the ceiling function rounds a real number to the nearest integer in the direction of positive infinity. For example,

$$\left\lceil -\tfrac{1}{2} \right\rceil = 0, \quad \left\lceil \tfrac{1}{2} \right\rceil = 1, \quad \text{and} \quad \lceil 1 \rceil = 1.$$

Some identities involving the floor and ceiling functions that are often useful include the following:

$$\begin{aligned}
\lfloor x+n \rfloor &= \lfloor x \rfloor + n \quad \text{for } x \in \mathbb{R} \text{ and } n \in \mathbb{Z}; \\
\lceil x+n \rceil &= \lceil x \rceil + n \quad \text{for } x \in \mathbb{R} \text{ and } n \in \mathbb{Z}; \\
\lceil x \rceil &= -\lfloor -x \rfloor \quad \text{for } x \in \mathbb{R}; \\
\lfloor x \rfloor &= -\lceil -x \rceil \quad \text{for } x \in \mathbb{R}; \\
\left\lceil \frac{m}{n} \right\rceil &= \left\lfloor \frac{m+n-1}{n} \right\rfloor = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 \quad \text{for } m,n \in \mathbb{Z} \text{ and } n > 0; \quad \text{and} \\
\left\lfloor \frac{m}{n} \right\rfloor &= \left\lceil \frac{m-n+1}{n} \right\rceil = \left\lceil \frac{m+1}{n} \right\rceil - 1 \quad \text{for } m,n \in \mathbb{Z} \text{ and } n > 0.
\end{aligned}$$

The **remainder function**, denoted mod, is a function that takes two integers m and n and yields the remainder obtained after dividing m by n, where the remainder is defined to always be nonnegative. For example, $\operatorname{mod}(10,3) = 1$, $\operatorname{mod}(-3,2) = 1$, and $\operatorname{mod}(8,2) = 0$. The mod function can be expressed in terms of the floor function as

$$\operatorname{mod}(m,n) = m - n\lfloor m/n \rfloor \quad \text{for } n,m \in \mathbb{Z} \text{ and } n \neq 0.$$

3.5.11 Unit-Impulse Function

In systems theory, one elementary function of fundamental importance is the unit-impulse function. Instead of defining this function explicitly, it is defined in terms of its properties. In particular, the **unit-impulse function** (also known as **Dirac delta function** or **delta function**) is denoted as δ and defined as the function with the following two properties:

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and} \tag{3.22a}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1. \tag{3.22b}$$

From these properties, we can see that the function is zero everywhere, except at $t = 0$ where it is undefined. Indeed, this is an unusual function. Although it is zero everywhere except at a single point, it has a nonzero integral. Technically, the unit-impulse function is not a function in the ordinary sense. Rather, it is what is known as a **generalized function**.

Graphically, we represent the unit-impulse function as shown in Figure 3.20. Since the function assumes an infinite value at the origin, we cannot plot the true value of the function. Instead, we use a vertical arrow to represent this infinite value. To show the strength of the impulse, its weight is also indicated. In Figure 3.21, we plot a scaled and time-shifted version of the unit-impulse function.

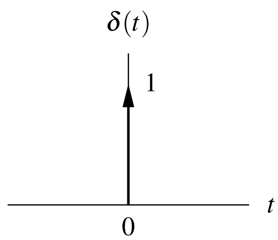


Figure 3.20: Unit impulse function.

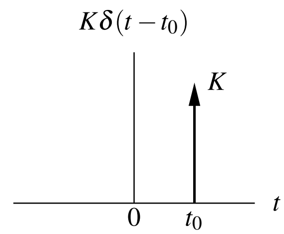


Figure 3.21: Scaled and time-shifted unit impulse function.

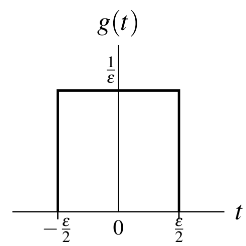


Figure 3.22: Unit-impulse function as limit of rectangular pulse.

We can view the unit-impulse function δ as a limiting case involving a rectangular pulse. More specifically, let us define the following rectangular pulse function

$$g(t) = \begin{cases} \frac{1}{\varepsilon} & |t| < \frac{\varepsilon}{2} \\ 0 & \text{otherwise.} \end{cases}$$

This function is plotted in Figure 3.22. Clearly, the area under the curve is unity for any choice of ε. The function δ is obtained by taking the following limit:

$$\delta(t) = \lim_{\varepsilon \to 0} g(t).$$

Thus, δ can be viewed as a limiting case of a rectangular pulse where the pulse width becomes infinitesimally small and the pulse height becomes infinitely large in such a way that the integral of the resulting function remains unity.

Informally, one can also think of the unit-impulse function δ as the derivative of the unit-step function u. Strictly speaking, however, the derivative of u does not exist in the ordinary sense, since u is discontinuous at 0. To be more precise, δ is what is called the **generalized derivative** of u. The generalized derivative is essentially an extension of the notion of (ordinary) derivative, which can be well defined even for functions with discontinuities.

The unit-impulse function has two important properties that follow from its operational definition in (3.22). These properties are given by the theorems below.

Theorem 3.3 (Equivalence property)**.** *For any function x that is continuous at the point t_0,*

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0) \quad \text{for all } t. \tag{3.23}$$

*This result is known as the **equivalence property**. This property is illustrated graphically in Figure 3.23.*

Proof. The proof follows immediately from the fact that the unit-impulse function is only nonzero at a single point. ■

Theorem 3.4 (Sifting property)**.** *For any function x that is continuous at the point t_0,*

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0). \tag{3.24}$$

*This result is known as the **sifting property**.*

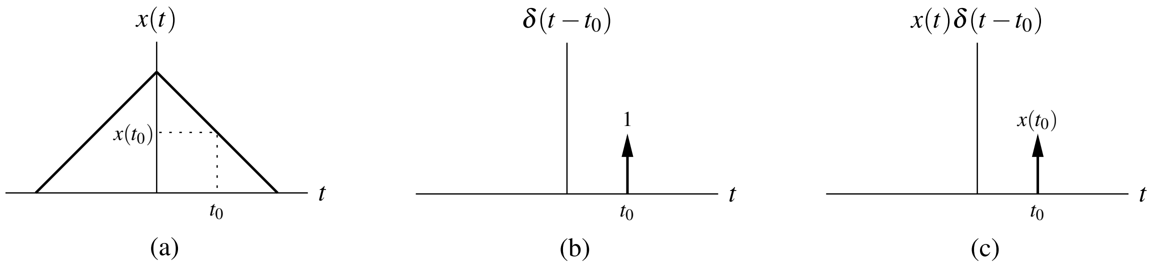


Figure 3.23: Graphical interpretation of equivalence property. (a) A function x; (b) a time-shifted unit-impulse function; and (c) the product of the these two functions.

Proof. From (3.22b), we can write

$$\int_{-\infty}^{\infty} \delta(t-t_0)dt = 1.$$

Multiplying both sides of the preceding equation by $x(t_0)$ yields

$$\int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)dt = x(t_0).$$

Then, by using the equivalence property in (3.23), we can write

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0). \qquad \blacksquare$$

As we shall see later, the equivalence and sifting properties are extremely helpful. Lastly, note that it follows from the definition of δ that integrating this function over any interval not containing the origin will result in the value of zero.

Example 3.9 (Sifting property example)**.** Evaluate the integral

$$\int_{-\infty}^{\infty} \sin(t)\delta\left(t-\tfrac{\pi}{4}\right)dt.$$

Solution. Using the sifting property of the unit impulse function, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \sin(t)\delta\left(t-\tfrac{\pi}{4}\right)dt &= \sin\left(\tfrac{\pi}{4}\right)\\ &= \tfrac{1}{\sqrt{2}}.\end{aligned} \qquad \blacksquare$$

Example 3.10 (Sifting property example)**.** Evaluate the integral

$$\int_{-\infty}^{\infty} \sin(2\pi t)\delta(4t-1)dt.$$

Solution. First, we observe that the integral to be evaluated does not quite have the same form as (3.24). So, we need to perform a change of variable. Let $\tau = 4t$ so that $t = \frac{1}{4}\tau$ and $dt = \frac{1}{4}d\tau$. Performing the change of variable, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \sin(2\pi t)\delta(4t-1)dt &= \int_{-\infty}^{\infty} \tfrac{1}{4}\sin\left[2\pi\left(\tfrac{1}{4}\tau\right)\right]\delta(\tau-1)d\tau\\ &= \int_{-\infty}^{\infty} \tfrac{1}{4}\sin\left(\tfrac{\pi}{2}\tau\right)\delta(\tau-1)d\tau.\end{aligned}$$

Now the integral has the desired form, and we can use the sifting property of the unit-impulse function to write

$$\begin{aligned}\int_{-\infty}^{\infty} \sin(2\pi t)\delta(4t-1)dt &= \left[\tfrac{1}{4}\sin\left(\tfrac{\pi}{2}\tau\right)\right]\Big|_{\tau=1}\\ &= \tfrac{1}{4}\sin\left(\tfrac{\pi}{2}\right)\\ &= \tfrac{1}{4}.\end{aligned}$$

■

Example 3.11. Evaluate the integral $\int_{-\infty}^{t}(\tau^2+1)\delta(\tau-2)d\tau$.

Solution. Using the equivalence property of the delta function given by (3.23), we can write

$$\begin{aligned}\int_{-\infty}^{t}(\tau^2+1)\delta(\tau-2)d\tau &= \int_{-\infty}^{t}(2^2+1)\delta(\tau-2)d\tau\\ &= 5\int_{-\infty}^{t}\delta(\tau-2)d\tau.\end{aligned}$$

Using the defining properties of the delta function given by (3.22), we have that

$$\begin{aligned}\int_{-\infty}^{t}\delta(\tau-2)d\tau &= \begin{cases}1 & t \geq 2\\ 0 & t < 2\end{cases}\\ &= u(t-2).\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}\int_{-\infty}^{t}(\tau^2+1)\delta(\tau-2)d\tau &= \begin{cases}5 & t \geq 2\\ 0 & t < 2\end{cases}\\ &= 5u(t-2).\end{aligned}$$

■

3.6 Representation of Arbitrary Functions Using Elementary Functions

In the earlier sections, we introduced a number of elementary functions. Often in signal analysis, it is convenient to represent arbitrary functions in terms of elementary functions. Here, we consider how the unit-step function can be exploited in order to obtain alternative representations of functions.

Example 3.12 (Rectangular function)**.** Show that the rect function can be expressed in terms of u as

$$\operatorname{rect} t = u\left(t+\tfrac{1}{2}\right) - u\left(t-\tfrac{1}{2}\right).$$

Solution. Using the definition of u and time-shift transformations, we have

$$u\left(t+\tfrac{1}{2}\right) = \begin{cases}1 & t \geq -\tfrac{1}{2}\\ 0 & \text{otherwise}\end{cases} \quad \text{and} \quad u\left(t-\tfrac{1}{2}\right) = \begin{cases}1 & t \geq \tfrac{1}{2}\\ 0 & \text{otherwise.}\end{cases}$$

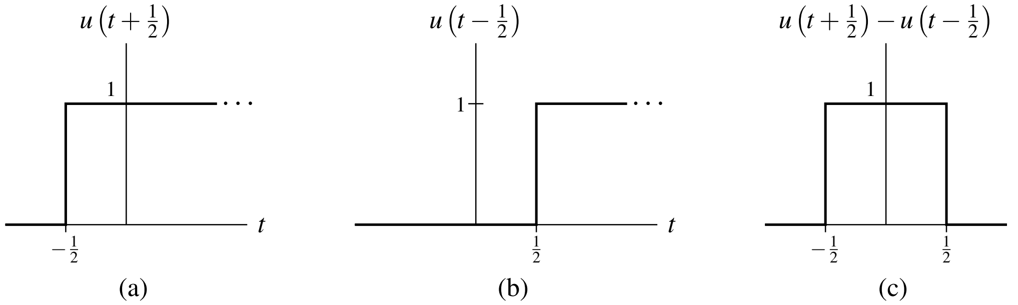


Figure 3.24: Representing the rectangular function using unit-step functions. (a) A shifted unit-step function, (b) another shifted unit-step function, and (c) their difference (which is the rectangular function).

Thus, we have

$$\begin{aligned} u\left(t+\tfrac{1}{2}\right)-u\left(t-\tfrac{1}{2}\right) &= \begin{cases} 0-0 & t<-\frac{1}{2} \\ 1-0 & -\frac{1}{2}\leq t<\frac{1}{2} \\ 1-1 & t\geq\frac{1}{2} \end{cases} \\ &= \begin{cases} 0 & t<-\frac{1}{2} \\ 1 & -\frac{1}{2}\leq t<\frac{1}{2} \\ 0 & t\geq\frac{1}{2} \end{cases} \\ &= \begin{cases} 1 & -\frac{1}{2}\leq t<\frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \operatorname{rect} t. \end{aligned}$$

Graphically, we have the scenario depicted in Figure 3.24. ■

The result in the preceding example can be generalized. For a rectangular pulse x of height 1 with a rising edge at a and falling edge at b, one can show that

$$\begin{aligned} x(t) &= u(t-a)-u(t-b) \\ &= \begin{cases} 1 & a\leq t<b \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Example 3.13 (Piecewise-linear function)**.** Consider the piecewise-linear function x given by

$$x(t) = \begin{cases} t & 0\leq t<1 \\ 1 & 1\leq t<2 \\ 3-t & 2\leq t<3 \\ 0 & \text{otherwise.} \end{cases}$$

Find a single expression for $x(t)$ (involving unit-step functions) that is valid for all t.

Solution. A plot of x is shown in Figure 3.25(a). We consider each segment of the piecewise-linear function separately. The first segment (i.e., for $0\leq t<1$) can be expressed as

$$v_1(t) = t[u(t)-u(t-1)].$$

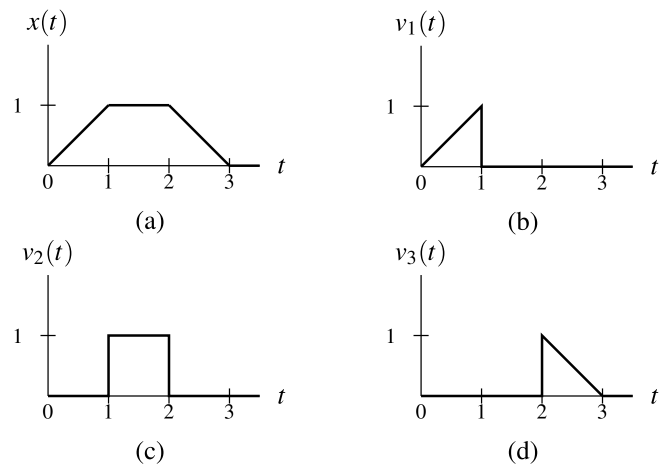


Figure 3.25: Representing a piecewise-linear function using unit-step functions. (a) The function x. (b), (c), and (d) Three functions whose sum is x.

This function is plotted in Figure 3.25(b). The second segment (i.e., for $1 \le t < 2$) can be expressed as

$$v_2(t) = u(t-1) - u(t-2).$$

This function is plotted in Figure 3.25(c). The third segment (i.e., for $2 \le t < 3$) can be expressed as

$$v_3(t) = (3-t)[u(t-2) - u(t-3)].$$

This function is plotted in Figure 3.25(d). Now, we observe that $x = v_1 + v_2 + v_3$. That is, we have

$$\begin{aligned} x(t) &= v_1(t) + v_2(t) + v_3(t) \\ &= t[u(t) - u(t-1)] + [u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)] \\ &= tu(t) + (1-t)u(t-1) + (3-t-1)u(t-2) + (t-3)u(t-3) \\ &= tu(t) + (1-t)u(t-1) + (2-t)u(t-2) + (t-3)u(t-3). \end{aligned}$$

Thus, we have found a single expression for $x(t)$ that is valid for all t. ■

Example 3.14 (Piecewise-polynomial function)**.** Consider the piecewise-polynomial function x given by

$$x(t) = \begin{cases} 1 & 0 \le t < 1 \\ (t-2)^2 & 1 \le t < 3 \\ 4-t & 3 \le t < 4 \\ 0 & \text{otherwise.} \end{cases}$$

Find a single expression for $x(t)$ (involving unit-step functions) that is valid for all t.

Solution. A plot of x is shown in Figure 3.26(a). We consider each segment of the piecewise-polynomial function separately. The first segment (i.e., for $0 \le t < 1$) can be written as

$$v_1(t) = u(t) - u(t-1).$$

This function is plotted in Figure 3.26(b). The second segment (i.e., for $1 \le t < 3$) can be written as

$$v_2(t) = (t-2)^2[u(t-1) - u(t-3)] = (t^2 - 4t + 4)[u(t-1) - u(t-3)].$$

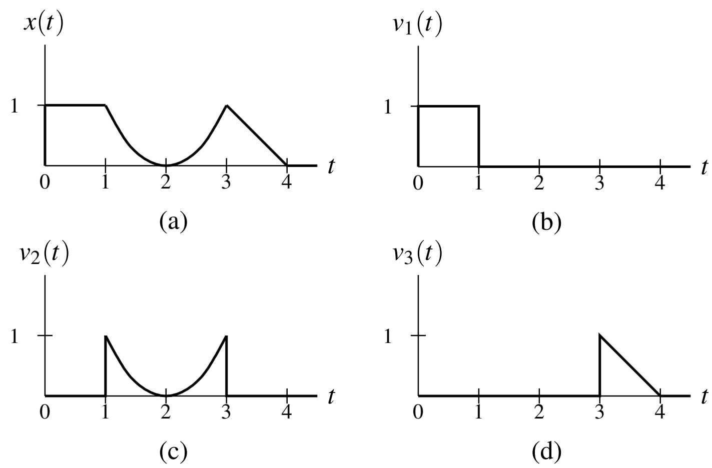


Figure 3.26: Representing a piecewise-polynomial function using unit-step functions. (a) The function x; and (b), (c), and (d) three functions whose sum is x.

This function is plotted in Figure 3.26(c). The third segment (i.e., for $3 \leq t < 4$) can be written as

$$v_3(t) = (4-t)[u(t-3) - u(t-4)].$$

This function is plotted in Figure 3.26(d). Now, we observe that $x = v_1 + v_2 + v_3$. So, we have

$$\begin{aligned}
x(t) &= v_1(t) + v_2(t) + v_3(t) \\
&= [u(t) - u(t-1)] + (t^2 - 4t + 4)[u(t-1) - u(t-3)] + (4-t)[u(t-3) - u(t-4)] \\
&= u(t) + (t^2 - 4t + 4 - 1)u(t-1) + (4 - t - [t^2 - 4t + 4])u(t-3) - (4-t)u(t-4) \\
&= u(t) + (t^2 - 4t + 3)u(t-1) + (-t^2 + 3t)u(t-3) + (t-4)u(t-4).
\end{aligned}$$

Thus, we have found a single expression for $x(t)$ that is valid for all t. ■

Example 3.15 (Periodic function)**.** Consider the periodic function x shown in Figure 3.27(a). Find a single expression for $x(t)$ (involving unit-step functions) that is valid for all t.

Solution. We begin by finding an expression for a single period of x. Let us denote this expression as v. We can then write:

$$v(t) = u(t + \tfrac{1}{2}) - u(t - \tfrac{1}{2}).$$

This function is plotted in Figure 3.27(b). In order to obtain the periodic function x, we must repeat v every 2 units (since the period of x is 2). This can be accomplished by adding an infinite number of shifted copies of v as given by

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} v(t-2k) \\
&= \sum_{k=-\infty}^{\infty} [u(t + \tfrac{1}{2} - 2k) - u(t - \tfrac{1}{2} - 2k)].
\end{aligned}$$

Thus, we have found a single expression for x that is valid for all t. ■

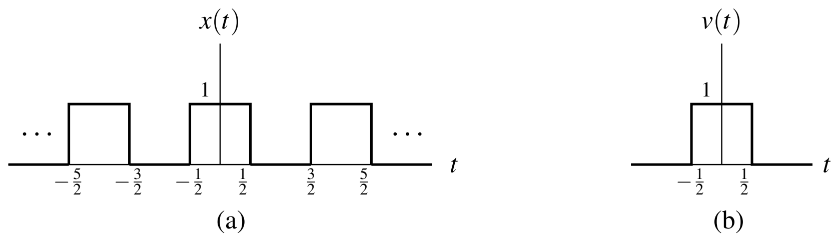


Figure 3.27: Representing a periodic function using unit-step functions. (a) The periodic function x; and (b) a function v that consists of a single period of x.

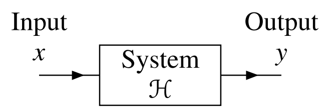


Figure 3.28: Block diagram of system.

3.7 Continuous-Time Systems

Suppose that we have a system with input x and output y. Such a system can be described mathematically by the equation

$$y = \mathcal{H}x, \tag{3.25}$$

where $\mathcal{H}$ is an operator (i.e., transformation) representing the system. The operator $\mathcal{H}$ simply maps the input function x to the output function y. Alternatively, we sometimes express the relationship (3.25) using the notation

$$x \xrightarrow{\mathcal{H}} y.$$

Furthermore, if clear from the context, the operator $\mathcal{H}$ is often omitted, yielding the abbreviated notation

$$x \to y.$$

Note that the symbols "$\to$" and "$=$" have very different meanings. For example, the notation $x \to y$ does not in any way imply that $x = y$. The symbol "$\to$" should be read as "produces" (not as "equals"). That is, "$x \to y$" should be read as "the input x produces the output y".

3.7.1 Block Diagram Representation

Suppose that we have a system defined by the operator $\mathcal{H}$ with input x and output y. Often, we represent such a system using a block diagram as shown in Figure 3.28.

3.7.2 Interconnection of Systems

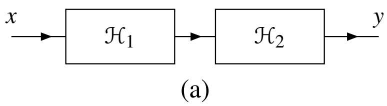
Systems may be interconnected in a number of ways. Two basic types of connections are as shown in Figure 3.29. The first type of connection, as shown in Figure 3.29(a), is known as a **series** or **cascade** connection. In this case, the overall system is defined by

$$y = \mathcal{H}_2\mathcal{H}_1x. \tag{3.26}$$

The second type of connection, as shown in Figure 3.29(b), is known as a **parallel** connection. In this case, the overall system is defined by

$$y = \mathcal{H}_1x + \mathcal{H}_2x. \tag{3.27}$$

The system equations in (3.26) and (3.27) cannot be simplified further unless the definitions of the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ are known.



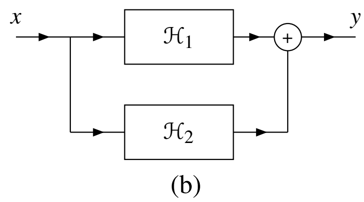


Figure 3.29: Interconnection of systems. (a) Series interconnection of the systems $\mathcal{H}_1$ and $\mathcal{H}_2$. (b) Parallel interconnection of the systems $\mathcal{H}_1$ and $\mathcal{H}_2$.

3.8 Properties of Systems

In what follows, we will define a number of important properties that a system may possess. These properties are useful in classifying systems, as well as characterizing their behavior.

3.8.1 Memory

A system $\mathcal{H}$ is said to be **memoryless** if, for every real constant t_0, $\mathcal{H}x(t)$ does not depend on $x(t)$ for some $t \neq t_0$. In other words, a memoryless system is such that the value of its output at any given point in time can depend on the value of its input at only the *same* point in time. A system that is not memoryless is said to have **memory**. Although simple, a memoryless system is not very flexible, since its current output value cannot rely on past or future values of the input.

Example 3.16 (Ideal amplifier)**.** Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(t) = Ax(t)$$

and A is a nonzero real constant.

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = Ax(t_0).$$

Thus, $\mathcal{H}x(t_0)$ depends on $x(t)$ only for $t = t_0$. Therefore, the system is memoryless. ■

Example 3.17 (Ideal integrator)**.** Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = \int_{-\infty}^{t_0} x(\tau)d\tau.$$

Thus, $\mathcal{H}x(t_0)$ depends on $x(t)$ for $-\infty < t \leq t_0$. So, $\mathcal{H}x(t_0)$ is dependent on $x(t)$ for some $t \neq t_0$ (e.g., $t_0 - 1$). Therefore, the system has memory (i.e., is not memoryless). ■

Example 3.18. Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(t) = e^{x(t)}.$$

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = e^{x(t_0)}.$$

Thus, $\mathcal{H}x(t_0)$ depends on $x(t)$ only for $t = t_0$. Therefore, the system is memoryless. ■

Example 3.19. Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(t) = \mathrm{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = \tfrac{1}{2}\left[x(t_0) - x(-t_0)\right].$$

Thus, for any x and any real t_0, we have that $\mathcal{H}x(t_0)$ depends on $x(t)$ for $t = t_0$ and $t = -t_0$. Since $\mathcal{H}x(t_0)$ depends on $x(t)$ for some $t \neq t_0$, the system has memory (i.e., the system is not memoryless). ■

3.8.2 Causality

A system $\mathcal{H}$ is said to be **causal** if, for every real constant t_0, $\mathcal{H}x(t)$ does not depend on $x(t)$ for some $t > t_0$. In other words, a causal system is such that the value of its output at any given point in time can depend on the value of its input at only the *same or earlier* points in time (i.e., *not later* points in time). A memoryless system is always causal, although the converse is not necessarily true.

If the independent variable represents time, a system must be causal in order to be physically realizable. Noncausal systems can sometimes be useful in practice, however, as the independent variable need not always represent time.

Example 3.20 (Ideal integrator)**.** Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Solution. Consider the calculation of $\mathcal{H}x(t_0)$ for arbitrary t_0. We have

$$\mathcal{H}x(t_0) = \int_{-\infty}^{t_0} x(\tau)d\tau.$$

Thus, we can see that $\mathcal{H}x(t_0)$ depends only on $x(t)$ for $-\infty < t \leq t_0$. Since all of the values in this interval are less than or equal to t_0, the system is causal. ■

Example 3.21. Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(t) = \int_{t-1}^{t+1} x(\tau)d\tau.$$

Solution. Consider the calculation of $\mathcal{H}x(t_0)$ for arbitrary t_0. We have

$$\mathcal{H}x(t_0) = \int_{t_0-1}^{t_0+1} x(\tau)d\tau.$$

Thus, we can see that $\mathcal{H}x(t_0)$ only depends on $x(t)$ for $t_0 - 1 \leq t \leq t_0 + 1$. Since some of the values in this interval are greater than t_0 (e.g., $t_0 + 1$), the system is not causal. ■

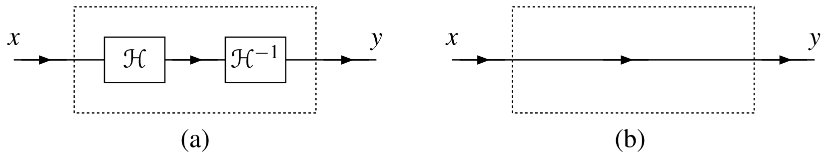


Figure 3.30: Systems that are equivalent (assuming $\mathcal{H}^{-1}$ exists). (a) First and (b) second system.

Example 3.22. Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(t) = (t+1)e^{x(t-1)}.$$

Solution. Consider the calculation of $\mathcal{H}x(t_0)$ for arbitrary t_0. We have

$$\mathcal{H}x(t_0) = (t_0+1)e^{x(t_0-1)}.$$

Thus, we can see that $\mathcal{H}x(t_0)$ depends only on $x(t)$ for $t = t_0 - 1$. Since $t_0 - 1 \leq t_0$, the system is causal. ■

Example 3.23. Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(t) = \operatorname{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

Solution. For any x and any real constant t_0, we have that $\mathcal{H}x(t_0)$ depends only on $x(t)$ for $t = t_0$ and $t = -t_0$. Suppose that $t_0 = -1$. In this case, we have that $\mathcal{H}x(t_0)$ (i.e., $\mathcal{H}x(-1)$) depends on $x(t)$ for $t = 1$ but $t = 1 > t_0$. Therefore, the system is not causal. ■

3.8.3 Invertibility

The **inverse** of a system $\mathcal{H}$ is a system $\mathcal{G}$ such that, for every function x,

$$\mathcal{G}\mathcal{H}x = x$$

(i.e., the system formed by the cascade interconnection of $\mathcal{H}$ followed by $\mathcal{G}$ is a system whose input and output are equal). In other words, the effect of $\mathcal{H}$ is cancelled by $\mathcal{G}$. As a matter of notation, the inverse of $\mathcal{H}$ is denoted $\mathcal{H}^{-1}$. The relationship between a system and its inverse is illustrated in Figure 3.30. The two systems in this figure must be equivalent, due to the relationship between $\mathcal{H}$ and $\mathcal{H}^{-1}$ (i.e., $\mathcal{H}^{-1}$ cancels $\mathcal{H}$).

A system $\mathcal{H}$ is said to be **invertible** if it has a corresponding inverse system (i.e., its inverse exists). An invertible system must be such that its input x can always be uniquely determined from its output $\mathcal{H}x$. From this definition, it follows that an invertible system will always produce distinct outputs from any two distinct inputs.

To show that a system is invertible, we simply find the inverse system. To show that a system is not invertible, it is sufficient to find two distinct inputs to that system that result in identical outputs. In practical terms, invertible systems are nice in the sense that their effects can be undone.

Example 3.24. Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(t) = x(t - t_0)$$

and t_0 is a real constant.

Solution. Let $y = \mathcal{H}x$. By substituting $t+t_0$ for t in $y(t) = x(t-t_0)$, we obtain

$$\begin{aligned} y(t+t_0) &= x(t+t_0-t_0) \\ &= x(t). \end{aligned}$$

Thus, we have shown that

$$x(t) = y(t+t_0).$$

This, however, is simply the equation of the inverse system $\mathcal{H}^{-1}$. In particular, we have that

$$x(t) = \mathcal{H}^{-1}y(t)$$

where

$$\mathcal{H}^{-1}y(t) = y(t+t_0).$$

Thus, we have found $\mathcal{H}^{-1}$. Therefore, the system $\mathcal{H}$ is invertible. ■

Example 3.25. Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(t) = \sin[x(t)].$$

Solution. Consider an input of the form $x(t) = 2\pi k$ where k is an arbitrary integer. The response $\mathcal{H}x$ to such an input is given by

$$\begin{aligned} \mathcal{H}x(t) &= \sin[x(t)] \\ &= \sin(2\pi k) \\ &= 0. \end{aligned}$$

Thus, we have found an infinite number of distinct inputs (i.e., $x(t) = 2\pi k$ for $k = 0, \pm 1, \pm 2, \ldots$) that all result in the same output. Therefore, the system is not invertible. ■

Example 3.26. Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(t) = 3x(3t+3).$$

Solution. Let $y = \mathcal{H}x$. From the definition of $\mathcal{H}$, we can write

$$\begin{aligned} & y(t) = 3x(3t+3) \\ \Rightarrow\quad & y\left(\tfrac{1}{3}t-1\right) = 3x(t) \\ \Rightarrow\quad & x(t) = \tfrac{1}{3}y\left(\tfrac{1}{3}t-1\right). \end{aligned}$$

In other words, $\mathcal{H}^{-1}$ is given by $\mathcal{H}^{-1}y(t) = \frac{1}{3}y\left(\frac{1}{3}t-1\right)$. Since we have just found $\mathcal{H}^{-1}$, $\mathcal{H}^{-1}$ exists. Therefore, the system $\mathcal{H}$ is invertible. ■

Example 3.27. Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(t) = \operatorname{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

Solution. Consider the response $\mathcal{H}x$ of the system to an input x of the form

$$x(t) = \alpha,$$

where α is a real constant. We have that

$$\begin{aligned}\mathcal{H}x(t) &= \tfrac{1}{2}[x(t) - x(-t)] \\ &= \tfrac{1}{2}(\alpha - \alpha) \\ &= 0.\end{aligned}$$

Therefore, any constant input yields the same zero output. This, however, implies that distinct inputs can yield identical outputs. Therefore, the system is not invertible. ■

3.8.4 Bounded-Input Bounded-Output (BIBO) Stability

Although stability can be defined in numerous ways, in systems theory, we are often most interested in bounded-input bounded-output (BIBO) stability.

A system $\mathcal{H}$ is **BIBO stable** if, for every bounded function x, $\mathcal{H}x$ is also bounded (i.e., $|x(t)| < \infty$ for all t implies that $|\mathcal{H}x(t)| < \infty$ for all t). In other words, a BIBO stable system is such that it guarantees to always produce a bounded output as long as its input is bounded.

To prove that a system is BIBO stable, we must show that every bounded input leads to a bounded output. To show that a system is not BIBO stable, we simply need to find one counterexample (i.e., a single bounded input that leads to an unbounded output).

In practical terms, a BIBO stable system is well behaved in the sense that, as long as the system input remains finite for all time, the output will also remain finite for all time. Usually, a system that is not BIBO stable will have serious safety issues. For example, a portable music player with a battery input of 3.7 volts and headset output of ∞ volts would result in one vaporized human (and likely one big lawsuit as well).

Example 3.28 (Squarer)**.** Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(t) = x^2(t).$$

Solution. Suppose that the input x is bounded such that (for all t)

$$|x(t)| \leq A,$$

where A is a finite real constant. Squaring both sides of the inequality, we obtain

$$|x(t)|^2 \leq A^2.$$

Interchanging the order of the squaring and magnitude operations on the left-hand side of the inequality, we have

$$\left|x^2(t)\right| \leq A^2.$$

Using the fact that $\mathcal{H}x(t) = x^2(t)$, we can write

$$|\mathcal{H}x(t)| \leq A^2.$$

Since A is finite, A^2 is also finite. Thus, we have that $\mathcal{H}x$ is bounded (i.e., $|\mathcal{H}x(t)| \leq A^2 < \infty$ for all t). Therefore, the system is BIBO stable. ■

Example 3.29 (Ideal integrator)**.** Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Solution. Suppose that we choose the input $x = u$ (where u denotes the unit-step function). Clearly, u is bounded (i.e., $|u(t)| \leq 1$ for all t). Calculating the response $\mathcal{H}x$ to this input, we have

$$\begin{aligned}\mathcal{H}x(t) &= \int_{-\infty}^{t} u(\tau)d\tau \\ &= \int_{0}^{t} d\tau \\ &= [\tau]|_0^t \\ &= t.\end{aligned}$$

From this result, however, we can see that as $t \to \infty$, $\mathcal{H}x(t) \to \infty$. Thus, the output $\mathcal{H}x$ is unbounded for the bounded input x. Therefore, the system is not BIBO stable. ■

Example 3.30. Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(t) = \operatorname{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

Solution. Suppose that x is bounded. Then, $x(-t)$ is also bounded. Since the difference of two bounded functions is bounded, $x(t) - x(-t)$ is bounded. Multiplication of a bounded function by a finite constant yields a bounded result. So, the function $\frac{1}{2}[x(t) - x(-t)]$ is bounded. Thus, $\mathcal{H}x(t)$ is bounded. Since a bounded input must yield a bounded output, the system is BIBO stable. ■

Example 3.31 (Ideal differentiator)**.** Determine whether the system $\mathcal{H}$ is BIBO stable, where

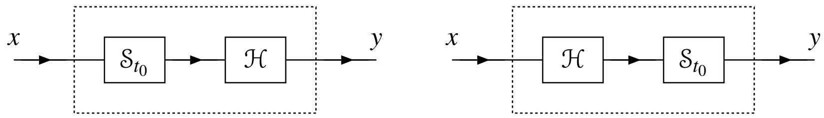
$$\mathcal{H}x(t) = \mathcal{D}x(t)$$

and $\mathcal{D}$ denotes the derivative operator.

Solution. Consider the input $x(t) = \sin\left(t^2\right)$. Clearly, x is bounded, since the sine function is bounded. In particular, $|x(t)| \leq 1$ for all real t. Now, consider the response of the system to the input x. We have

$$\begin{aligned}\mathcal{H}x(t) &= \mathcal{D}x(t) \\ &= \mathcal{D}\left\{\sin(t^2)\right\}(t) \\ &= 2t\cos(t^2).\end{aligned}$$

Clearly, $\mathcal{H}x$ is unbounded, since $|\mathcal{H}x(t)|$ grows without bound as $|t| \to \infty$. Thus, the output is not bounded for some bounded input. Therefore, the system is not BIBO stable. ■



(a) (b)

Figure 3.31: Systems that are equivalent if $\mathcal{H}$ is time invariant (i.e., $\mathcal{H}$ commutes with $\mathcal{S}_{t_0}$). (a) A system that first time shifts by t_0 and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}\mathcal{S}_{t_0}x$); and (b) a system that first applies $\mathcal{H}$ and then time shifts by t_0 (i.e., $y = \mathcal{S}_{t_0}\mathcal{H}x$).

3.8.5 Time Invariance

A system $\mathcal{H}$ is said to be **time invariant (TI)** (or **shift invariant (SI)**) if, for every function x and every real number t_0, the following condition holds:

$$\mathcal{H}x(t-t_0) = \mathcal{H}x'(t) \text{ for all } t \quad \text{where} \quad x'(t) = x(t-t_0)$$

(i.e., $\mathcal{H}$ *commutes with time shifts*). In other words, a system is time invariant, if a time shift (i.e., advance or delay) in the input function results in an identical time shift in the output function. A system that is not time invariant is said to be **time varying** (or **shift varying**). In effect, time invariance means that the two systems shown in Figure 3.31 are equivalent, where $\mathcal{S}_{t_0}$ denotes an operator that applies a time shift of t_0 to a function (i.e., $\mathcal{S}_{t_0}x(t) = x(t-t_0)$).

In simple terms, a time invariant system is a system whose behavior does not change with respect to time. Practically speaking, compared to time-varying systems, time-invariant systems are much easier to design and analyze, since their behavior does not change with respect to time.

Example 3.32. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = \sin[x(t)].$$

Solution. Let $x'(t) = x(t-t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we can easily deduce that

$$\begin{aligned} \mathcal{H}x(t-t_0) &= \sin[x(t-t_0)] \quad \text{and} \\ \mathcal{H}x'(t) &= \sin[x'(t)] \\ &= \sin[x(t-t_0)]. \end{aligned}$$

Since $\mathcal{H}x(t-t_0) = \mathcal{H}x'(t)$ for all x and t_0, the system is time invariant. ■

Example 3.33. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = tx(t).$$

Solution. Let $x'(t) = x(t-t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} \mathcal{H}x(t-t_0) &= (t-t_0)x_1(t-t_0) \quad \text{and} \\ \mathcal{H}x'(t) &= tx'(t) \\ &= tx(t-t_0). \end{aligned}$$

Since $\mathcal{H}x(t-t_0) = \mathcal{H}x'(t)$ does not hold for all x and t_0, the system is not time invariant (i.e., the system is time varying). ■

Example 3.34. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = \sum_{k=-10}^{10} x(t-k).$$

Solution. Let $x'(t) = x(t-t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we can easily deduce that

$$\begin{aligned}
\mathcal{H}x(t-t_0) &= \sum_{k=-10}^{10} x(t-t_0-k) \quad \text{and} \\
\mathcal{H}x'(t) &= \sum_{k=-10}^{10} x'(t-k) \\
&= \sum_{k=-10}^{10} x(t-k-t_0) \\
&= \sum_{k=-10}^{10} x(t-t_0-k).
\end{aligned}$$

Since $\mathcal{H}x(t-t_0) = \mathcal{H}x'(t)$ for all x and t_0, the system is time invariant. ■

Example 3.35. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = \mathrm{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

Solution. Let $x'(t) = x(t-t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
\mathcal{H}x(t-t_0) &= \tfrac{1}{2}[x(t-t_0) - x(-(t-t_0))] \\
&= \tfrac{1}{2}[x(t-t_0) - x(-t+t_0)] \quad \text{and} \\
\mathcal{H}x'(t) &= \tfrac{1}{2}[x'(t) - x'(-t)] \\
&= \tfrac{1}{2}[x(t-t_0) - x(-t-t_0)].
\end{aligned}$$

Since $\mathcal{H}x(t-t_0) = \mathcal{H}x'(t)$ does not hold for all x and t_0, the system is not time invariant. ■

3.8.6 Linearity

Two of the most and frequently-occurring mathematical operations are addition and scalar multiplication. For this reason, it is often extremely helpful to know if these operations commute with the operation performed by a given system. The system properties to be introduced next relate to this particular issue.

A system $\mathcal{H}$ is said to be **additive** if, for all functions x_1 and x_2, the following condition holds:

$$\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$$

(i.e., $\mathcal{H}$ *commutes with addition*). Essentially, a system $\mathcal{H}$ being additive means that the two systems shown in Figure 3.32 are equivalent.

A system $\mathcal{H}$ is said to be **homogeneous** if, for every function x and every complex constant a, the following condition holds:

$$\mathcal{H}(ax) = a\mathcal{H}x$$

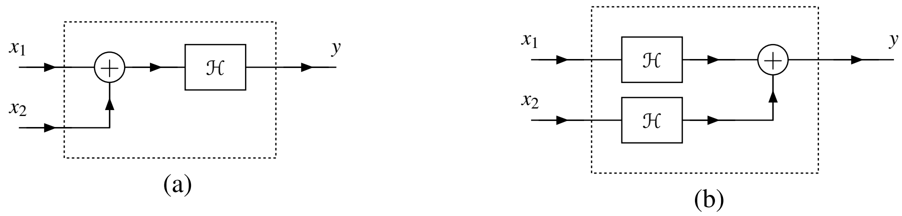


Figure 3.32: Systems that are equivalent if $\mathcal{H}$ is additive (i.e., $\mathcal{H}$ commutes with addition). (a) A system that first performs addition and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}(x_1 + x_2)$); and (b) a system that first applies $\mathcal{H}$ and then performs addition (i.e., $y = \mathcal{H}x_1 + \mathcal{H}x_2$).

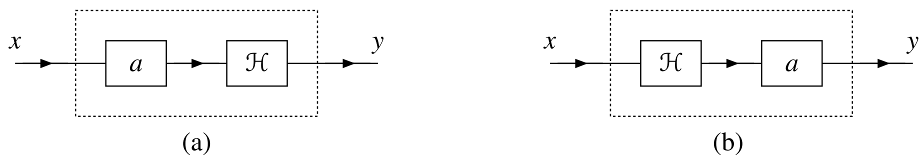


Figure 3.33: Systems that are equivalent if $\mathcal{H}$ is homogeneous (i.e., $\mathcal{H}$ commutes with scalar multiplication). (a) A system that first performs scalar multiplication and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}(ax)$); and (b) a system that first applies $\mathcal{H}$ and then performs scalar multiplication (i.e., $y = a\mathcal{H}x$).

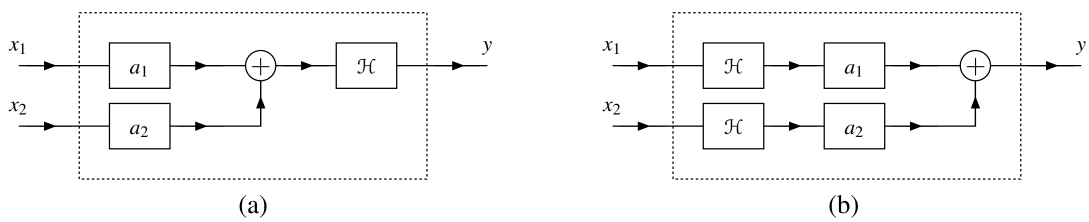


Figure 3.34: Systems that are equivalent if $\mathcal{H}$ is linear (i.e., $\mathcal{H}$ commutes with linear combinations). (a) A system that first computes a linear combination and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}(a_1x_1 + a_2x_2)$); and (b) a system that first applies $\mathcal{H}$ and then computes a linear combination (i.e., $y = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$).

(i.e., $\mathcal{H}$ *commutes with scalar multiplication*). Essentially, a system $\mathcal{H}$ being homogeneous means that the two systems shown in Figure 3.33 are equivalent.

The additivity and homogeneity properties can be combined into a single property known as superposition. In particular, a system $\mathcal{H}$ is said to have the **superposition** property, if for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$$

(i.e., $\mathcal{H}$ *commutes with linear combinations*). A system that is both additive and homogeneous (or equivalently satisfies superposition) is said to be **linear**. Essentially, a system $\mathcal{H}$ being linear means that the two systems shown in Figure 3.34 are equivalent. To show that a system is linear, we can show that it possesses both the additivity and homogeneity properties, or we can simply show that the superposition property holds. Practically speaking, linear systems are much easier to design and analyze than nonlinear systems.

Example 3.36. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = tx(t).$$

Solution. Let $x'(t) = a_1x_1(t) + a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we can write

$$\begin{aligned} a_1\mathcal{H}x_1(t)+a_2\mathcal{H}x_2(t) &= a_1tx_1(t)+a_2tx_2(t) \quad \text{and} \\ \mathcal{H}x'(t) &= tx'(t) \\ &= t\left[a_1x_1(t)+a_2x_2(t)\right] \\ &= a_1tx_1(t)+a_2tx_2(t). \end{aligned}$$

Since $\mathcal{H}(a_1x_1+a_2x_2) = a_1\mathcal{H}x_1+a_2\mathcal{H}x_2$ for all x_1, x_2, a_1, and a_2, the superposition property holds and the system is linear. ■

Example 3.37. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = |x(t)|.$$

Solution. Let $x'(t) = a_1x_1(t)+a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} a_1\mathcal{H}x_1(t)+a_2\mathcal{H}x_2(t) &= a_1|x_1(t)|+a_2|x_2(t)| \quad \text{and} \\ \mathcal{H}x'(t) &= \left|x'(t)\right| \\ &= |a_1x_1(t)+a_2x_2(t)|. \end{aligned}$$

At this point, we recall the triangle inequality (i.e., for $a,b \in \mathbb{C}$, $|a+b| \le |a|+|b|$). Thus, $\mathcal{H}(a_1x_1+a_2x_2) = a_1\mathcal{H}x_1+a_2\mathcal{H}x_2$ cannot hold for all x_1, x_2, a_1, and a_2 due, in part, to the triangle inequality. For example, this condition fails to hold for

$$a_1 = -1, \quad x_1(t) = 1, \quad a_2 = 0, \quad \text{and} \quad x_2(t) = 0,$$

in which case

$$a_1\mathcal{H}x_1(t)+a_2\mathcal{H}x_2(t) = -1 \quad \text{and} \quad \mathcal{H}x'(t) = 1.$$

Therefore, the superposition property does not hold and the system is not linear. ■

Example 3.38. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = \operatorname{Odd}(x)(t) = \tfrac{1}{2}\left[x(t)-x(-t)\right].$$

Solution. Let $x'(t) = a_1x_1(t)+a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} a_1\mathcal{H}x_1(t)+a_2\mathcal{H}x_2(t) &= \tfrac{1}{2}a_1[x_1(t)-x_1(-t)]+\tfrac{1}{2}a_2[x_2(t)-x_2(-t)] \quad \text{and} \\ \mathcal{H}x'(t) &= \tfrac{1}{2}[x'(t)-x'(-t)] \\ &= \tfrac{1}{2}[a_1x_1(t)+a_2x_2(t)-[a_1x_1(-t)+a_2x_2(-t)]] \\ &= \tfrac{1}{2}[a_1x_1(t)-a_1x_1(-t)+a_2x_2(t)-a_2x_2(-t)] \\ &= \tfrac{1}{2}a_1[x_1(t)-x_1(-t)]+\tfrac{1}{2}a_2[x_2(t)-x_2(-t)]. \end{aligned}$$

Since $\mathcal{H}(a_1x_1+a_2x_2) = a_1\mathcal{H}x_1+a_2\mathcal{H}x_2$ for all x_1, x_2, a_1, and a_2, the system is linear. ■

Example 3.39. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = x(t)x(t-1).$$

Solution. Let $x'(t) = a_1x_1(t) + a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) &= a_1x_1(t)x_1(t-1) + a_2x_2(t)x_2(t-1) \quad \text{and} \\
\mathcal{H}x'(t) &= x'(t)x'(t-1) \\
&= [a_1x_1(t) + a_2x_2(t)][a_1x_1(t-1) + a_2x_2(t-1)] \\
&= a_1^2x_1(t)x_1(t-1) + a_1a_2x_1(t)x_2(t-1) + a_1a_2x_1(t-1)x_2(t) + a_2^2x_2(t)x_2(t-1).
\end{aligned}$$

Clearly, the expressions for $\mathcal{H}(a_1x_1 + a_2x_2)$ and $a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$ are quite different. Consequently, these expressions are not equal for many choices of a_1, a_2, x_1, and x_2 (e.g., $a_1 = 2$, $a_2 = 0$, $x_1(t) = 1$, and $x_2(t) = 0$). Therefore, the superposition property does not hold and the system is not linear. ■

Example 3.40 (Ideal integrator)**.** A system $\mathcal{H}$ is defined by the equation

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Determine whether this system is additive and/or homogeneous. Determine whether this system is linear.

Solution. First, we consider the additivity property. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
\mathcal{H}x_1(t) + \mathcal{H}x_2(t) &= \int_{-\infty}^{t} x_1(\tau)d\tau + \int_{-\infty}^{t} x_2(\tau)d\tau \quad \text{and} \\
\mathcal{H}(x_1 + x_2)(t) &= \int_{-\infty}^{t} (x_1 + x_2)(\tau)d\tau \\
&= \int_{-\infty}^{t} [x_1(\tau) + x_2(\tau)]d\tau \\
&= \int_{-\infty}^{t} x_1(\tau)d\tau + \int_{-\infty}^{t} x_2(\tau)d\tau.
\end{aligned}$$

Since $\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$ for all x_1 and x_2, the system is additive.

Second, we consider the homogeneity property. Let a denote an arbitrary complex constant. From the definition of $\mathcal{H}$, we can write

$$\begin{aligned}
a\mathcal{H}x(t) &= a\int_{-\infty}^{t} x(\tau)d\tau \quad \text{and} \\
\mathcal{H}(ax)(t) &= \int_{-\infty}^{t} (ax)(\tau)d\tau \\
&= \int_{-\infty}^{t} ax(\tau)d\tau \\
&= a\int_{-\infty}^{t} x(\tau)d\tau.
\end{aligned}$$

Since $\mathcal{H}(ax) = a\mathcal{H}x$ for all x and a, the system is homogeneous.

Lastly, we consider the linearity property. Since the system is both additive and homogeneous, it is linear. ■

Example 3.41. A system $\mathcal{H}$ is given by

$$\mathcal{H}x(t) = \operatorname{Re}[x(t)].$$

Determine whether this system is additive and/or homogeneous. Determine whether this system is linear.

Solution. First, we check if the additivity property is satisfied. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
\mathcal{H}x_1(t) + \mathcal{H}x_2(t) &= \operatorname{Re}[x_1(t)] + \operatorname{Re}[x_2(t)] \quad \text{and} \\
\mathcal{H}(x_1 + x_2)(t) &= \operatorname{Re}[(x_1 + x_2)(t)] \\
&= \operatorname{Re}[x_1(t) + x_2(t)] \\
&= \operatorname{Re}[x_1(t)] + \operatorname{Re}[x_2(t)].
\end{aligned}$$

Since $\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$ for all x_1 and x_2, the system is additive.

Second, we check if the homogeneity property is satisfied. Let a denote an arbitrary complex constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
a\mathcal{H}x(t) &= a \operatorname{Re} x(t) \quad \text{and} \\
\mathcal{H}(ax)(t) &= \operatorname{Re}[(ax)(t)] \\
&= \operatorname{Re}[ax(t)].
\end{aligned}$$

In order for $\mathcal{H}$ to be homogeneous, $a\mathcal{H}x(t) = \mathcal{H}(ax)(t)$ must hold for all x and all complex a. Suppose that $a = j$ and x is not identically zero (i.e., x is not the function $x(t) = 0$). In this case, we have

$$\begin{aligned}
a\mathcal{H}x(t) &= j \operatorname{Re}[x(t)] \quad \text{and} \\
\mathcal{H}(ax)(t) &= \operatorname{Re}[(jx)(t)] \\
&= \operatorname{Re}[jx(t)] \\
&= \operatorname{Re}[j(\operatorname{Re}[x(t)] + j\operatorname{Im}[x(t)])] \\
&= \operatorname{Re}(-\operatorname{Im}[x(t)] + j\operatorname{Re}[x(t)]) \\
&= -\operatorname{Im}[x(t)].
\end{aligned}$$

Thus, the quantities $\mathcal{H}(ax)$ and $a\mathcal{H}x$ are clearly not equal. Therefore, the system is not homogeneous.

Lastly, we consider the linearity property. Since the system does not possess both the additivity and homogeneity properties, it is not linear. ■

3.8.7 Eigenfunctions

An **eigenfunction** of a system $\mathcal{H}$ is a function x that satisfies

$$\mathcal{H}x = \lambda x,$$

for some complex constant λ, which is called an **eigenvalue**. Essentially, a system behaves as an ideal amplifier (i.e., performs amplitude scaling) when presented with one of its eigenfunctions as input. The significance of the eigenfunction property cannot be overstated. No matter how complicated a system might be, it exhibits extremely simple behavior for its eigenfunctions. We can often exploit this simplicity to reduce the complexity of solving many types of problems involving systems. In fact, as we will see later, eigenfunctions essentially form the basis for many of the mathematical tools that we use for studying systems.

Example 3.42. Consider the system $\mathcal{H}$ characterized by the equation

$$\mathcal{H}x(t) = \mathcal{D}^2 x(t),$$

where $\mathcal{D}$ denotes the derivative operator. For each function x given below, determine if x is an eigenfunction of $\mathcal{H}$, and if it is, find the corresponding eigenvalue.
(a) $x(t) = \cos(2t)$; and
(b) $x(t) = t^3$.

Solution. (a) We have

$$\begin{aligned}\mathcal{H}x(t) &= \mathcal{D}^2\{\cos(2t)\}(t) \\ &= \mathcal{D}\{-2\sin(2t)\}(t) \\ &= -4\cos(2t) \\ &= -4x(t).\end{aligned}$$

Therefore, x is an eigenfunction of $\mathcal{H}$ with the eigenvalue -4.

(b) We have

$$\begin{aligned}\mathcal{H}x(t) &= \mathcal{D}^2\{t^3\}(t) \\ &= \mathcal{D}\{3t^2\}(t) \\ &= 6t \\ &= \tfrac{6}{t^2}x(t).\end{aligned}$$

Therefore, x is not an eigenfunction of $\mathcal{H}$. ■

Example 3.43 (Ideal amplifier)**.** Consider the system $\mathcal{H}$ given by

$$\mathcal{H}x(t) = ax(t),$$

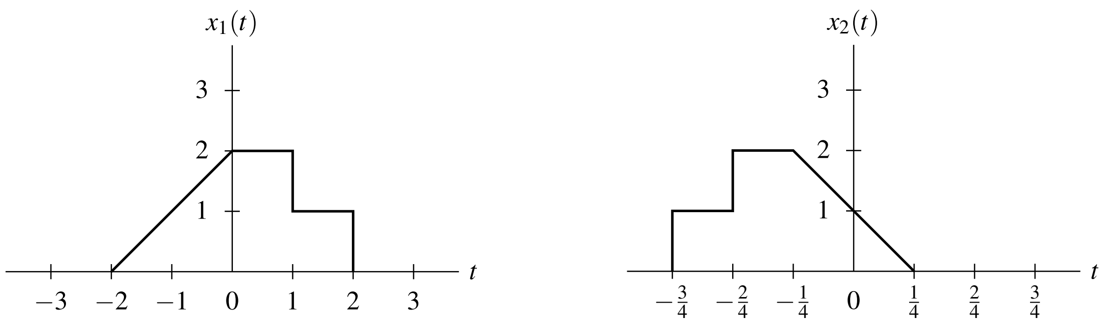
where a is a complex constant. Clearly, every function is an eigenfunction of $\mathcal{H}$ with eigenvalue a. ■

3.9 Exercises

3.9.1 Exercises Without Answer Key

3.1 Identify the independent- and dependent-variable transformations that must be applied to the function x in order to obtain each function y given below. Choose the transformations such that time shifting precedes time scaling and amplitude scaling precedes amplitude shifting. Be sure to clearly indicate the order in which the transformations are to be applied.
(a) $y(t) = x(2t-1)$;
(b) $y(t) = x(\frac{1}{2}t+1)$;
(c) $y(t) = 2x(-\frac{1}{2}t+1)+3$;
(d) $y(t) = -\frac{1}{2}x(-t+1)-1$; and
(e) $y(t) = -3x(2[t-1])-1$;
(f) $y(t) = x(7[t+3])$.

3.2 Given the functions x_1 and x_2 shown in the figures below, express x_2 in terms of x_1.

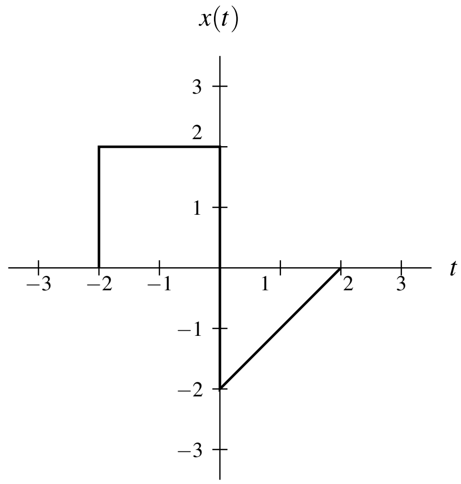


3.3 Suppose that we have two functions x and y related as

$$y(t) = x(at-b),$$

where a and b are real constants and $a \neq 0$.
(a) Show that y can be formed by first time shifting x by b and then time scaling the result by a.
(b) Show that y can also be formed by first time scaling x by a and then time shifting the result by $\frac{b}{a}$.

3.4 Given the function x shown in the figure below, plot and label each of the following functions:
(a) $x(t-1)$;
(b) $x(2t)$;
(c) $x(-t)$;
(d) $x(2t+1)$; and
(e) $\frac{1}{4}x(-\frac{1}{2}t+1)-\frac{1}{2}$.



3.5 Determine if each function x given below is periodic, and if it is, find its fundamental period.
(a) $x(t) = \cos(2\pi t) + \sin(5t)$;
(b) $x(t) = [\cos(4t - \frac{\pi}{3})]^2$;
(c) $x(t) = e^{j2\pi t} + e^{j3\pi t}$;
(d) $x(t) = 1 + \cos(2t) + e^{j5t}$;
(e) $x(t) = \cos(14t - 1) + \cos(77t - 3)$;
(f) $x(t) = \cos(et) + \sin(42t)$; and
(g) $x(t) = |\sin(\pi t)|$.

3.6 If the function x is T-periodic, show that the function y is also T-periodic in the case that:
(a) $y(t) = cx(t)$, where c is a complex constant;
(b) $y(t) = x(t) + c$, where c is a complex constant; and
(c) $y(t) = x(t - c)$, where c is a real constant.

3.7 Let y be the function given by

$$y(t) = \sum_{k=-\infty}^{\infty} x(t - Tk),$$

where x is any arbitrary function and T is a strictly positive real constant. Show that y is T periodic.

3.8 Determine whether each function x given below is even, odd, or neither even nor odd.
(a) $x(t) = t^3$;
(b) $x(t) = t^3 |t|$;
(c) $x(t) = |t^3|$;
(d) $x(t) = \cos(2\pi t)\sin(2\pi t)$;
(e) $x(t) = e^{j2\pi t}$; and
(f) $x(t) = \frac{1}{2}[e^t + e^{-t}]$.

3.9 Prove each of the following assertions:
(a) The sum of two even functions is even.
(b) The sum of two odd functions is odd.
(c) The sum of an even function and an odd function, where neither function is identically zero, is neither even nor odd.
(d) The product of two even functions is even.
(e) The product of two odd functions is even.
(f) The product of an even function and an odd function is odd.

3.10 Show that, if x is an odd function, then

$$\int_{-A}^{A} x(t)dt = 0,$$

where A is a positive real constant.

3.11 Show that, for any function x,

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_\mathsf{e}^2(t)dt + \int_{-\infty}^{\infty} x_\mathsf{o}^2(t)dt,$$

where x_e and x_o denote the even and odd parts of x, respectively.

3.12 Let x denote a function with derivative y.
(a) Show that if x is even then y is odd.
(b) Show that if x is odd then y is even.

3.13 For an arbitrary function x with even and odd parts x_e and x_o, respectively, show that:
(a) $\int_{-\infty}^{\infty} x_\mathsf{e}(t)x_\mathsf{o}(t)dt = 0$; and
(b) $\int_{-\infty}^{\infty} x(t)dt = \int_{-\infty}^{\infty} x_\mathsf{e}(t)dt$.

3.14 Show that:
(a) if a function x is T-periodic and even, then $\int_0^T x(t)dt = 2\int_0^{T/2} x(t)dt$.
(b) if a function x is T-periodic and odd, then $\int_0^T x(t)dt = 0$.

3.15 Show that the only function that is both even and odd is the zero function (i.e., the function x satisfying $x(t) = 0$ for all t).

3.16 Suppose that the function h is causal and has the even part h_e given by

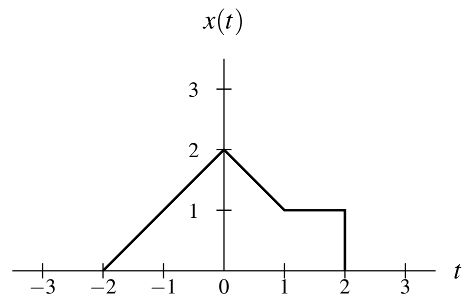
$$h_\mathsf{e}(t) = t[u(t) - u(t-1)] + u(t-1) \quad \text{for } t \geq 0.$$

Find h.

3.17 Let Ex denote the energy of the function x. Show that:
(a) $E\{ax\} = a^2 Ex$, where a is a real constant;
(b) $Ex' = Ex$, where $x'(t) = x(t - t_0)$ and t_0 is a real constant; and
(c) $Ex' = \frac{1}{|a|}Ex$, where $x'(t) = x(at)$ and a is a nonzero real constant.

3.18 Fully simplify each of the expressions below.
(a) $\int_{-\infty}^{\infty} \sin\left(2t + \frac{\pi}{4}\right)\delta(t)dt$;
(b) $\int_{-\infty}^{t} \cos(\tau)\delta(\tau + \pi)d\tau$;
(c) $\int_{-\infty}^{\infty} x(t)\delta(at - b)dt$, where a and b are real constants and $a \neq 0$;
(d) $\int_0^2 e^{j2t}\delta(t-1)dt$;
(e) $\int_{-\infty}^{t} \delta(\tau)d\tau$; and
(f) $\int_0^{\infty} \tau^2 \cos(\tau)\delta(\tau + 42)d\tau$.

3.19 Suppose that we have the function x shown in the figure below. Use unit-step functions to find a single expression for $x(t)$ that is valid for all t.



3.20 For each function x given below, find a single expression for x (i.e., an expression that does not involve multiple cases). Group similar unit-step function terms together in the expression for x.

(a) $x(t) = \begin{cases} -t-3 & -3 \le t < -2 \\ -1 & -2 \le t < -1 \\ t^3 & -1 \le t < 1 \\ 1 & 1 \le t < 2 \\ -t+3 & 2 \le t < 3 \\ 0 & \text{otherwise;} \end{cases}$

(b) $x(t) = \begin{cases} -1 & t < -1 \\ t & -1 \le t < 1 \\ 1 & t \ge 1; \end{cases}$

(c) $x(t) = \begin{cases} 4t+4 & -1 \le t < -\frac{1}{2} \\ 4t^2 & -\frac{1}{2} \le t < \frac{1}{2} \\ -4t+4 & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise;} \end{cases}$

3.21 Determine whether each system $\mathcal{H}$ given below is memoryless.
(a) $\mathcal{H}x(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;
(b) $\mathcal{H}x(t) = \operatorname{Even}(x)(t)$;
(c) $\mathcal{H}x(t) = x(t-1)+1$;
(d) $\mathcal{H}x(t) = \int_{t}^{\infty} x(\tau)d\tau$;
(e) $\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)\delta(\tau)d\tau$;
(f) $\mathcal{H}x(t) = tx(t)$; and
(g) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$.

3.22 Determine whether each system $\mathcal{H}$ given below is causal.
(a) $\mathcal{H}x(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;
(b) $\mathcal{H}x(t) = \operatorname{Even}(x)(t)$;
(c) $\mathcal{H}x(t) = x(t-1)+1$;
(d) $\mathcal{H}x(t) = \int_{t}^{\infty} x(\tau)d\tau$;
(e) $\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)\delta(\tau)d\tau$; and
(f) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau$.

3.23 For each system $\mathcal{H}$ given below, determine if $\mathcal{H}$ is invertible, and if it is, specify its inverse.
(a) $\mathcal{H}x(t) = x(at-b)$ where a and b are real constants and $a \neq 0$;
(b) $\mathcal{H}x(t) = e^{x(t)}$, where x is a real function;
(c) $\mathcal{H}x(t) = \operatorname{Even}(x)(t) - \operatorname{Odd}(x)(t)$;
(d) $\mathcal{H}x(t) = \mathcal{D}x(t)$, where $\mathcal{D}$ denotes the derivative operator; and
(e) $\mathcal{H}x(t) = x^2(t)$.

3.24 Determine whether each system $\mathcal{H}$ given below is BIBO stable.
(a) $\mathcal{H}x(t) = \int_t^{t+1} x(\tau)d\tau$ [Hint: For any function f, $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\,dx$.];
(b) $\mathcal{H}x(t) = \frac{1}{2}x^2(t) + x(t)$;
(c) $\mathcal{H}x(t) = 1/x(t)$;
(d) $\mathcal{H}x(t) = e^{-|t|}x(t)$; and
(e) $\mathcal{H}x(t) = \left(\frac{1}{t-1}\right)x(t)$.

3.25 Determine whether each system $\mathcal{H}$ given below is time invariant.
(a) $\mathcal{H}x(t) = \mathcal{D}x(t)$; where $\mathcal{D}$ denotes the derivative operator;
(b) $\mathcal{H}x(t) = \operatorname{Even}(x)(t)$;
(c) $\mathcal{H}x(t) = \int_t^{t+1} x(\tau-\alpha)d\tau$, where α is a constant;
(d) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$, where h is an arbitrary (but fixed) function;
(e) $\mathcal{H}x(t) = x(-t)$; and
(f) $\mathcal{H}x(t) = \int_{-\infty}^{2t} x(\tau)d\tau$.

3.26 Determine whether each system $\mathcal{H}$ given below is linear.
(a) $\mathcal{H}x(t) = \int_{t-1}^{t+1} x(\tau)d\tau$;
(b) $\mathcal{H}x(t) = e^{x(t)}$;
(c) $\mathcal{H}x(t) = \operatorname{Even}(x)(t)$;
(d) $\mathcal{H}x(t) = x^2(t)$; and
(e) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$, where h is an arbitrary (but fixed) function.

3.27 Determine whether each system $\mathcal{H}$ given below is additive and/or homogeneous.
(a) $\mathcal{H}x(t) = x^*(t)$; and
(b) $\mathcal{H}x(t) = \operatorname{Im}[x(t)]$.

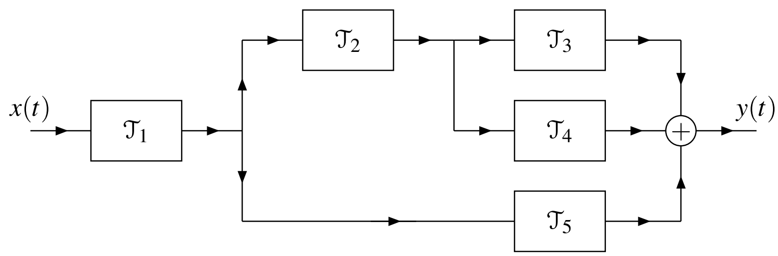
3.28 Show that if a system $\mathcal{H}$ is either additive or homogeneous, it has the property that, for a function x that is identically zero (i.e., $x(t) = 0$ for all t), $\mathcal{H}x$ is identically zero (i.e., $\mathcal{H}x(t) = 0$ for all t).

3.29 Given the function

$$x(t) = u(t+2) + u(t+1) + u(t) - 2u(t-1) - u(t-2),$$

find and sketch $y(t) = x(-4t-1)$.

3.30 For the system shown in the figure below, express the output y in terms of the input x and the transformations $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_5$.



3.31 Let z denote a function that is identically zero (i.e., $z(t) = 0$ for all t). Show that a linear system $\mathcal{H}$ is invertible if and only if $\mathcal{H}x = z$ implies that $x = z$ (i.e., the only input x that produces an output $y = z$ is $x = z$).

3.32 For each system $\mathcal{H}$ and the functions $\{x_k\}$ given below, determine if each of the x_k is an eigenfunction of $\mathcal{H}$, and if it is, also state the corresponding eigenvalue.
(a) $\mathcal{H}x(t) = x^2(t)$, $x_1(t) = a$, $x_2(t) = e^{-at}$, and $x_3(t) = \cos t$, where a is a complex constant;
(b) $\mathcal{H}x(t) = \mathcal{D}x(t)$, $x_1(t) = e^{at}$, $x_2(t) = e^{at^2}$, and $x_3(t) = 42$, where $\mathcal{D}$ denotes the derivative operator and a is a real constant;
(c) $\mathcal{H}x(t) = \int_{t-1}^{t} x(\tau)d\tau$, $x_1(t) = e^{at}$, $x_2(t) = t$, and $x_3(t) = \sin t$, where a is a nonzero complex constant; and
(d) $\mathcal{H}x(t) = |x(t)|$, $x_1(t) = a$, $x_2(t) = t$, $x_3(t) = t^2$, where a is a strictly positive real constant.

3.9.2 Exercises With Answer Key

3.33 For each function x given below, determine whether x is periodic, and if it is, find its fundamental period T.
(a) $x(t) = 3\cos(\sqrt{2}t) + 7\cos(2t)$;
(b) $x(t) = [3\cos(2t)]^3$; and
(c) $x(t) = 7\cos(35t+3) + 5\sin(15t-2)$.

Short Answer. (a) not periodic; (b) π-periodic; (c) $\left(\frac{2\pi}{5}\right)$-periodic

3.34 Simplify each of the following expressions:
(a) $\frac{(\omega^2+1)\delta(\omega-1)}{\omega^2+9}$;
(b) $\frac{\sin(k\omega)\delta(\omega)}{\omega}$;
(c) $\int_{-\infty}^{\infty} e^{t-1}\cos[\frac{\pi}{2}(t-5)]\delta(t-3)dt$;
(d) $\int_{-\infty}^{\infty} \delta(2t-3)\sin(\pi t)dt$; and
(e) $\int_{t}^{\infty} (\tau^2+1)\delta(\tau-2)d\tau$.

Short Answer. (a) $\frac{1}{5}\delta(\omega-1)$; (b) $k\delta(\omega)$; (c) $-e^2$; (d) $-\frac{1}{2}$; (e) $5u(2-t)$.

3.35 Determine whether each system $\mathcal{H}$ given below is memoryless.
(a) $\mathcal{H}x(t) = u[x(t)]$;
(b) $\mathcal{H}x(t) = x[u(t)]$;
(c) $\mathcal{H}x(t) = 42$;
(d) $\mathcal{H}x(t) = x(t^2)$;
(e) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} 2x(\tau)\delta(\tau-t)d\tau$;
(f) $\mathcal{H}x(t) = [x(t+1)]^{-1}$; and
(g) $\mathcal{H}x(t) = x(-t)$.

Short Answer. (a) memoryless; (b) has memory; (c) memoryless; (d) has memory; (e) memoryless; (f) has memory; (g) has memory

3.36 Determine whether each system $\mathcal{H}$ given below is causal.
(a) $\mathcal{H}x(t) = x(at)$, where a is a nonzero real constant;
(b) $\mathcal{H}x(t) = tu(t)x(t)$;
(c) $\mathcal{H}x(t) = x(t-a)$, where a is a strictly negative real constant;
(d) $\mathcal{H}x(t) = [x(t+1)]^{-1}$; and
(e) $\mathcal{H}x(t) = x(-t)$.

Short Answer. (a) causal if and only if $a = 1$; (b) causal; (c) not causal; (d) not causal; (e) not causal

3.37 Determine whether each system $\mathcal{H}$ given below is invertible.
(a) $\mathcal{H}x(t) = \cos[x(t)]$;
(b) $\mathcal{H}x(t) = x * x(t)$, where $f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$; and
(c) $\mathcal{H}x(t) = \operatorname{Even} x(t)$.

Short Answer. (a) not invertible; (b) not invertible; (c) not invertible

3.38 Determine whether each system $\mathcal{H}$ given below is BIBO stable.
(a) $\mathcal{H}x(t) = u(t)x(t)$;
(b) $\mathcal{H}x(t) = \ln x(t)$;
(c) $\mathcal{H}x(t) = e^{x(t)}$;
(d) $\mathcal{H}x(t) = e^{t}x(t)$;
(e) $\mathcal{H}x(t) = \cos[x(t)]$;
(f) $\mathcal{H}x(t) = x * x(t)$, where $f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$; and
(g) $\mathcal{H}x(t) = 3x(3t+3)$.

Short Answer. (a) BIBO stable; (b) not BIBO stable; (c) BIBO stable; (d) not BIBO stable; (e) BIBO stable (if x is real valued or complex valued); (f) not BIBO stable; (g) BIBO stable

3.39 Determine whether each system $\mathcal{H}$ given below is time invariant.
(a) $\mathcal{H}x(t) = \int_{-4}^{4} x(\tau)d\tau$;
(b) $\mathcal{H}x(t) = \mathcal{D}\{x^2\}(t)$, where $\mathcal{D}$ denotes the derivative operator;
(c) $\mathcal{H}x(t) = tu(t)x(t)$;
(d) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$; and
(e) $\mathcal{H}x(t) = 3x(3t+3)$.

Short Answer. (a) not time invariant; (b) time invariant; (c) not time invariant; (d) not time invariant; (e) not time invariant

3.40 Determine whether each system $\mathcal{H}$ given below is linear.
(a) $\mathcal{H}x(t) = \frac{1}{3}[x(t)-2]$;
(b) $\mathcal{H}x(t) = \mathcal{D}x(t)$, where $\mathcal{D}$ denotes the derivative operator;
(c) $\mathcal{H}x(t) = tu(t)x(t)$;
(d) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$;
(e) $\mathcal{H}x(t) = t^2\mathcal{D}^2x(t) + t\mathcal{D}x(t)$, where $\mathcal{D}$ denotes the derivative operator; and
(f) $\mathcal{H}x(t) = 3x(3t+3)$.

Short Answer. (a) not linear; (b) linear; (c) linear; (d) not linear; (e) linear; (f) linear

3.41 Determine whether each system $\mathcal{H}$ given below is additive and/or homogeneous.
(a) $\mathcal{H}x(t) = \frac{x^2(t)}{\mathcal{D}x(t)}$, where $\mathcal{D}$ denotes the derivative operator.

Short Answer. (a) homogeneous but not additive

3.42 For each system $\mathcal{H}$ and the functions $\{x_k\}$ given below, determine if each of the x_k is an eigenfunction of $\mathcal{H}$, and if it is, also state the corresponding eigenvalue.
(a) $\mathcal{H}x(t) = \mathcal{D}^2 x(t)$, $x_1(t) = \cos t$, $x_2(t) = \sin t$, and $x_3(t) = 42$, where $\mathcal{D}$ denotes the derivative operator;
(b) $\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau$, $x_1(t) = e^{2t}$, and $x_2(t) = e^t u(-t)$; and
(c) $\mathcal{H}x(t) = t^2\mathcal{D}^2 x(t) + t\mathcal{D}x(t)$ and $x_1(t) = t^k$, where k is an integer constant such that $k \geq 2$, and $\mathcal{D}$ denotes the derivative operator.

Short Answer. (a) x_1 is an eigenfunction with eigenvalue -1, x_2 is an eigenfunction with eigenvalue -1, x_3 is an eigenfunction with eigenvalue 0; (b) x_1 is an eigenfunction with eigenvalue $\frac{1}{2}$, x_2 is not an eigenfunction; (c) x_1 is an eigenfunction with eigenvalue k^2.

Chapter 4

Continuous-Time Linear Time-Invariant Systems

4.1 Introduction

In the previous chapter, we identified a number of properties that a system may possess. Two of these properties were linearity and time invariance. In this chapter, we focus our attention exclusively on systems with both of these properties. Such systems are referred to as **linear time-invariant (LTI)** systems.

4.2 Continuous-Time Convolution

In the context of LTI systems, a mathematical operation known as convolution turns out to be particularly important. The **convolution** of the functions x and h, denoted $x*h$, is defined as the function

$$x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \tag{4.1}$$

Herein, the asterisk (or star) symbol (i.e., "$*$") will be used to denote convolution, not multiplication. It is important to make a distinction between convolution and multiplication, since these two operations are quite different and do not generally yield the same result.

Notationally, $x*h$ denotes a function, namely the function that results from convolving x and h. In contrast, $x*h(t)$ denotes the function $x*h$ evaluated at t. Although we could equivalently write $x*h(t)$ with an extra pair of brackets as $(x*h)(t)$, we usually omit this extra pair of brackets, since doing so does not introduce any ambiguity and leads to more compact notation. That is, there is only one sensible way to group operations in the expression $x*h(t)$. The grouping $x*[h(t)]$ would not make sense since a convolution requires two functions as operands and $h(t)$ is not a function, but rather the value of h evaluated at t. Thus, the only sensible way to interpret the expression $x*h(t)$ is as $(x*h)(t)$.

Since the convolution operation is used extensively in system theory, we need some practical means for evaluating a convolution integral. Suppose that, for the given functions x and h, we wish to compute

$$x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Of course, we could naively attempt to compute $x*h$ by evaluating $x*h(t)$ as a separate integral for each possible value of t. This approach, however, is not feasible, as t can assume an infinite number of values, and therefore, an infinite number of integrals would need to be evaluated. Instead, we consider a slightly different approach. Let us redefine the integrand in terms of the intermediate function w_t where

$$w_t(\tau) = x(\tau)h(t-\tau).$$

(Note that $w_t(\tau)$ is implicitly a function of t.) This means that we need to compute

$$x*h(t) = \int_{-\infty}^{\infty} w_t(\tau)d\tau.$$

Now, we observe that, for most functions x and h of practical interest, the form of $w_t(\tau)$ typically remains fixed over particular ranges of t. Thus, we can compute the convolution result $x*h$ by first identifying each of the distinct expressions for $w_t(\tau)$ and the range over which each expression is valid. Then, for each range, we evaluate an integral. In this way, we typically only need to compute a small number of integrals instead of the infinite number required with the naive approach suggested above.

The above discussion leads us to propose the following general approach for computing a convolution:

1. Plot $x(\tau)$ and $h(t-\tau)$ as a function of τ.
2. Initially, consider an arbitrarily large negative value for t. This will result in $h(t-\tau)$ being shifted very far to the left on the time axis.
3. Write the mathematical expression for $w_t(\tau)$.
4. Increase t gradually until the expression for $w_t(\tau)$ changes form. Record the interval over which the expression for $w_t(\tau)$ was valid.
5. Repeat steps 3 and 4 until t is an arbitrarily large positive value. This corresponds to $h(t-\tau)$ being shifted very far to the right on the time axis.
6. For each of the intervals identified above, integrate $w_t(\tau)$ in order to find an expression for $x*h(t)$. This will yield an expression for $x*h(t)$ for each interval.
7. The results for the various intervals can be combined in order to obtain the convolution result $x*h(t)$ for all t.

Example 4.1. Compute the convolution $x*h$ where

$$x(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = e^{-t}u(t).$$

Solution. We begin by plotting the functions x and h as shown in Figures 4.1(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of h. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 4.1(c). Second, we time-shift the resulting function by t to obtain $h(t-\tau)$ as shown in Figure 4.1(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of x and h, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.1(e) to (h).

First, we consider the case of $t < -1$. From Figure 4.1(e), we can see that

$$x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.2}$$

Second, we consider the case of $-1 \le t < 0$. From Figure 4.1(f), we can see that

$$\begin{aligned} x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{t} -e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{t} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{t} \\ &= -e^{-t}[e^{t}-e^{-1}] \\ &= e^{-t-1}-1. \end{aligned} \tag{4.3}$$

Third, we consider the case of $0 \le t < 1$. From Figure 4.1(g), we can see that

$$\begin{aligned} x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{t} e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{t} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{t} \\ &= -e^{-t}[1-e^{-1}] + e^{-t}[e^{t}-1] \\ &= e^{-t}[e^{-1}-1+e^{t}-1] \\ &= 1+(e^{-1}-2)e^{-t}. \end{aligned} \tag{4.4}$$

Fourth, we consider the case of $t \ge 1$. From Figure 4.1(h), we can see that

$$\begin{aligned} x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{1} e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{1} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{1} \\ &= e^{-t}[e^{-1}-1+e-1] \\ &= (e-2+e^{-1})e^{-t}. \end{aligned} \tag{4.5}$$

Combining the results of (4.2), (4.3), (4.4), and (4.5), we have that

$$x*h(t) = \begin{cases} 0 & t < -1 \\ e^{-t-1}-1 & -1 \le t < 0 \\ (e^{-1}-2)e^{-t}+1 & 0 \le t < 1 \\ (e-2+e^{-1})e^{-t} & 1 \le t. \end{cases}$$

The convolution result $x*h$ is plotted in Figure 4.1(i). ■

Example 4.2. Compute the convolution $x*h$, where

$$x(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Solution. We begin by plotting the functions x and h as shown in Figures 4.2(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of $h(\tau)$. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 4.2(c). Second, we time-shift the resulting function by t to obtain $h(t-\tau)$ as shown in Figure 4.2(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of x and h, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.2(e) to (h).

First, we consider the case of $t < 0$. From Figure 4.2(e), we can see that

$$x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.6}$$

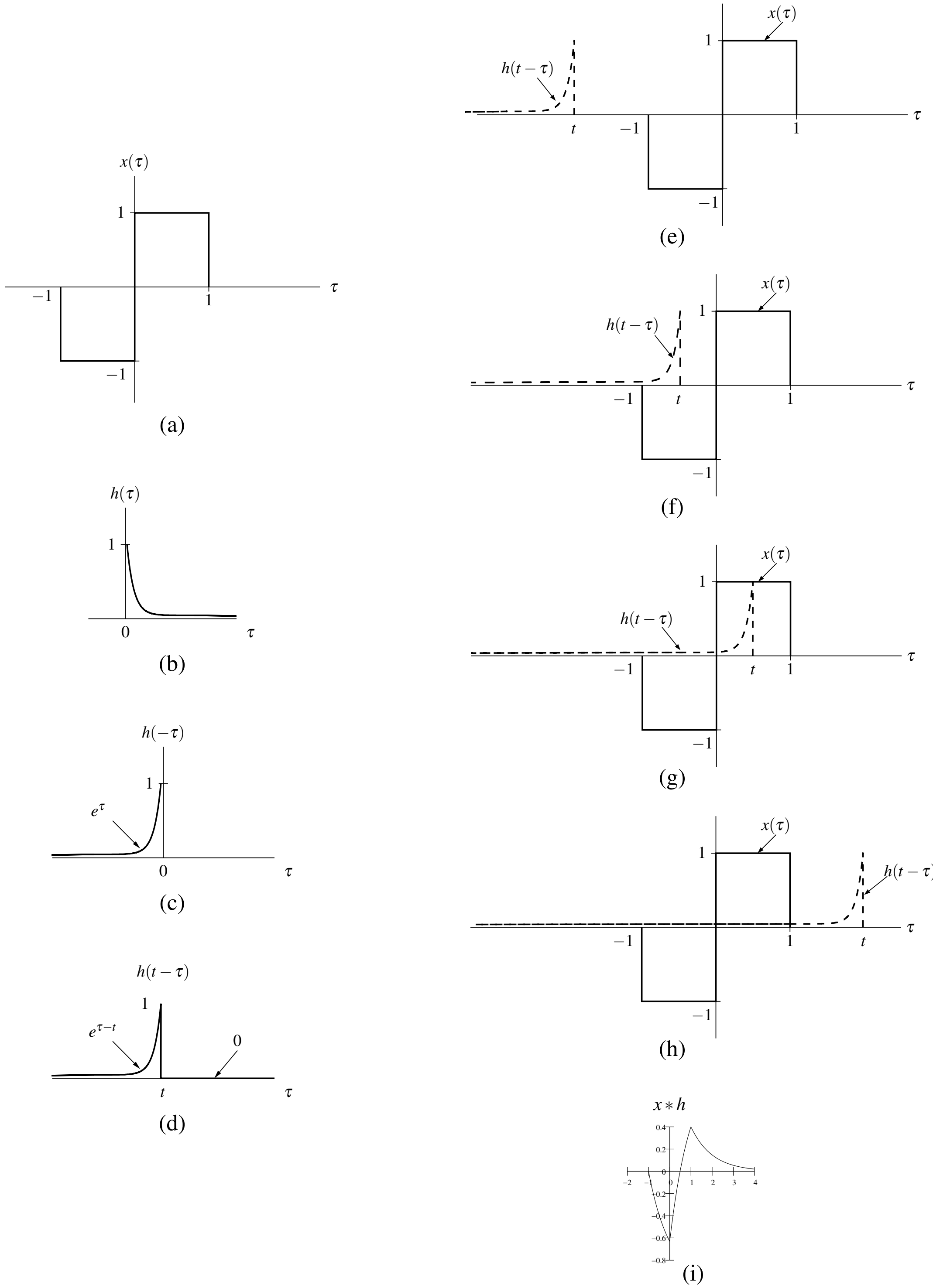


Figure 4.1: Evaluation of the convolution $x * h$. (a) The function x; (b) the function h; plots of (c) $h(-\tau)$ and (d) $h(t-\tau)$ versus τ; the functions associated with the product in the convolution integral for (e) $t < -1$, (f) $-1 \leq t < 0$, (g) $0 \leq t < 1$, and (h) $t \geq 1$; and (i) the convolution result $x * h$.

Second, we consider the case of $0 \le t < 1$. From Figure 4.2(f), we can see that

$$\begin{aligned} x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_0^t (t-\tau)d\tau \\ &= [t\tau - \tfrac{1}{2}\tau^2]|_0^t \\ &= t^2 - \tfrac{1}{2}t^2 \\ &= \tfrac{1}{2}t^2. \end{aligned} \tag{4.7}$$

Third, we consider the case of $1 \le t < 2$. From Figure 4.2(g), we can see that

$$\begin{aligned} x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^{1} (t-\tau)d\tau \\ &= [t\tau - \tfrac{1}{2}\tau^2]_{t-1}^{1} \\ &= t - \tfrac{1}{2}(1)^2 - [t(t-1) - \tfrac{1}{2}(t-1)^2] \\ &= t - \tfrac{1}{2} - [t^2 - t - \tfrac{1}{2}(t^2 - 2t + 1)] \\ &= -\tfrac{1}{2}t^2 + t. \end{aligned} \tag{4.8}$$

Fourth, we consider the case of $t \ge 2$. From Figure 4.2(h), we can see that

$$x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.9}$$

Combining the results of (4.6), (4.7), (4.8), and (4.9), we have that

$$x*h(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}t^2 & 0 \le t < 1 \\ -\frac{1}{2}t^2 + t & 1 \le t < 2 \\ 0 & t \ge 2. \end{cases}$$

The convolution result $x*h$ is plotted in Figure 4.2(i). ■

Example 4.3. Compute the quantity $x*h$, where

$$x(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t < 1 \\ -t+2 & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases} \quad \text{and} \quad h(t) = u(t) - u(t-1).$$

Solution. Due to the somewhat ugly nature of the expressions for $x(t)$ and $h(t)$, this problem can be more easily solved if we use the graphical interpretation of convolution to guide us. We begin by plotting the functions x and h, as shown in Figures 4.3(a) and (b), respectively.

Next, we need to determine $h(t-\tau)$, the time-reversed and time-shifted version of $h(\tau)$. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 4.3(c). Second, we time-shift the resulting signal by t to obtain $h(t-\tau)$ as shown in Figure 4.3(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of x and h, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.3(e) to (i).

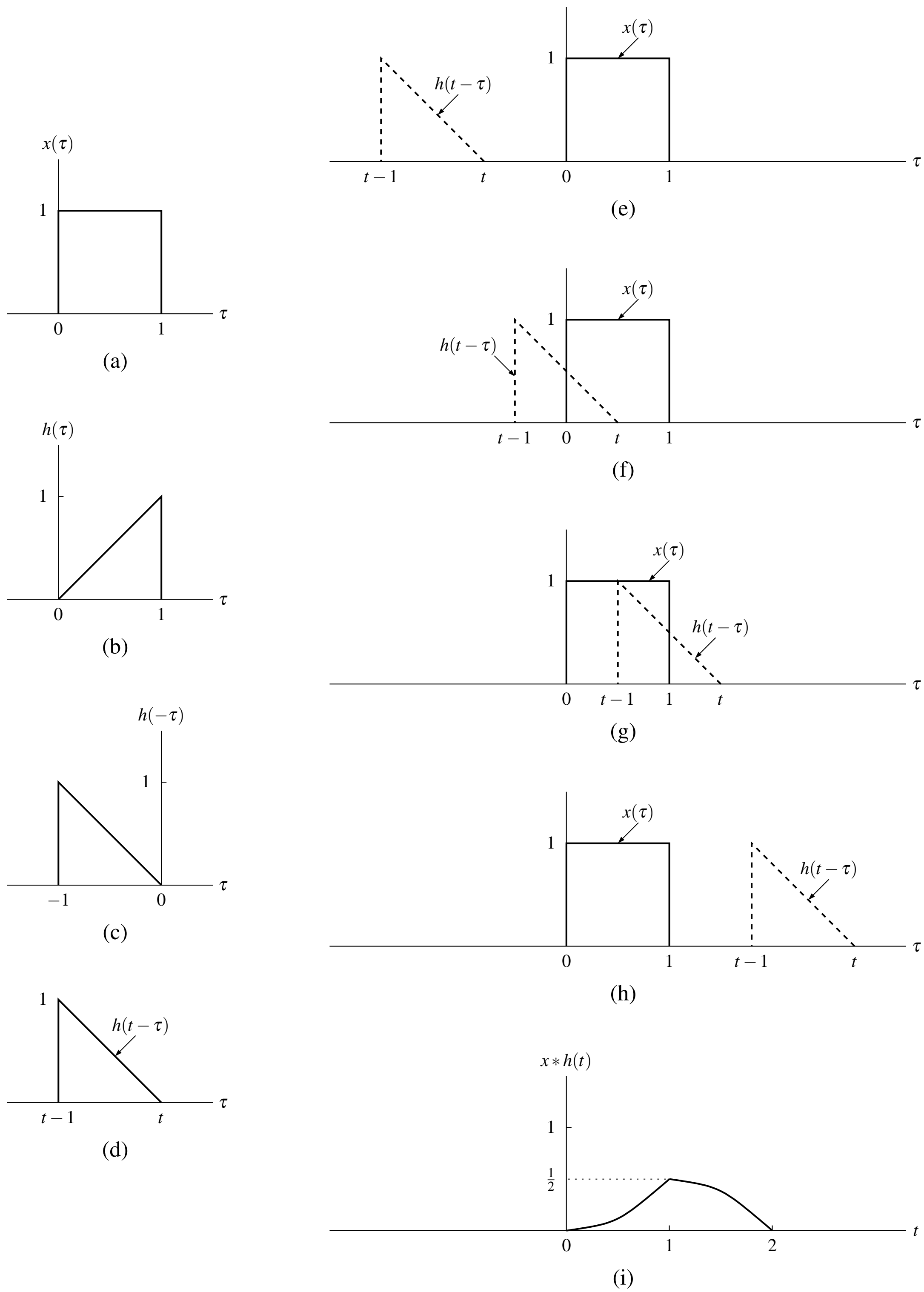


Figure 4.2: Evaluation of the convolution $x*h$. (a) The function x; (b) the function h; plots of (c) $h(-\tau)$ and (d) $h(t-\tau)$ versus τ; the functions associated with the product in the convolution integral for (e) $t<0$, (f) $0 \leq t < 1$, (g) $1 \leq t < 2$, and (h) $t \geq 2$; and (i) the convolution result $x*h$.

First, we consider the case of $t < 0$. From Figure 4.3(e), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.10}$$

Second, we consider the case of $0 \le t < 1$. From Figure 4.3(f), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{0}^{t} \tau d\tau \\ &= [\tfrac{1}{2}\tau^2]_0^t \\ &= \tfrac{1}{2}t^2. \end{aligned} \tag{4.11}$$

Third, we consider the case of $1 \le t < 2$. From Figure 4.3(g), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^{1} \tau d\tau + \int_{1}^{t} (-\tau+2)d\tau \\ &= [\tfrac{1}{2}\tau^2]_{t-1}^1 + [-\tfrac{1}{2}\tau^2 + 2\tau]_1^t \\ &= \tfrac{1}{2} - [\tfrac{1}{2}(t-1)^2] - \tfrac{1}{2}t^2 + 2t - [-\tfrac{1}{2} + 2] \\ &= -t^2 + 3t - \tfrac{3}{2}. \end{aligned} \tag{4.12}$$

Fourth, we consider the case of $2 \le t < 3$. From Figure 4.3(h), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^{2} (-\tau+2)d\tau \\ &= [-\tfrac{1}{2}\tau^2 + 2\tau]_{t-1}^2 \\ &= 2 - [-\tfrac{1}{2}t^2 + 3t - \tfrac{5}{2}] \\ &= \tfrac{1}{2}t^2 - 3t + \tfrac{9}{2}. \end{aligned} \tag{4.13}$$

Lastly, we consider the case of $t \ge 3$. From Figure 4.3(i), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.14}$$

Combining the results of (4.10), (4.11), (4.12), (4.13), and (4.14) together, we have that

$$x * h(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}t^2 & 0 \le t < 1 \\ -t^2 + 3t - \frac{3}{2} & 1 \le t < 2 \\ \frac{1}{2}t^2 - 3t + \frac{9}{2} & 2 \le t < 3 \\ 0 & t \ge 3. \end{cases}$$

The convolution result $x * h$ is plotted in Figure 4.3(j). ■

Example 4.4. Compute the convolution $x * h$, where

$$x(t) = e^{-at}u(t), \quad h(t) = u(t),$$

and a is a strictly positive real constant.

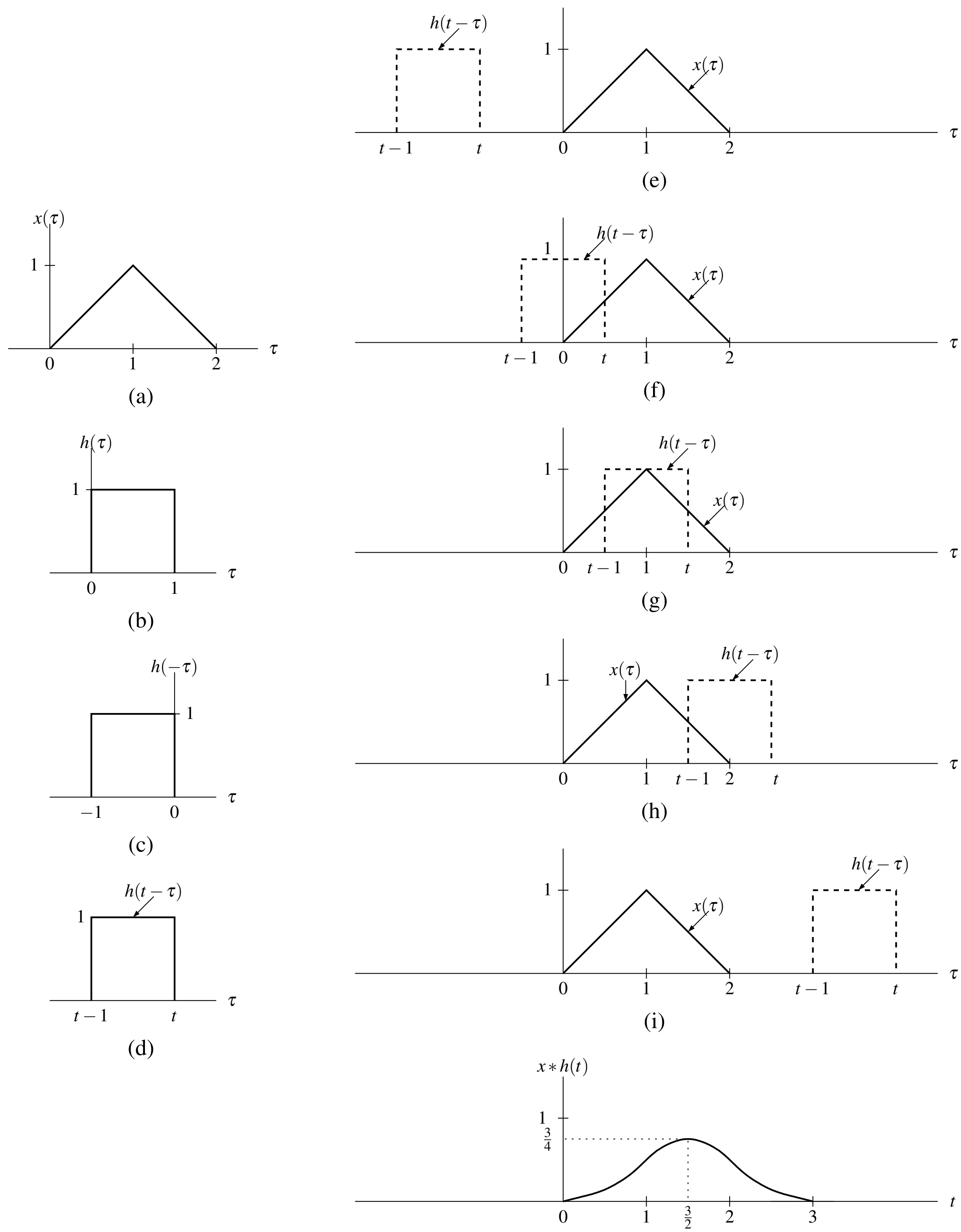


Figure 4.3: Evaluation of the convolution $x*h$. (a) The function x; (b) the function h; plots of (c) $h(-\tau)$ and (d) $h(t-\tau)$ versus τ; the functions associated with the product in the convolution integral for (e) $t<0$, (f) $0 \leq t < 1$, (g) $1 \leq t < 2$, (h) $2 \leq t < 3$, and (i) $t \geq 3$; and (j) the convolution result $x*h$.

Solution. Since x and h are relatively simple functions, we will solve this problem without the aid of graphs. Our objective in this example is twofold. First, we want to show that it is possible, if one is very careful, to perform simple convolutions without using graphs as aids. Second, we would like to show that this is actually somewhat tricky to do correctly, and probably it would have been better to draw graphs for guidance in this example in order to reduce the likelihood of errors.

From the definition of convolution, we have

$$\begin{aligned} x*h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau. \end{aligned} \tag{4.15}$$

Since

$$u(\tau)u(t-\tau) = \begin{cases} 1 & 0 \leq \tau \text{ and } \tau \leq t \\ 0 & \text{otherwise,} \end{cases}$$

the integrand can only be nonzero if $0 \leq \tau$ and $\tau \leq t$ (which necessarily requires that $t \geq 0$). So, if $t < 0$, the integrand will be zero, and $x*h(t) = 0$. Now, let us consider the case of $t > 0$. From (4.15), we have

$$\begin{aligned} x*h(t) &= \int_0^t e^{-a\tau}d\tau \\ &= [-\tfrac{1}{a}e^{-a\tau}]_0^t \\ &= \tfrac{1}{a}(1-e^{-at}). \end{aligned}$$

Thus, we have

$$\begin{aligned} x*h(t) &= \begin{cases} \frac{1}{a}(1-e^{-at}) & t > 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \tfrac{1}{a}(1-e^{-at})u(t). \end{aligned}$$

Note that, as the above solution illustrates, computing a convolution without graphs as aids can be somewhat tricky to do correctly, even when the functions being convolved are relatively simple like the ones in this example. If some steps in the above solution are unclear, it would be helpful to sketch graphs to assist in the convolution computation. For example, the use of graphs, like those shown in Figure 4.4, would likely make the above convolution much easier to compute correctly. ■

4.3 Properties of Convolution

Since convolution is frequently employed in the study of LTI systems, it is important for us to know some of its basic properties. In what follows, we examine some of these properties.

Theorem 4.1 (Commutativity of convolution)**.** *Convolution is commutative. That is, for any two functions x and h,*

$$x*h = h*x. \tag{4.16}$$

In other words, the result of a convolution is not affected by the order of its operands.

Proof. We now provide a proof of the commutative property stated above. To begin, we expand the left-hand side of (4.16) as follows:

$$x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

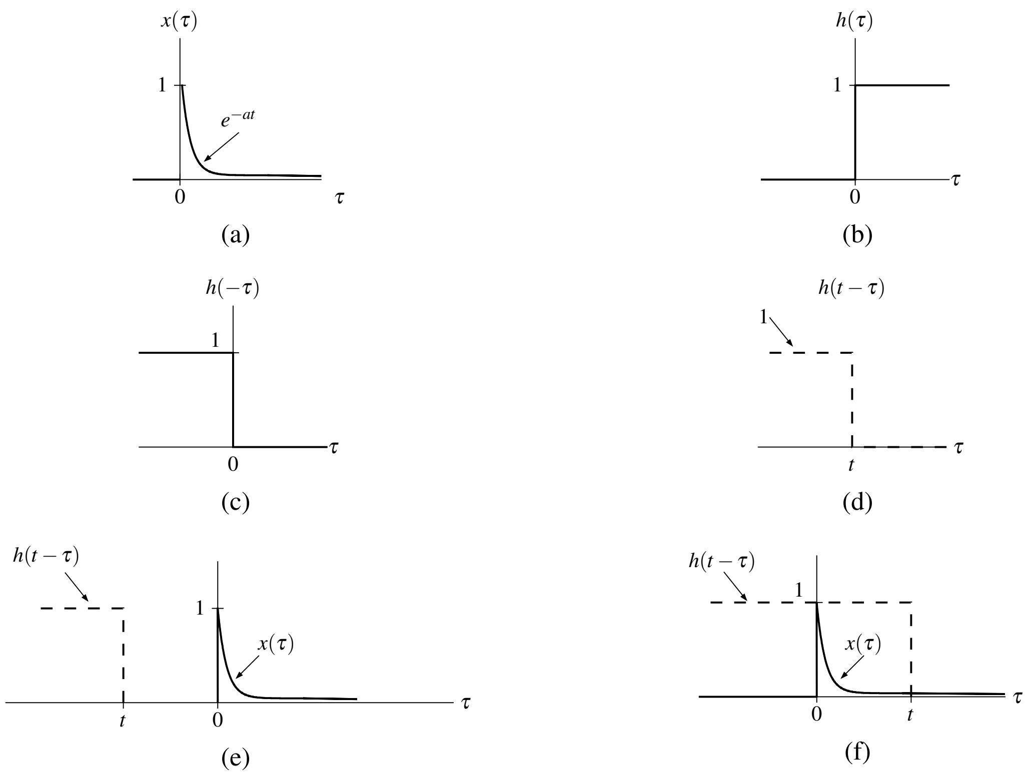


Figure 4.4: Evaluation of the convolution $x * h$. (a) The function x; (b) the function h; plots of (c) $h(-\tau)$ and (d) $h(t-\tau)$ versus τ; and the functions associated with the product in the convolution integral for (e) $t < 0$ and (f) $t > 0$.

Next, we perform a change of variable. Let $v = t - \tau$ which implies that $\tau = t - v$ and $d\tau = -dv$. Using this change of variable, we can rewrite the previous equation as

$$\begin{aligned} x * h(t) &= \int_{t+\infty}^{t-\infty} x(t-v)h(v)(-dv) \\ &= \int_{\infty}^{-\infty} x(t-v)h(v)(-dv) \\ &= \int_{-\infty}^{\infty} x(t-v)h(v)dv \\ &= \int_{-\infty}^{\infty} h(v)x(t-v)dv \\ &= h * x(t). \end{aligned}$$

(Note that, above, we used the fact that, for any function f, $\int_a^b f(x)dx = -\int_b^a f(x)dx$.) Thus, we have proven that convolution is commutative. ■

Theorem 4.2 (Associativity of convolution)**.** *Convolution is associative. That is, for any three functions x, h_1, and h_2,*

$$(x * h_1) * h_2 = x * (h_1 * h_2). \tag{4.17}$$

In other words, the final result of multiple convolutions does not depend on how the convolution operations are grouped.

Proof. To begin, we use the definition of the convolution operation to expand the left-hand side of (4.17) as follows:

$$\begin{aligned} ([x * h_1] * h_2)(t) &= \int_{-\infty}^{\infty} [x * h_1(v)]h_2(t-v)dv \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)h_1(v-\tau)d\tau \right) h_2(t-v)dv. \end{aligned}$$

Now, we change the order of integration to obtain

$$([x * h_1] * h_2)(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)h_1(v-\tau)h_2(t-v)dvd\tau.$$

Pulling the factor of $x(\tau)$ out of the inner integral yields

$$([x * h_1] * h_2)(t) = \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h_1(v-\tau)h_2(t-v)dvd\tau.$$

Next, we perform a change of variable. Let $\lambda = v - \tau$ which implies that $v = \lambda + \tau$ and $d\lambda = dv$. Using this change of variable, we can write

$$\begin{aligned} ([x * h_1] * h_2)(t) &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty-\tau}^{\infty-\tau} h_1(\lambda)h_2(t-\lambda-\tau)d\lambda d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h_1(\lambda)h_2(t-\lambda-\tau)d\lambda d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h_1(\lambda)h_2([t-\tau]-\lambda)d\lambda \right) d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) \, [h_1 * h_2(t-\tau)] \, d\tau \\ &= (x * [h_1 * h_2])(t). \end{aligned}$$

Thus, we have proven that convolution is associative. ■

Theorem 4.3 (Distributivity of convolution)**.** *Convolution is distributive. That is, for any three functions x, h_1, and h_2,*

$$x*(h_1+h_2) = x*h_1 + x*h_2. \tag{4.18}$$

In other words, convolution can be distributed across addition.

Proof. The proof of this property is relatively simple. Expanding the left-hand side of (4.18), we have

$$\begin{aligned}(x*[h_1+h_2])(t) &= \int_{-\infty}^{\infty} x(\tau)[h_1(t-\tau)+h_2(t-\tau)]d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h_2(t-\tau)d\tau \\ &= x*h_1(t) + x*h_2(t).\end{aligned}$$

Thus, we have shown that convolution is distributive. ■

The identity for an operation defined on elements of a set is often extremely helpful to know. Consider the operations of addition and multiplication as defined for real numbers. For any real number a, $a+0=a$. Since adding zero to a has no effect (i.e., the result is a), we call 0 the **additive identity**. For any real number a, $1 \cdot a = a$. Since multiplying a by 1 has no effect (i.e., the result is a), we call 1 the **multiplicative identity**. Imagine for a moment how difficult arithmetic would be if we did not know that $a+0=a$ or $1 \cdot a = a$. For this reason, identity values are clearly of fundamental importance.

Earlier, we were introduced to a new operation known as convolution. So, in light of the above, it is natural to wonder if there is a convolutional identity. In fact, there is, as given by the theorem below.

Theorem 4.4 (Convolutional identity)**.** *For any function x,*

$$x*\delta = x. \tag{4.19}$$

In other words, δ is the convolutional identity (i.e., convolving any function x with δ simply yields x).

Proof. Suppose that we have an arbitrary function x. From the definition of convolution, we can write

$$x*\delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau.$$

Now, let us employ a change of variable. Let $\lambda = -\tau$ so that $\tau = -\lambda$ and $d\tau = -d\lambda$. Applying the change of variable, we obtain

$$\begin{aligned}x*\delta(t) &= \int_{-(-\infty)}^{-(\infty)} x(-\lambda)\delta(t+\lambda)(-1)d\lambda \\ &= \int_{\infty}^{-\infty} x(-\lambda)\delta(t+\lambda)(-1)d\lambda \\ &= \int_{-\infty}^{\infty} x(-\lambda)\delta(\lambda+t)d\lambda.\end{aligned} \tag{4.20}$$

From the equivalence property of δ, we can rewrite the preceding equation as

$$\begin{aligned}x*\delta(t) &= \int_{-\infty}^{\infty} x(-[-t])\delta(\lambda+t)d\lambda \\ &= \int_{-\infty}^{\infty} x(t)\delta(\lambda+t)d\lambda.\end{aligned}$$

Factoring $x(t)$ out of the integral, we obtain

$$x*\delta(t) = x(t)\int_{-\infty}^{\infty} \delta(\lambda+t)d\lambda.$$

Since $\int_{-\infty}^{\infty}\delta(\lambda)d\lambda = 1$ implies that $\int_{-\infty}^{\infty}\delta(\lambda+t)d\lambda = 1$, we have

$$x*\delta(t) = x(t).$$

Thus, δ is the convolutional identity (i.e., $x*\delta = x$). (Alternatively, we could have directly applied the sifting property to (4.20) to show the desired result.) ■

4.4 Periodic Convolution

The convolution of two periodic functions is usually not well defined. This motivates an alternative notion of convolution for periodic signals known as periodic convolution. The **periodic convolution** of the T-periodic functions x and h, denoted $x \circledast h$, is defined as

$$x \circledast h(t) = \int_T x(\tau)h(t-\tau)d\tau,$$

where $\int_T$ denotes integration over an interval of length T. The periodic convolution and (linear) convolution of the T-periodic functions x and h are related as

$$x \circledast h(t) = x_0 * h(t) \quad \text{where} \quad x(t) = \sum_{k=-\infty}^{\infty} x_0(t-kT)$$

(i.e., $x_0(t)$ equals $x(t)$ over a single period of x and is zero elsewhere).

4.5 Characterizing LTI Systems and Convolution

As a matter of terminology, the **impulse response** h of a system $\mathcal{H}$ is defined as

$$h = \mathcal{H}\delta.$$

In other words, the impulse response of a system is the output that it produces when presented with δ as an input. As it turns out, a LTI system has a very special relationship between its input, output, and impulse response, as given by the theorem below.

Theorem 4.5 (LTI systems and convolution)**.** *A LTI system $\mathcal{H}$ with impulse response h is such that*

$$\mathcal{H}x = x * h.$$

In other words, a LTI system computes a convolution. In particular, the output of the system is given by the convolution of the input and impulse response.

Proof. To begin, we assume that $\mathcal{H}$ is LTI (i.e., $\mathcal{H}$ is both linear and time invariant). Using the fact that δ is the convolutional identity, we can write

$$\mathcal{H}x = \mathcal{H}\{x * \delta\}.$$

From the definition of convolution, we have

$$\mathcal{H}x = \mathcal{H}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(\cdot-\tau)d\tau\right\}.$$

Since $\mathcal{H}$ is linear, we can pull the integral and $x(\tau)$ (which is a constant with respect to the operation performed by $\mathcal{H}$) outside $\mathcal{H}$ to obtain

$$\mathcal{H}x = \int_{-\infty}^{\infty} x(\tau)\mathcal{H}\{\delta(\cdot-\tau)\}d\tau. \tag{4.21}$$

Since $\mathcal{H}$ is time invariant, we can interchange the order of $\mathcal{H}$ and the time shift of δ by τ. That is, we have

$$\mathcal{H}\{\delta(\cdot-\tau)\} = h(\cdot-\tau).$$

Thus, we can rewrite (4.21) as

$$\begin{aligned}\mathcal{H}x &= \int_{-\infty}^{\infty} x(\tau)h(\cdot-\tau)d\tau \\ &= x * h.\end{aligned}$$

Thus, we have shown that $\mathcal{H}x = x * h$, where $h = \mathcal{H}\delta$. ■

By Theorem 4.5 above, the behavior of a LTI system is completely characterized by its impulse response. That is, if the impulse response of a system is known, we can determine the response of the system to *any* input. Consequently, the impulse response provides a very powerful tool for the study of LTI systems.

Example 4.5. Consider a LTI system $\mathcal{H}$ with impulse response

$$h(t) = u(t). \tag{4.22}$$

Show that $\mathcal{H}$ is characterized by the equation

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau \tag{4.23}$$

(i.e., $\mathcal{H}$ corresponds to an ideal integrator).

Solution. Since the system is LTI, we have that

$$\mathcal{H}x(t) = x * h(t).$$

Substituting (4.22) into the preceding equation, and simplifying we obtain

$$\begin{aligned}
\mathcal{H}x(t) &= x * h(t) \\
&= x * u(t) \\
&= \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau \\
&= \int_{-\infty}^{t} x(\tau)u(t-\tau)d\tau + \int_{t^+}^{\infty} x(\tau)u(t-\tau)d\tau \\
&= \int_{-\infty}^{t} x(\tau)d\tau.
\end{aligned}$$

Therefore, the system with the impulse response h given by (4.22) is, in fact, the ideal integrator given by (4.23). ■

Example 4.6. Consider a LTI system $\mathcal{H}$ with impulse response h, where

$$h(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find and plot the response y of the system to the input x given by

$$x(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Solution. Plots of x and h are given in Figures 4.5(a) and (b), respectively. Since the system is LTI, we know that

$$y(t) = x * h(t).$$

Thus, in order to find the response y of the system to the input x, we simply need to compute the convolution $x * h$.

We begin by plotting the functions x and h as shown in Figures 4.5(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of $h(\tau)$. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 4.5(c). Second, we time-shift the resulting signal by t to obtain $h(t-\tau)$ as shown in Figure 4.5(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of x

and h, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.5(e) to (h).

First, we consider the case of $t < 0$. From Figure 4.5(e), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.24}$$

Second, we consider the case of $0 \leq t < 1$. From Figure 4.5(f), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{0}^{t} d\tau \\ &= [\tau]|_0^t \\ &= t. \end{aligned} \tag{4.25}$$

Third, we consider the case of $1 \leq t < 2$. From Figure 4.5(g), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^{1} d\tau \\ &= [\tau]|_{t-1}^{1} \\ &= 1-(t-1) \\ &= 2-t. \end{aligned} \tag{4.26}$$

Fourth, we consider the case of $t \geq 2$. From Figure 4.5(h), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.27}$$

Combining the results of (4.24), (4.25), (4.26), and (4.27), we have that

$$x * h(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 2-t & 1 \leq t < 2 \\ 0 & t \geq 2. \end{cases}$$

The convolution result $x * h$ is plotted in Figure 4.5(i). The response y of the system to the specified input is simply $x * h$. ■

4.6 Step Response of LTI Systems

The **step response** s of a system $\mathcal{H}$ is defined as

$$s = \mathcal{H}u$$

(i.e., the step response of a system is the output it produces for a unit-step function input). In the case of a LTI system, it turns out that the step response is closely related to the impulse response, as given by the theorem below.

Theorem 4.6. *The step response s and impulse response h of a LTI system are related as*

$$h(t) = \frac{ds(t)}{dt} \quad \textit{and} \quad s(t) = \int_{-\infty}^{t} h(\tau)d\tau.$$

That is, the impulse response h is the derivative of the step response s.

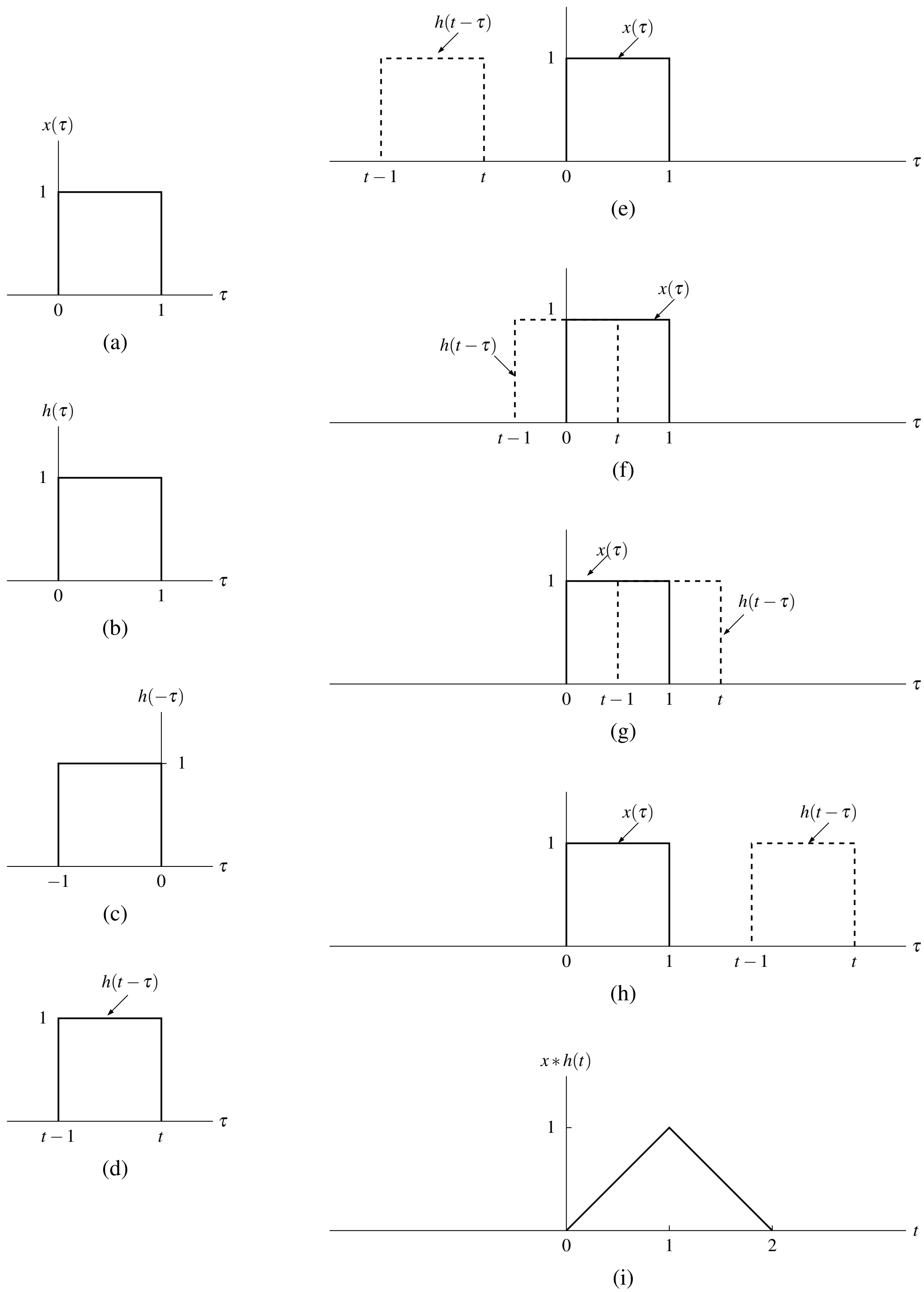


Figure 4.5: Evaluation of the convolution $x*h$. (a) The function x; (b) the function h; plots of (c) $h(-\tau)$ and (d) $h(t-\tau)$ versus τ; the functions associated with the product in the convolution integral for (e) $t<0$, (f) $0 \le t < 1$, (g) $1 \le t < 2$, and (h) $t \ge 2$; and (i) the convolution result $x*h$.

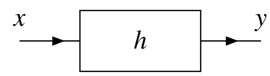


Figure 4.6: Block diagram representation of continuous-time LTI system with input x, output y, and impulse response h.

Proof. Using the fact that $s = u * h$, we can write

$$\begin{aligned}
s(t) &= u * h(t) \\
&= h * u(t) \\
&= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau \\
&= \int_{-\infty}^{t} h(\tau)d\tau.
\end{aligned}$$

Thus, s can be obtained by integrating h. Taking the derivative of s, we obtain

$$\begin{aligned}
\frac{ds(t)}{dt} &= \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{-\infty}^{t+\Delta t} h(\tau)d\tau - \int_{-\infty}^{t} h(\tau)d\tau\right] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\int_{t}^{t+\Delta t} h(\tau)d\tau \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(h(t)\Delta t\right) \\
&= h(t).
\end{aligned}$$

Thus, h is the derivative of s. ∎

The step response is often of great practical interest, since it can be used to determine the impulse response of a LTI system. In particular, the impulse response can be determined from the step response via differentiation. From a practical point of view, the step response is more useful for characterizing a system based on experimental measurements. Obviously, we cannot directly measure the impulse response of a system because we cannot (in the real world) generate a unit-impulse function or an accurate approximation thereof. We can, however, produce a reasonably good approximation of the unit-step function in the real world. Thus, we can measure the step response and from it determine the impulse response.

4.7 Block Diagram Representation of Continuous-Time LTI Systems

Frequently, it is convenient to represent continuous-time LTI systems in block diagram form. Since a LTI system is completely characterized by its impulse response, we often label such a system with its impulse response in a block diagram. That is, we represent a LTI system with input x, output y, and impulse response h, as shown in Figure 4.6.

4.8 Interconnection of Continuous-Time LTI Systems

Suppose that we have a LTI system with input x, output y, and impulse response h. We know that x and y are related as $y = x * h$. In other words, the system can be viewed as performing a convolution operation. From the properties of convolution introduced earlier, we can derive a number of equivalences involving the impulse responses of series- and parallel-interconnected systems.

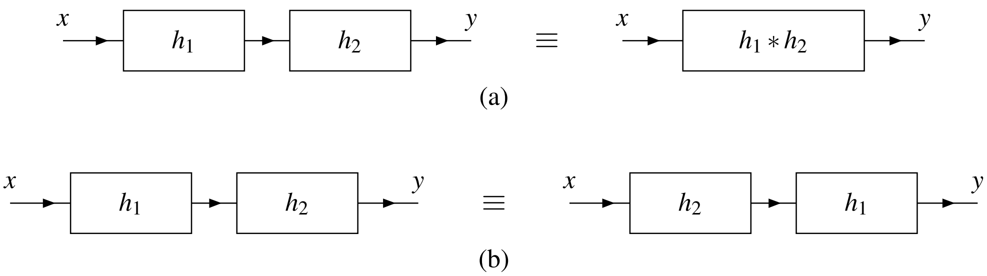


Figure 4.7: Equivalences for the series interconnection of continuous-time LTI systems. The (a) first and (b) second equivalences.

Consider two LTI systems with impulse responses h_1 and h_2 that are connected in a series configuration, as shown on the left-side of Figure 4.7(a). From the block diagram on the left side of Figure 4.7(a), we have

$$y = (x * h_1) * h_2.$$

Due to the associativity of convolution, however, this is equivalent to

$$y = x * (h_1 * h_2).$$

Thus, the series interconnection of two LTI systems behaves as a single LTI system with impulse response $h_1 * h_2$. In other words, we have the equivalence shown in Figure 4.7(a).

Consider two LTI systems with impulse responses h_1 and h_2 that are connected in a series configuration, as shown on the left-side of Figure 4.7(b). From the block diagram on the left side of Figure 4.7(b), we have

$$y = (x * h_1) * h_2.$$

Due to the associativity and commutativity of convolution, this is equivalent to

$$\begin{aligned} y &= x * (h_1 * h_2) \\ &= x * (h_2 * h_1) \\ &= (x * h_2) * h_1. \end{aligned}$$

Thus, interchanging the two LTI systems does not change the behavior of the overall system with input x and output y. In other words, we have the equivalence shown in Figure 4.7(b).

Consider two LTI systems with impulse responses h_1 and h_2 that are connected in a parallel configuration, as shown on the left-side of Figure 4.8. From the block diagram on the left side of Figure 4.8, we have

$$y = x * h_1 + x * h_2.$$

Due to convolution being distributive, however, this equation can be rewritten as

$$y = x * (h_1 + h_2).$$

Thus, the parallel interconnection of two LTI systems behaves as a single LTI system with impulse response $h_1 + h_2$. In other words, we have the equivalence shown in Figure 4.8.

Example 4.7. Consider the system with input x, output y, and impulse response h as shown in Figure 4.9. Each subsystem in the block diagram is LTI and labelled with its impulse response. Find h.

Solution. From the left half of the block diagram, we can write

$$\begin{aligned} v(t) &= x(t) + x * h_1(t) + x * h_2(t) \\ &= x * \delta(t) + x * h_1(t) + x * h_2(t) \\ &= (x * [\delta + h_1 + h_2])(t). \end{aligned}$$

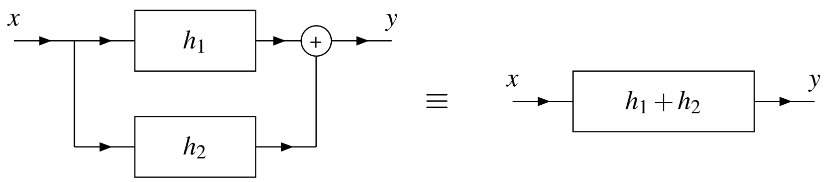


Figure 4.8: Equivalence for the parallel interconnection of continuous-time LTI systems.

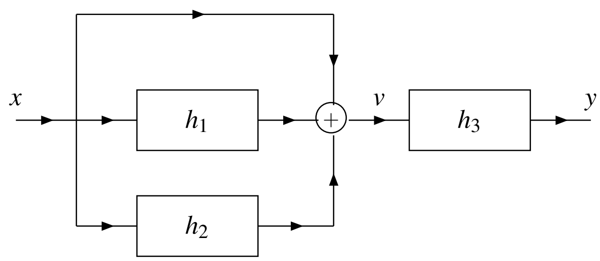


Figure 4.9: System interconnection example.

Similarly, from the right half of the block diagram, we can write

$$y(t) = v * h_3(t).$$

Substituting the expression for v into the preceding equation we obtain

$$\begin{aligned} y(t) &= v * h_3(t) \\ &= (x * [\delta + h_1 + h_2]) * h_3(t) \\ &= x * [h_3 + h_1 * h_3 + h_2 * h_3](t). \end{aligned}$$

Thus, the impulse response h of the overall system is

$$h(t) = h_3(t) + h_1 * h_3(t) + h_2 * h_3(t).$$ ■

4.9 Properties of Continuous-Time LTI Systems

In the previous chapter, we introduced a number of properties that might be possessed by a system (e.g., memory, causality, BIBO stability, and invertibility). Since a LTI system is completely characterized by its impulse response, one might wonder if there is a relationship between some of the properties introduced previously and the impulse response. In what follows, we explore some of these relationships.

4.9.1 Memory

The first system property to be considered is memory.

Theorem 4.7 (Memorylessness of LTI system)**.** *A LTI system with impulse response h is memoryless if and only if*

$$h(t) = 0 \textit{ for all } t \neq 0.$$

Proof. Recall that a system is memoryless if its output y at any arbitrary time depends only on the value of its input x at that same time. Suppose now that we have a LTI system with input x, output y, and impulse response h. The output y at some arbitrary time t_0 is given by

$$\begin{aligned} y(t_0) &= x * h(t_0) \\ &= h * x(t_0) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t_0 - \tau)d\tau. \end{aligned}$$

Consider the integral in the above equation. In order for the system to be memoryless, the result of the integration is allowed to depend on $x(t)$ only for $t = t_0$. This, however, is only possible if

$$h(t) = 0 \quad \text{for all } t \neq 0.$$ ■

From the preceding theorem, it follows that a memoryless LTI system must have an impulse response h of the form

$$h(t) = K\delta(t) \tag{4.28}$$

where K is a complex constant. As a consequence of this fact, we also have that all memoryless LTI systems must have an input-output relation of the form

$$\begin{aligned} y(t) &= x * (K\delta)(t) \\ &= K(x * \delta)(t) \\ &= Kx(t). \end{aligned}$$

In other words, a memoryless LTI system must be an ideal amplifier (i.e., a system that simply performs amplitude scaling).

Example 4.8. Consider the LTI system with the impulse response h given by

$$h(t) = e^{-at}u(t),$$

where a is a real constant. Determine whether this system has memory.

Solution. The system has memory since $h(t) \neq 0$ for some $t \neq 0$ (e.g., $h(1) = e^{-a} \neq 0$). ■

Example 4.9. Consider the LTI system with the impulse response h given by

$$h(t) = \delta(t).$$

Determine whether this system has memory.

Solution. Clearly, h is only nonzero at the origin. This follows immediately from the definition of the unit-impulse function δ. Therefore, the system is memoryless (i.e., does not have memory). ■

4.9.2 Causality

The next system property to be considered is causality.

Theorem 4.8 (Causality of LTI system)**.** *A LTI system with impulse response h is causal if and only if*

$$h(t) = 0 \textit{ for all } t < 0.$$

(i.e., h is causal).

Proof. Recall that a system is causal if its output y at any arbitrary time t_0 does not depend on its input x at a time later than t_0. Suppose that we have the LTI system with input x, output y, and impulse response h. The value of the output y at t_0 is given by

$$\begin{aligned} y(t_0) &= x * h(t_0) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t_0-\tau)d\tau \\ &= \int_{-\infty}^{t_0} x(\tau)h(t_0-\tau)d\tau + \int_{t_0^+}^{\infty} x(\tau)h(t_0-\tau)d\tau. \end{aligned} \tag{4.29}$$

In order for the expression for $y(t_0)$ in (4.29) not to depend on $x(t)$ for $t > t_0$, we must have that

$$h(t) = 0 \quad \text{for } t < 0 \tag{4.30}$$

(i.e., h is causal). In this case, (4.29) simplifies to

$$y(t_0) = \int_{-\infty}^{t_0} x(\tau)h(t_0-\tau)d\tau.$$

Clearly, the result of this integration does not depend on $x(t)$ for $t > t_0$ (since τ varies from $-\infty$ to t_0). Therefore, a LTI system is causal if its impulse response h satisfies (4.30). ■

Example 4.10. Consider the LTI system with impulse response h given by

$$h(t) = e^{-at}u(t),$$

where a is a real constant. Determine whether this system is causal.

Solution. Clearly, $h(t) = 0$ for $t < 0$ (due to the $u(t)$ factor in the expression for $h(t)$). Therefore, the system is causal. ■

Example 4.11. Consider the LTI system with impulse response h given by

$$h(t) = \delta(t+t_0),$$

where t_0 is a strictly positive real constant. Determine whether this system is causal.

Solution. From the definition of δ, we can easily deduce that $h(t) = 0$ except at $t = -t_0$. Since $-t_0 < 0$, the system is not causal. ■

4.9.3 Invertibility

The next system property to be considered is invertibility.

Theorem 4.9 (Inverse of LTI system)**.** *Let $\mathcal{H}$ be a LTI system with impulse response h. If the inverse $\mathcal{H}^{-1}$ of $\mathcal{H}$ exists, $\mathcal{H}^{-1}$ is LTI and has an impulse response h_{inv} that satisfies*

$$h * h_{\mathsf{inv}} = \delta.$$

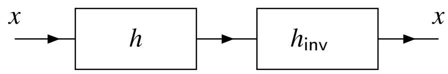


Figure 4.10: System in cascade with its inverse.

Proof. To begin, we need to show that the inverse of a LTI system, if it exists, must also be LTI. This part of the proof, however, is left as an exercise for the reader in Exercise 4.15. (The general approach to take for this problem is to show that: 1) the inverse of a linear system, if it exists, is linear; and 2) the inverse of a time-invariant system, if it exists, is time invariant.) We assume that this part of the proof has been demonstrated and proceed.

Suppose now that the inverse system $\mathcal{H}^{-1}$ exists. We have that

$$\mathcal{H}x = x * h \quad \text{and} \quad \mathcal{H}^{-1}x = x * h_{\mathsf{inv}}.$$

From the definition of an inverse system, we have that, for every function x,

$$\mathcal{H}^{-1}\mathcal{H}x = x.$$

Expanding the left-hand side of the preceding equation, we obtain

$$\begin{aligned} &\mathcal{H}^{-1}[x * h] = x \\ \Leftrightarrow \quad & x * h * h_{\mathsf{inv}} = x. \end{aligned} \tag{4.31}$$

This relationship is expressed diagrammatically in Figure 4.10. Since the unit-impulse function is the convolutional identity, we can equivalently rewrite (4.31) as

$$x * h * h_{\mathsf{inv}} = x * \delta.$$

This equation, however, must hold for arbitrary x. Thus, by comparing the left- and right-hand sides of this equation, we conclude

$$h * h_{\mathsf{inv}} = \delta. \tag{4.32}$$

Therefore, if $\mathcal{H}^{-1}$ exists, it must have an impulse response h_{inv} that satisfies (4.32). This completes the proof. ■

From the preceding theorem, we have the following result:

Theorem 4.10 (Invertibility of LTI system)**.** *A LTI system $\mathcal{H}$ with impulse response h is invertible if and only if there exists a function h_{inv} satisfying*

$$h * h_{\mathsf{inv}} = \delta.$$

Proof. The proof follows immediately from the result of Theorem 4.9 by simply observing that $\mathcal{H}$ being invertible is equivalent to the existence of $\mathcal{H}^{-1}$. ■

Example 4.12. Consider the LTI system $\mathcal{H}$ with impulse response h given by

$$h(t) = A\delta(t - t_0),$$

where A and t_0 are real constants and $A \neq 0$. Determine if $\mathcal{H}$ is invertible, and if it is, find the impulse response h_{inv} of the system $\mathcal{H}^{-1}$.

Solution. If the system $\mathcal{H}^{-1}$ exists, its impulse response h_{inv} is given by the solution to the equation

$$h * h_{\mathsf{inv}} = \delta. \tag{4.33}$$

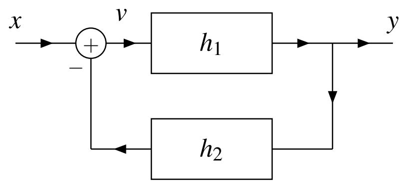


Figure 4.11: Feedback system with input x and output y.

So, let us attempt to solve this equation for h_{inv}. Substituting the given function h into (4.33) and using straightforward algebraic manipulation, we can write

$$\begin{aligned} & h * h_{\mathsf{inv}}(t) = \delta(t) \\ \Rightarrow \quad & \int_{-\infty}^{\infty} h(\tau) h_{\mathsf{inv}}(t-\tau) d\tau = \delta(t) \\ \Rightarrow \quad & \int_{-\infty}^{\infty} A\delta(\tau - t_0) h_{\mathsf{inv}}(t-\tau) d\tau = \delta(t) \\ \Rightarrow \quad & \int_{-\infty}^{\infty} \delta(\tau - t_0) h_{\mathsf{inv}}(t-\tau) d\tau = \tfrac{1}{A}\delta(t). \end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the integral expression in the preceding equation to obtain

$$\begin{aligned} & h_{\mathsf{inv}}(t-\tau)\big|_{\tau=t_0} = \tfrac{1}{A}\delta(t) \\ \Rightarrow \quad & h_{\mathsf{inv}}(t-t_0) = \tfrac{1}{A}\delta(t). \end{aligned} \tag{4.34}$$

Substituting $t+t_0$ for t in the preceding equation yields

$$\begin{aligned} h_{\mathsf{inv}}([t+t_0]-t_0) &= \tfrac{1}{A}\delta(t+t_0) \quad \Leftrightarrow \\ h_{\mathsf{inv}}(t) &= \tfrac{1}{A}\delta(t+t_0). \end{aligned}$$

Since $A \neq 0$, the function h_{inv} is always well defined. Thus, $\mathcal{H}^{-1}$ exists and consequently $\mathcal{H}$ is invertible. ■

Example 4.13. Consider the system with the input x and output y as shown in Figure 4.11. Each subsystem in the block diagram is LTI and labelled with its impulse response. Use the notion of an inverse system in order to express y in terms of x.

Solution. From Figure 4.11, we can write:

$$v = x - y * h_2 \quad \text{and} \tag{4.35}$$

$$y = v * h_1. \tag{4.36}$$

Substituting (4.35) into (4.36), and simplifying we obtain

$$\begin{aligned} & y = [x - y * h_2] * h_1 \\ \Rightarrow \quad & y = x * h_1 - y * h_2 * h_1 \\ \Rightarrow \quad & y + y * h_2 * h_1 = x * h_1 \\ \Rightarrow \quad & y * \delta + y * h_2 * h_1 = x * h_1 \\ \Rightarrow \quad & y * [\delta + h_2 * h_1] = x * h_1. \end{aligned} \tag{4.37}$$

For convenience, we now define the function g as

$$g = \delta + h_2 * h_1. \tag{4.38}$$

So, we can rewrite (4.37) as

$$y * g = x * h_1. \tag{4.39}$$

Thus, we have almost solved for y in terms of x. To complete the solution, we need to eliminate g from the left-hand side of the equation. To do this, we use the notion of an inverse system. Consider the inverse of the system with impulse response g. This inverse system has an impulse response g_{inv} given by

$$g * g_{\mathsf{inv}} = \delta. \tag{4.40}$$

This relationship follows from the definition of an inverse system. Now, we use g_{inv} in order to simplify (4.39) as follows:

$$\begin{aligned} & y * g = x * h_1 \\ \Rightarrow \quad & y * g * g_{\mathsf{inv}} = x * h_1 * g_{\mathsf{inv}} \\ \Rightarrow \quad & y * \delta = x * h_1 * g_{\mathsf{inv}} \\ \Rightarrow \quad & y = x * h_1 * g_{\mathsf{inv}}. \end{aligned}$$

Thus, we can express the output y in terms of the input x as

$$y = x * h_1 * g_{\mathsf{inv}},$$

where g_{inv} is given by (4.40) and g is given by (4.38). ■

4.9.4 BIBO Stability

The last system property to be considered is BIBO stability.

Theorem 4.11 (BIBO stability of LTI system)**.** *A LTI system with impulse response h is BIBO stable if and only if*

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty \tag{4.41}$$

(i.e., h is absolutely integrable).

Proof. Recall that a system is BIBO stable if, for every bounded input, the system produces a bounded output. Suppose that we have a LTI system with input x, output y, and impulse response h.

First, we consider the sufficiency of (4.41) for BIBO stability. Assume that $|x(t)| \leq A < \infty$ for all t (i.e., x is bounded). We can write

$$\begin{aligned} y(t) &= x * h(t) \\ &= h * x(t) \\ &= \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau. \end{aligned}$$

By taking the magnitude of both sides of the preceding equation, we obtain

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau \right|. \tag{4.42}$$

One can show, for any two functions f_1 and f_2, that

$$\left|\int_{-\infty}^{\infty} f_1(t) f_2(t)dt\right| \leq \int_{-\infty}^{\infty} |f_1(t) f_2(t)|\,dt.$$

Using this inequality, we can rewrite (4.42) as

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|\,d\tau = \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\,d\tau.$$

We know (by assumption) that $|x(t)| \leq A$ for all t, so we can replace $|x(t)|$ by its bound A in the above inequality to obtain

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\,d\tau \leq \int_{-\infty}^{\infty} A\,|h(\tau)|\,d\tau = A\int_{-\infty}^{\infty} |h(\tau)|\,d\tau. \tag{4.43}$$

Thus, we have

$$|y(t)| \leq A\int_{-\infty}^{\infty} |h(\tau)|\,d\tau. \tag{4.44}$$

Since A is finite, we can deduce from (4.44) that y is bounded if

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty \tag{4.45}$$

(i.e., h is absolutely integrable). Thus, the absolute integrability of the impulse response h is a sufficient condition for BIBO stability.

Now, we consider the necessity of (4.41) for BIBO stability. Suppose that h is not absolutely integrable. That is, suppose that

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \infty.$$

If such is the case, we can show that the system is not BIBO stable. To begin, consider the particular input x given by

$$x(t) = e^{-j\arg[h(-t)]}.$$

Since $\left|e^{j\theta}\right| = 1$ for all real θ, x is bounded (i.e., $|x(t)| \leq 1$ for all t). The output y is given by

$$\begin{aligned} y(t) &= x*h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-j\arg[h(-\tau)]}h(t-\tau)d\tau. \end{aligned} \tag{4.46}$$

Now, let us consider the output value $y(t)$ at $t=0$. From (4.46), we have

$$y(0) = \int_{-\infty}^{\infty} e^{-j\arg[h(-\tau)]}h(-\tau)d\tau. \tag{4.47}$$

Since $e^{-j\arg z}z = |z|$ for all complex z, $e^{-j\arg[h(-\tau)]}h(-\tau) = |h(-\tau)|$, and we can simplify (4.47) to obtain

$$\begin{aligned} y(0) &= \int_{-\infty}^{\infty} |h(-\tau)|\,d\tau \\ &= \int_{-\infty}^{\infty} |h(\tau)|\,d\tau \\ &= \infty. \end{aligned}$$

Thus, we have shown that the bounded input x will result in an unbounded output y (where $y(t)$ is unbounded for $t=0$). Thus, the absolute integrability of h is also necessary for BIBO stability. This completes the proof. ■

Example 4.14. Consider the LTI system with impulse response h given by

$$h(t) = e^{at}u(t),$$

where a is a real constant. Determine for what values of a the system is BIBO stable.

Solution. We need to determine for what values of a the impulse response h is absolutely integrable. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} \left|e^{at}u(t)\right| dt \\
&= \int_{-\infty}^{0} 0\,dt + \int_{0}^{\infty} e^{at}\,dt \\
&= \int_{0}^{\infty} e^{at}\,dt \\
&= \begin{cases} \int_0^\infty e^{at}\,dt & a \neq 0 \\ \int_0^\infty 1\,dt & a = 0 \end{cases} \\
&= \begin{cases} \left[\frac{1}{a}e^{at}\right]\Big|_0^\infty & a \neq 0 \\ [t]\big|_0^\infty & a = 0. \end{cases}
\end{aligned}$$

Now, we simplify the preceding equation for each of the cases $a \neq 0$ and $a = 0$. Suppose that $a \neq 0$. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= \left[\tfrac{1}{a}e^{at}\right]\Big|_0^\infty \\
&= \tfrac{1}{a}\left(e^{a\infty} - 1\right).
\end{aligned}$$

We can see that the result of the above integration is finite if $a < 0$ and infinite if $a > 0$. In particular, if $a < 0$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= 0 - \tfrac{1}{a} \\
&= -\tfrac{1}{a}.
\end{aligned}$$

Suppose now that $a = 0$. In this case, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= [t]\big|_0^\infty \\
&= \infty.
\end{aligned}$$

Thus, we have shown that

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \begin{cases} -\frac{1}{a} & a < 0 \\ \infty & a \geq 0. \end{cases}$$

In other words, the impulse response h is absolutely integrable if and only if $a < 0$. Consequently, the system is BIBO stable if and only if $a < 0$. ■

Example 4.15. Consider the LTI system with input x and output y defined by

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

(i.e., an ideal integrator). Determine whether this system is BIBO stable.

Solution. First, we find the impulse response h of the system. We have

$$\begin{aligned} h(t) &= \int_{-\infty}^{t} \delta(\tau)d\tau \\ &= \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \\ &= u(t). \end{aligned}$$

Using this expression for h, we now check to see if h is absolutely integrable. We have

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} |u(t)|\,dt \\ &= \int_{0}^{\infty} 1\,dt \\ &= \infty. \end{aligned}$$

Thus, h is not absolutely integrable. Therefore, the system is not BIBO stable. ■

4.10 Eigenfunctions of Continuous-Time LTI Systems

Earlier, in Section 3.8.7, we were introduced to notion of eigenfunctions of systems. Given that eigenfunctions have the potential to simplify the mathematics associated with systems, it is natural to wonder what eigenfunctions LTI systems might have. In this regard, the following theorem is enlightening.

Theorem 4.12 (Eigenfunctions of LTI systems)**.** *For an arbitrary LTI system $\mathcal{H}$ with impulse response h and a function of the form $x(t) = e^{st}$, where s is an arbitrary complex constant (i.e., x is an arbitrary complex exponential), the following holds:*

$$\mathcal{H}x(t) = H(s)e^{st},$$

where

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau. \tag{4.48}$$

That is, x is an eigenfunction of $\mathcal{H}$ with the corresponding eigenvalue $H(s)$.

Proof. To begin, we observe that a system $\mathcal{H}$ is LTI if and only if it computes a convolution (i.e., $\mathcal{H}x = x * h$ for some h). We have

$$\begin{aligned} \mathcal{H}x(t) &= x * h(t) \\ &= h * x(t) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau \\ &= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \\ &= H(s)e^{st}. \end{aligned}$$

■

As a matter of terminology, the function H that appears in the preceding theorem (i.e., Theorem 4.12) is referred to as the **system function** (or **transfer function**) of the system $\mathcal{H}$. The system function completely characterizes the behavior of a LTI system. Consequently, system functions are often useful when working with LTI systems. As it turns out, an integral of the form appearing in (4.48) is of great importance, as it defines what is called the Laplace transform. We will study the Laplace transform in great depth later in Chapter 7.

Note that a LTI system can have eigenfunctions other than complex exponentials. For example, the system in Example 3.43 is LTI and has every function as an eigenfunction. Also, a system that has every complex exponential function as an eigenfunction is not necessarily LTI. This is easily demonstrated by the example below.

Example 4.16. Let S denote the set of all complex exponential functions (i.e., S is the set of all functions x of the form $x(t) = ae^{st}$ where $a, s \in \mathbb{C}$). Consider the system $\mathcal{H}$ given by

$$\mathcal{H}x = \begin{cases} x & x \in S \\ 1 & \text{otherwise.} \end{cases}$$

For any function $x \in S$, we have $\mathcal{H}x = x$, implying that x is an eigenfunction of $\mathcal{H}$ with eigenvalue 1. Therefore, every complex exponential function is an eigenfunction of $\mathcal{H}$.

Now, we show that $\mathcal{H}$ is not linear. In what follows, let a denote an arbitrary complex constant. Consider the function $x(t) = t$. Clearly, $x \notin S$. Since $x \notin S$, we have $\mathcal{H}x = 1$, which implies that

$$a\mathcal{H}x = a.$$

Next, consider the function $ax(t) = at$. Since $ax \notin S$, we have

$$\mathcal{H}(ax) = 1.$$

From the above equations, however, we conclude that $\mathcal{H}(ax) = a\mathcal{H}x$ only in the case that $a = 1$. Therefore, $\mathcal{H}$ is not homogeneous and consequently not linear. So, $\mathcal{H}$ is an example of a system that has every complex exponential as an eigenfunction, but is not LTI. ■

Let us now consider an application of eigenfunctions. Since convolution can often be quite painful to handle at the best of times, let us exploit eigenfunctions in order to devise a means to avoid having to deal with convolution directly in certain circumstances.

Suppose that we have a LTI system $\mathcal{H}$ with input x, output y, impulse response h, and system function H. Suppose now that we can express some arbitrary input signal x as a sum of complex exponentials as follows:

$$x(t) = \sum_k a_k e^{s_k t}.$$

(As it turns out, many functions can be expressed in this way.) From the eigenfunction properties of LTI systems, the response of the system to the input $a_k e^{s_k t}$ is $a_k H(s_k) e^{s_k t}$. By using this knowledge and the superposition property, we can write

$$\begin{aligned} y(t) &= \mathcal{H}x(t) \\ &= \mathcal{H}\left\{\sum_k a_k e^{s_k t}\right\}(t) \\ &= \sum_k a_k \mathcal{H}\{e^{s_k t}\}(t) \\ &= \sum_k a_k H(s_k) e^{s_k t}. \end{aligned}$$

Thus, we have that

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}. \tag{4.49}$$

Thus, if an input to a LTI system can be represented as a linear combination of complex exponentials, the output can also be represented as linear combination of the same complex exponentials. Furthermore, observe that the relationship between the input $x(t) = \sum_k a_k e^{s_k t}$ and output y in (4.49) does not involve convolution (such as in the equation $y = x * h$). In fact, the formula for y is identical to that for x except for the insertion of a constant multiplicative factor $H(s_k)$. In effect, we have used eigenfunctions to replace convolution with the much simpler operation of multiplication by a constant.

Example 4.17. Consider the LTI system $\mathcal{H}$ with the impulse response h given by

$$h(t) = \delta(t-1).$$

(a) Find the system function H of the system $\mathcal{H}$. (b) Use the system function H to determine the response y of the system $\mathcal{H}$ to the particular input x given by

$$x(t) = e^t \cos(\pi t).$$

Solution. (a) We find the system function H using (4.48). Substituting the given function h into (4.48), we obtain

$$\begin{aligned} H(s) &= \int_{-\infty}^{\infty} h(t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} \delta(t-1) e^{-st} dt \\ &= \left[e^{-st}\right]\Big|_{t=1} \\ &= e^{-s}. \end{aligned}$$

(b) We can rewrite x to obtain

$$\begin{aligned} x(t) &= e^t \cos(\pi t) \\ &= e^t \left[\tfrac{1}{2}\left(e^{j\pi t} + e^{-j\pi t}\right)\right] \\ &= \tfrac{1}{2} e^{(1+j\pi)t} + \tfrac{1}{2} e^{(1-j\pi)t}. \end{aligned}$$

So, the input x is now expressed in the form

$$x(t) = \sum_{k=0}^{1} a_k e^{s_k t},$$

where

$$a_k = \tfrac{1}{2} \text{ for } k \in \{0,1\} \quad \text{and} \quad s_k = \begin{cases} 1 + j\pi & k = 0 \\ 1 - j\pi & k = 1. \end{cases}$$

Now, we use H and the eigenfunction properties of LTI systems to find y. Calculating y, we have

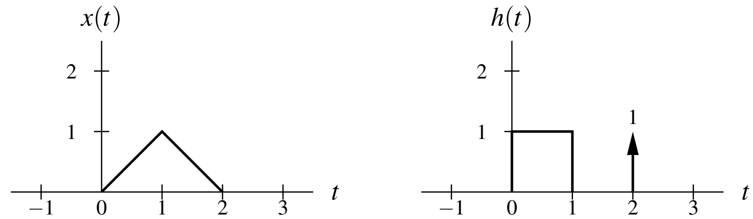
$$\begin{aligned} y(t) &= \sum_{k=0}^{1} a_k H(s_k) e^{s_k t} \\ &= a_0 H(s_0) e^{s_0 t} + a_1 H(s_1) e^{s_1 t} \\ &= \tfrac{1}{2} H(1+j\pi) e^{(1+j\pi)t} + \tfrac{1}{2} H(1-j\pi) e^{(1-j\pi)t} \\ &= \tfrac{1}{2} e^{-(1+j\pi)} e^{(1+j\pi)t} + \tfrac{1}{2} e^{-(1-j\pi)} e^{(1-j\pi)t} \\ &= \tfrac{1}{2} e^{t-1+j\pi t-j\pi} + \tfrac{1}{2} e^{t-1-j\pi t+j\pi} \\ &= \tfrac{1}{2} e^{t-1} e^{j\pi(t-1)} + \tfrac{1}{2} e^{t-1} e^{-j\pi(t-1)} \\ &= e^{t-1}\left[\tfrac{1}{2}\left(e^{j\pi(t-1)} + e^{-j\pi(t-1)}\right)\right] \\ &= e^{t-1} \cos\left[\pi(t-1)\right]. \end{aligned}$$

Observe that the output y is just the input x time shifted by 1. This is not a coincidence because, as it turns out, a LTI system with the system function $H(s) = e^{-s}$ is an ideal unit delay (i.e., a system that performs a time shift of 1). ■

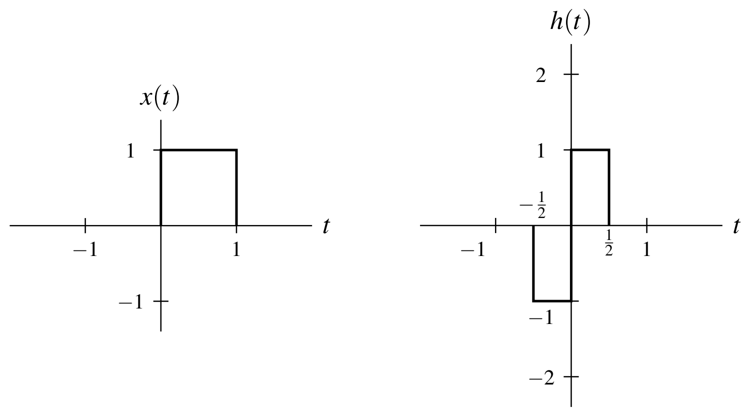
4.11 Exercises

4.11.1 Exercises Without Answer Key

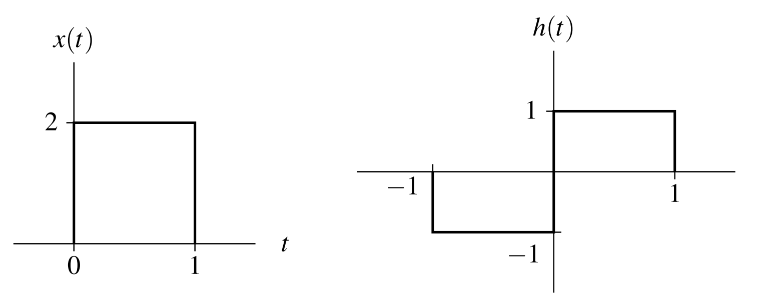
4.1 Using the graphical method, for each pair of functions x and h given in the figures below, compute $x * h$.



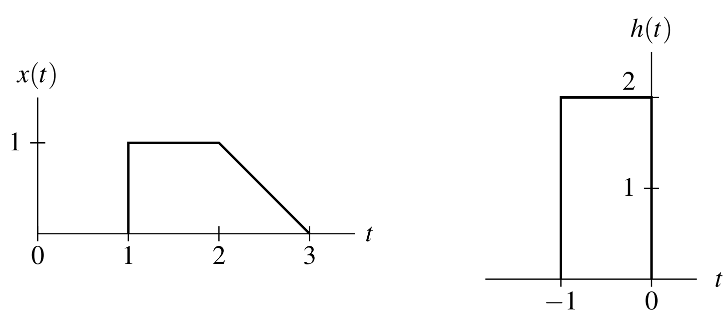
(a)



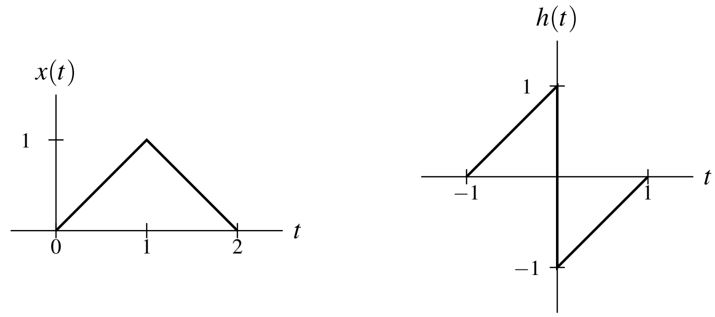
(b)



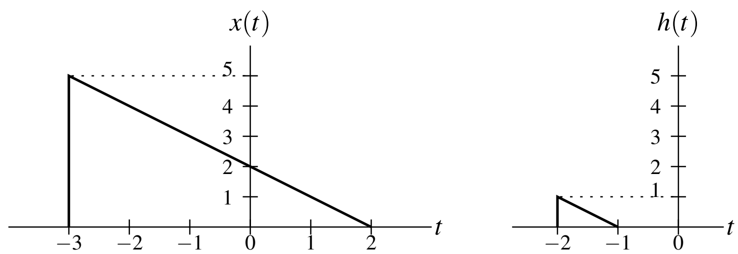
(c)



(d)



(e)



(f)

4.2 For each pair of functions x and h given below, compute $x * h$.
(a) $x(t) = e^{at}u(-t)$ and $h(t) = e^{-at}u(t)$, where a is a (strictly) positive real constant;
(b) $x(t) = e^{-j\omega_0 t}u(t)$ and $h(t) = e^{j\omega_0 t}u(t)$ where ω_0 is a (strictly) positive real constant;
(c) $x(t) = u(t-2)$ and $h(t) = u(t+3)$;
(d) $x(t) = u(t)$ and $h(t) = e^{-2t}u(t-1)$; and
(e) $x(t) = u(t-1) - u(t-2)$ and $h(t) = e^{t}u(-t)$.

4.3 Using the graphical method, compute $x * h$ for each pair of functions x and h given below.
(a) $x(t) = e^{t}u(-t)$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$
(b) $x(t) = e^{-|t|}$ and $h(t) = \operatorname{rect}(\frac{1}{3}[t-\frac{1}{2}])$;
(c) $x(t) = e^{-t}u(t)$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$
(d) $x(t) = \operatorname{rect}\left(\frac{1}{2}t\right)$ and $h(t) = e^{2-t}u(t-2)$;
(e) $x(t) = e^{-|t|}$ and $h(t) = \begin{cases} t+2 & -2 \le t < -1 \\ 0 & \text{otherwise;} \end{cases}$
(f) $x(t) = e^{-|t|}$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$
(g) $x(t) = \begin{cases} 1-\frac{1}{4}t & 0 \le t < 4 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$
(h) $x(t) = \operatorname{rect}\left(\frac{1}{4}t\right)$ and $h(t) = \begin{cases} 2-t & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$ and
(i) $x(t) = e^{-t}u(t)$ and $h(t) = \begin{cases} t-2 & 2 \le t < 4 \\ 0 & \text{otherwise.} \end{cases}$

4.4 Let x, y, h, and v be functions such that $y = x * h$ and

$$v(t) = \int_{-\infty}^{\infty} x(-\tau - b)h(\tau + at)d\tau,$$

where a and b are real constants. Express v in terms of y.

4.5 Consider the convolution $y = x * h$. Assuming that the convolution y exists, prove that each of the following assertions is true:
(a) If x is periodic, then y is periodic.
(b) If x is even and h is odd, then y is odd.

4.6 From the definition of convolution, show that if $y = x * h$, then $y'(t) = x * h'(t)$, where y' and h' denote the derivatives of y and h, respectively.

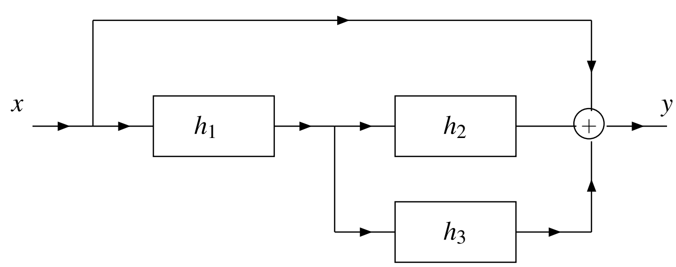
4.7 Let x and h be functions satisfying

$$x(t) = 0 \quad \text{for } t < A_1 \text{ or } t > A_2, \quad \text{and}$$
$$h(t) = 0 \quad \text{for } t < B_1 \text{ or } t > B_2$$

(i.e., x and h are finite duration). Determine for which values of t the quantity $x * h(t)$ must be zero.

4.8 Find the impulse response of the LTI system $\mathcal{H}$ characterized by each of the equations below.
(a) $\mathcal{H}x(t) = \int_{-\infty}^{t+1} x(\tau)d\tau$;
(b) $\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau+5)e^{\tau-t+1}u(t-\tau-2)d\tau$;
(c) $\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)v(t-\tau)d\tau$; and
(d) $\mathcal{H}x(t) = \int_{t-1}^{t} x(\tau)d\tau$.

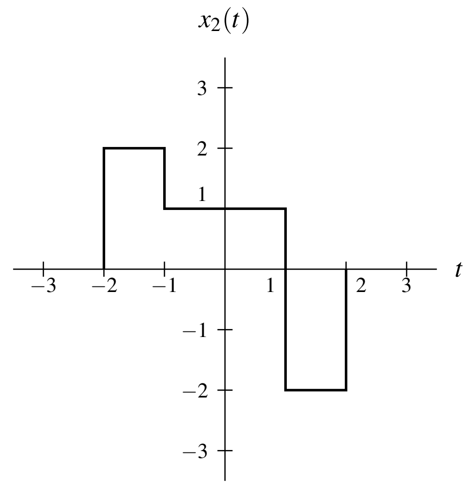
4.9 Consider the system with input x and output y as shown in the figure below. Each system in the block diagram is LTI and labelled with its impulse response.



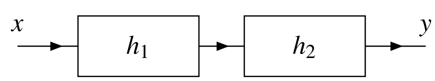
(a) Find the impulse response h of the overall system in terms of h_1, h_2, and h_3.
(b) Determine the impulse response h in the specific case that

$$h_1(t) = \delta(t+1), \quad h_2(t) = \delta(t), \quad \text{and} \quad h_3(t) = \delta(t).$$

4.10 Consider a LTI system whose response to the function $x_1(t) = u(t) - u(t-1)$ is the function y_1. Determine the response y_2 of the system to the input x_2 shown in the figure below in terms of y_1.



4.11 Consider the system shown in the figure below with input x and output y. This system is formed by the series interconnection of two LTI systems with the impulse responses h_1 and h_2.



For each pair of h_1 and h_2 given below, find the output y if the input $x(t) = u(t)$.
(a) $h_1(t) = \delta(t)$ and $h_2(t) = \delta(t)$;
(b) $h_1(t) = \delta(t+1)$ and $h_2(t) = \delta(t+1)$; and
(c) $h_1(t) = e^{-3t}u(t)$ and $h_2(t) = \delta(t)$.

4.12 Determine whether the LTI system with each impulse response h given below is causal and/or memoryless.
(a) $h(t) = (t+1)u(t-1)$;
(b) $h(t) = 2\delta(t+1)$;
(c) $h(t) = \frac{\omega_c}{\pi}\operatorname{sinc}(\omega_c t)$;
(d) $h(t) = e^{-4t}u(t-1)$;
(e) $h(t) = e^{t}u(-1-t)$;
(f) $h(t) = e^{-3|t|}$; and
(g) $h(t) = 3\delta(t)$.

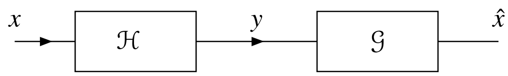
4.13 Determine whether the LTI system with each impulse response h given below is BIBO stable.
(a) $h(t) = e^{at}u(-t)$ where a is a strictly positive real constant;
(b) $h(t) = (1/t)u(t-1)$;
(c) $h(t) = e^{t}u(t)$;
(d) $h(t) = \delta(t-10)$;
(e) $h(t) = \operatorname{rect} t$; and
(f) $h(t) = e^{-|t|}$.

4.14 Suppose that we have two LTI systems with impulse responses

$$h_1(t) = \tfrac{1}{2}\delta(t-1) \quad \text{and} \quad h_2(t) = 2\delta(t+1).$$

Determine whether these systems are inverses of one another.

4.15 Consider the system shown in the figure below, where $\mathcal{H}$ is a LTI system and $\mathcal{G}$ is known to be the inverse system of $\mathcal{H}$. Let $y_1 = \mathcal{H}x_1$ and $y_2 = \mathcal{H}x_2$.



(a) Determine the response of the system $\mathcal{G}$ to the input $y'(t) = a_1 y_1(t) + a_2 y_2(t)$, where a_1 and a_2 are complex constants.
(b) Determine the response of the system $\mathcal{G}$ to the input $y_1'(t) = y_1(t - t_0)$, where t_0 is a real constant.
(c) Using the results of the previous parts of this question, determine whether the system $\mathcal{G}$ is linear and/or time invariant.

4.16 Suppose that we have the systems $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$, whose responses to a complex exponential input $x(t) = e^{j2t}$ are given by

$$\mathcal{H}_1 x(t) = 2e^{j2t}, \quad \mathcal{H}_2 x(t) = te^{j2t}, \quad \mathcal{H}_3 x(t) = e^{j2t + \pi/3}, \quad \text{and} \quad \mathcal{H}_4 x(t) = \cos(2t).$$

Indicate which of these systems cannot be LTI.

4.17 Show that, for any function x, $x * v(t) = x(t - t_0)$, where $v(t) = \delta(t - t_0)$ and t_0 is an arbitrary real constant.

4.11.2 Exercises With Answer Key

4.18 Using the graphical method, compute $x * h$ for each pair of functions x and h given below.

(a) $x(t) = 2\operatorname{rect}(t - \frac{1}{2})$ and $h(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 \le t < 1 \\ 0 & \text{otherwise;} \end{cases}$

(b) $x(t) = u(t-1)$ and $h(t) = \begin{cases} t+1 & -1 \le t < 0 \\ t-1 & 0 \le t < 1 \\ 0 & \text{otherwise;} \end{cases}$

(c) $x(t) = \begin{cases} t-2 & 1 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \operatorname{rect}\left[\frac{1}{2}(t+2)\right]$;

(d) $x(t) = \operatorname{rect}\left[\frac{1}{3}\left(t - \frac{3}{2}\right)\right]$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(e) $x(t) = \begin{cases} \frac{1}{4}(t-1)^2 & 1 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(f) $x(t) = \begin{cases} 2\cos\left(\frac{\pi}{4}t\right) & 0 \le t < 2 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} 2-t & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(g) $x(t) = e^{-|t|}$ and $h(t) = \operatorname{rect}\left[\frac{1}{2}(t-2)\right]$;

(h) $x(t) = \begin{cases} \frac{1}{2}t - \frac{1}{2} & 1 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} -t-1 & -2 \le t < -1 \\ 0 & \text{otherwise;} \end{cases}$

(i) $x(t) = e^{-|t|}$ and $h(t) = \operatorname{tri}\left[\frac{1}{2}(t-3)\right]$;

(j) $x(t) = \begin{cases} \frac{1}{4}t - \frac{1}{4} & 1 \le t < 5 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} \frac{3}{2} - \frac{1}{2}t & 1 \le t < 3 \\ 0 & \text{otherwise;} \end{cases}$

(k) $x(t) = \operatorname{rect}\left(\frac{1}{20}t\right)$ and $h(t) = \begin{cases} t-1 & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(l) $x(t) = \begin{cases} 1 - \frac{1}{100}t & 0 \le t < 100 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = e^{-t}u(t-1)$;

(m) $x(t) = \operatorname{rect}\left(\frac{1}{20}t\right)$ and $h(t) = \begin{cases} 1-(t-2)^2 & 1 \le t < 3 \\ 0 & \text{otherwise;} \end{cases}$

(n) $x(t) = e^{-t}u(t)$ and $h(t) = e^{-3t}u(t-2)$;

(o) $x(t) = e^{-|t|}$ and $h(t) = \operatorname{rect}\left(t - \frac{3}{2}\right)$;

(p) $x(t) = e^{-2t}u(t)$ and $h(t) = \operatorname{rect}\left(t - \frac{5}{2}\right)$;

(q) $x(t) = u(t-1)$ and $h(t) = \begin{cases} \sin[\pi(t-1)] & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(r) $x(t) = u(t)$ and $h(t) = \operatorname{rect}\left(\frac{1}{4}[t-4]\right)$;

(s) $x(t) = e^{-t}u(t)$ and $h(t) = e^{2-2t}u(t-1)$;

(t) $x(t) = e^{-3t}$ and $h(t) = u(t+1)$; and

(u) $x(t) = \begin{cases} 2-t & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases}$ and $h(t) = \begin{cases} -t-2 & -3 \le t < -2 \\ 0 & \text{otherwise.} \end{cases}$

Short Answer. (a) $x * h(t) = \begin{cases} \int_0^{t+1} -2d\tau & -1 \le t < 0 \\ \int_0^t 2d\tau + \int_t^1 -2d\tau & 0 \le t < 1 \\ \int_{t-1}^1 2d\tau & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(b) $x * h(t) = \begin{cases} \int_1^{t+1} (-\tau+t+1)d\tau & 0 \le t < 1 \\ \int_1^t (-\tau+t-1)d\tau + \int_t^{t+1} (-\tau+t+1)d\tau & 1 \le t < 2 \\ \int_{t-1}^t (-\tau+t-1)d\tau + \int_t^{t+1} (-\tau+t+1)d\tau & t \ge 2 \\ 0 & \text{otherwise;} \end{cases}$

(c) $x * h(t) = \begin{cases} \int_1^{t+3} (\tau-2)d\tau & -2 \le t < 0 \\ \int_{t+1}^3 (\tau-2)d\tau & 0 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(d) $x * h(t) = \begin{cases} \int_0^{t-1} (t-\tau-1)d\tau & 1 \le t < 2 \\ \int_{t-2}^{t-1} (t-\tau-1)d\tau & 2 \le t < 4 \\ \int_{t-2}^3 (t-\tau-1)d\tau & 4 \le t < 5 \\ 0 & \text{otherwise;} \end{cases}$

(e) $x * h(t) = \begin{cases} \int_1^{t-1} \frac{1}{4}(\tau-1)^2(t-\tau-1)d\tau & 2 \le t < 3 \\ \int_{t-2}^{t-1} \frac{1}{4}(\tau-1)^2(t-\tau-1)d\tau & 3 \le t < 4 \\ \int_{t-2}^3 \frac{1}{4}(\tau-1)^2(t-\tau-1)d\tau & 4 \le t < 5 \\ 0 & \text{otherwise;} \end{cases}$

(f) $x * h(t) = \begin{cases} \int_0^{t-1} 2\cos\left(\frac{\pi}{4}\tau\right)(\tau-t+2)d\tau & 1 \le t < 2 \\ \int_{t-2}^{t-1} 2\cos\left(\frac{\pi}{4}\tau\right)(\tau-t+2)d\tau & 2 \le t < 3 \\ \int_{t-2}^2 2\cos\left(\frac{\pi}{4}\tau\right)(\tau-t+2)d\tau & 3 \le t < 4 \\ 0 & \text{otherwise;} \end{cases}$

(g) $x * h(t) = \begin{cases} \int_{t-1}^{t-3} e^{\tau}d\tau & t < 1 \\ \int_{t-3}^0 e^{\tau}d\tau + \int_0^{t-1} e^{-\tau}d\tau & 1 \le t < 3 \\ \int_{t-3}^{t-1} e^{-\tau}d\tau & t \ge 3; \end{cases}$

(h) $x * h(t) = \begin{cases} \int_1^{t+2} \left(\frac{1}{2}\tau - \frac{1}{2}\right)(\tau - t - 1)d\tau & -1 \le t < 0 \\ \int_{t+1}^{t+2} \left(\frac{1}{2}\tau - \frac{1}{2}\right)(\tau - t - 1)d\tau & 0 \le t < 1 \\ \int_{t+1}^{3} \left(\frac{1}{2}\tau - \frac{1}{2}\right)(\tau - t - 1)d\tau & 1 \le t < 2 \\ 0 & \text{otherwise;} \end{cases}$

(i) $x * h(t) = \begin{cases} \int_{t-4}^{t-3} e^{\tau}(\tau - t + 4)d\tau + \int_{t-3}^{t-2} e^{\tau}(t - \tau - 2)d\tau & t < 2 \\ \int_{t-4}^{t-3} e^{\tau}(\tau - t + 4)d\tau + \int_{t-3}^{0} e^{\tau}(t - \tau - 2)d\tau + \int_{0}^{t-2} e^{-\tau}(t - \tau - 2)d\tau & 2 \le t < 3 \\ \int_{t-4}^{0} e^{\tau}(\tau - t + 4)d\tau + \int_{0}^{t-3} e^{-\tau}(t - \tau + 4)d\tau + \int_{t-3}^{t-2} e^{-\tau}(t - \tau - 2)d\tau & 3 \le t < 4 \\ \int_{t-4}^{t-3} e^{-\tau}(\tau - t + 4)d\tau + \int_{t-3}^{t-2} e^{-\tau}(t - \tau - 2)d\tau & t \ge 4; \end{cases}$

(j) $x * h(t) = \begin{cases} \int_1^{t+1} \left(\frac{1}{4}\tau - \frac{1}{4}\right)\left(\frac{1}{2}\tau - \frac{1}{2}t + \frac{3}{2}\right)d\tau & 2 \le t < 4 \\ \int_{t-3}^{t-1} \left(\frac{1}{4}\tau - \frac{1}{4}\right)\left(\frac{1}{2}\tau - \frac{1}{2}t + \frac{3}{2}\right)d\tau & 4 \le t < 6 \\ \int_{t-3}^{5} \left(\frac{1}{4}\tau - \frac{1}{4}\right)\left(\frac{1}{2}\tau - \frac{1}{2}t + \frac{3}{2}\right)d\tau & 6 \le t < 8 \\ 0 & \text{otherwise;} \end{cases}$

(k) $x * h(t) = \begin{cases} \int_{-10}^{t-1} (t - \tau - 1)d\tau & -9 \le t < -8 \\ \int_{t-2}^{t-1} (t - \tau - 1)d\tau & -8 \le t < 11 \\ \int_{10}^{t-1} (t - \tau - 1)d\tau & 11 \le t < 12 \\ 0 & \text{otherwise;} \end{cases}$

(l) $x * h(t) = \begin{cases} 0 & t < 1 \\ \int_0^{t-1} \left(1 - \frac{1}{100}\tau\right) e^{\tau - t} d\tau & 1 \le t < 101 \\ \int_0^{100} \left(1 - \frac{1}{100}\tau\right) e^{\tau - t} d\tau & t \ge 101; \end{cases}$

(m) $x * h(t) = \begin{cases} \int_{-10}^{t-1} [1 - (t - \tau - 2)^2]d\tau & -9 \le t < -7 \\ \int_{t-3}^{t-1} [1 - (t - \tau - 2)^2]d\tau & -7 \le t < 11 \\ \int_{t-3}^{10} [1 - (t - \tau - 2)^2]d\tau & 11 \le t < 13 \\ 0 & \text{otherwise;} \end{cases}$

(n) $x * h(t) = \begin{cases} \int_0^{t-2} e^{-\tau} e^{3\tau - 3t} d\tau & t \ge 2 \\ 0 & \text{otherwise;} \end{cases}$

(o) $x * h(t) = \begin{cases} \int_{t-2}^{t-1} e^{\tau} d\tau & t < 1 \\ \int_{t-2}^{0} e^{\tau} d\tau + \int_0^{t-1} e^{-\tau} d\tau & 1 \le t < 2 \\ \int_{t-2}^{t-1} e^{-\tau} d\tau & t \ge 2; \end{cases}$

(p) $x * h(t) = \begin{cases} 0 & t < 2 \\ \int_0^{t-2} e^{-2\tau} d\tau & 2 \le t < 3 \\ \int_{t-3}^{t-2} e^{-2\tau} d\tau & t \ge 3; \end{cases}$

(q) $x * h(t) = \begin{cases} 0 & t < 2 \\ \int_1^{t-1} \sin(\pi t - \pi\tau - \pi)d\tau & 2 \le t < 3 \\ \int_{t-2}^{t-1} \sin(\pi t - \pi\tau - \pi)d\tau & t \ge 3; \end{cases}$

(r) $x * h(t) = \begin{cases} 0 & t < 2 \\ \int_0^{t-2} 1 d\tau & 2 \le t < 6 \\ \int_{t-6}^{t-2} 1 d\tau & t \ge 6; \end{cases}$

(s) $x * h(t) = \begin{cases} \int_0^{t-1} e^{\tau - 2t + 2} d\tau & t \ge 1 \\ 0 & \text{otherwise;} \end{cases}$

(t) $x * h(t) = \begin{cases} \int_0^{t+1} e^{-3\tau} d\tau & t \ge -1 \\ 0 & \text{otherwise;} \end{cases}$

(u) $x*h(t) = \begin{cases} \frac{1}{6}t^3 - t - \frac{2}{3} & -2 \le t < -1 \\ -\frac{1}{6}t^3 & -1 \le t < 0 \\ 0 & \text{otherwise} \end{cases}$

4.19 Using the graphical method, compute $x*h$ for each pair of functions x and h given below.
(a) $x(t) = \operatorname{rect}\left(\frac{1}{2a}t\right)$ and $h(t) = \operatorname{rect}\left(\frac{1}{2a}t\right)$; and
(b) $x(t) = \operatorname{rect}\left(\frac{1}{a}t\right)$ and $h(t) = \operatorname{rect}\left(\frac{1}{a}t\right)$.

Short Answer. (a) $x*h(t) = 2a\operatorname{tri}\left(\frac{1}{4a}t\right)$; (b) $x*h(t) = a\operatorname{tri}\left(\frac{1}{2a}t\right)$.

4.20 Determine whether the LTI system with each impulse response h given below is causal.
(a) $h(t) = u(t+1) - u(t-1)$; and
(b) $h(t) = e^{-5t}u(t-1)$.

Short Answer. (a) not causal (b) causal

4.21 Determine whether the LTI system with each impulse response h given below is BIBO stable.
(a) $h(t) = u(t-1) - u(t-2)$.

Short Answer. (a) BIBO stable

4.22 Compute the convolution $x*h$, where

$$x(t) = \cos(3t) \quad \text{and} \quad h(t) = u(t).$$

Short Answer. $x*h(t) = \frac{1}{3}\sin(3t)$

Chapter 5

Continuous-Time Fourier Series

5.1 Introduction

One very important tool in the study of signals and systems is the Fourier series. A very large class of functions can be represented using Fourier series, namely most practically useful periodic functions. The Fourier series represents a periodic function as a (possibly infinite) linear combination of complex sinusoids. This is often desirable since complex sinusoids are easy functions with which to work. For example, complex sinusoids are easy to integrate and differentiate. Also, complex sinusoids have important properties in relation to LTI systems. In particular, complex sinusoids are eigenfunctions of LTI systems. Therefore, the response of a LTI system to a complex sinusoid is the same complex sinusoid multiplied by a complex constant.

5.2 Definition of Continuous-Time Fourier Series

Suppose that we have a set of **harmonically-related** complex sinusoids of the form

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t} \quad k = 0, \pm 1, \pm 2, \ldots.$$

The fundamental frequency of the kth complex sinusoid ϕ_k is $k\omega_0$, an integer multiple of ω_0. Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of ω_0, a linear combination of these complex sinusoids must be periodic. More specifically, a linear combination of these complex sinusoids is periodic with period $T = \frac{2\pi}{\omega_0}$.

Suppose that we can represent a periodic complex-valued function x as a linear combination of harmonically-related complex sinusoids as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \tag{5.1}$$

Such a representation is known as a **Fourier series**. More specifically, this is the **complex exponential form** of the Fourier series. As a matter of terminology, we refer to (5.1) as the **Fourier series synthesis equation**. The terms in the summation for $k = 1$ and $k = -1$ are known as the fundamental frequency components or **first harmonic components**, and have the fundamental frequency ω_0. More generally, the terms in the summation for $k = K$ and $k = -K$ are called the Kth **harmonic components**, and have the fundamental frequency $K\omega_0$. Since the complex sinusoids are harmonically related, the function x is periodic with period $T = \frac{2\pi}{\omega_0}$ (and frequency ω_0).

Since we often work with Fourier series, it is sometimes convenient to have an abbreviated notation to indicate that a function is associated with particular Fourier series coefficients. If a function x has the Fourier series coefficient sequence c, we sometimes indicate this using the notation

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k.$$

5.3 Determining the Fourier Series Representation of a Continuous-Time Periodic Function

Given an arbitrary periodic function x, we need some means for finding its corresponding Fourier series representation. In other words, we need a method for calculating the Fourier series coefficient sequence c. Such a method is given by the theorem below.

Theorem 5.1 (Fourier series analysis equation)**.** *The Fourier series coefficient sequence c of a periodic function x with fundamental period T is given by*

$$c_k = \tfrac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt, \tag{5.2}$$

where $\int_T$ denotes integration over an interval of length T.

Proof. We begin with the definition of the Fourier series in (5.1). Multiplying both sides of this equation by $e^{-jn\omega_0 t}$ yields

$$\begin{aligned} x(t)e^{-jn\omega_0 t} &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-jn\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} c_k e^{j(k-n)\omega_0 t}. \end{aligned}$$

As a matter of notation, we use $\int_T$ to denote the integral over an arbitrary interval of length T (i.e., the interval (t_0, t_0+T) for arbitrary t_0). Integrating both sides of this equation over one period T of x, we obtain

$$\int_T x(t)e^{-jn\omega_0 t} dt = \int_T \sum_{k=-\infty}^{\infty} c_k e^{j(k-n)\omega_0 t} dt.$$

Reversing the order of integration and summation yields

$$\int_T x(t)e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} c_k \left(\int_T e^{j(k-n)\omega_0 t} dt \right). \tag{5.3}$$

Now, we note that the following identity holds:

$$\int_T e^{j(k-n)\omega_0 t} dt = \begin{cases} T & k = n \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

(The proof of this fact is left as an exercise for the reader in Exercise A.11.) Substituting (5.4) into (5.3), we obtain

$$\int_T x(t)e^{-jn\omega_0 t} dt = c_n T. \tag{5.5}$$

Rearranging, we obtain

$$c_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt. \qquad \blacksquare$$

As a matter of terminology, we refer to (5.2) as the **Fourier series analysis equation**.

Suppose that we have a complex-valued periodic function x with period T and Fourier series coefficient sequence c. One can easily show that the coefficient c_0 is the average value of x over a single period T. The proof is trivial. Consider the Fourier series analysis equation given by (5.2). Substituting $k = 0$ into this equation, we obtain

$$\begin{aligned} c_0 &= \left[\tfrac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \right] \Bigg|_{k=0} \\ &= \tfrac{1}{T} \int_T x(t) e^0 dt \\ &= \tfrac{1}{T} \int_T x(t) dt. \end{aligned}$$

Thus, c_0 is simply the average value of x over a single period.

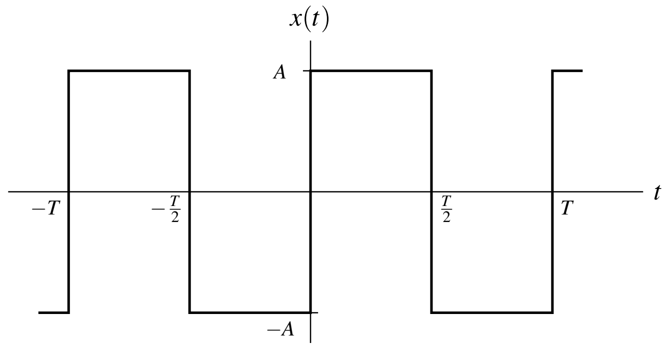


Figure 5.1: Periodic square wave.

Example 5.1 (Fourier series of a periodic square wave). Find the Fourier series representation of the periodic square wave x shown in Figure 5.1.

Solution. Let us consider the single period of $x(t)$ for $0 \leq t < T$. For this range of t, we have

$$x(t) = \begin{cases} A & 0 \leq t < \frac{T}{2} \\ -A & \frac{T}{2} \leq t < T. \end{cases}$$

Let $\omega_0 = \frac{2\pi}{T}$. From the Fourier series analysis equation, we have

$$\begin{aligned} c_k &= \tfrac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\ &= \tfrac{1}{T} \left(\int_0^{T/2} A e^{-jk\omega_0 t} dt + \int_{T/2}^{T} (-A) e^{-jk\omega_0 t} dt \right) \\ &= \begin{cases} \dfrac{1}{T} \left(\left[\dfrac{-A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_0^{T/2} + \left[\dfrac{A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_{T/2}^{T} \right) & k \neq 0 \\ \dfrac{1}{T} \left([At]|_0^{T/2} + [-At]|_{T/2}^{T} \right) & k = 0. \end{cases} \end{aligned}$$

Now, we simplify the expression for c_k for each of the cases $k \neq 0$ and $k = 0$ in turn. First, suppose that $k \neq 0$. We have

$$\begin{aligned} c_k &= \frac{1}{T} \left(\left[\frac{-A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_0^{T/2} + \left[\frac{A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_{T/2}^{T} \right) \\ &= \frac{-A}{j2\pi k} \left(\left[e^{-jk\omega_0 t} \right] \Big|_0^{T/2} - \left[e^{-jk\omega_0 t} \right] \Big|_{T/2}^{T} \right) \\ &= \frac{jA}{2\pi k} \left(\left[e^{-j\pi k} - 1 \right] - \left[e^{-j2\pi k} - e^{-j\pi k} \right] \right) \\ &= \frac{jA}{2\pi k} \left[2e^{-j\pi k} - e^{-j2\pi k} - 1 \right] \\ &= \frac{jA}{2\pi k} \left[2(e^{-j\pi})^k - (e^{-j2\pi})^k - 1 \right]. \end{aligned}$$

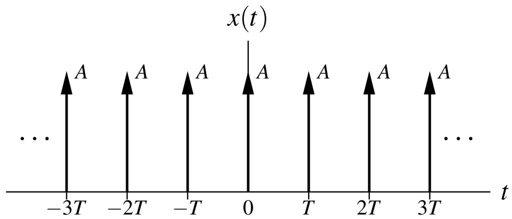


Figure 5.2: Periodic impulse train.

Now, we observe that $e^{-j\pi} = -1$ and $e^{-j2\pi} = 1$. So, we have

$$\begin{aligned} c_k &= \frac{jA}{2\pi k}[2(-1)^k - 1^k - 1] \\ &= \frac{jA}{2\pi k}[2(-1)^k - 2] \\ &= \frac{jA}{\pi k}[(-1)^k - 1] \\ &= \begin{cases} \dfrac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even}, k \neq 0. \end{cases} \end{aligned}$$

Now, suppose that $k = 0$. We have

$$\begin{aligned} c_0 &= \tfrac{1}{T}\left([At]\big|_0^{T/2} + [-At]\big|_{T/2}^{T} \right) \\ &= \tfrac{1}{T}\left[\tfrac{AT}{2} - \tfrac{AT}{2}\right] \\ &= 0. \end{aligned}$$

Thus, the Fourier series of x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j(2\pi/T)kt},$$

where

$$c_k = \begin{cases} \dfrac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even}. \end{cases}$$

■

Example 5.2 (Fourier series of a periodic impulse train)**.** Consider the periodic impulse train x shown in Figure 5.2. Find the Fourier series representation of x.

Solution. Let $\omega_0 = \frac{2\pi}{T}$. Let us consider the single period of $x(t)$ for $-\frac{T}{2} \leq t < \frac{T}{2}$. From the Fourier series analysis equation, we have

$$\begin{aligned} c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\ &= \tfrac{1}{T}\int_{-T/2}^{T/2} A\delta(t)e^{-jk\omega_0 t}dt \\ &= \tfrac{A}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0 t}dt. \end{aligned}$$

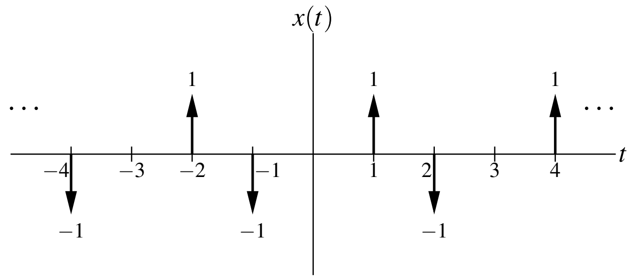


Figure 5.3: Periodic impulse train.

Using the sifting property of the unit-impulse function, we can simplify the above result to obtain

$$c_k = \tfrac{A}{T}.$$

Thus, the Fourier series for x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} \tfrac{A}{T} e^{j(2\pi/T)kt}. \qquad \blacksquare$$

Example 5.3. Consider the periodic function x with fundamental period $T = 3$ as shown in Figure 5.3. Find the Fourier series representation of x.

Solution. The function x has the fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$. Let us consider the single period of $x(t)$ for $-\frac{T}{2} \leq t < \frac{T}{2}$ (i.e., $-\frac{3}{2} \leq t < \frac{3}{2}$). From the Fourier series analysis equation, we have

$$\begin{aligned}
c_k &= \tfrac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\
&= \tfrac{1}{3} \int_{-3/2}^{3/2} x(t) e^{-j(2\pi/3)kt} dt \\
&= \tfrac{1}{3} \int_{-3/2}^{3/2} \left[-\delta(t+1) + \delta(t-1)\right] e^{-j(2\pi/3)kt} dt \\
&= \tfrac{1}{3} \left[\int_{-3/2}^{3/2} -\delta(t+1) e^{-j(2\pi/3)kt} dt + \int_{-3/2}^{3/2} \delta(t-1) e^{-j(2\pi/3)kt} dt\right] \\
&= \tfrac{1}{3} \left[\int_{-\infty}^{\infty} -\delta(t+1) e^{-j(2\pi/3)kt} dt + \int_{-\infty}^{\infty} \delta(t-1) e^{-j(2\pi/3)kt} dt\right] \\
&= \tfrac{1}{3} \left(\left[-e^{-j(2\pi/3)kt}\right]\Big|_{t=-1} + \left[e^{-j(2\pi/3)kt}\right]\Big|_{t=1} \right) \\
&= \tfrac{1}{3} \left[-e^{-jk(2\pi/3)(-1)} + e^{-jk(2\pi/3)(1)}\right] \\
&= \tfrac{1}{3} \left[e^{-j(2\pi/3)k} - e^{j(2\pi/3)k}\right] \\
&= \tfrac{1}{3} \left[2j \sin\left(-\tfrac{2\pi}{3}k\right)\right] \\
&= \tfrac{2j}{3} \sin\left(-\tfrac{2\pi}{3}k\right) \\
&= -\tfrac{2j}{3} \sin\left(\tfrac{2\pi}{3}k\right).
\end{aligned}$$

Thus, x has the Fourier series representation

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} -\tfrac{2j}{3} \sin\left(\tfrac{2\pi}{3}k\right) e^{j(2\pi/3)kt}. \end{aligned}$$ ■

Example 5.4 (Fourier series of an even real function)**.** Let x be an arbitrary periodic real function that is even. Let c denote the Fourier series coefficient sequence for x. Show that

- c is real (i.e., $\mathrm{Im}\{c_k\} = 0$ for all k);
- c is even (i.e., $c_k = c_{-k}$ for all k); and
- $c_0 = \frac{1}{T}\int_0^T x(t)dt$.

Solution. From the Fourier series analysis equation (5.2) and using Euler's relation, we can write

$$\begin{aligned} c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\ &= \tfrac{1}{T}\int_T \left(x(t)\left[\cos(-k\omega_0 t) + j\sin(-k\omega_0 t)\right]\right)dt. \end{aligned}$$

Since cos and sin are even and odd functions, respectively, we can rewrite the above equation as

$$\begin{aligned} c_k &= \tfrac{1}{T}\int_T \left(x(t)\left[\cos(k\omega_0 t) - j\sin(k\omega_0 t)\right]\right)dt \\ &= \tfrac{1}{T}\left[\int_T x(t)\cos(k\omega_0 t)dt - j\int_T x(t)\sin(k\omega_0 t)dt\right]. \end{aligned}$$

Consider the first integral above (i.e., the one involving the cos function). Since x is even and $\cos(k\omega_0 t)$ is even, we have that $x(t)\cos(k\omega_0 t)$ is even. Thus, $\int_T x(t)\cos(k\omega_0 t)dt = 2\int_0^{T/2} x(t)\cos(k\omega_0 t)dt$. Consider the second integral above (i.e., the one involving the sin function). Since x is even and $\sin(k\omega_0 t)$ is odd, we have that $x(t)\sin(k\omega_0 t)$ is odd. If we integrate an odd periodic function over one period (or an integer multiple thereof), the result is zero. Therefore, the second integral is zero. Combining these results, we can write

$$\begin{aligned} c_k &= \tfrac{1}{T}\left[2\int_0^{T/2} x(t)\cos(k\omega_0 t)dt\right] \\ &= \tfrac{2}{T}\int_0^{T/2} x(t)\cos(k\omega_0 t)dt. \end{aligned} \tag{5.6}$$

Since x is real, the quantity c_k must also be real. Thus, we have that $\mathrm{Im}(c_k) = 0$.

Consider now the expression for c_{-k}. We substitute $-k$ for k in (5.6) to obtain

$$c_{-k} = \tfrac{2}{T}\int_0^{T/2} x(t)\cos(-k\omega_0 t)dt.$$

Since cos is an even function, we can simplify this expression to obtain

$$\begin{aligned} c_{-k} &= \tfrac{2}{T}\int_0^{T/2} x(t)\cos(k\omega_0 t)dt \\ &= c_k. \end{aligned}$$

Thus, $c_k = c_{-k}$.

Consider now the quantity c_0. Substituting $k=0$ into (5.6), we can write

$$\begin{aligned} c_0 &= \tfrac{2}{T}\int_0^{T/2} x(t)\cos(0)dt \\ &= \tfrac{2}{T}\int_0^{T/2} x(t)dt \\ &= \tfrac{2}{T}\left[\tfrac{1}{2}\int_0^{T} x(t)dt\right] \\ &= \tfrac{1}{T}\int_0^{T} x(t)dt. \end{aligned}$$

Thus, $c_0 = \frac{1}{T}\int_0^T x(t)dt$. ■

Example 5.5 (Fourier series of an odd real function)**.** Let x be a periodic real function that is odd. Let c denote the Fourier series coefficient sequence for x. Show that

- c is purely imaginary (i.e., $\operatorname{Re}\{c_k\} = 0$ for all k);
- c is odd (i.e., $c_k = -c_{-k}$ for all k); and
- $c_0 = 0$.

Solution. From the Fourier series analysis equation (5.2) and Euler's formula, we can write

$$\begin{aligned} c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\ &= \tfrac{1}{T}\left[\int_T x(t)\left[\cos(-k\omega_0 t) + j\sin(-k\omega_0 t)\right]dt\right]. \end{aligned}$$

Since cos and sin are even and odd functions, respectively, we can rewrite the above equation as

$$\begin{aligned} c_k &= \tfrac{1}{T}\left[\int_T x(t)\left[\cos(k\omega_0 t) - j\sin(k\omega_0 t)\right]dt\right] \\ &= \tfrac{1}{T}\left[\int_T x(t)\cos(k\omega_0 t)dt - j\int_T x(t)\sin(k\omega_0 t)dt\right]. \end{aligned}$$

Consider the first integral above (i.e., the one involving the cos function). Since x is odd and $\cos(k\omega_0 t)$ is even, we have that $x(t)\cos(k\omega_0 t)$ is odd. If we integrate an odd periodic function over a single period (or an integer multiple thereof), the result is zero. Therefore, the first integral is zero. Consider the second integral above (i.e., the one involving the sin function). Since x is odd and $\sin(k\omega_0 t)$ is odd, we have that $x(t)\sin(k\omega_0 t)$ is even. Thus, $\int_T x(t)\sin(k\omega_0 t)dt = 2\int_0^{T/2} x(t)\sin(k\omega_0 t)dt$. Combining these results, we can write

$$\begin{aligned} c_k &= \tfrac{-j}{T}\int_T x(t)\sin(k\omega_0 t)dt \\ &= \tfrac{-j2}{T}\int_0^{T/2} x(t)\sin(k\omega_0 t)dt. \end{aligned} \tag{5.7}$$

Since x is real, the result of the integration is real, and consequently c_k is purely imaginary. Thus, $\operatorname{Re}(c_k) = 0$.

Consider the quantity c_{-k}. Substituting $-k$ for k in (5.7), we obtain

$$\begin{aligned} c_{-k} &= \tfrac{-j2}{T}\int_0^{T/2} x(t)\sin(-k\omega_0 t)dt \\ &= \tfrac{-j2}{T}\int_0^{T/2} x(t)[-\sin(k\omega_0 t)]dt \\ &= \tfrac{j2}{T}\int_0^{T/2} x(t)\sin(k\omega_0 t)dt \\ &= -c_k. \end{aligned}$$

Thus, $c_k = -c_{-k}$.

Consider now the quantity c_0. Substituting $k = 0$ in the expression (5.7), we have

$$\begin{aligned} c_0 &= \tfrac{-j2}{T} \int_0^{T/2} x(t) \sin(0) dt \\ &= 0. \end{aligned}$$

Thus, $c_0 = 0$. ■

5.4 Convergence of Continuous-Time Fourier Series

So far we have assumed that a given periodic function x can be represented by a Fourier series. Since a Fourier series consists of an infinite number of terms (infinitely many of which may be nonzero), we need to more carefully consider the issue of convergence. That is, we want to know under what circumstances the Fourier series of x converges (in some sense) to x.

Suppose that we have an arbitrary periodic function x. This function has the Fourier series representation given by (5.1) and (5.2). Let x_N denote the finite series

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

(i.e., x_N is a Fourier series truncated after the Nth harmonic components). The approximation error is given by

$$e_N(t) = x(t) - x_N(t).$$

Let us also define the mean-squared error (MSE) as

$$E_N = \frac{1}{T} \int_T |e_N(t)|^2 \, dt.$$

Before we can proceed further, we need to more precisely specify what we mean by convergence. This is necessary because convergence can be defined in more than one way. For example, two common types of convergence are: pointwise and MSE. In the case of pointwise convergence, the error goes to zero at every point. If convergence is pointwise and the rate of convergence is the same everywhere, we call this uniform convergence. In the case of MSE convergence, the MSE goes to zero, which does not necessarily imply that the error goes to zero at every point.

Now, we introduce a few important results regarding the convergence of Fourier series for various types of periodic functions. The first result that we consider is for the case of continuous functions as given below.

Theorem 5.2 (Convergence of Fourier series (continuous case))**.** *If a periodic function x is continuous and its Fourier series coefficient sequence c is absolutely summable (i.e., $\sum_{k=-\infty}^{\infty} |c_k| < \infty$), then the Fourier series representation of x converges uniformly (i.e., converges pointwise and at the same rate everywhere).*

Proof. A rigorous proof of this theorem is somewhat involved and omitted here. ■

In other words, in the above theorem, we have that if x is continuous, then as $N \to \infty$, $e_N(t) \to 0$ for all t. Often, however, we must work with functions that are not continuous. For example, many useful periodic functions are not continuous (e.g., a square wave). Consequently, we must consider the matter of convergence for functions with discontinuities.

Another important result regarding convergence applies to functions that have finite energy over a single period. Mathematically, a function x has finite energy over a single period if it satisfies

$$\int_T |x(t)|^2 \, dt < \infty.$$

In the case of such a function, we have the following important result.

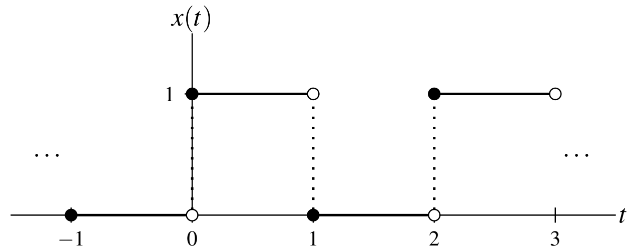


Figure 5.4: Periodic function x.

Theorem 5.3 (Convergence of Fourier series (finite-energy case))**.** *If the periodic function x has finite energy in a single period (i.e., $\int_T |x(t)|^2 dt < \infty$), the Fourier series converges in the MSE sense.*

Proof. A rigorous proof of this theorem is somewhat involved and omitted here. ■

In other words, in the above theorem, we have that if x is of finite energy, then as $N \to \infty$, $E_N \to 0$.

The last important result regarding convergence that we shall consider relates to what are known as the Dirichlet conditions. The Dirichlet[1] conditions for the periodic function x are as follows:

1. Over a single period, x is absolutely integrable (i.e., $\int_T |x(t)|\, dt < \infty$).
2. Over a single period, x has a finite number of maxima and minima (i.e., x is of bounded variation).
3. Over any finite interval, x has a finite number of discontinuities, each of which is finite.

Theorem 5.4 (Convergence of Fourier series (Dirichlet case))**.** *If x is a periodic function satisfying the Dirichlet conditions, then:*

1. *The Fourier series converges pointwise everywhere to x, except at the points of discontinuity of x.*
2. *At each point t_a of discontinuity of x, the Fourier series converges to $\frac{1}{2}(x(t_a^-)+x(t_a^+))$, where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the function on the left- and right-hand sides of the discontinuity, respectively.*

In other words, if the Dirichlet conditions are satisfied, then as $N \to \infty$, $e_N(t) \to 0$ for all t except at discontinuities. Furthermore, at each discontinuity, the Fourier series converges to the average of the function values on the left- and right-hand side of the discontinuity.

Example 5.6. Consider the periodic function x with period $T = 2$ as shown in Figure 5.4. Let $\hat{x}$ denote the Fourier series representation of x (i.e., $\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, where $\omega_0 = \pi$). Determine the values $\hat{x}(0)$ and $\hat{x}(1)$.

Solution. We begin by observing that x satisfies the Dirichlet conditions. Consequently, Theorem 5.4 applies. Thus, we have that

$$\begin{aligned}
\hat{x}(0) &= \tfrac{1}{2}\left[x(0^-)+x(0^+)\right] \\
&= \tfrac{1}{2}(0+1) \\
&= \tfrac{1}{2} \quad \text{and} \\
\hat{x}(1) &= \tfrac{1}{2}\left[x(1^-)+x(1^+)\right] \\
&= \tfrac{1}{2}(1+0) \\
&= \tfrac{1}{2}.
\end{aligned}$$

■

[1] Pronounced Dee-ree-klay.

Although many functions of practical interest satisfy the Dirichlet conditions, not all functions satisfy these conditions. In what follows, some examples of functions that violate these conditions are given.

Consider the 1-periodic function x defined by

$$x(t) = t^{-1} \text{ for } 0 \leq t < 1 \quad \text{and} \quad x(t) = x(t+1).$$

A plot of x is shown in Figure 5.5(a). The function x violates the first Dirichlet condition, since x is not absolutely integrable over a single period.

Consider the 1-periodic function x defined by

$$x(t) = \sin\left(2\pi t^{-1}\right) \text{ for } 0 < t \leq 1 \quad \text{and} \quad x(t) = x(t+1).$$

A plot of x is shown in Figure 5.5(b). Since x has an infinite number of minima and maxima over a single period, x violates the second Dirichlet condition.

Consider the 1-periodic function x shown in Figure 5.5(c). As t goes from 0 to 1, $x(t)$ traces out a sequence of steps in a staircase, where each step is half the size of the previous step. As it turns out, the number of steps in this staircase between 0 and 1 is infinite. Thus, x violates the third Dirichlet condition, as x has an infinite number of discontinuities over a single period.

One might wonder how the Fourier series converges for periodic functions with discontinuities. Let us consider the periodic square wave from Example 5.1. In Figure 5.6, we have plotted the truncated Fourier series x_N for the square wave (with period $T = 1$ and amplitude $A = 1$) for several values of N. At the discontinuities of x, we can see that the series appears to converge to the average of the function values on either side of the discontinuity. In the vicinity of a discontinuity, however, the truncated series x_N exhibits ripples and the peak amplitude of the ripples does not seem to decrease with increasing N. As it turns out, as N increases, the ripples get compressed towards the discontinuity, but, for any finite N, the peak amplitude of the ripples remains constant. This behavior is known as **Gibbs phenomenon**.

5.5 Properties of Continuous-Time Fourier Series

Fourier series representations possess a number of important properties. In the sections that follow, we introduce a number of these properties. For convenience, these properties are also summarized later in Table 5.1 (on page 129).

5.5.1 Linearity

Arguably, the most important property of Fourier series is linearity, as introduced below.

Theorem 5.5 (Linearity). *Let x and y denote two periodic functions with period T and frequency $\omega_0 = \frac{2\pi}{T}$. If*

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k \quad \textit{and} \quad y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k,$$

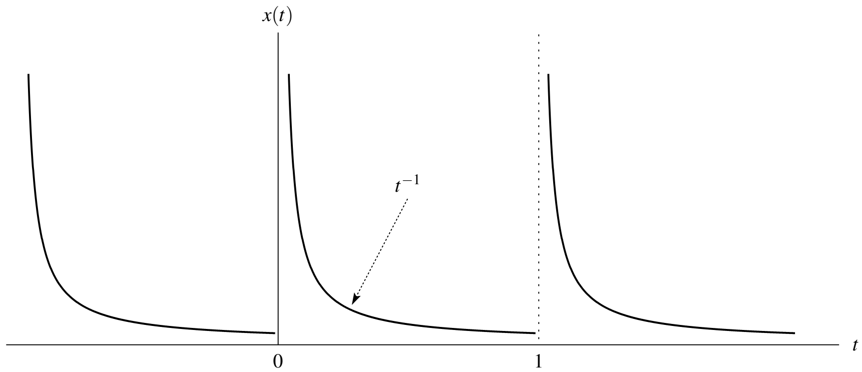
then

$$Ax(t) + By(t) \overset{\text{CTFS}}{\longleftrightarrow} Aa_k + Bb_k,$$

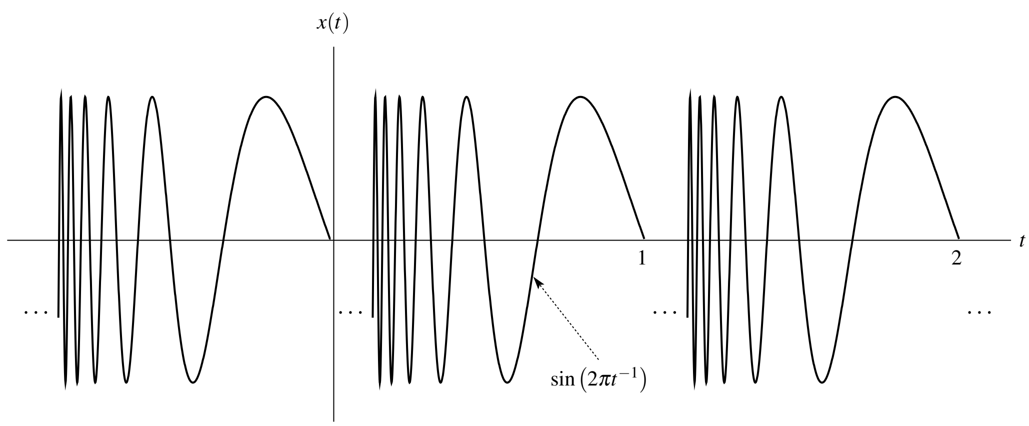
where A and B are complex constants. In other words, a linear combination of functions produces the same linear combination of their Fourier series coefficients.

Proof. To prove the above property, we proceed as follows. First, we express x and y in terms of their corresponding Fourier series as

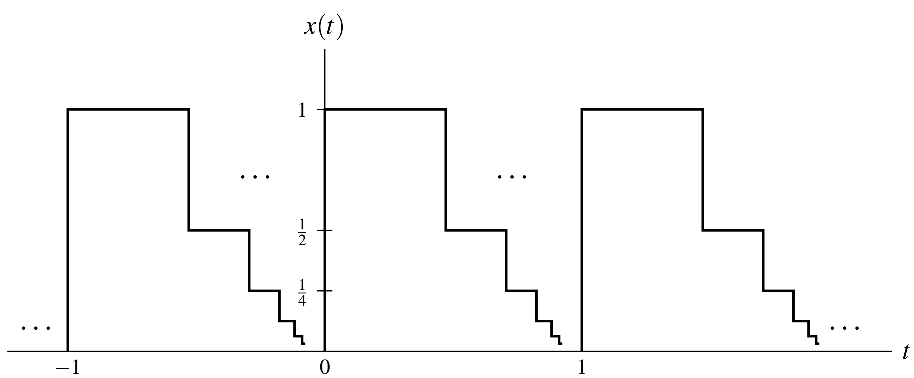
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{and} \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}.$$



(a)



(b)



(c)

Figure 5.5: Examples of functions that violate the Dirichlet conditions. (a) A function that is not absolutely integrable over a single period. (b) A function that has an infinite number of maxima and minima over a single period. (c) A function that has an infinite number of discontinuities over a single period.

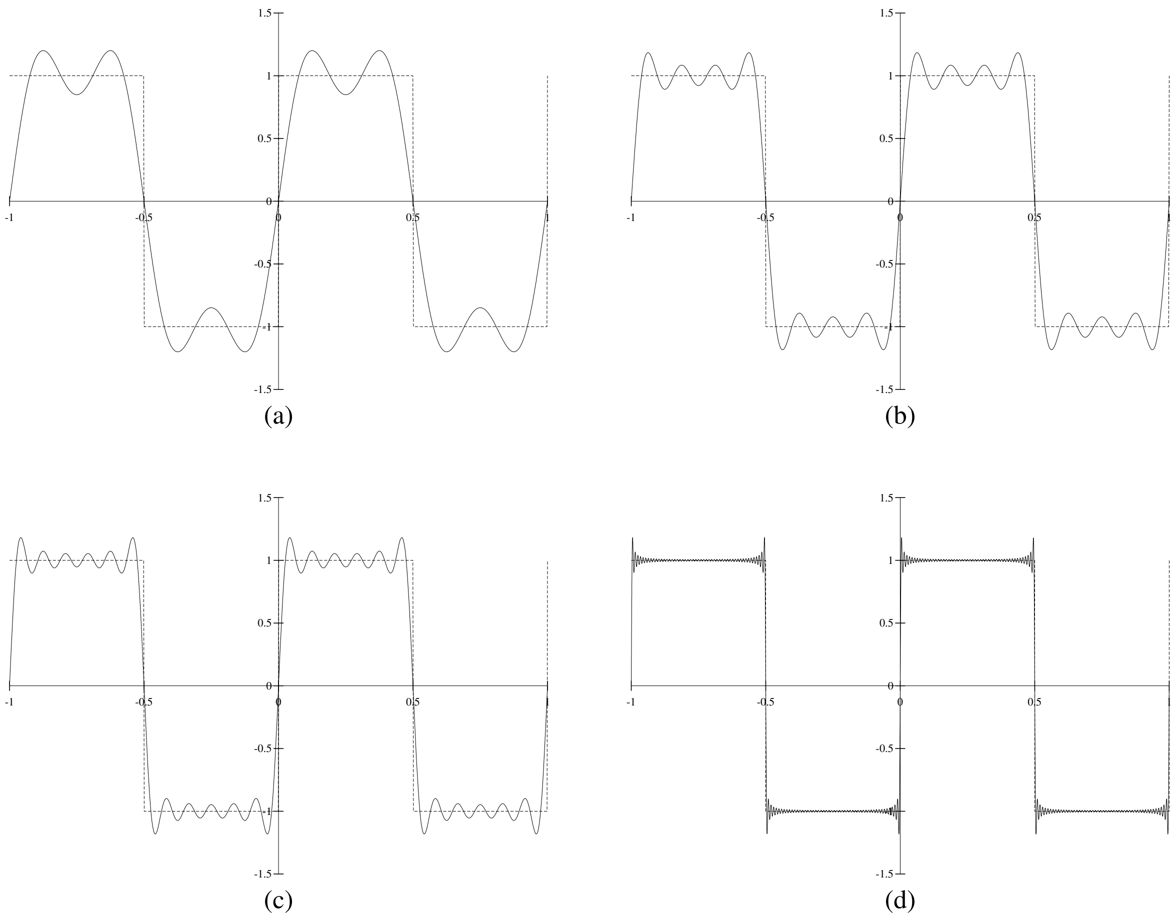


Figure 5.6: Gibbs phenomenon. The Fourier series for the periodic square wave truncated after the Nth harmonic components for (a) $N = 3$, (b) $N = 7$, (c) $N = 11$, and (d) $N = 101$.

Now, we determine the Fourier series of $Ax+By$. We have

$$\begin{aligned}
Ax(t)+By(t) &= A\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + B\sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} Aa_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} Bb_k e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} (Aa_k + Bb_k) e^{jk\omega_0 t}.
\end{aligned}$$

Therefore, we have that $Ax(t)+By(t) \stackrel{\text{CTFS}}{\longleftrightarrow} Aa_k + Bb_k$. ■

5.5.2 Time Shifting (Translation)

The next property of Fourier series to be introduced is the time-shifting (i.e., translation) property, as given below.

Theorem 5.6 (Time shifting (i.e., translation))**.** *Let x denote a periodic function with period T and frequency $\omega_0 = \frac{2\pi}{T}$. If*

$$x(t) \stackrel{\text{CTFS}}{\longleftrightarrow} a_k,$$

then

$$x(t-t_0) \stackrel{\text{CTFS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k,$$

where t_0 is a real constant.

Proof. To prove the time-shifting property, we proceed as follows. The Fourier series of x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{5.8}$$

We express $x(t-t_0)$ in terms of its Fourier series, and then use algebraic manipulation to obtain

$$\begin{aligned}
x(t-t_0) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t-t_0)} \\
&= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jk\omega_0 t_0} \\
&= \sum_{k=-\infty}^{\infty} \left(a_k e^{-jk\omega_0 t_0}\right) e^{jk\omega_0 t}. \qquad (5.9)
\end{aligned}$$

Comparing (5.8) and (5.9), we have that $x(t-t_0) \stackrel{\text{CTFS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k$. ■

From the above theorem, we can see that time shifting a periodic function does not change the magnitude of its Fourier series coefficients (since $\left|e^{j\theta}\right| = 1$ for all real θ).

5.5.3 Frequency Shifting (Modulation)

The next property of Fourier series to be introduced is the frequency-shifting (i.e., modulation) property, as given below.

Theorem 5.7 (Frequency shifting (i.e., modulation))**.** *Let x denote a periodic function with period T and frequency $\omega_0 = \frac{2\pi}{T}$. If*

$$x(t) \stackrel{\text{CTFS}}{\longleftrightarrow} a_k,$$

then

$$e^{jM(2\pi/T)t}x(t) = e^{jM\omega_0 t}x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_{k-M},$$

where M is an integer constant.

Proof. To prove the frequency-shifting property, we proceed as follows. We have

$$\begin{aligned} e^{j(2\pi/T)Mt}x(t) &= e^{j(2\pi/T)Mt}\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)kt} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)Mt}e^{j(2\pi/T)kt} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)(k+M)t}. \end{aligned}$$

Now, we employ a change of variable. Let $k' = k+M$ so that $k = k'-M$. Applying the change of variable and dropping the primes, we obtain

$$e^{j(2\pi/T)Mt}x(t) = \sum_{k=-\infty}^{\infty} a_{k-M}e^{j(2\pi/T)kt}.$$

Now, we observe that the right-hand side of this equation is a Fourier series. Therefore, the Fourier series coefficient sequence for $e^{j(2\pi/T)Mt}x(t)$ (i.e., the left-hand side of the equation) is $a'(k) = a_{k-M}$. ∎

5.5.4 Time Reversal (Reflection)

The next property of Fourier series to be introduced is the time-reversal (i.e., reflection) property, as given below.

Theorem 5.8 (Time reversal (i.e., reflection)). *Let x denote a periodic function with period T and frequency $\omega_0 = \frac{2\pi}{T}$. If*

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k,$$

then

$$x(-t) \overset{\text{CTFS}}{\longleftrightarrow} a_{-k}.$$

Proof. To prove the time-reversal property, we proceed in the following manner. The Fourier series of x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{5.10}$$

Now, we consider the Fourier series expansion of $x(-t)$. The Fourier series in this case is given by

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(-t)}. \tag{5.11}$$

Now, we employ a change of variable. Let $l = -k$ so that $k = -l$. Performing this change of variable, we can rewrite (5.11) to obtain

$$\begin{aligned} x(-t) &= \sum_{l=-\infty}^{\infty} a_{-l}e^{j(-l)\omega_0(-t)} \\ &= \sum_{l=-\infty}^{\infty} a_{-l}e^{jl\omega_0 t}. \end{aligned} \tag{5.12}$$

Comparing (5.10) and (5.12), we have that $x(-t) \overset{\text{CTFS}}{\longleftrightarrow} a_{-k}$. ∎

In other words, the above theorem states that time reversing a function time reverses the corresponding sequence of Fourier series coefficients.

5.5.5 Conjugation

The next property of Fourier series to be introduced is the conjugation property, as given below.

Theorem 5.9 (Conjugation)**.** *For a T-periodic function x with Fourier series coefficient sequence c,*

$$x^*(t) \overset{\text{CTFS}}{\longleftrightarrow} c^*_{-k}.$$

Proof. From the definition of a Fourier series, we have

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Taking the complex conjugate of both sides of the preceding equation, we obtain

$$\begin{aligned} x^*(t) &= \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} \left(c_k e^{jk\omega_0 t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}. \end{aligned}$$

Replacing k by $-k$ in the summation of the preceding equation, we obtain

$$x^*(t) = \sum_{k=-\infty}^{\infty} c^*_{-k} e^{jk\omega_0 t}.$$

Thus, the Fourier series coefficient sequence c' of x^* is $c'_k = c^*_{-k}$. ■

In other words, the above theorem states that conjugating a function has the effect of time reversing and conjugating the corresponding Fourier series coefficient sequence.

5.5.6 Periodic Convolution

The next property of Fourier series to be introduced is the periodic-convolution property, as given below.

Theorem 5.10 (Periodic convolution)**.** *Let x and y be T-periodic functions given by*

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \textit{and} \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where $\omega_0 = \frac{2\pi}{T}$*. Let* $z(t) = x \circledast y(t)$*, where*

$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

The sequences a, b, and c are related as

$$c_k = T a_k b_k.$$

Proof. In what follows, let δ denote the delta sequence (i.e., $\delta(n)$ is 1 for $n = 0$ and 0 elsewhere). From the definition of periodic convolution, we have

$$\begin{aligned}
x \circledast y(t) &= \int_T x(\tau) y(t-\tau) d\tau \\
&= \int_T \left(\sum_{\ell=-\infty}^{\infty} a_\ell e^{j\omega_0 \ell \tau} \right) \left(\sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 k(t-\tau)} \right) d\tau \\
&= \int_T \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_\ell b_k e^{j\omega_0 \ell \tau} e^{j\omega_0 k(t-\tau)} d\tau \\
&= \int_T \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_\ell b_k e^{j\omega_0 kt} e^{j\omega_0 (\ell-k)\tau} d\tau \\
&= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_\ell b_k e^{j\omega_0 kt} \int_T e^{j\omega_0 (\ell-k)\tau} d\tau \\
&= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_\ell b_k e^{j\omega_0 kt} T \delta(\ell - k) \\
&= \sum_{k=-\infty}^{\infty} a_k b_k e^{j\omega_0 kt} T \\
&= \sum_{k=-\infty}^{\infty} T a_k b_k e^{j\omega_0 kt}. \qquad (5.13)
\end{aligned}$$

In the above simplification, we used the fact that

$$\begin{aligned}
\int_T e^{j(2\pi/T)kt} dt &= \begin{cases} T & k = 0 \\ 0 & \text{otherwise} \end{cases} \\
&= T\delta(k).
\end{aligned}$$

Now, we simply observe that the right-hand side of (5.13) is a Fourier series. Therefore, the Fourier series coefficient sequence c of $x \circledast y$ is given by $c_k = T a_k b_k$. ■

5.5.7 Multiplication

The next property of Fourier series to be considered is the multiplication property, as given below.

Theorem 5.11 (Multiplication)**.** *Let x and y be T-periodic functions given by*

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \textit{and} \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where $\omega_0 = \frac{2\pi}{T}$. Let $z(t) = x(t)y(t)$, where

$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

The sequences a, b, and c are related as

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

(i.e., c is the DT convolution of a and b).

Proof. From the Fourier series analysis equation, we can write

$$c_k = \tfrac{1}{T}\int_T x(t)y(t)e^{-j(2\pi/T)kt}\,dt.$$

Replacing x by its Fourier series representation, we obtain

$$\begin{aligned} c_k &= \tfrac{1}{T}\int_T \left(\sum_{n=-\infty}^{\infty} a_n e^{j(2\pi/T)nt}\right) y(t)e^{-j(2\pi/T)kt}\,dt \\ &= \tfrac{1}{T}\int_T \sum_{n=-\infty}^{\infty} a_n e^{j(2\pi/T)nt} y(t)e^{-j(2\pi/T)kt}\,dt. \end{aligned}$$

Reversing the order of the summation and integration, we have

$$\begin{aligned} c_k &= \tfrac{1}{T}\sum_{n=-\infty}^{\infty}\int_T a_n e^{j(2\pi/T)nt} y(t)e^{-j(2\pi/T)kt}\,dt \\ &= \sum_{n=-\infty}^{\infty} a_n \left(\tfrac{1}{T}\int_T y(t)e^{-j(2\pi/T)(k-n)t}\,dt\right). \end{aligned}$$

Observing that the expression on the preceding line in the large pair of parenthesis is simply the formula for computing the $(k-n)$th Fourier series coefficient of y, we conclude

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}.$$

■

5.5.8 Parseval's Relation

Another important property of Fourier series relates to the energy of functions and sequences, as given by the theorem below.

Theorem 5.12 (Parseval's relation)**.** *A periodic function x and its Fourier series coefficient sequence c satisfy the relationship*

$$\tfrac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

(i.e., the energy in x and the energy in c are equal).

Proof. Let x, y, and z denote T-periodic functions with the Fourier series given by

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \\ y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}, \quad \text{and} \\ z(t) = x(t)y(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \end{aligned}$$

From the multiplication property of Fourier series (i.e., Theorem 5.11), we have

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}. \tag{5.14}$$

Now, let $y(t) = x^*(t)$ so that $z(t) = x(t)x^*(t) = |x(t)|^2$. From the conjugation property of Fourier series (i.e., Theorem 5.9), since $y(t) = x^*(t)$, we know

$$b_k = a^*_{-k}.$$

So, we can rewrite (5.14) as

$$\begin{aligned} c_k &= \sum_{n=-\infty}^{\infty} a_n a^*_{-(k-n)} \\ &= \sum_{n=-\infty}^{\infty} a_n a^*_{n-k}. \end{aligned} \tag{5.15}$$

From the Fourier series analysis equation, we have

$$c_k = \tfrac{1}{T} \int_T |x(t)|^2 e^{-jk\omega_0 t} dt. \tag{5.16}$$

Equating (5.15) and (5.16), we obtain

$$\tfrac{1}{T} \int_T |x(t)|^2 e^{-jk\omega_0 t} dt = \sum_{n=-\infty}^{\infty} a_n a^*_{n-k}.$$

Letting $k = 0$ in the preceding equation yields

$$\tfrac{1}{T} \int_T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} a_n a^*_n = \sum_{n=-\infty}^{\infty} |a_n|^2. \qquad \blacksquare$$

The above theorem is simply stating that the amount of energy in x (i.e., $\frac{1}{T}\int_T |x(t)|^2 dt$) and the amount of energy in the Fourier series coefficient sequence c (i.e., $\sum_{k=-\infty}^{\infty} |c_k|^2$) are equal. In other words, the transformation between a function and its Fourier series coefficient sequence preserves energy.

5.5.9 Even and Odd Symmetry

Fourier series preserves signal symmetry. In other words, we have the result below.

Theorem 5.13 (Even/odd symmetry)**.** *For a T-periodic function x with Fourier series coefficient sequence c, the following properties hold:*

x is even if and only if c is even; and
x is odd if and only if c is odd.

Proof. The proof is left as an exercise for the reader in Exercise 5.3. $\blacksquare$

In other words, the above theorem states that the even/odd symmetry properties of x and c always match (i.e., Fourier series preserve symmetry).

5.5.10 Real Functions

Consider the Fourier series representation of the periodic function x given by (5.1). In the most general case, x is a complex-valued function, but let us now suppose that x is real valued. In the case of real-valued functions, an important relationship exists between the Fourier series coefficients c_k and c_{-k} as given by the theorem below.

Theorem 5.14 (Fourier series of real-valued function)**.** *Let x be a periodic function with Fourier series coefficient sequence c. The function x is real valued if and only if*

$$c_k = c^*_{-k} \text{ for all } k \tag{5.17}$$

(i.e., c is conjugate symmetric).

Proof. Suppose that we can represent x in the form of a Fourier series, as given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \tag{5.18}$$

Taking the complex conjugate of both sides of the preceding equation, we obtain

$$\begin{aligned} x^*(t) &= \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} \left(c_k e^{jk\omega_0 t} \right)^* \\ &= \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}. \end{aligned}$$

Replacing k by $-k$ in the summation of the preceding equation, we obtain

$$x^*(t) = \sum_{k=-\infty}^{\infty} c_{-k}^* e^{jk\omega_0 t}. \tag{5.19}$$

Suppose now that x is real valued. Then, $x^* = x$ and the right-hand sides of (5.18) and (5.19) must be equal, implying that $c_k = c_{-k}^*$ for all k.

Suppose now that $c_k = c_{-k}^*$ for all k. Then, the right-hand sides of (5.18) and (5.19) must be equal, implying that $x^* = x$ (i.e., x is real valued). ■

Using the relationship in (5.17), we can derive two alternative forms of the Fourier series for the case of real-valued functions. We begin by rewriting (5.1) in a slightly different form. In particular, we rearrange the summation to obtain

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} \right].$$

Substituting $c_k = c_{-k}^*$ from (5.17), we obtain

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[c_k e^{jk\omega_0 t} + c_k^* e^{-jk\omega_0 t} \right].$$

Now, we observe that the two terms inside the summation are complex conjugates of each other. So, we can rewrite the equation as

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}(c_k e^{jk\omega_0 t}). \tag{5.20}$$

Let us now rewrite c_k in polar form as

$$c_k = |c_k| e^{j\theta_k},$$

where θ_k is real (i.e., $\theta_k = \arg c_k$). Substituting this expression for c_k into (5.20) yields

$$\begin{aligned} x(t) &= c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left[|c_k| e^{j(k\omega_0 t + \theta_k)} \right] \\ &= c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left(|c_k| \left[\cos(k\omega_0 t + \theta_k) + j\sin(k\omega_0 t + \theta_k) \right] \right) \\ &= c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left[|c_k| \cos(k\omega_0 t + \theta_k) + j|c_k| \sin(k\omega_0 t + \theta_k) \right]. \end{aligned}$$

Finally, further simplification yields

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k| \cos(k\omega_0 t + \theta_k)$$

(where $\theta_k = \arg c_k$). This is known as the **combined trigonometric form** of a Fourier series.

A second alternative form of the Fourier series can be obtained by expressing c_k in Cartesian form as

$$c_k = \tfrac{1}{2}(a_k - jb_k).$$

where a_k and b_k are real. Substituting this expression for c_k into (5.20) from earlier yields

$$\begin{aligned}
x(t) &= c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left[\tfrac{1}{2}(a_k - jb_k)e^{jk\omega_0 t}\right] \\
&= c_0 + \sum_{k=1}^{\infty} \operatorname{Re}\left[(a_k - jb_k)\left(\cos[k\omega_0 t] + j\sin[k\omega_0 t]\right)\right] \\
&= c_0 + \sum_{k=1}^{\infty} \operatorname{Re}\left[a_k\cos(k\omega_0 t) + ja_k\sin(k\omega_0 t) - jb_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)\right].
\end{aligned}$$

Further simplification yields

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)\right]$$

(where $a_k = \operatorname{Re}(2c_k)$ and $b_k = -\operatorname{Im}(2c_k)$). This is known as the **trigonometric form** of a Fourier series.

By comparing the various forms of the Fourier series introduced above, we can see that the quantities c_k, a_k, b_k, and θ_k are related as

$$2c_k = a_k - jb_k \quad \text{and} \quad c_k = |c_k|e^{j\theta_k}.$$

(Recall that a_k, b_k, and θ_k are real and c_k is complex.) Note that each of the trigonometric and combined-trigonometric forms of Fourier series only involve real quantities, whereas the the exponential form involves some complex quantities. For this reason, the trigonometric and combined-trigonometric forms may sometimes be preferred when dealing with Fourier series of real-valued functions.

As noted earlier in Theorem 5.14, the Fourier series of a real-valued function has a special structure. In particular, a function x is real valued if and only if its Fourier series coefficient sequence c satisfies $c_k = c_{-k}^*$ for all k (i.e., c is conjugate symmetric). Thus, for a real-valued function, the negative-indexed Fourier series coefficients are *redundant*, as they are completely determined by the nonnegative-indexed coefficients. From properties of complex numbers, one can show that

$$c_k = c_{-k}^* \text{ for all } k$$

is equivalent to

$$|c_k| = |c_{-k}| \text{ for all } k \quad \text{and} \quad \arg c_k = -\arg c_{-k} \text{ for all } k$$

(i.e., $|c_k|$ is *even* and $\arg c_k$ is *odd*). Note that x being real valued does *not* necessarily imply that c is real.

5.6 Fourier Series and Frequency Spectra

The Fourier series represents a function in terms of harmonically-related complex sinusoids. In this sense, the Fourier series captures information about the frequency content of a function. Each complex sinusoid is associated with a particular frequency (which is some integer multiple of the fundamental frequency). So, these coefficients indicate at which frequencies the information/energy in a function is concentrated. For example, if only the Fourier series

Table 5.1: Properties of CT Fourier series

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha a_k + \beta b_k$
Translation	$x(t - t_0)$	$e^{-jk(2\pi/T)t_0} a_k$
Modulation	$e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Reflection	$x(-t)$	a_{-k}
Conjugation	$x^*(t)$	a^*_{-k}
Periodic Convolution	$x \circledast y(t)$	$Ta_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{n=-\infty}^{\infty} a_n b_{k-n}$

Property	
Parseval's relation	$\frac{1}{T}\int_T \lvert x(t)\rvert^2 \, dt = \sum_{k=-\infty}^{\infty} \lvert a_k \rvert^2$
Even symmetry	x is even $\Leftrightarrow$ a even
Odd symmetry	x is odd $\Leftrightarrow$ a odd
Real	x is real $\Leftrightarrow$ a is conjugate symmetric

coefficients for the low-order harmonics have large magnitudes, then the function is mostly associated with low frequencies. On the other hand, if a function has many large magnitude coefficients for high-order harmonics, then the function has a considerable amount of information/energy associated with high frequencies. In this way, the Fourier series representation provides a means for measuring the frequency content of a function. The distribution of the energy/information in a function over different frequencies is referred to as the **frequency spectrum** of the function.

To gain further insight into the role played by the Fourier series coefficients c_k in the context of the frequency spectrum of the function x, it is helpful to write the Fourier series with the c_k expressed in polar form as follows:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} |c_k| e^{j \arg c_k} e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} |c_k| e^{j(k\omega_0 t + \arg c_k)}. \end{aligned}$$

Clearly (from the last line of the above equation), the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $k\omega_0$ that has had its amplitude scaled by a factor of $|c_k|$ and has been time-shifted by an amount that depends on $\arg c_k$. For a given k, the larger $|c_k|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{jk\omega_0 t}$, and therefore the larger the contribution the kth term (which is associated with frequency $k\omega_0$) will make to the overall summation. In this way, we can use $|c_k|$ as a measure of how much information a function x has at the frequency $k\omega_0$.

Various ways exist to illustrate the frequency spectrum of a function. Typically, we plot the Fourier series coefficients as a function of frequency. Since, in general, the Fourier series coefficients are complex valued, we usually display this information using two plots. One plot shows the magnitude of the coefficients as a function of frequency. This is called the **magnitude spectrum**. The other plot shows the arguments of the coefficients as a function of frequency. In this context, the argument is referred to as the phase, and the plot is called the **phase spectrum**.

Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, we only have values to plot for these particular frequencies. In other words, the frequency spectrum is discrete in the independent variable (i.e., frequency). For this reason, we use a stem graph to plot such functions. Due to the general appearance of the graph (i.e., a number of vertical lines at various frequencies) we refer to such spectra as **line spectra**.

Recall that, for a real function x, the Fourier series coefficient sequence c is conjugate symmetric (i.e., $c_k = c^*_{-k}$ for all k). This, however, implies that $|c_k| = |c_{-k}|$ and $\arg c_k = -\arg c_{-k}$. Since $|c_k| = |c_{-k}|$, the magnitude spectrum of a real function is always even. Similarly, since $\arg c_k = -\arg c_{-k}$, the phase spectrum of a real function is always odd.

Example 5.7. Consider the 1-periodic function x with the Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt},$$

where

$$c_k = \begin{cases} \frac{j}{20} & k=-7 \\ \frac{2j}{20} & k=-5 \\ \frac{4j}{20} & k=-3 \\ \frac{13j}{20} & k=-1 \\ -\frac{13j}{20} & k=1 \\ -\frac{4j}{20} & k=3 \\ -\frac{2j}{20} & k=5 \\ -\frac{j}{20} & k=7 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we have

$$x(t) = \tfrac{j}{20}e^{-j14\pi t} + \tfrac{2j}{20}e^{-j10\pi t} + \tfrac{4j}{20}e^{-j6\pi t} + \tfrac{13j}{20}e^{-j2\pi t} \\ - \tfrac{13j}{20}e^{j2\pi t} - \tfrac{4j}{20}e^{j6\pi t} - \tfrac{2j}{20}e^{j10\pi t} - \tfrac{j}{20}e^{j14\pi t}.$$

Clearly, this Fourier series has 8 (nonzero) terms (i.e., 8 nonzero Fourier-series coefficients). Moreover, since c is conjugate symmetric, x is real. In particular, from Euler's relation, we have

$$x(t) = \tfrac{13}{10}\sin(2\pi t) + \tfrac{4}{10}\sin(6\pi t) + \tfrac{2}{10}\sin(10\pi t) + \tfrac{1}{10}\sin(14\pi t).$$

A plot of x is shown in Figure 5.7(a).

Suppose that we want to approximate x by keeping only 4 of the 8 terms in the Fourier series. Furthermore, we want to do this in a manner that yields the most faithful reproduction of x (i.e., minimizes error). Since terms with larger magnitude Fourier series coefficients make a more significant contribution to the Fourier series sum, it makes sense intuitively that these are the terms that we should select. Thus, the best approximation would be obtained by choosing the terms for $k \in \{-3,-1,1,3\}$. A plot of the resulting approximation is shown in Figure 5.7(b). Clearly, this approximation faithfully reproduces the general trends in the original function x. In fact, it can be shown that this choice is the one that minimizes mean-squared error.

Suppose instead that we chose the 4 terms with the smallest magnitude (nonzero) Fourier series coefficients. Intuitively, this choice should be a very poor one, since it keeps the terms that make the least significant contribution to the Fourier-series sum. The terms with the smallest magnitude (nonzero) coefficients correspond to the terms for $k \in \{-7,-5,5,7\}$. If these terms are kept, we obtain the approximation shown in Figure 5.7(c). Clearly, this approximation is a very poor one, failing to capture even the general trends in the original function x.

This example helps to illustrate that the most dominant terms in the Fourier series sum are the ones with the largest magnitude coefficients. In this sense, the Fourier series coefficient magnitudes can be used to quantify how much each term in the Fourier series contributes to the overall Fourier series sum. ■

Example 5.8. The periodic square wave x in Example 5.1 has fundamental period T, fundamental frequency ω_0, and the Fourier series coefficient sequence given by

$$c_k = \begin{cases} \frac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even,} \end{cases}$$

where A is a positive constant. Find and plot the magnitude and phase spectra of x. Determine at what frequency (or frequencies) x has the most information.

Solution. First, we compute the magnitude spectrum of x, which is given by $|c_k|$. We have

$$\begin{aligned} |c_k| &= \begin{cases} \left|\frac{-j2A}{\pi k}\right| & k \text{ odd} \\ 0 & k \text{ even} \end{cases} \\ &= \begin{cases} \frac{2A}{\pi|k|} & k \text{ odd} \\ 0 & k \text{ even.} \end{cases} \end{aligned}$$

Next, we compute the phase spectrum of x, which is given by $\arg c_k$. Using the fact that $\arg 0 = 0$ and $\arg \frac{-j2A}{\pi k} = -\frac{\pi}{2}\operatorname{sgn} k$, we have

$$\begin{aligned} \arg c_k &= \begin{cases} \arg \frac{-j2A}{\pi k} & k \text{ odd} \\ \arg 0 & k \text{ even} \end{cases} \\ &= \begin{cases} \frac{\pi}{2} & k \text{ odd}, k < 0 \\ -\frac{\pi}{2} & k \text{ odd}, k > 0 \\ 0 & k \text{ even} \end{cases} \\ &= \begin{cases} -\frac{\pi}{2}\operatorname{sgn} k & k \text{ odd} \\ 0 & k \text{ even.} \end{cases} \end{aligned}$$

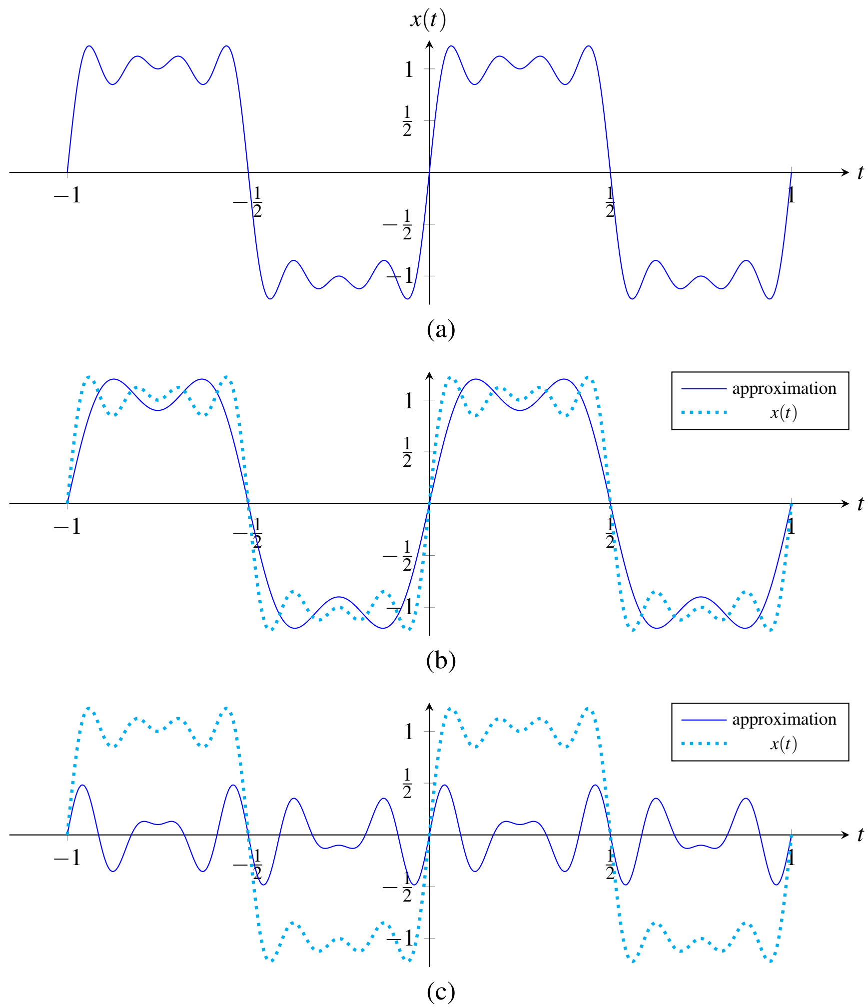


Figure 5.7: Approximation of the Fourier series for the function x. (a) The function x. (b) The approximation obtained by taking the 4 terms in the Fourier series with the largest magnitude coefficients. (c) The approximation obtained by taking the 4 terms in the Fourier series with the smallest magnitude (nonzero) coefficients.

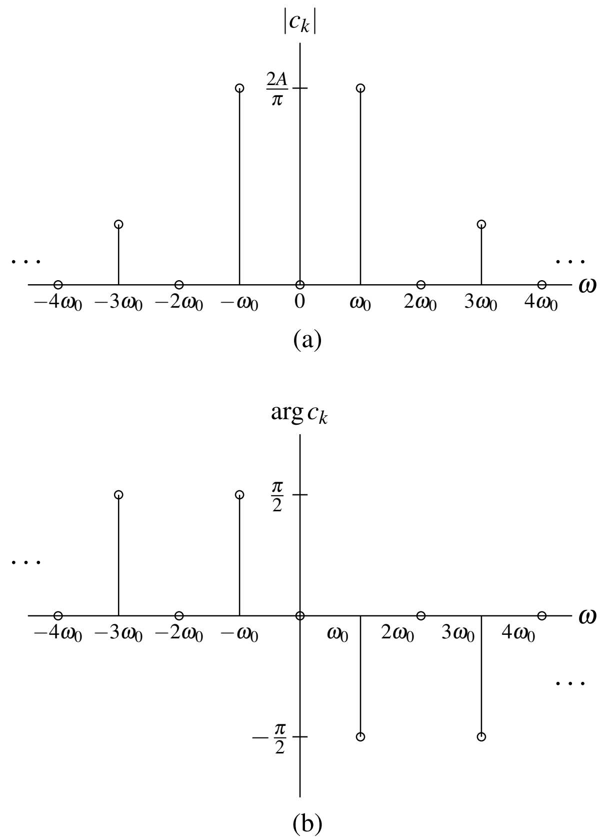


Figure 5.8: Frequency spectrum of the periodic square wave. (a) Magnitude spectrum and (b) phase spectrum.

The magnitude and phase spectra of x are plotted in Figures 5.8(a) and (b), respectively. Note that the magnitude spectrum is an even function, while the phase spectrum is an odd function. This is what we should expect, since x is real. Since $|c_k|$ is largest for $k = -1$ and $k = 1$, the function x has the most information at frequencies $-\omega_0$ and ω_0. ■

Example 5.9. The periodic impulse train x in Example 5.2 has fundamental period T, fundamental frequency ω_0, and the Fourier series coefficient sequence c given by

$$c_k = \frac{A}{T},$$

where A is a positive real constant. Find and plot the magnitude and phase spectra of x.

Solution. We have $|c_k| = \frac{A}{T}$ and $\arg c_k = 0$. The magnitude and phase spectra of x are plotted in Figures 5.9(a) and (b), respectively. ■

5.7 Fourier Series and LTI Systems

From earlier, in Theorem 4.12, we know that complex exponentials are eigenfunctions of LTI systems. Since complex sinusoids are a special case of complex exponentials, it follows that complex sinusoids are also eigenfunctions of LTI systems. In other words, we have the result below.

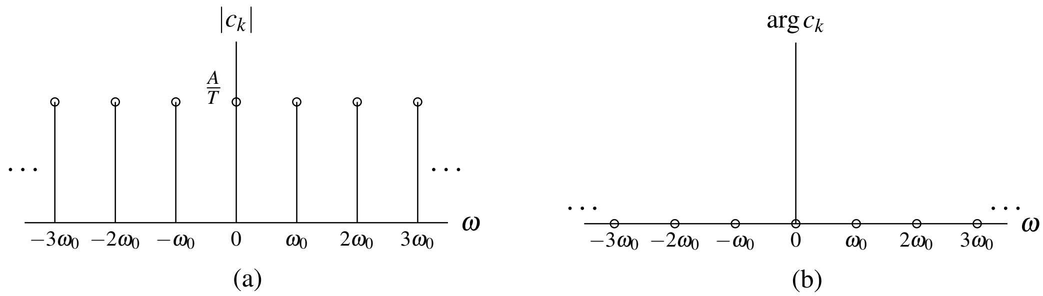


Figure 5.9: Frequency spectrum for the periodic impulse train. (a) Magnitude spectrum and (b) phase spectrum.

Corollary 5.1. *For an arbitrary LTI system $\mathcal{H}$ with impulse response h and a function of the form $x(t) = e^{j\omega t}$, where ω is an arbitrary real constant (i.e., x is an arbitrary complex sinusoid), the following holds:*

$$\mathcal{H}x(t) = H(\omega)e^{j\omega t},$$

where

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt. \tag{5.21}$$

That is, x is an eigenfunction of $\mathcal{H}$ with the corresponding eigenvalue $H(\omega)$.

The preceding result (i.e., Corollary 5.1) is simply a special case of Theorem 4.12 for $s = j\omega$. Note that, in order to obtain more convenient notation, the function H in Corollary 5.1 is defined differently from the function H in Theorem 4.12. In particular, letting H_F and H_L denote the function H that appears in each of Corollary 5.1 and Theorem 4.12, respectively, we have the relationship $H_\mathsf{F}(\omega) = H_\mathsf{L}(j\omega)$.

As a matter of terminology, the function H in (5.21) is referred to as the **frequency response** of the system $\mathcal{H}$. The frequency response completely characterizes the behavior of a LTI system. Consequently, the frequency response is often useful when working with LTI systems. As it turns out, an integral of the form appearing on the right-hand side of (5.21) is of great importance, as it defines what is called the (CT) Fourier transform. We will study the (CT) Fourier transform in great depth later in Chapter 6.

Let us now consider an application of eigenfunctions. Since convolution can often be quite painful to handle at the best of times, let us exploit eigenfunctions in order to devise a means to avoid having to deal with convolution directly in certain circumstances. Suppose now that we have a periodic function x represented in terms of a Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Using (5.21) and the superposition property, we can determine the system response y to the input x as follows:

$$\begin{aligned}
y(t) &= \mathcal{H}x(t) \\
&= \mathcal{H}\left\{\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right\}(t) \\
&= \sum_{k=-\infty}^{\infty} \mathcal{H}\left\{c_k e^{jk\omega_0 t}\right\}(t) \\
&= \sum_{k=-\infty}^{\infty} c_k \mathcal{H}\left\{e^{jk\omega_0 t}\right\}(t) \\
&= \sum_{k=-\infty}^{\infty} c_k H(k\omega_0)e^{jk\omega_0 t}.
\end{aligned}$$

Therefore, we can view a LTI system as an entity that operates on the individual coefficients of a Fourier series. In particular, the system forms its output by multiplying each Fourier series coefficient by the value of the frequency response function at the frequency to which the Fourier series coefficient corresponds. In other words, if

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$$

then

$$y(t) \overset{\text{CTFS}}{\longleftrightarrow} H(k\omega_0)c_k.$$

Example 5.10. Consider a LTI system with the frequency response

$$H(\omega) = e^{-j\omega/4}.$$

Find the response y of the system to the input x, where

$$x(t) = \tfrac{1}{2}\cos(2\pi t).$$

Solution. To begin, we rewrite x as

$$x(t) = \tfrac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}).$$

Thus, the Fourier series for x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$ and

$$c_k = \begin{cases} \frac{1}{4} & k \in \{-1, 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we can write

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t} \\ &= c_{-1}H(-\omega_0)e^{-j\omega_0 t} + c_1 H(\omega_0) e^{j\omega_0 t} \\ &= \tfrac{1}{4}H(-2\pi)e^{-j2\pi t} + \tfrac{1}{4}H(2\pi)e^{j2\pi t} \\ &= \tfrac{1}{4}e^{j\pi/2}e^{-j2\pi t} + \tfrac{1}{4}e^{-j\pi/2}e^{j2\pi t} \\ &= \tfrac{1}{4}[e^{-j(2\pi t-\pi/2)} + e^{j(2\pi t-\pi/2)}] \\ &= \tfrac{1}{4}\left(2\cos(2\pi t - \tfrac{\pi}{2})\right) \\ &= \tfrac{1}{2}\cos(2\pi t - \tfrac{\pi}{2}) \\ &= \tfrac{1}{2}\cos\left(2\pi\left[t - \tfrac{1}{4}\right]\right). \end{aligned}$$

Observe that $y(t) = x\left(t - \frac{1}{4}\right)$. This is not a coincidence because, as it turns out, a LTI system with the frequency response $H(\omega) = e^{-\omega/4}$ is an ideal delay of $\frac{1}{4}$ (i.e., a system that performs a time shift of $\frac{1}{4}$). ■

5.8 Filtering

In some applications, we want to change the relative amplitude of the frequency components of a function or possibly eliminate some frequency components altogether. This process of modifying the frequency components of a function is referred to as **filtering**. Various types of filters exist. Frequency-selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies. Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

An **ideal lowpass filter** eliminates all frequency components with a frequency greater in magnitude than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the cutoff frequency. A plot of this frequency response is shown in Figure 5.10(a).

The **ideal highpass filter** eliminates all frequency components with a frequency lesser in magnitude than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the cutoff frequency. A plot of this frequency response is shown in Figure 5.10(b).

An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

where the limits of the passband are ω_{c1} and ω_{c2}. A plot of this frequency response is shown in Figure 5.10(c).

Example 5.11 (Lowpass filtering)**.** Suppose that we have a LTI system with input x, output y, and frequency response H, where

$$H(\omega) = \begin{cases} 1 & |\omega| \leq 3\pi \\ 0 & \text{otherwise.} \end{cases}$$

Further, suppose that the input x is the periodic function

$$x(t) = 1 + 2\cos(2\pi t) + \cos(4\pi t) + \tfrac{1}{2}\cos(6\pi t).$$

(a) Find the Fourier series representation of x. (b) Use this representation in order to find the response y of the system to the input x. (c) Plot the frequency spectra of x and y.

Solution. (a) We begin by finding the Fourier series representation of x. Using Euler's formula, we can re-express x as

$$\begin{aligned}
x(t) &= 1 + 2\cos(2\pi t) + \cos(4\pi t) + \tfrac{1}{2}\cos(6\pi t) \\
&= 1 + 2\left[\tfrac{1}{2}(e^{j2\pi t} + e^{-j2\pi t})\right] + \left[\tfrac{1}{2}(e^{j4\pi t} + e^{-j4\pi t})\right] + \tfrac{1}{2}\left[\tfrac{1}{2}(e^{j6\pi t} + e^{-j6\pi t})\right] \\
&= 1 + e^{j2\pi t} + e^{-j2\pi t} + \tfrac{1}{2}[e^{j4\pi t} + e^{-j4\pi t}] + \tfrac{1}{4}[e^{j6\pi t} + e^{-j6\pi t}] \\
&= \tfrac{1}{4}e^{-j6\pi t} + \tfrac{1}{2}e^{-j4\pi t} + e^{-j2\pi t} + 1 + e^{j2\pi t} + \tfrac{1}{2}e^{j4\pi t} + \tfrac{1}{4}e^{j6\pi t} \\
&= \tfrac{1}{4}e^{j(-3)(2\pi)t} + \tfrac{1}{2}e^{j(-2)(2\pi)t} + e^{j(-1)(2\pi)t} + e^{j(0)(2\pi)t} + e^{j(1)(2\pi)t} + \tfrac{1}{2}e^{j(2)(2\pi)t} + \tfrac{1}{4}e^{j(3)(2\pi)t}.
\end{aligned}$$

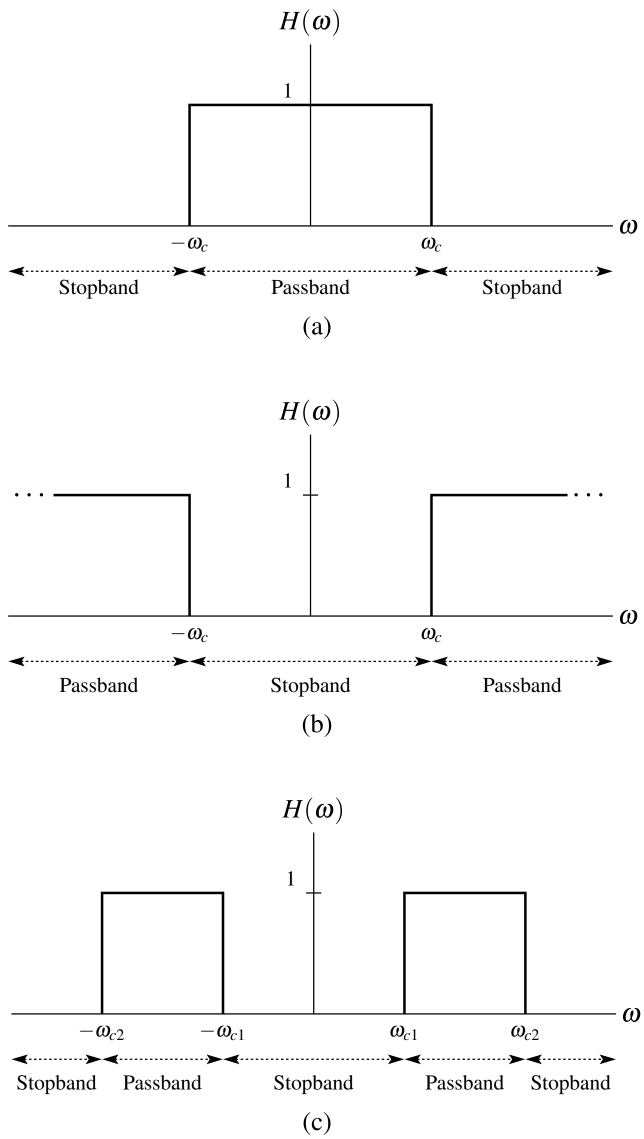


Figure 5.10: Frequency responses of (a) ideal lowpass, (b) ideal highpass, and (c) ideal bandpass filters.

From the last line of the preceding equation, we deduce that $\omega_0 = 2\pi$, since a larger value for ω_0 would imply that some Fourier series coefficient indices are noninteger, which clearly makes no sense. Thus, we have that the Fourier series of x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$ and

$$a_k = \begin{cases} 1 & k \in \{-1,0,1\} \\ \frac{1}{2} & k \in \{-2,2\} \\ \frac{1}{4} & k \in \{-3,3\} \\ 0 & \text{otherwise.} \end{cases}$$

(b) Since the system is LTI, we know that the output y has the form

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where

$$b_k = a_k H(k\omega_0).$$

Using the results from above, we can calculate the b_k as follows:

$$\begin{aligned} b_0 &= a_0 H([0][2\pi]) = 1(1) = 1, \\ b_1 &= a_1 H([1][2\pi]) = 1(1) = 1, \\ b_{-1} &= a_{-1} H([-1][2\pi]) = 1(1) = 1, \\ b_2 &= a_2 H([2][2\pi]) = \tfrac{1}{2}(0) = 0, \\ b_{-2} &= a_{-2} H([-2][2\pi]) = \tfrac{1}{2}(0) = 0, \\ b_3 &= a_3 H([3][2\pi]) = \tfrac{1}{4}(0) = 0, \quad \text{and} \\ b_{-3} &= a_{-3} H([-3][2\pi]) = \tfrac{1}{4}(0) = 0. \end{aligned}$$

Thus, we have

$$b_k = \begin{cases} 1 & k \in \{-1,0,1\} \\ 0 & \text{otherwise.} \end{cases}$$

(c) Lastly, we plot the frequency spectra of x and y in Figures 5.11(a) and (b), respectively. The frequency response H is superimposed on the plot of the frequency spectrum of x for illustrative purposes. ■

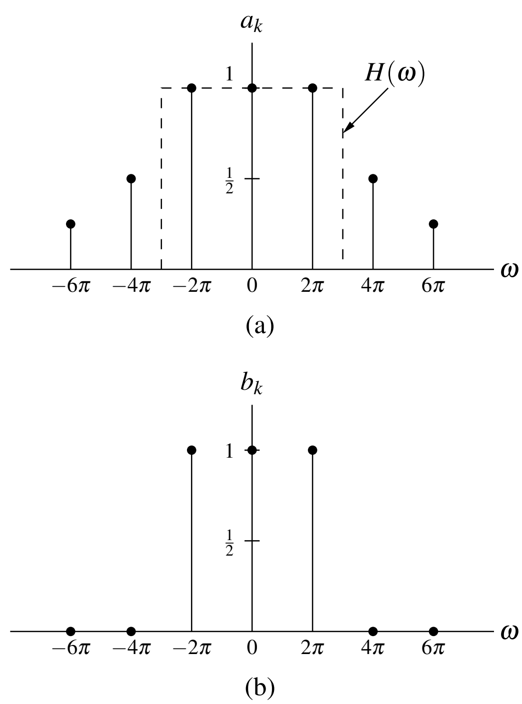


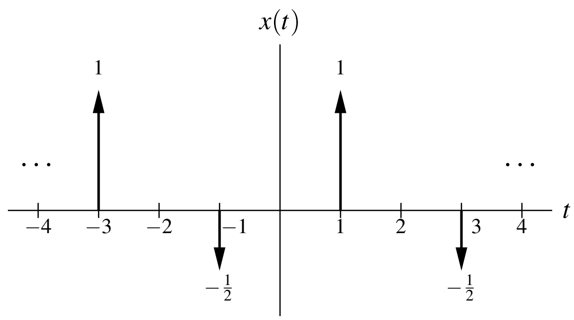
Figure 5.11: Frequency spectra of the (a) input function x and (b) output function y.

5.9 Exercises

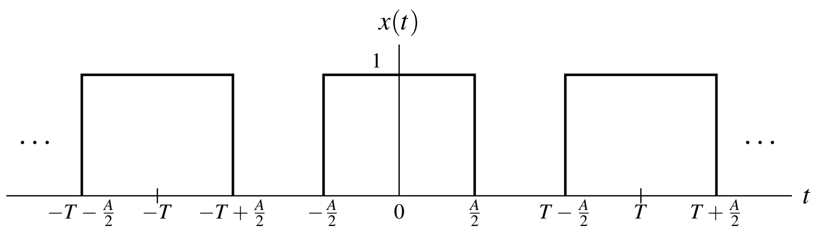
5.9.1 Exercises Without Answer Key

5.1 For each case below, find the Fourier series representation (in complex exponential form) of the function x, explicitly identifying the fundamental period of x and the Fourier series coefficient sequence c.
(a) $x(t) = 1 + \cos(\pi t) + \sin^2(\pi t)$;
(b) $x(t) = \cos(4t)\sin(t)$; and
(c) $x(t) = |\sin(2\pi t)|$. [Hint: $\int e^{ax}\sin(bx)dx = \frac{e^{ax}[a\sin(bx) - b\cos(bx)]}{a^2+b^2} + C$, where a and b are arbitrary complex and nonzero real constants, respectively.]

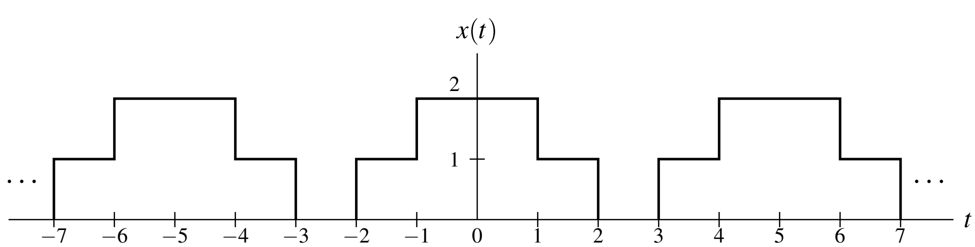
5.2 For each of the periodic functions shown in the figures below, find the corresponding Fourier series coefficient sequence.



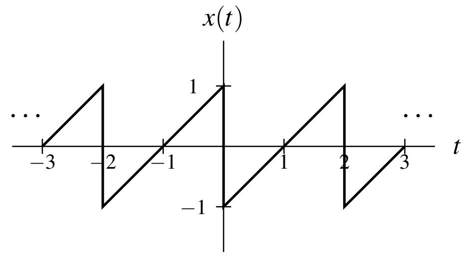
(a)



(b)



(c)



(d)

5.3 Show that, for a complex-valued periodic function x with the Fourier series coefficient sequence c:
(a) x is even if and only if c is even; and
(b) x is odd if and only if c is odd.

5.4 Let x be a T-periodic function x with the Fourier-series coefficient sequence c. Determine the Fourier series coefficient sequence d of the function $y(t) = \mathcal{D}x(t)$, where $\mathcal{D}$ denotes the derivative operator.

5.5 Let x be a periodic function with the Fourier series coefficient sequence c given by

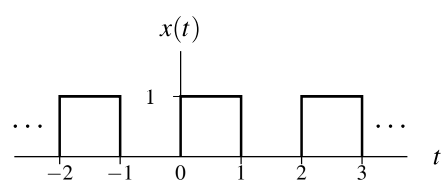
$$c_k = \begin{cases} 1 & k = 0 \\ j\left(\frac{1}{2}\right)^{|k|} & \text{otherwise.} \end{cases}$$

Using Fourier series properties as appropriate, determine if each of the following assertions is true:
(a) x is real;
(b) x is even;
(c) $\frac{d}{dt}x(t)$ is even. [Hint: Consider Exercise 5.4 first.]

5.6 A periodic function x with period T and Fourier series coefficient sequence c is said to be odd harmonic if $c_k = 0$ for all even k.
(a) Show that if x is odd harmonic, then $x(t) = -x(t - \frac{T}{2})$ for all t.
(b) Show that if $x(t) = -x(t - \frac{T}{2})$ for all t, then x is odd harmonic.

5.7 Let x be a periodic function with fundamental period T and Fourier series coefficient sequence c. Find the Fourier series coefficient sequence c' of each of the following functions x' in terms of c:
(a) $x' = \text{Even}(x)$
(b) $x' = \text{Re}(x)$.

5.8 Find the Fourier series coefficient sequence c of the periodic function x shown in the figure below. Plot the frequency spectrum of x, including the first five harmonics.



5.9 Consider a LTI system with frequency response

$$H(\omega) = \begin{cases} 1 & |\omega| \geq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Using frequency-domain methods, find the output y of the system if the input x is given by

$$x(t) = 1 + 2\cos(2t) + 2\cos(4t) + \tfrac{1}{2}\cos(6t).$$

5.9.2 Exercises With Answer Key

5.10 Find the Fourier series coefficient sequence c of each periodic function x given below with fundamental period T.
(a) $x(t) = e^{-t}$ for $-1 \leq t < 1$ and $T = 2$;
(b) $x(t) = \operatorname{rect}(t - \frac{3}{2}) - \operatorname{rect}(t + \frac{3}{2})$ for $-\frac{5}{2} \leq t < \frac{5}{2}$ and $T = 5$; and
(c) $x(t) = e^{-2|t|}$ for $-2 \leq t < 2$ and $T = 4$.

Short Answer. (a) $c_k = \frac{(-1)^k(e-e^{-1})}{j2\pi k+2}$; (b) $c_k = \begin{cases} \frac{1}{j\pi k}\left(\cos\left(\frac{2\pi k}{5}\right) - \cos\left(\frac{4\pi k}{5}\right)\right) & k \neq 0 \\ 0 & k = 0; \end{cases}$ (c) $c_k = \frac{e^{-4}(-1)^k - 1}{(2-jk\pi/2)(-2-jk\pi/2)}$

5.10 MATLAB Exercises

5.101 Consider the periodic function x shown in Figure B of Exercise 5.2, where $T = 1$ and $A = \frac{1}{2}$. We can show that x has the Fourier series representation

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

where $c_k = \frac{1}{2}\operatorname{sinc}\left(\frac{\pi k}{2}\right)$ and $\omega_0 = 2\pi$. Let $\hat{x}_N(t)$ denote the above infinite series truncated after the Nth harmonic component. That is,

$$\hat{x}_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

(a) Use MATLAB to plot $\hat{x}_N(t)$ for $N = 1, 5, 10, 50, 100$. You should see that as N increases, $\hat{x}_N$ converges to x. [Hint: You may find the `sym`, `symsum`, `subs`, and `ezplot` functions useful for this problem. Note that the MATLAB `sinc` function does not compute the sinc function as defined herein. Instead, the MATLAB `sinc` function computes the normalized sinc function as defined by (3.21).]
(b) By examining the graphs obtained in part (a), answer the following: As $N \to \infty$, does $\hat{x}_N$ converge to x uniformly (i.e., at the same rate everywhere)? If not, where is the rate of convergence slower?
(c) The function x is not continuous everywhere. For example, x has a discontinuity at $\frac{1}{4}$. As $N \to \infty$, to what value does $\hat{x}_N$ appear to converge at this point? Again, deduce your answer from the graphs obtained in part (a).

Chapter 6

Continuous-Time Fourier Transform

6.1 Introduction

The (CT) Fourier series provides an extremely useful representation for periodic functions. Often, however, we need to deal with functions that are not periodic. A more general tool than the Fourier series is needed in this case. In this chapter, we will introduce a tool for representing arbitrary (i.e., possibly aperiodic) functions, known as the Fourier transform.

6.2 Development of the Continuous-Time Fourier Transform for Aperiodic Functions

As demonstrated earlier, the Fourier series is an extremely useful function representation. Unfortunately, this representation can only be used for periodic functions, since a Fourier series is inherently periodic. Many functions, however, are not periodic. Therefore, one might wonder if we can somehow use the Fourier series to develop a representation for aperiodic functions. As it turns out, this is possible. In order to understand why, we must make the following key observation. An aperiodic function can be viewed as a periodic function with a period of infinity. By viewing an aperiodic function as this limiting case of a periodic function where the period is infinite, we can use the Fourier series to develop a more general function representation that can be used in the aperiodic case. (In what follows, our development of the Fourier transform is not completely rigorous, as we assume that various integrals, summations, and limits converge. Such assumptions are not valid in all cases. Our development is mathematically sound, however, provided that the Fourier transform of the function being considered exists.)

Suppose that we have an aperiodic function x. From x, let us define the function x_T as

$$x_T(t) = \begin{cases} x(t) & -\frac{T}{2} \leq t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

In other words, $x_T(t)$ is identical to $x(t)$ over the interval $-\frac{T}{2} \leq t < \frac{T}{2}$ and zero otherwise. Let us now repeat the portion of $x_T(t)$ for $-\frac{T}{2} \leq t < \frac{T}{2}$ to form a T-periodic function $\tilde{x}$. That is, we define $\tilde{x}$ as

$$\tilde{x}(t) = x_T(t) \ \text{ for } -\tfrac{T}{2} \leq t < \tfrac{T}{2} \quad \text{and} \quad \tilde{x}(t) = \tilde{x}(t+T).$$

In Figures 6.1 and 6.2, we provide illustrative examples of the functions x, x_T, and $\tilde{x}$.

Before proceeding further, we make two important observations that we will use later. First, from the definition of x_T, we have

$$\lim_{T\to\infty} x_T(t) = x(t). \tag{6.2}$$

Second, from the definition of x_T and $\tilde{x}$, we have

$$\lim_{T\to\infty} \tilde{x}(t) = x(t). \tag{6.3}$$

Now, let us consider the function $\tilde{x}$. Since $\tilde{x}$ is periodic, we can represent it using a Fourier series as

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{6.4}$$

where

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}\,dt \tag{6.5}$$

and $\omega_0 = \frac{2\pi}{T}$. Since $x_T(t) = \tilde{x}(t)$ for $-\frac{T}{2} \le t < \frac{T}{2}$, we can rewrite (6.5) as

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-jk\omega_0 t}\,dt.$$

Furthermore, since $x_T(t) = 0$ for $t < -\frac{T}{2}$ and $t \ge \frac{T}{2}$, we can rewrite the preceding expression for a_k as

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x_T(t)e^{-jk\omega_0 t}\,dt.$$

Substituting this expression for a_k into (6.4) and rearranging, we obtain the following Fourier series representation for $\tilde{x}$:

$$\begin{aligned}
\tilde{x}(t) &= \sum_{k=-\infty}^{\infty}\left[\frac{1}{T}\int_{-\infty}^{\infty} x_T(\tau)e^{-jk\omega_0\tau}\,d\tau\right]e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty}\left[\frac{\omega_0}{2\pi}\int_{-\infty}^{\infty} x_T(\tau)e^{-jk\omega_0\tau}\,d\tau\right]e^{jk\omega_0 t} \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_T(\tau)e^{-jk\omega_0\tau}\,d\tau\right]e^{jk\omega_0 t}\omega_0.
\end{aligned}$$

Substituting the above expression for $\tilde{x}$ into (6.3), we obtain

$$x(t) = \lim_{T\to\infty}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_T(\tau)e^{-jk\omega_0\tau}\,d\tau\right]e^{jk\omega_0 t}\omega_0. \tag{6.6}$$

Now, we must evaluate the above limit. As $T \to \infty$, we have that $\omega_0 \to 0$. Thus, in the limit above, $k\omega_0$ becomes a continuous variable which we denote as ω, ω_0 becomes the infinitesimal $d\omega$, and the summation becomes an integral. This is illustrated in Figure 6.3. Also, as $T \to \infty$, we have that $x_T \to x$. Combining these results, we can rewrite (6.6) to obtain

$$x(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega,$$

where

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt.$$

Thus, we have found a representation of the aperiodic function x in terms of complex sinusoids at all frequencies. We call this the Fourier transform representation of the function x.

6.3 Generalized Fourier Transform

In the previous section, we used a limiting process involving the analysis and synthesis equations for Fourier series in order to develop a new mathematical tool known as the Fourier transform. As it turns out, many functions of

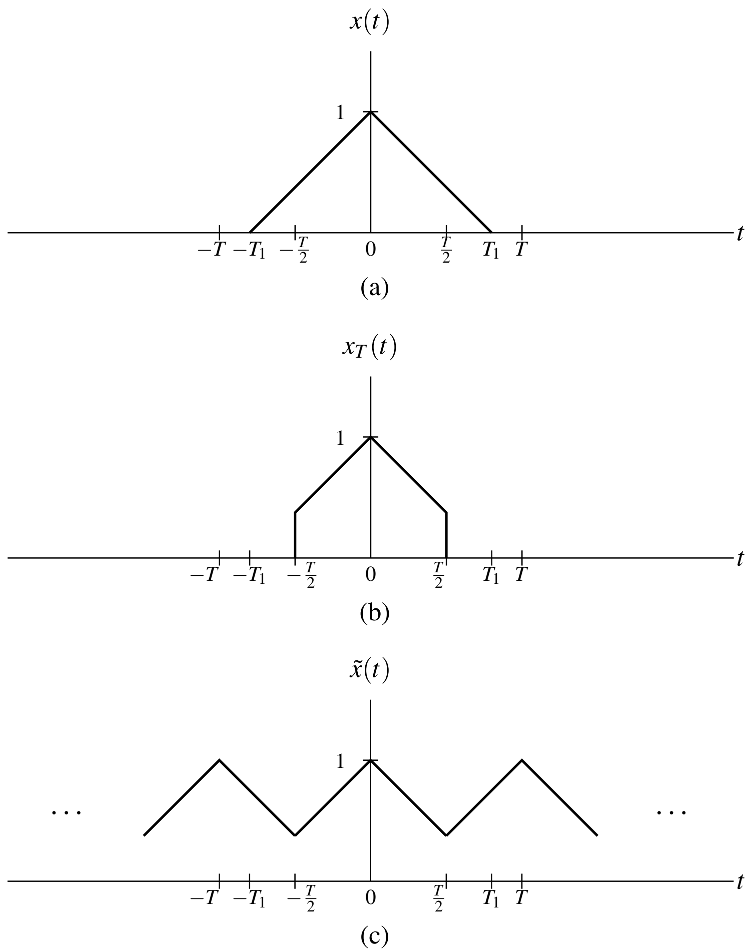


Figure 6.1: An example of the functions used in the derivation of the Fourier transform representation, where $T_1 > \frac{T}{2}$. (a) An aperiodic function x; (b) the function x_T; and (c) the T-periodic function $\tilde{x}$.

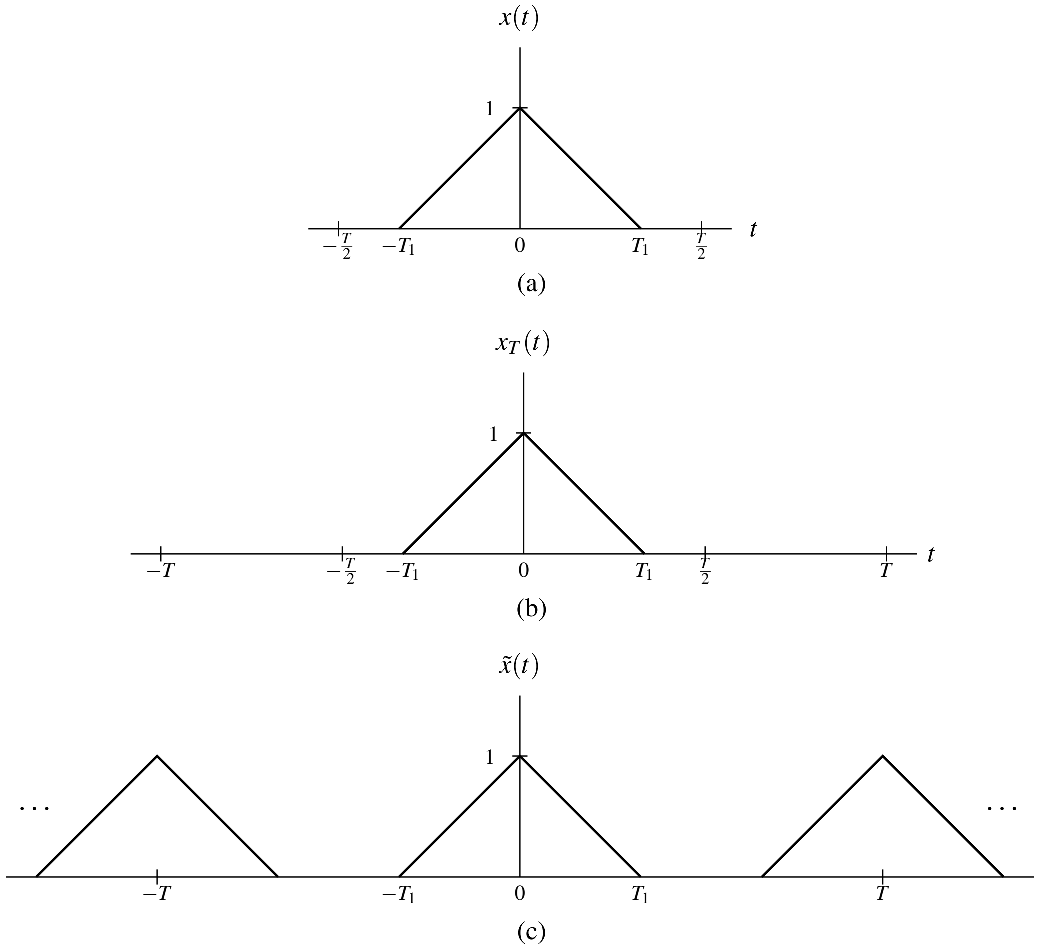


Figure 6.2: An example of the functions used in the derivation of the Fourier transform representation, where $T_1 < \frac{T}{2}$. (a) An aperiodic function x; (b) the function x_T; and (c) the T-periodic function $\tilde{x}$.

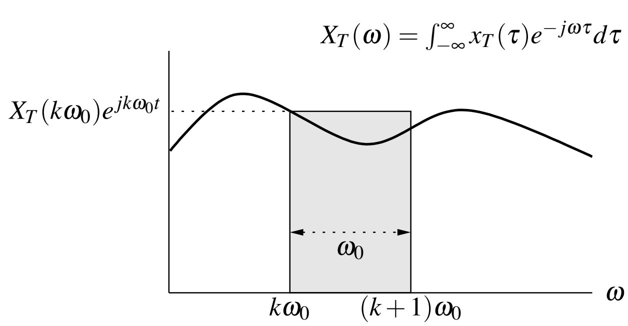


Figure 6.3: Integral obtained in the derivation of the Fourier transform representation.

practical interest do not have a Fourier transform in the sense of the definition developed previously. That is, for a given function x, the Fourier transform integral

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

may fail to converge, in which case the Fourier transform X of x does not exist. For example, the preceding integral does not converge if x is any of the following (as well as many other possibilities):

- a nonzero constant function;
- a periodic function (e.g., a real or complex sinusoid);
- the unit-step function (i.e., u); or
- the signum function (i.e., sgn).

Functions such these are of great practical interest, however. Therefore, it is highly desirable to have a mathematical tool that can handle such functions. This motivates the development of what is called the **generalized Fourier transform**. The generalized Fourier transform exists for periodic functions, nonzero constant functions, and many other types of functions as well. The underlying math associated with the generalized Fourier transform is quite complicated. So, we will not attempt to formally develop the generalized Fourier transform here. Although not entirely correct, one can think of the generalized Fourier transform as being defined by the same formulas as the classical Fourier transform. So, for this and other reasons, we can mostly ignore the distinction between the generalized Fourier transform and classical Fourier transform, and think of them as being one and the same. In what follows, we will avoid making a distinction between the classical Fourier transform and generalized Fourier transform, except in a very small number of places where it is beneficial to do so. The main disadvantage of not formally introducing the generalized Fourier transform is that some results presented later (which actually rely on the use of the generalized Fourier transform) must be accepted on faith since their proof would require formal knowledge of the generalized Fourier transform, which is not introduced herein. As long as the generalized Fourier transform is used, both periodic and aperiodic functions can be handled, and in this sense we have a more general tool than Fourier series (which require periodic functions). Later, when we discuss the Fourier transform of periodic functions, we will implicitly be using the generalized Fourier transform in that context. In fact, in much of what follows, when we speak of the Fourier transform, we are often referring to the generalized Fourier transform.

6.4 Definition of the Continuous-Time Fourier Transform

Earlier, we developed the Fourier transform representation of a function. This representation expresses a function in terms of complex sinusoids at all frequencies. More formally, the **Fourier transform** of the function x, denoted as $\mathcal{F}x$ or X, is defined as

$$\mathcal{F}x(\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt. \tag{6.7}$$

Similarly, the inverse Fourier transform of X, denoted as $\mathcal{F}^{-1}X$ or x, is given by

$$\mathcal{F}^{-1}X(t) = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega. \tag{6.8}$$

We refer to (6.7) as the **Fourier transform analysis equation** and (6.8) as the **Fourier transform synthesis equation**. To denote that a function x has the Fourier transform X, we can write

$$x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega).$$

As a matter of terminology, x and X are said to constitute a **Fourier transform pair**.

Example 6.1 (Fourier transform of the unit-impulse function)**.** Find the Fourier transform X of the function

$$x(t) = A\delta(t - t_0),$$

where A and t_0 are real constants. Then, from this result, write the Fourier transform representation of x.

Solution. From the definition of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} A\delta(t-t_0)e^{-j\omega t}\,dt \\ &= A\int_{-\infty}^{\infty} \delta(t-t_0)e^{-j\omega t}\,dt. \end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the above result to obtain

$$\begin{aligned} X(\omega) &= A\left[e^{-j\omega t}\right]\Big|_{t=t_0} \\ &= Ae^{-j\omega t_0}. \end{aligned}$$

Thus, we have shown that

$$A\delta(t-t_0) \overset{\text{CTFT}}{\longleftrightarrow} Ae^{-j\omega t_0}.$$

From the Fourier transform analysis and synthesis equations, we have that the Fourier transform representation of x is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega, \quad \text{where} \quad X(\omega) = Ae^{-j\omega t_0}. \qquad \blacksquare$$

Example 6.2 (Inverse Fourier transform of the unit-impulse function)**.** Find the inverse Fourier transform x of the function

$$X(\omega) = 2\pi A\delta(\omega - \omega_0),$$

where A and ω_0 are real constants.

Solution. From the definition of the inverse Fourier transform, we can write

$$\begin{aligned} x(t) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi A\delta(\omega-\omega_0)e^{j\omega t}\,d\omega \\ &= A\int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{j\omega t}\,d\omega. \end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the preceding equation to obtain

$$x(t) = Ae^{j\omega_0 t}.$$

Thus, we have that

$$Ae^{j\omega_0 t} \overset{\text{CTFT}}{\longleftrightarrow} 2\pi A\delta(\omega-\omega_0). \qquad \blacksquare$$

Example 6.3 (Fourier transform of the rectangular function)**.** Find the Fourier transform X of the function

$$x(t) = \operatorname{rect} t.$$

Solution. From the definition of the Fourier transform, we can write

$$X(\omega) = \int_{-\infty}^{\infty} \operatorname{rect}(t) e^{-j\omega t} dt.$$

From the definition of the rectangular function, we can simplify this equation to obtain

$$\begin{aligned} X(\omega) &= \int_{-1/2}^{1/2} \operatorname{rect}(t) e^{-j\omega t} dt \\ &= \int_{-1/2}^{1/2} e^{-j\omega t} dt. \end{aligned}$$

Evaluating the integral and simplifying, we have

$$\begin{aligned} X(\omega) &= \left[-\tfrac{1}{j\omega} e^{-j\omega t} \right] \Big|_{-1/2}^{1/2} \\ &= \tfrac{1}{j\omega} \left(e^{j\omega/2} - e^{-j\omega/2} \right) \\ &= \tfrac{1}{j\omega} \left[2j \sin\left(\tfrac{1}{2}\omega\right) \right] \\ &= \tfrac{2}{\omega} \sin\left(\tfrac{1}{2}\omega\right) \\ &= \left[\sin\left(\tfrac{1}{2}\omega\right) \right] / \left(\tfrac{1}{2}\omega\right) \\ &= \operatorname{sinc}\left(\tfrac{1}{2}\omega\right). \end{aligned}$$

Thus, we have shown that

$$\operatorname{rect} t \overset{\text{CTFT}}{\longleftrightarrow} \operatorname{sinc}\left(\tfrac{1}{2}\omega\right). \qquad \blacksquare$$

6.5 Remarks on Notation Involving the Fourier Transform

Each of the Fourier transform operator $\mathcal{F}$ and inverse Fourier transform operator $\mathcal{F}^{-1}$ map a function to a function. Consequently, the operand for each of these operators must be a function (and not a number). Consider the unnamed function that maps t to $e^{-|t|}$ as shown in Figure 6.4. Suppose that we would like to write an expression that denotes the Fourier transform of this function. At first, we might be inclined to write "$\mathcal{F}\{e^{-|t|}\}$". Strictly speaking, however, this notation is not correct, since the Fourier transform operator requires a function as an operand and "$e^{-|t|}$" (strictly speaking) denotes a number (i.e., the value of the function in the figure evaluated at t). Essentially, the cause of our problems here is that the function in question does not have a name (such as "x") by which it can be referred. To resolve this problem, we could define a function x using the equation $x(t) = e^{-|t|}$ and then write the Fourier transform as "$\mathcal{F}x$". Unfortunately, introducing a new function name just for the sake of strictly correct notation is often undesirable as it frequently leads to overly verbose writing.

One way to avoid overly verbose writing when referring to functions without names is offered by dot notation, introduced earlier in Section 2.1. Again, consider the function from Figure 6.4 that maps t to $e^{-|t|}$. Using strictly correct notation, we could write the Fourier transform of this function as "$\mathcal{F}\left\{e^{-|\cdot|}\right\}$". In other words, we can indicate that an expression refers to a function (as opposed to the value of function) by using the interpunct symbol (as discussed in Section 2.1). Some examples of the use of dot notation can be found below in Example 6.4. Dot notation is often extremely beneficial when one wants to employ precise (i.e., strictly correct) notation without being overly verbose.

Example 6.4 (Dot notation)**.** Several examples of the use of dot notation are as follows:

1. To denote the Fourier transform of the function x defined by the equation $x(t) = e^{t^2}$ (without the need to introduce the named function x), we can write: $\mathcal{F}\left\{e^{(\cdot)^2}\right\}$.

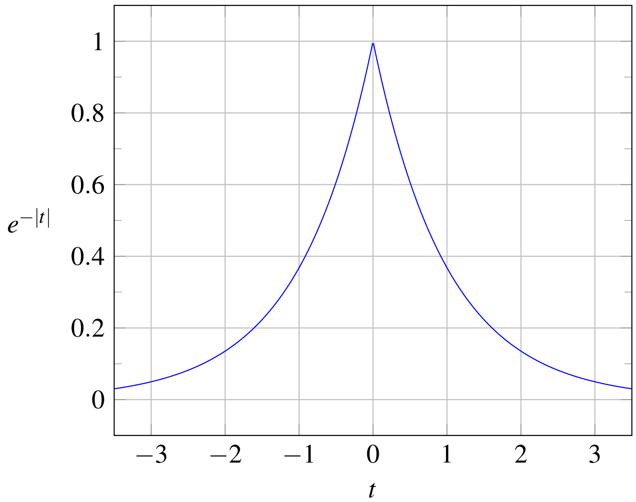


Figure 6.4: A plot of $e^{-|t|}$ versus t.

2. To denote the Fourier transform of the function x defined by the equation $x(t) = e^{t^2}$ evaluated at $2\omega - 3$ (without the need to introduce the named function x), we can write: $\mathcal{F}\left\{e^{(\cdot)^2}\right\}(2\omega-3)$.

3. To denote the inverse Fourier transform of the function X defined by the equation $X(\omega) = \frac{1}{j\omega}$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1}{j(\cdot)}\right\}$.

4. To denote the inverse Fourier transform of the function X defined by the equation $X(\omega) = \frac{1}{j\omega}$ evaluated at $t-3$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1}{j(\cdot)}\right\}(t-3)$. ■

If the reader is comfortable with dot notation, the author would encourage the reader to use it when appropriate. Since some readers may find the dot notation to be confusing, however, this book (for the most part) attempts to minimize the use of dot notation. Instead, as a compromise solution, this book adopts the following notational conventions in order to achieve conciseness and a reasonable level of clarity without the need to use dot notation pervasively:

- unless indicated otherwise, in an expression for the operand of the Fourier transform operator $\mathcal{F}$, the variable "t" is assumed to be the independent variable for the function to which the Fourier transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "t" were a "$\cdot$");
- unless indicated otherwise, in an expression for the operand of the inverse Fourier transform operator $\mathcal{F}^{-1}$, the variable "ω" is assumed to be the independent variable for the function to which the inverse Fourier transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "ω" were a "$\cdot$")

Some examples of using these book-sanctioned notational conventions can be found below in Example 6.5. Admittedly, these book-sanctioned conventions are not ideal, as they abuse mathematical notation somewhat, but they seem to be the best compromise in order to accommodate those who may prefer not to use dot notation.

Example 6.5 (Book-sanctioned notation)**.** Several examples of using the notational conventions that are employed throughout most of this book (as described above) are as follows:

1. To denote the Fourier transform of the function x defined by the equation $x(t) = e^{t^2}$ (without the need to introduce the named function x), we can write: $\mathcal{F}\{e^{t^2}\}$.

2. To denote the Fourier transform of the function x defined by the equation $x(t) = e^{t^2}$ evaluated at $2\omega - 3$ (without the need to introduce the named function x), we can write: $\mathcal{F}\{e^{t^2}\}(2\omega-3)$.

3. To denote the inverse Fourier transform of the function X defined by the equation $X(\omega) = \frac{1}{j\omega}$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1}{j\omega}\right\}$.

4. To denote the inverse Fourier transform of the function X defined by the equation $X(\omega) = \frac{1}{j\omega}$ evaluated at $t-3$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1}{j\omega}\right\}(t-3)$. ■

Since applying the Fourier transform operator or inverse Fourier transform operator to a function yields another function, we can evaluate this other function at some value. Again, consider the function from Figure 6.4 that maps t to $e^{-|t|}$. To denote the value of the Fourier transform of this function evaluated at $\omega-1$, we would write "$\mathcal{F}\{e^{-|\cdot|}\}(\omega-1)$" using dot notation or "$\mathcal{F}\{e^{-|t|}\}(\omega-1)$" using the book-sanctioned notational conventions described above.

6.6 Convergence of the Continuous-Time Fourier Transform

When deriving the Fourier transform representation earlier, we implicitly made some assumptions about the convergence of the integrals and other expressions involved. These assumptions are not always valid. For this reason, a more careful examination of the convergence properties of the Fourier transform is in order.

Suppose that we have an arbitrary function x. This function has the Fourier transform representation $\hat{x}$ given by

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega, \quad \text{where} \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

Now, we need to concern ourselves with the convergence properties of this representation. In other words, we want to know when $\hat{x}$ is a valid representation of x. In our earlier derivation of the Fourier transform, we relied heavily on the Fourier series. Therefore, one might expect that the convergence of the Fourier transform representation is closely related to the convergence properties of Fourier series. This is, in fact, the case. The convergence properties of the Fourier transform are very similar to the convergence properties of the Fourier series (as studied in Section 5.4).

The first important result concerning convergence relates to continuous functions as stated by the theorem below.

Theorem 6.1 (Convergence of the Fourier transform (continuous case))**.** *If a function x is continuous and absolutely integrable (i.e., $\int_{-\infty}^{\infty}|x(t)|\,dt < \infty$) and the Fourier transform X of x is absolutely integrable (i.e., $\int_{-\infty}^{\infty}|X(\omega)|\,d\omega < \infty$), then the Fourier transform representation of x converges pointwise (i.e., $x(t) = \hat{x}(t)$ for all t).*

Proof. A rigorous proof of this result is beyond the scope of this book and is therefore omitted here. ■

Since, in practice, we often encounter functions with discontinuities (e.g., a rectangular pulse), the above result is sometimes of limited value. This motivates us to consider additional results concerning convergence.

The next important result concerning convergence relates to finite-energy functions as stated by the theorem below.

Theorem 6.2 (Convergence of Fourier transform (finite-energy case))**.** *If a function x is of finite energy (i.e., $\int_{-\infty}^{\infty}|x(t)|^2\,dt < \infty$), then its Fourier transform representation $\hat{x}$ converges in the MSE sense.*

Proof. A rigorous proof of this result is beyond the scope of this book and is therefore omitted here. ■

In other words, the preceding theorem states that, if x is of finite energy, then

$$E = \int_{-\infty}^{\infty}|\hat{x}(t) - x(t)|^2\,dt = 0.$$

Although x and $\hat{x}$ may differ at individual points, the energy E in the difference is zero.

The last result concerning convergence that we shall consider relates to what are known as the Dirichlet conditions. The Dirichlet conditions for the function x are as follows:

1. The function x is absolutely integrable (i.e., $\int_{-\infty}^{\infty}|x(t)|\,dt < \infty$).

2. The function x has a finite number of maxima and minima on any finite interval (i.e., is of bounded variation).

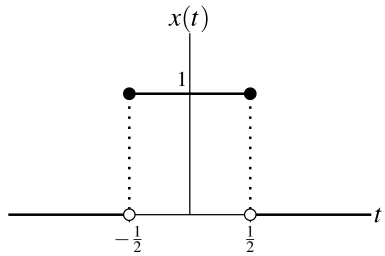


Figure 6.5: Function x.

3. The function x has a finite number of discontinuities on any finite interval, and each discontinuity is itself finite.

For a function satisfying the Dirichlet conditions, we have the important convergence result stated below.

Theorem 6.3 (Convergence of Fourier transform (Dirichlet case))**.** *If a function x satisfies the Dirichlet conditions, then its Fourier transform representation $\hat{x}$ converges pointwise everywhere except at points of discontinuity. Furthermore, at each discontinuity point t_a, we have that*

$$\hat{x}(t_a) = \tfrac{1}{2}\left[x\left(t_a^+\right) + x\left(t_a^-\right)\right],$$

where $x\left(t_a^-\right)$ and $x\left(t_a^+\right)$ denote the values of the function x on the left- and right-hand sides of the discontinuity, respectively.

Proof. A rigorous proof of this result is beyond the scope of this book and is therefore omitted here. ■

In other words, the preceding theorem states that, if a function x satisfies the Dirichlet conditions, then the Fourier transform representation $\hat{x}$ is such that $\hat{x}(t) = x(t)$ for all t, except at points of discontinuity where $\hat{x}(t)$ equals the average value of x on the two sides of the discontinuity.

The finite-energy and Dirichlet conditions mentioned above are only sufficient conditions for the convergence of the Fourier transform representation. They are not necessary. In other words, a function may violate these conditions and still have a valid Fourier transform representation.

Example 6.6. Consider the function x shown in Figure 6.5. Let $\hat{x}$ denote the Fourier transform representation of x (i.e., $\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$, where X denotes the Fourier transform of x). Determine the values $\hat{x}(-\frac{1}{2})$ and $\hat{x}(\frac{1}{2})$.

Solution. We begin by observing that x satisfies the Dirichlet conditions. Consequently, Theorem 6.3 applies. Thus, we have that

$$\begin{aligned}
\hat{x}(-\tfrac{1}{2}) &= \tfrac{1}{2}\left[x(-\tfrac{1}{2}^-) + x(-\tfrac{1}{2}^+)\right] \\
&= \tfrac{1}{2}(0+1) \\
&= \tfrac{1}{2} \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\hat{x}(\tfrac{1}{2}) &= \tfrac{1}{2}\left[x(\tfrac{1}{2}^-) + x(\tfrac{1}{2}^+)\right] \\
&= \tfrac{1}{2}(1+0) \\
&= \tfrac{1}{2}.
\end{aligned}$$

■

6.7 Properties of the Continuous-Time Fourier Transform

The Fourier transform has a number of important properties. In the sections that follow, we introduce several of these properties. For convenience, these properties are also later summarized in Table 6.1 (on page 169). Also, for convenience, several Fourier-transform pairs are given later in Table 6.2 (on page 173). In what follows, we will sometimes refer to transform pairs in this table.

6.7.1 Linearity

Arguably, the most important property of the Fourier transform is linearity, as introduced below.

Theorem 6.4 (Linearity)**.** *If* $x_1(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ *and* $x_2(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$*, then*

$$a_1x_1(t) + a_2x_2(t) \stackrel{\text{CTFT}}{\longleftrightarrow} a_1X_1(\omega) + a_2X_2(\omega),$$

where a_1 *and* a_2 *are arbitrary complex constants.*

Proof. To prove the above property, we proceed as follows. Let $y(t) = a_1x_1(t) + a_2x_2(t)$ and let $Y = \mathcal{F}y$. We have

$$\begin{aligned}
Y(\omega) &= \int_{-\infty}^{\infty} [a_1x_1(t) + a_2x_2(t)]e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{\infty} a_1x_1(t)e^{-j\omega t}\,dt + \int_{-\infty}^{\infty} a_2x_2(t)e^{-j\omega t}\,dt \\
&= a_1\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}\,dt + a_2\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}\,dt \\
&= a_1X_1(\omega) + a_2X_2(\omega).
\end{aligned}$$

Thus, we have shown that the linearity property holds. ■

Example 6.7 (Linearity property of the Fourier transform)**.** Using properties of the Fourier transform and the transform pair

$$e^{j\omega_0 t} \stackrel{\text{CTFT}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0),$$

find the Fourier transform X of the function

$$x(t) = A\cos(\omega_0 t),$$

where A and ω_0 are real constants.

Solution. We recall that $\cos\alpha = \frac{1}{2}[e^{j\alpha} + e^{-j\alpha}]$ for any real α. Thus, we can write

$$\begin{aligned}
X(\omega) &= (\mathcal{F}\{A\cos(\omega_0 t)\})(\omega) \\
&= \left(\mathcal{F}\{\tfrac{A}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})\}\right)(\omega).
\end{aligned}$$

Then, we use the linearity property of the Fourier transform to obtain

$$X(\omega) = \tfrac{A}{2}\mathcal{F}\{e^{j\omega_0 t}\}(\omega) + \tfrac{A}{2}\mathcal{F}\{e^{-j\omega_0 t}\}(\omega).$$

Using the given Fourier transform pair, we can further simplify the above expression for $X(\omega)$ as follows:

$$\begin{aligned}
X(\omega) &= \tfrac{A}{2}[2\pi\delta(\omega + \omega_0)] + \tfrac{A}{2}[2\pi\delta(\omega - \omega_0)] \\
&= A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].
\end{aligned}$$

Thus, we have shown that

$$A\cos(\omega_0 t) \stackrel{\text{CTFT}}{\longleftrightarrow} A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].$$ ■

Example 6.8 (Fourier transform of the unit-step function)**.** Using properties of the Fourier transform and the transform pairs

$$1 \stackrel{\text{CTFT}}{\longleftrightarrow} 2\pi\delta(\omega) \quad \text{and} \quad \operatorname{sgn} t \stackrel{\text{CTFT}}{\longleftrightarrow} \tfrac{2}{j\omega},$$

find the Fourier transform X of the function $x = u$.

Solution. First, we observe that x can be expressed in terms of the signum function as

$$x(t) = u(t) = \tfrac{1}{2} + \tfrac{1}{2}\operatorname{sgn} t.$$

Taking the Fourier transform of both sides of this equation yields

$$X(\omega) = \left(\mathcal{F}\{\tfrac{1}{2} + \tfrac{1}{2}\operatorname{sgn} t\}\right)(\omega).$$

Using the linearity property of the Fourier transform, we can rewrite this as

$$X(\omega) = \tfrac{1}{2}\mathcal{F}\{1\}(\omega) + \tfrac{1}{2}\mathcal{F}\{\operatorname{sgn} t\}(\omega).$$

Evaluating the two Fourier transforms using the given transform pairs, we obtain

$$\begin{aligned} X(\omega) &= \tfrac{1}{2}[2\pi\delta(\omega)] + \tfrac{1}{2}(\tfrac{2}{j\omega}) \\ &= \pi\delta(\omega) + \tfrac{1}{j\omega}. \end{aligned}$$

Thus, we have shown that

$$u(t) \stackrel{\text{CTFT}}{\longleftrightarrow} \pi\delta(\omega) + \tfrac{1}{j\omega}.$$

■

6.7.2 Time-Domain Shifting (Translation)

The next property of the Fourier transform to be introduced is the time-domain shifting (i.e., translation) property, as given below.

Theorem 6.5 (Time-domain shifting (i.e., translation))**.** *If* $x(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X(\omega)$*, then*

$$x(t - t_0) \stackrel{\text{CTFT}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega),$$

where t_0 *is an arbitrary real constant.*

Proof. To prove the above property, we proceed as follows. Let $y(t) = x(t - t_0)$ and let $Y = \mathcal{F}y$. From the definition of the Fourier transform, we have

$$Y(\omega) = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt.$$

Now, we use a change of variable. Let $\lambda = t - t_0$ so that $t = \lambda + t_0$ and $dt = d\lambda$. Performing the change of variable and simplifying, we obtain

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega(\lambda + t_0)} d\lambda \\ &= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} e^{-j\omega t_0} d\lambda \\ &= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} d\lambda \\ &= e^{-j\omega t_0} X(\omega). \end{aligned}$$

Thus, we have proven that the time-shifting property holds. ■

Example 6.9 (Time-domain shifting property of the Fourier transform)**.** Find the Fourier transform X of the function

$$x(t) = A\cos(\omega_0 t + \theta),$$

where A, ω_0, and θ are real constants.

Solution. Let $v(t) = A\cos(\omega_0 t)$ so that $x(t) = v(t + \frac{\theta}{\omega_0})$. Also, let $V = \mathcal{F}v$. From Table 6.2, we have that

$$\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)].$$

Using this transform pair and the linearity property of the Fourier transform, we have that

$$\begin{aligned} V(\omega) &= \mathcal{F}\{A\cos(\omega_0 t)\}(\omega) \\ &= A\mathcal{F}\{\cos(\omega_0 t)\}(\omega) \\ &= A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \end{aligned}$$

From the definition of v and the time-shifting property of the Fourier transform, we have

$$\begin{aligned} X(\omega) &= e^{j\omega\theta/\omega_0} V(\omega) \\ &= e^{j\omega\theta/\omega_0} A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \end{aligned}$$

Thus, we have shown that

$$A\cos(\omega_0 t + \theta) \overset{\text{CTFT}}{\longleftrightarrow} A\pi e^{j\omega\theta/\omega_0}[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].$$ ■

6.7.3 Frequency-Domain Shifting (Modulation)

The next property of the Fourier transform to be introduced is the frequency-domain shifting (i.e., modulation) property, as given below.

Theorem 6.6 (Frequency-domain shifting (i.e., modulation))**.** *If* $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$*, then*

$$e^{j\omega_0 t} x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega - \omega_0),$$

where ω_0 *is an arbitrary real constant.*

Proof. To prove the above property, we proceed as follows. Let $y(t) = e^{j\omega_0 t}$ and let $Y = \mathcal{F}y$. From the definition of the Fourier transform and straightforward algebraic manipulation, we can write

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t} dt \\ &= X(\omega - \omega_0). \end{aligned}$$

Thus, we have shown that the frequency-domain shifting property holds. ■

Example 6.10 (Frequency-domain shifting property of the Fourier transform)**.** Find the Fourier transform X of the function

$$x(t) = \cos(\omega_0 t)\cos(20\pi t),$$

where ω_0 is a real constant.

Solution. Recall that $\cos\alpha = \frac{1}{2}[e^{j\alpha}+e^{-j\alpha}]$ for any real α. Using this relationship and the linearity property of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \mathcal{F}\{\cos(\omega_0 t)(\tfrac{1}{2})(e^{j20\pi t}+e^{-j20\pi t})\}(\omega) \\ &= \mathcal{F}\{\tfrac{1}{2}e^{j20\pi t}\cos(\omega_0 t)+\tfrac{1}{2}e^{-j20\pi t}\cos(\omega_0 t)\}(\omega) \\ &= \tfrac{1}{2}\mathcal{F}\{e^{j20\pi t}\cos(\omega_0 t)\}(\omega)+\tfrac{1}{2}\mathcal{F}\{e^{-j20\pi t}\cos(\omega_0 t)\}(\omega). \end{aligned}$$

From Table 6.2, we have that

$$\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)].$$

From this transform pair and the frequency-domain shifting property of the Fourier transform, we have

$$\begin{aligned} X(\omega) &= \tfrac{1}{2}\left(\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega-20\pi)+\tfrac{1}{2}\left(\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega+20\pi) \\ &= \tfrac{1}{2}\left[\pi\left[\delta(v-\omega_0)+\delta(v+\omega_0)\right]\right]\big|_{v=\omega-20\pi}+\tfrac{1}{2}\left[\pi\left[\delta(v-\omega_0)+\delta(v+\omega_0)\right]\right]\big|_{v=\omega+20\pi} \\ &= \tfrac{1}{2}\left(\pi[\delta(\omega+\omega_0-20\pi)+\delta(\omega-\omega_0-20\pi)]\right)+\tfrac{1}{2}\left(\pi[\delta(\omega+\omega_0+20\pi)+\delta(\omega-\omega_0+20\pi)]\right) \\ &= \tfrac{\pi}{2}\left[\delta(\omega+\omega_0-20\pi)+\delta(\omega-\omega_0-20\pi)+\delta(\omega+\omega_0+20\pi)+\delta(\omega-\omega_0+20\pi)\right]. \end{aligned}$$

■

6.7.4 Time- and Frequency-Domain Scaling (Dilation)

The next property of the Fourier transform to be introduced is the time/frequency-scaling (i.e., dilation) property, as given below.

Theorem 6.7 (Time/frequency-domain scaling (i.e., dilation))**.** *If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then*

$$x(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{\omega}{a}\right),$$

where a is an arbitrary nonzero real constant.

Proof. To prove the above property, we proceed as follows. Let $y(t) = x(at)$ and let $Y = \mathcal{F}y$. From the definition of the Fourier transform, we can write

$$Y(\omega) = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt.$$

Now, we use a change of variable. Let $\lambda = at$ so that $t = \lambda/a$ and $dt = d\lambda/a$. Performing the change of variable (and being mindful of the change in the limits of integration), we obtain

$$\begin{aligned} Y(\omega) &= \begin{cases} \displaystyle\int_{-a(\infty)}^{a(\infty)} x(\lambda)e^{-j(\omega/a)\lambda}\left(\tfrac{1}{a}\right)d\lambda & a>0 \\ \displaystyle\int_{-a(\infty)}^{a(\infty)} x(\lambda)e^{-j(\omega/a)\lambda}\left(\tfrac{1}{a}\right)d\lambda & a<0 \end{cases} \\ &= \begin{cases} \displaystyle\int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}\left(\tfrac{1}{a}\right)d\lambda & a>0 \\ \displaystyle\int_{\infty}^{-\infty} x(\lambda)e^{-j(\omega/a)\lambda}\left(\tfrac{1}{a}\right)d\lambda & a<0 \end{cases} \\ &= \begin{cases} \displaystyle\tfrac{1}{a}\int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}d\lambda & a>0 \\ \displaystyle-\tfrac{1}{a}\int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}d\lambda & a<0. \end{cases} \end{aligned}$$

Combining the two cases (i.e., for $a < 0$ and $a > 0$), we obtain

$$\begin{aligned} Y(\omega) &= \tfrac{1}{|a|} \int_{-\infty}^{\infty} x(\lambda) e^{-j(\omega/a)\lambda} d\lambda \\ &= \tfrac{1}{|a|} X\left(\tfrac{\omega}{a}\right). \end{aligned}$$

Thus, we have shown that the time/frequency-scaling property holds. ∎

Example 6.11 (Time scaling property of the Fourier transform)**.** Using the Fourier transform pair

$$\operatorname{rect} t \overset{\text{CTFT}}{\longleftrightarrow} \operatorname{sinc}\left(\frac{\omega}{2}\right),$$

find the Fourier transform X of the function

$$x(t) = \operatorname{rect}(at),$$

where a is a nonzero real constant.

Solution. Let $v(t) = \operatorname{rect} t$ so that $x(t) = v(at)$. Also, let $V = \mathcal{F}v$. From the given transform pair, we know that

$$V(\omega) = \mathcal{F}\{\operatorname{rect} t\}(\omega) = \operatorname{sinc}\left(\frac{\omega}{2}\right). \tag{6.9}$$

From the definition of v and the time-scaling property of the Fourier transform, we have

$$X(\omega) = \frac{1}{|a|} V\left(\frac{\omega}{a}\right).$$

Substituting the expression for V in (6.9) into the preceding equation, we have

$$X(\omega) = \frac{1}{|a|} \operatorname{sinc}\left(\frac{\omega}{2a}\right).$$

Thus, we have shown that

$$\operatorname{rect}(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} \operatorname{sinc}\left(\frac{\omega}{2a}\right).$$ ∎

6.7.5 Conjugation

The next property of the Fourier transform to be introduced is the conjugation property, as given below.

Theorem 6.8 (Conjugation)**.** *If* $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, *then*

$$x^*(t) \overset{\text{CTFT}}{\longleftrightarrow} X^*(-\omega).$$

Proof. To prove the above property, we proceed as follows. Let $y(t) = x^*(t)$ and let $Y = \mathcal{F}y$. From the definition of the Fourier transform, we have

$$Y(\omega) = \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt.$$

From the properties of conjugation, we can rewrite this equation as

$$\begin{aligned} Y(\omega) &= \left[\left(\int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt\right)^*\right]^* \\ &= \left[\int_{-\infty}^{\infty} [x(t)^*]^* \left(e^{-j\omega t}\right)^* dt\right]^* \\ &= \left[\int_{-\infty}^{\infty} x(t) e^{-j(-\omega)t} dt\right]^* \\ &= X^*(-\omega). \end{aligned}$$

Thus, we have shown that the conjugation property holds. ∎

Example 6.12 (Fourier transform of a real function)**.** Let X denote the Fourier transform of the function x. Show that, if x is real, then X is conjugate symmetric (i.e., $X(\omega) = X^*(-\omega)$ for all ω).

Solution. From the conjugation property of the Fourier transform, we have

$$\mathcal{F}\{x^*(t)\}(\omega) = X^*(-\omega).$$

Since x is real, we can replace x^* with x to yield

$$\mathcal{F}x(\omega) = X^*(-\omega),$$

or equivalently

$$X(\omega) = X^*(-\omega). \qquad \blacksquare$$

6.7.6 Duality

The next property of the Fourier transform to be introduced is the duality property, as given below.

Theorem 6.9 (Duality)**.** *If* $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$*, then*

$$X(t) \overset{\text{CTFT}}{\longleftrightarrow} 2\pi x(-\omega).$$

Proof. To prove the above property, we proceed as follows. From the Fourier transform synthesis equation, we have

$$x(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)e^{j\lambda t}d\lambda.$$

Substituting $-\omega$ for t, we obtain

$$x(-\omega) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)e^{-j\lambda\omega}d\lambda.$$

Now, we multiply both sides of the equation by 2π to yield

$$\begin{aligned} 2\pi x(-\omega) &= \int_{-\infty}^{\infty} X(\lambda)e^{-j\lambda\omega}d\lambda \\ &= \mathcal{F}X(\omega). \end{aligned}$$

Thus, we have shown that the duality property holds. $\blacksquare$

The duality property stated in the preceding theorem follows from the high degree of similarity in the equations for the forward and inverse Fourier transforms, given by (6.7) and (6.8), respectively. To make this similarity more obvious, we can rewrite the forward and inverse Fourier transform equations, respectively, as

$$X(\lambda) = \int_{-\infty}^{\infty} x(\theta)e^{-j\theta\lambda}d\theta \quad \text{and} \quad x(\lambda) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\theta)e^{j\theta\lambda}d\theta.$$

Observe that these two equations are identical except for: 1) a factor of 2π; and 2) a different sign in the parameter for the exponential function. Consequently, if we were to accidentally use one equation in place of the other, we would obtain an almost correct result. In fact, this almost correct result could be made to be correct by compensating for the above two differences (i.e., the factor of 2π and the sign difference in the exponential function). This is, in effect, what the duality property states.

Although the relationship $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$ only directly provides us with the Fourier transform X of x, the duality property allows us to indirectly infer the Fourier transform of X. Consequently, the duality property can be used to effectively double the number of Fourier transform pairs that we know.

Example 6.13 (Fourier transform of the sinc function)**.** Using the transform pair

$$\operatorname{rect} t \overset{\text{CTFT}}{\longleftrightarrow} \operatorname{sinc}\left(\frac{\omega}{2}\right),$$

find the Fourier transform X of the function

$$x(t) = \operatorname{sinc}\left(\frac{t}{2}\right).$$

Solution. From the given Fourier transform pair, we have

$$v(t) = \operatorname{rect} t \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad V(\omega) = \operatorname{sinc}\left(\frac{\omega}{2}\right).$$

By duality, we have

$$V(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad \mathcal{F}V(\omega) = 2\pi v(-\omega) = 2\pi \operatorname{rect}(-\omega) = 2\pi \operatorname{rect} \omega.$$

Thus, we have

$$V(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad \mathcal{F}V(\omega) = 2\pi \operatorname{rect} \omega.$$

Observing that $V = x$ and $\mathcal{F}V = X$, we can rewrite the preceding relationship as

$$x(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad X(\omega) = 2\pi \operatorname{rect} \omega.$$

Thus, we have shown that

$$X(\omega) = 2\pi \operatorname{rect} \omega.$$

■

6.7.7 Time-Domain Convolution

The next property of the Fourier transform to be introduced is the time-domain convolution property, as given below.

Theorem 6.10 (Time-domain convolution)**.** *If* $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ *and* $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$*, then*

$$x_1 * x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)X_2(\omega).$$

Proof. The proof of this property is as follows. Let $y(t) = x_1 * x_2(t)$ and let $Y = \mathcal{F}y$. From the definition of the Fourier transform and convolution, we have

$$\begin{aligned}
Y(\omega) &= \int_{-\infty}^{\infty} [x_1 * x_2(t)]\, e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau \right] e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) e^{-j\omega t} d\tau dt.
\end{aligned}$$

Changing the order of integration, we obtain

$$Y(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) e^{-j\omega t} dt d\tau.$$

Now, we use a change of variable. Let $\lambda = t - \tau$ so that $t = \lambda + \tau$ and $d\lambda = dt$. Applying the change of variable and simplifying, we obtain

$$\begin{aligned}
Y(\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(\lambda)e^{-j\omega(\lambda+\tau)}d\lambda d\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(\lambda)e^{-j\omega\lambda}e^{-j\omega\tau}d\lambda d\tau \\
&= \int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}\left[\int_{-\infty}^{\infty} x_2(\lambda)e^{-j\omega\lambda}d\lambda\right]d\tau \\
&= \left[\int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}d\tau\right]\left[\int_{-\infty}^{\infty} x_2(\lambda)e^{-j\omega\lambda}d\lambda\right] \\
&= X_1(\omega)X_2(\omega).
\end{aligned}$$

Thus, we have shown that the time-domain convolution property holds. ■

The time-domain convolution property of the Fourier transform has important practical implications. Since the Fourier transform effectively converts a convolution into a multiplication, the Fourier transform can be used as a means to avoid directly dealing with convolution operations. This is often extremely helpful when solving problems involving LTI systems, for example, since such problems almost inevitably involve convolution (due to the fact that a LTI system computes a convolution).

Example 6.14 (Time-domain convolution property of the Fourier transform)**.** With the aid of Table 6.2, find the Fourier transform X of the function

$$x(t) = x_1 * x_2(t),$$

where

$$x_1(t) = e^{-2t}u(t) \quad \text{and} \quad x_2(t) = u(t).$$

Solution. Let X_1 and X_2 denote the Fourier transforms of x_1 and x_2, respectively. From the time-domain convolution property of the Fourier transform, we know that

$$\begin{aligned}
X(\omega) &= (\mathcal{F}\{x_1 * x_2\})(\omega) \\
&= X_1(\omega)X_2(\omega). \qquad (6.10)
\end{aligned}$$

From Table 6.2, we know that

$$\begin{aligned}
X_1(\omega) &= \left(\mathcal{F}\{e^{-2t}u(t)\}\right)(\omega) \\
&= \tfrac{1}{2+j\omega} \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
X_2(\omega) &= \mathcal{F}u(\omega) \\
&= \pi\delta(\omega) + \tfrac{1}{j\omega}.
\end{aligned}$$

Substituting these expressions for $X_1(\omega)$ and $X_2(\omega)$ into (6.10), we obtain

$$\begin{aligned}
X(\omega) &= [\tfrac{1}{2+j\omega}](\pi\delta(\omega) + \tfrac{1}{j\omega}) \\
&= \tfrac{\pi}{2+j\omega}\delta(\omega) + \tfrac{1}{j\omega}(\tfrac{1}{2+j\omega}) \\
&= \tfrac{\pi}{2+j\omega}\delta(\omega) + \tfrac{1}{j2\omega-\omega^2}.
\end{aligned}$$

From the equivalence property of the delta function, we have

$$X(\omega) = \tfrac{\pi}{2}\delta(\omega) + \tfrac{1}{j2\omega-\omega^2}.$$ ■

6.7.8 Time-Domain Multiplication

The next property of the Fourier transform to be introduced is the time-domain multiplication (or frequency-domain convolution) property, as given below.

Theorem 6.11 (Time-domain multiplication)**.** *If* $x_1(t) \xleftrightarrow{\text{CTFT}} X_1(\omega)$ *and* $x_2(t) \xleftrightarrow{\text{CTFT}} X_2(\omega)$*, then*

$$x_1(t)x_2(t) \xleftrightarrow{\text{CTFT}} \tfrac{1}{2\pi} X_1 * X_2(\omega) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\theta)X_2(\omega-\theta)d\theta.$$

Proof. To prove the above property, we proceed as follows. Let $Y(\omega) = \frac{1}{2\pi}X_1 * X_2(\omega)$ and let $y = \mathcal{F}^{-1}Y$. From the definition of the inverse Fourier transform, we have

$$\begin{aligned}
y(t) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \left[\tfrac{1}{2\pi}X_1 * X_2(\omega)\right] e^{j\omega t} d\omega \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \tfrac{1}{2\pi}X_1(\lambda)X_2(\omega-\lambda)d\lambda\right] e^{j\omega t} d\omega \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{2\pi}X_1(\lambda)X_2(\omega-\lambda)e^{j\omega t}d\lambda d\omega.
\end{aligned}$$

Reversing the order of integration, we obtain

$$y(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{2\pi}X_1(\lambda)X_2(\omega-\lambda)e^{j\omega t}d\omega d\lambda.$$

Now, we employ a change of variable. Let $v = \omega - \lambda$ so that $\omega = v + \lambda$ and $dv = d\omega$. Applying the change of variable and simplifying yields

$$\begin{aligned}
y(t) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{2\pi}X_1(\lambda)X_2(v)e^{j(v+\lambda)t}dv d\lambda \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{2\pi}X_1(\lambda)X_2(v)e^{jvt}e^{j\lambda t}dv d\lambda \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)e^{j\lambda t}\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X_2(v)e^{jvt}dv\right] d\lambda \\
&= \left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)e^{j\lambda t}d\lambda\right]\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X_2(v)e^{jvt}dv\right] \\
&= \left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)e^{j\omega t}d\omega\right]\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega)e^{j\omega t}d\omega\right] \\
&= x_1(t)x_2(t).
\end{aligned}$$

Thus, we have shown that the frequency-domain convolution property holds. ∎

From the time-domain multiplication property in the preceding theorem, we can see that the Fourier transform effectively converts a multiplication operation into a convolution operation (with a scale factor of $\frac{1}{2\pi}$). Since convolution is significantly more complicated than multiplication, we normally prefer to avoid using this property in a manner that would result in the introduction of additional convolution operations into our work.

Example 6.15 (Frequency-domain convolution property)**.** Let x and y be functions related as

$$y(t) = x(t)\cos(\omega_c t),$$

where ω_c is a nonzero real constant. Let $Y = \mathcal{F}y$ and $X = \mathcal{F}x$. Find an expression for Y in terms of X.

Solution. To allow for simpler notation in what follows, we define

$$v(t) = \cos(\omega_c t)$$

and let V denote the Fourier transform of v. From Table 6.2, we have that

$$V(\omega) = \pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)].$$

From the definition of v, we have

$$y(t) = x(t)v(t).$$

Taking the Fourier transform of both sides of this equation, we have

$$Y(\omega) = \mathcal{F}\{x(t)v(t)\}(\omega).$$

Using the frequency-domain convolution property of the Fourier transform, we obtain

$$\begin{aligned} Y(\omega) &= \tfrac{1}{2\pi} X * V(\omega) \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)V(\omega-\lambda)d\lambda. \end{aligned}$$

Substituting the above expression for V, we obtain

$$\begin{aligned} Y(\omega) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)(\pi[\delta(\omega-\lambda-\omega_c)+\delta(\omega-\lambda+\omega_c)])d\lambda \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} X(\lambda)[\delta(\omega-\lambda-\omega_c)+\delta(\omega-\lambda+\omega_c)]d\lambda \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\omega-\lambda-\omega_c)d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\omega-\lambda+\omega_c)d\lambda\right] \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\lambda-\omega+\omega_c)d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\lambda-\omega-\omega_c)d\lambda\right] \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta[\lambda-(\omega-\omega_c)]d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta[\lambda-(\omega+\omega_c)]d\lambda\right] \\ &= \tfrac{1}{2}[X(\omega-\omega_c)+X(\omega+\omega_c)] \\ &= \tfrac{1}{2}X(\omega-\omega_c)+\tfrac{1}{2}X(\omega+\omega_c). \end{aligned}$$

■

6.7.9 Time-Domain Differentiation

The next property of the Fourier transform to be introduced is the time-domain differentiation property, as given below.

Theorem 6.12 (Time-domain differentiation). *If* $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, *then*

$$\frac{dx(t)}{dt} \overset{\text{CTFT}}{\longleftrightarrow} j\omega X(\omega).$$

Proof. To prove the above property, we proceed as follows. Let $Y(\omega) = j\omega X(\omega)$ and let $y = \mathcal{F}^{-1}Y$. We begin by using the definition of the inverse Fourier transform to write

$$x(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$$

Now, we differentiate both sides of the preceding equation with respect to t and simplify to obtain

$$\begin{aligned} \tfrac{dx(t)}{dt} &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(\omega)e^{j\omega t}d\omega \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{j\omega t}d\omega \\ &= y(t). \end{aligned}$$

Thus, we have shown that the time-differentiation property holds. ■

By repeated application of the preceding theorem, we can obtain the more general result that

$$\left(\tfrac{d}{dt}\right)^n x(t) \overset{\text{CTFT}}{\longleftrightarrow} (j\omega)^n X(\omega).$$

The time-domain differentiation property of the Fourier transform has important practical implications. Since the Fourier transform effectively converts differentiation into multiplication (by $j\omega$), the Fourier transform can be used as a means to avoid directly dealing with differentiation operations. This can often be beneficial when working with differential and integro-differential equations.

Example 6.16 (Time-domain differentiation property)**.** Find the Fourier transform X of the function

$$x(t) = \tfrac{d}{dt}\delta(t).$$

Solution. Taking the Fourier transform of both sides of the given equation for x yields

$$X(\omega) = \mathcal{F}\left\{\tfrac{d}{dt}\delta(t)\right\}(\omega).$$

Using the time-domain differentiation property of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \mathcal{F}\left\{\tfrac{d}{dt}\delta(t)\right\}(\omega) \\ &= j\omega\mathcal{F}\delta(\omega). \end{aligned}$$

Evaluating the Fourier transform of δ using Table 6.2, we obtain

$$\begin{aligned} X(\omega) &= j\omega(1) \\ &= j\omega. \end{aligned}$$

■

6.7.10 Frequency-Domain Differentiation

The next property of the Fourier transform to be introduced is the frequency-domain differentiation property, as given below.

Theorem 6.13 (Frequency-domain differentiation)**.** *If* $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$*, then*

$$tx(t) \overset{\text{CTFT}}{\longleftrightarrow} j\tfrac{d}{d\omega}X(\omega).$$

Proof. To prove the above property, we proceed as follows. Let $y(t) = tx(t)$ and let $Y = \mathcal{F}y$. From the definition of the Fourier transform, we can write

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

Now, we differentiate both sides of this equation with respect to ω and simplify to obtain

$$\begin{aligned} \tfrac{d}{d\omega}X(\omega) &= \int_{-\infty}^{\infty} x(t)(-jt)e^{-j\omega t}dt \\ &= -j\int_{-\infty}^{\infty} tx(t)e^{-j\omega t}dt \\ &= -jY(\omega). \end{aligned}$$

Multiplying both sides of the preceding equation by j yields

$$j\tfrac{d}{d\omega}X(\omega) = Y(\omega).$$

Thus, we have shown that the frequency-domain differentiation property holds. ■

Example 6.17 (Frequency-domain differentiation property)**.** Find the Fourier transform X of the function

$$x(t) = t\cos(\omega_0 t),$$

where ω_0 is a nonzero real constant.

Solution. Taking the Fourier transform of both sides of the equation for x yields

$$X(\omega) = \mathcal{F}\{t\cos(\omega_0 t)\}(\omega).$$

From the frequency-domain differentiation property of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \mathcal{F}\{t\cos(\omega_0 t)\}(\omega) \\ &= j\left(\mathcal{D}\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega), \end{aligned}$$

where $\mathcal{D}$ denotes the derivative operator. Evaluating the Fourier transform on the right-hand side using Table 6.2, we obtain

$$\begin{aligned} X(\omega) &= j\tfrac{d}{d\omega}\left[\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]\right] \\ &= j\pi\tfrac{d}{d\omega}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] \\ &= j\pi\tfrac{d}{d\omega}\delta(\omega-\omega_0)+j\pi\tfrac{d}{d\omega}\delta(\omega+\omega_0). \end{aligned}$$

■

6.7.11 Time-Domain Integration

The next property of the Fourier transform to be introduced is the time-domain integration property, as given below.

Theorem 6.14 (Time-domain integration)**.** *If* $x(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X(\omega)$*, then*

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\text{CTFT}}{\longleftrightarrow} \tfrac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega).$$

Proof. The above property can be proven as follows. Let $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$, $Y = \mathcal{F}y$, and $U = \mathcal{F}u$. First, we observe that

$$y(t) = x * u(t).$$

Taking the Fourier transform of both sides of the preceding equation and using the time-domain convolution property of the Fourier transform, we have

$$Y(\omega) = X(\omega)U(\omega). \tag{6.11}$$

From Example 6.8, we know that $u(t) \stackrel{\text{CTFT}}{\longleftrightarrow} \pi\delta(\omega)+\frac{1}{j\omega}$. Using this fact, we can rewrite (6.11) as

$$\begin{aligned} Y(\omega) &= X(\omega)\left[\pi\delta(\omega)+\tfrac{1}{j\omega}\right] \\ &= \tfrac{1}{j\omega}X(\omega)+\pi X(\omega)\delta(\omega). \end{aligned}$$

From the equivalence property of the delta function, we have

$$Y(\omega) = \tfrac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega).$$

Thus, we have shown that the time-domain integration property holds. ■

The time-domain integration property of the Fourier transform has important practical implications. Since the Fourier transform effectively converts integration into an operation involving division (by $j\omega$), the Fourier transform can be used as a means to avoid directly dealing with integration operations. This can often be beneficial when working with integral and integro-differential equations.

Example 6.18 (Time-domain integration property of the Fourier transform)**.** Use the time-domain integration property of the Fourier transform in order to find the Fourier transform X of the function $x = u$.

Solution. We begin by observing that x can be expressed in terms of an integral as

$$x(t) = u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau.$$

Now, we consider the Fourier transform of x. We have

$$X(\omega) = \mathcal{F}\left\{\int_{-\infty}^{t} \delta(\tau)d\tau\right\}(\omega).$$

From the time-domain integration property, we can write

$$X(\omega) = \tfrac{1}{j\omega}\mathcal{F}\delta(\omega) + \pi\mathcal{F}\delta(0)\delta(\omega).$$

Evaluating the two Fourier transforms on the right-hand side using Table 6.2, we obtain

$$\begin{aligned} X(\omega) &= \tfrac{1}{j\omega}(1) + \pi(1)\delta(\omega) \\ &= \tfrac{1}{j\omega} + \pi\delta(\omega). \end{aligned}$$

Thus, we have shown that $u(t) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega)$. ■

6.7.12 Parseval's Relation

The next property of the Fourier transform to be introduced, given below, relates to signal energy and is known as Parseval's relation.

Theorem 6.15 (Parseval's relation)**.** *If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then*

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega. \tag{6.12}$$

That is, the energy of x and energy of X are equal within a scaling factor of 2π. (Recall that the energy of a function x is given by $\int_{-\infty}^{\infty} |x(t)|^2 dt$.)

Proof. To prove the above relationship, we proceed as follows. Consider the left-hand side of (6.12) which we can write as

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)x^*(t)dt \\ &= \int_{-\infty}^{\infty} x(t)\mathcal{F}^{-1}\{\mathcal{F}(x^*)\}(t)dt. \end{aligned}$$

From the conjugation property of the Fourier transform, we have that $x^*(t) \overset{\text{CTFT}}{\longleftrightarrow} X^*(-\omega)$. So, we can rewrite the above equation as

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)\left(\mathcal{F}^{-1}\{X^*(-\omega)\}\right)(t)dt \\ &= \int_{-\infty}^{\infty} x(t)\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X^*(-\omega)e^{j\omega t}d\omega\right]dt. \end{aligned}$$

Now, we employ a change of variable (i.e., replace ω by $-\omega$) to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t) \left[\tfrac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega\right] dt \\ &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) X^*(\omega) e^{-j\omega t} d\omega dt.\end{aligned}$$

Reversing the order of integration and simplifying, we have

$$\begin{aligned}\int_{-\infty}^{\infty} |x(t)|^2 dt &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) X^*(\omega) e^{-j\omega t} dt d\omega \\ &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left(\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right) d\omega \\ &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) X(\omega) d\omega \\ &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega.\end{aligned}$$

Thus, Parseval's relation holds. ■

Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform preserves energy (up to a scale factor). For example, if we are solving a problem in the Fourier domain, we do not have to return to the time domain to compute energy, since we can do this directly in the Fourier domain by using Parseval's relation.

Example 6.19 (Energy of the sinc function)**.** Consider the function $x(t) = \operatorname{sinc}\left(\frac{1}{2}t\right)$, which has the Fourier transform X given by $X(\omega) = 2\pi \operatorname{rect} \omega$. Compute the energy of x.

Solution. We could directly compute the energy of x as

$$\begin{aligned}E &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= \int_{-\infty}^{\infty} \left|\operatorname{sinc}\left(\tfrac{1}{2}t\right)\right|^2 dt.\end{aligned}$$

This integral is not so easy to compute, however. Instead, we use Parseval's relation to write

$$\begin{aligned}E &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |2\pi \operatorname{rect} \omega|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-1/2}^{1/2} (2\pi)^2 d\omega \\ &= 2\pi \int_{-1/2}^{1/2} d\omega \\ &= 2\pi [\omega]\big|_{-1/2}^{1/2} \\ &= 2\pi [\tfrac{1}{2} + \tfrac{1}{2}] \\ &= 2\pi.\end{aligned}$$

Thus, we have

$$E = \int_{-\infty}^{\infty} \left|\operatorname{sinc}\left(\tfrac{1}{2}t\right)\right|^2 dt = 2\pi.$$ ■

6.7.13 Even/Odd Symmetry

The Fourier transform preserves symmetry. In other words, we have the result below.

Theorem 6.16 (Even/odd symmetry)**.** *For a function x with Fourier transform X, the following assertions hold:*

- *x is even if and only if X is even; and*
- *x is odd if and only if X is odd.*

Proof. First, we show that, if a function x is even/odd, then its Fourier transform X is even/odd. From the definition of the Fourier transform, we have

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

Since x is even/odd, we have that $x(t) = \pm x(-t)$, where the plus case and minus case in the "$\pm$" correspond to x being even and odd, respectively. Using this, we can rewrite the above expression for $X(\omega)$ as

$$X(\omega) = \int_{-\infty}^{\infty} \pm x(-t)e^{-j\omega t}dt.$$

Now, we employ a change of variable. Let $\lambda = -t$ so that $d\lambda = -dt$. Applying this change of variable, we obtain

$$\begin{aligned}
X(\omega) &= \int_{\infty}^{-\infty} \pm x(\lambda)e^{-j\omega(-\lambda)}(-1)d\lambda \\
&= \mp \int_{\infty}^{-\infty} x(\lambda)e^{j\omega\lambda}d\lambda \\
&= \pm \int_{-\infty}^{\infty} x(\lambda)e^{j\omega\lambda}d\lambda \\
&= \pm \int_{-\infty}^{\infty} x(\lambda)e^{-j(-\omega)\lambda}d\lambda \\
&= \pm X(-\omega).
\end{aligned}$$

Therefore, X is even/odd.

Next, we show that if X is even/odd, then x is even/odd. From the definition of the inverse Fourier transform, we have

$$x(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$$

Since X is even/odd, we have that $X(\omega) = \pm X(-\omega)$, where the plus case and minus case in the "$\pm$" correspond to X being even and odd, respectively. Using this, we can rewrite the above expression for $x(t)$ as

$$x(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \pm X(-\omega)e^{j\omega t}d\omega.$$

Now, we employ a change of variable. Let $\lambda = -\omega$ so that $d\lambda = -d\omega$. Applying this change of variable, we obtain

$$\begin{aligned}
x(t) &= \tfrac{1}{2\pi}\int_{\infty}^{-\infty} \pm X(\lambda)e^{-j\lambda t}(-1)d\lambda \\
&= \pm\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)e^{-j\lambda t}d\lambda \\
&= \pm\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)e^{j\lambda(-t)}d\lambda \\
&= \pm x(-t).
\end{aligned}$$

Therefore, x is even/odd. This completes the proof. ∎

In other words, the preceding theorem simply states that the forward and inverse Fourier transforms preserve even/odd symmetry.

6.7.14 Real Functions

As it turns out, the Fourier transform of a real-valued function has a special structure, as given by the theorem below.

Theorem 6.17 (Real-valued functions)**.** *A function x is real-valued if and only if its Fourier transform X satisfies*

$$X(\omega) = X^*(-\omega) \text{ for all } \omega$$

(i.e., X is conjugate symmetric).

Proof. From the definition of the Fourier transform, we have

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. \tag{6.13}$$

Substituting $-\omega$ for ω in the preceding equation, we have

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt.$$

Conjugating both sides of this equation, we obtain

$$X^*(-\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt. \tag{6.14}$$

First, we show that x being real-valued implies that X is conjugate symmetric. Suppose that x is real-valued. Since x is real-valued, we can replace x^* with x in (6.14) to yield

$$X^*(-\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

Observing that the right-hand side is simply $X(\omega)$, we have

$$X^*(-\omega) = X(\omega).$$

Thus, x being real-valued implies that X is conjugate symmetric.

Next, we show that X being conjugate symmetric implies that x is real-valued. Suppose that X is conjugate symmetric. Since X is conjugate symmetric, the expressions for $X(\omega)$ in (6.13) and $X^*(-\omega)$ in (6.14) must be equal. Thus, we can write

$$\begin{aligned} X(\omega) - X^*(-\omega) &= 0 \\ \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt - \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt &= 0 \\ \int_{-\infty}^{\infty} [x(t) - x^*(t)]e^{-j\omega t}dt &= 0. \end{aligned}$$

This implies that $x^* = x$. Therefore, x is real-valued. Thus, X being conjugate symmetric implies that x is real-valued. This completes the proof. ■

Suppose that X is the Fourier transform of a real-valued function x so that X is conjugate symmetric. From properties of complex numbers, we can show that that X being conjugate symmetric is equivalent to

$$|X(\omega)| = |X(-\omega)| \text{ for all } \omega \quad \text{and} \tag{6.15a}$$

$$\arg X(\omega) = -\arg X(-\omega) \text{ for all } \omega \tag{6.15b}$$

(i.e., the magnitude and argument of X are even and odd, respectively).

Since the Fourier transform X of a real-valued function x is conjugate symmetric, the graph of X for negative values is completely redundant and can be determined from the graph of X for nonnegative values. Lastly, note that x being real-valued does not necessarily imply that X is real-valued, since a conjugate-symmetric function need not be real-valued.

Table 6.1: Properties of the CT Fourier transform

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time-Domain Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency-Domain Shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(\omega)X_2(\omega)$
Time-Domain Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1 * X_2(\omega)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
Frequency-Domain Differentiation	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
Time-Domain Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$

Property	
Parseval's Relation	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Even	x is even $\Leftrightarrow$ X is even
Odd	x is odd $\Leftrightarrow$ X is odd
Real	x is real $\Leftrightarrow$ X is conjugate symmetric

6.8 Continuous-Time Fourier Transform of Periodic Functions

By making use of the generalized Fourier transform briefly discussed in Section 6.3, the Fourier transform can also be applied to periodic functions. In particular, the Fourier transform of a periodic function can be computed using the result below.

Theorem 6.18 (Fourier transform of a periodic function)**.** *Let x be a T-periodic function with frequency $\omega_0 = \frac{2\pi}{T}$ and Fourier series coefficient sequence a. Let x_T denote the function*

$$x_T(t) = \begin{cases} x(t) & -\frac{T}{2} \leq t < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

(i.e., x_T is a truncated/windowed version of the function x). (Note that x_T is a function equal to x over a single period and zero elsewhere.) Let X_T denote the Fourier transform of x_T. The Fourier transform X of x is given by

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0), \tag{6.16a}$$

or equivalently,

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0). \tag{6.16b}$$

Furthermore, a and X_T are related by

$$a_k = \tfrac{1}{T} X_T(k\omega_0). \tag{6.17}$$

Proof. Since x is T-periodic, we can express it using a Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{6.18a}$$

where

$$a_k = \tfrac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt. \tag{6.18b}$$

Consider the expression for a_k in (6.18b). Since $x_T(t) = x(t)$ for a single period of x and is zero otherwise, we can rewrite (6.18b) as

$$\begin{aligned} a_k &= \tfrac{1}{T} \int_{-\infty}^{\infty} x_T(t) e^{-jk\omega_0 t} dt \\ &= \tfrac{1}{T} X_T(k\omega_0). \end{aligned} \tag{6.19}$$

Thus, we have shown (6.17) to be correct.

Now, let us consider the Fourier transform X of x. By taking the Fourier transform of both sides of (6.18a), we obtain

$$\begin{aligned} X(\omega) &= \mathcal{F}\left\{ \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right\}(\omega) \\ &= \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right) e^{-j\omega t} dt. \end{aligned}$$

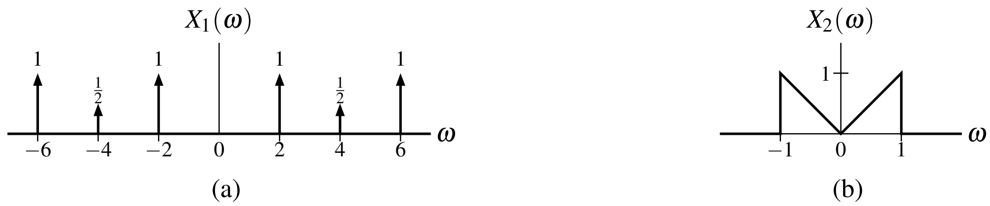


Figure 6.6: Frequency spectra. The frequency spectra (a) X_1 and (b) X_2.

Reversing the order of summation and integration, we have

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt \\ &= \sum_{k=-\infty}^{\infty} a_k \mathcal{F}\{e^{jk\omega_0 t}\}(\omega). \end{aligned} \tag{6.20}$$

From Table 6.2, we know that $e^{j\lambda t} \overset{\text{CTFT}}{\longleftrightarrow} 2\pi\delta(\omega-\lambda)$. So, we can simplify (6.20) to obtain

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} a_k [2\pi\delta(\omega - k\omega_0)] \\ &= \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0). \end{aligned} \tag{6.21}$$

Thus, we have shown (6.16a) to be correct. Furthermore, by substituting (6.19) into (6.21), we have

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} 2\pi[\tfrac{1}{T} X_T(k\omega_0)]\delta(\omega - k\omega_0) \\ &= \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0). \end{aligned}$$

Thus, we have shown (6.16b) to be correct. This completes the proof. ■

Theorem 6.18 above provides two formulas for computing the Fourier transform X of a periodic function x. One formula is in written terms of the Fourier series coefficient sequence a of x, while the other formula is in written in terms of the Fourier transform X_T of a function consisting of a single period of x. The choice of which formula to use would be driven by what information is available or most easily determined. For example, if the Fourier series coefficients of x were known, the use of (6.16b) would likely be preferred.

From Theorem 6.18, we can also make a few important observations. First, the Fourier transform of a periodic function is a series of impulse functions located at integer multiples of the fundamental frequency ω_0. The weight of each impulse is 2π times the corresponding Fourier series coefficient. Second, the Fourier series coefficient sequence a of the periodic function x is produced by sampling the Fourier transform of x_T at integer multiples of the fundamental frequency ω_0 and scaling the resulting sequence by $\frac{1}{T}$.

Example 6.20. Let X_1 and X_2 denote the Fourier transforms of x_1 and x_2, respectively. Suppose that X_1 and X_2 are as shown in Figures 6.6(a) and (b). Determine whether x_1 and x_2 are periodic.

Solution. We know that the Fourier transform X of a T-periodic function x must be of the form

$$X(\omega) = \sum_{k=-\infty}^{\infty} \alpha_k \delta(\omega - k\omega_0),$$

where $\omega_0 = \frac{2\pi}{T}$ and the $\{\alpha_k\}$ are complex constants. The spectrum X_1 does have this form, with $\omega_0 = 2$ and $T = \frac{2\pi}{2} = \pi$. Therefore, x_1 must be π-periodic. The spectrum X_2 does not have this form. Therefore, x_2 must not be periodic. ■

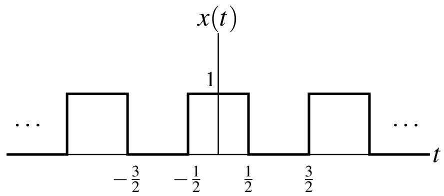


Figure 6.7: Periodic function x.

Example 6.21. Consider the periodic function x with fundamental period $T = 2$ as shown in Figure 6.7. Using the Fourier transform, find the Fourier series representation of x.

Solution. Let ω_0 denote the fundamental frequency of x. We have that $\omega_0 = \frac{2\pi}{T} = \pi$. Let $y(t) = \operatorname{rect} t$ (i.e., y corresponds to a single period of the periodic function x). Thus, we have that

$$x(t) = \sum_{k=-\infty}^{\infty} y(t-2k).$$

Let Y denote the Fourier transform of y. Taking the Fourier transform of y, we obtain

$$Y(\omega) = \mathcal{F}\{\operatorname{rect} t\}(\omega) = \operatorname{sinc}\left(\tfrac{1}{2}\omega\right).$$

Now, we seek to find the Fourier series representation of x, which has the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Using the Fourier transform, we have

$$\begin{aligned} c_k &= \tfrac{1}{T} Y(k\omega_0) \\ &= \tfrac{1}{2} \operatorname{sinc}\left(\tfrac{\omega_0}{2} k\right) \\ &= \tfrac{1}{2} \operatorname{sinc}\left(\tfrac{\pi}{2} k\right). \end{aligned}$$

■

6.9 More Fourier Transforms

Throughout this chapter, we have derived a number of Fourier transform pairs. Some of these and other important transform pairs are listed in Table 6.2. Using the various Fourier transform properties listed in Table 6.1 and the Fourier transform pairs listed in Table 6.2, we can determine (more easily) the Fourier transform of more complicated functions.

Example 6.22. Suppose that $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, $y(t) \overset{\text{CTFT}}{\longleftrightarrow} Y(\omega)$, and

$$y(t) = \mathcal{D}^2 x(t-2),$$

where $\mathcal{D}$ denotes the derivative operator. Express Y in terms of X.

Solution. Let $v_1(t) = \mathcal{D}^2 x(t)$ so that $y(t) = v_1(t-2)$. Now, we take the Fourier transforms of each of these two equations. Taking the Fourier transform of v_1 using the time-domain differentiation property of the Fourier transform, we obtain

$$\begin{aligned} V_1(\omega) &= (j\omega)^2 X(\omega) \\ &= -\omega^2 X(\omega). \end{aligned} \tag{6.22}$$

Table 6.2: Transform pairs for the CT Fourier transform

Pair	$x(t)$	$X(\omega)$
1	$\delta(t)$	1
2	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
3	1	$2\pi\delta(\omega)$
4	$\operatorname{sgn} t$	$\frac{2}{j\omega}$
5	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
6	$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
7	$\sin(\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
8	$\operatorname{rect}\left(\frac{t}{T}\right)$	$\lvert T\rvert \operatorname{sinc}\left(\frac{T\omega}{2}\right)$
9	$\frac{\lvert B\rvert}{\pi}\operatorname{sinc}(Bt)$	$\operatorname{rect}\left(\frac{\omega}{2B}\right)$
10	$e^{-at}u(t),\ \operatorname{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
11	$t^{n-1}e^{-at}u(t),\ \operatorname{Re}\{a\}>0$	$\frac{(n-1)!}{(a+j\omega)^n}$
12	$\operatorname{tri}\left(\frac{t}{T}\right)$	$\frac{\lvert T\rvert}{2}\operatorname{sinc}^2\left(\frac{T\omega}{4}\right)$

Taking the Fourier transform of y using the time-shifting property of the Fourier transform, we have

$$Y(\omega) = e^{-j2\omega}V_1(\omega). \tag{6.23}$$

Substituting (6.22) into (6.23), we obtain

$$\begin{aligned} Y(\omega) &= e^{-j2\omega}[-\omega^2 X(\omega)] \\ &= -e^{-j2\omega}\omega^2 X(\omega). \end{aligned}$$

■

Example 6.23. Suppose that $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, $y(t) \overset{\text{CTFT}}{\longleftrightarrow} Y(\omega)$, and

$$y(t) = x(at-b),$$

where a and b are real constants and $a \neq 0$. Express Y in terms of X.

Solution. We rewrite y as

$$y(t) = v_1(at)$$

where

$$v_1(t) = x(t-b).$$

We now take the Fourier transform of both sides of each of the preceding equations. Using the time-shifting property of the Fourier transform, we can write

$$V_1(\omega) = e^{-jb\omega}X(\omega). \tag{6.24}$$

Using the time-scaling property of the Fourier transform, we can write

$$Y(\omega) = \tfrac{1}{|a|}V_1\left(\tfrac{\omega}{a}\right). \tag{6.25}$$

Substituting the expression for $V_1(\omega)$ in (6.24) into (6.25), we obtain

$$Y(\omega) = \tfrac{1}{|a|}e^{-j(b/a)\omega}X\left(\tfrac{\omega}{a}\right).$$

■

Example 6.24. Consider the periodic function x given by

$$x(t) = \sum_{k=-\infty}^{\infty} x_0(t-kT),$$

where

$$x_0(t) = A\,\mathrm{rect}\left(\tfrac{2t}{T}\right)$$

and A and T are real constants with $T > 0$. Find the Fourier transform X of the function x.

Solution. From (6.16b), we know that

$$\begin{aligned} X(\omega) &= \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} x_0(t-kT)\right\}(\omega) \\ &= \sum_{k=-\infty}^{\infty} \omega_0 X_0(k\omega_0)\delta(\omega-k\omega_0). \end{aligned}$$

So, we need to find X_0. Using the linearity property of the Fourier transform and Table 6.2, we have

$$\begin{aligned} X_0(\omega) &= \mathcal{F}\left\{A \operatorname{rect}\left(\tfrac{2t}{T}\right)\right\}(\omega) \\ &= A\mathcal{F}\left\{\operatorname{rect}\left(\tfrac{2t}{T}\right)\right\}(\omega) \\ &= \tfrac{AT}{2} \operatorname{sinc}\left(\tfrac{\omega T}{4}\right). \end{aligned}$$

Thus, we have that

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} \omega_0 \left(\tfrac{AT}{2}\right) \operatorname{sinc}\left(\tfrac{k\omega_0 T}{4}\right) \delta(\omega - k\omega_0) \\ &= \sum_{k=-\infty}^{\infty} \pi A \operatorname{sinc}\left(\tfrac{\pi k}{2}\right) \delta(\omega - k\omega_0). \end{aligned} \qquad \blacksquare$$

Example 6.25. Find the Fourier transform X of the function x given by

$$x(t) = \int_{-\infty}^{t} e^{-(3+j2)\tau} u(\tau) d\tau.$$

Solution. We can rewrite x as

$$x(t) = \int_{-\infty}^{t} v_1(\tau) d\tau, \tag{6.26}$$

where

$$v_1(t) = e^{-j2t} v_2(t) \quad \text{and} \tag{6.27}$$

$$v_2(t) = e^{-3t} u(t). \tag{6.28}$$

Taking the Fourier transform of (6.26) (using the time-domain integration property of the Fourier transform), we have

$$X(\omega) = \tfrac{1}{j\omega} V_1(\omega) + \pi V_1(0) \delta(\omega). \tag{6.29}$$

Taking the Fourier transform of (6.27) (using the frequency-domain shifting property of the Fourier transform), we have

$$V_1(\omega) = V_2(\omega + 2). \tag{6.30}$$

Taking the Fourier transform of (6.28) by using Table 6.2 (i.e., the entry for $\mathcal{F}\{e^{-at}u(t)\}$), we have

$$V_2(\omega) = \frac{1}{3 + j\omega}. \tag{6.31}$$

Combining (6.29), (6.30), and (6.31), we obtain

$$\begin{aligned} X(\omega) &= \tfrac{1}{j\omega} V_1(\omega) + \pi V_1(0) \delta(\omega) \\ &= \tfrac{1}{j\omega} V_2(\omega + 2) + \pi V_2(2) \delta(\omega) \\ &= \frac{1}{j\omega}\left(\frac{1}{3 + j(\omega + 2)}\right) + \pi\left(\frac{1}{3 + j2}\right) \delta(\omega). \end{aligned} \qquad \blacksquare$$

Example 6.26. Let X and Y denote the Fourier transforms of x and y, respectively. Suppose that $y(t) = x(t)\cos(at)$, where a is a nonzero real constant. Find an expression for Y in terms of X.

Solution. Essentially, we need to take the Fourier transform of both sides of the given equation. There are two obvious ways in which to do this. One is to use the time-domain multiplication property of the Fourier transform, and another is to use the frequency-domain shifting property. We will solve this problem using each method in turn in order to show that the two approaches do not involve an equal amount of effort.

FIRST SOLUTION (USING AN UNENLIGHTENED APPROACH). We use the time-domain multiplication property. To allow for simpler notation in what follows, we define

$$v(t) = \cos(at)$$

and let V denote the Fourier transform of v. From Table 6.2, we have that

$$V(\omega) = \pi[\delta(\omega - a) + \delta(\omega + a)].$$

Taking the Fourier transform of both sides of the given equation, we obtain

$$\begin{aligned} Y(\omega) &= \mathcal{F}\{x(t)v(t)\}(\omega) \\ &= \tfrac{1}{2\pi} X * V(\omega) \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)V(\omega-\lambda)d\lambda. \end{aligned}$$

Substituting the above expression for V, we obtain

$$\begin{aligned} Y(\omega) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)(\pi[\delta(\omega-\lambda-a)+\delta(\omega-\lambda+a)])d\lambda \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} X(\lambda)[\delta(\omega-\lambda-a)+\delta(\omega-\lambda+a)]d\lambda \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\omega-\lambda-a)d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\omega-\lambda+a)d\lambda\right] \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\lambda-\omega+a)d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\lambda-\omega-a)d\lambda\right] \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta[\lambda-(\omega-a)]d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta[\lambda-(\omega+a)]d\lambda\right] \\ &= \tfrac{1}{2}[X(\omega-a)+X(\omega+a)] \\ &= \tfrac{1}{2}X(\omega-a)+\tfrac{1}{2}X(\omega+a). \end{aligned}$$

Note that the above solution is essentially identical to the one appearing earlier in Example 6.15 on page 161.

SECOND SOLUTION (USING AN ENLIGHTENED APPROACH). We use the frequency-domain shifting property. Taking the Fourier transform of both sides of the given equation, we obtain

$$\begin{aligned} Y(\omega) &= \mathcal{F}\{x(t)\cos(at)\}(\omega) \\ &= \mathcal{F}\left\{\tfrac{1}{2}(e^{jat}+e^{-jat})x(t)\right\}(\omega) \\ &= \tfrac{1}{2}\mathcal{F}\left\{e^{jat}x(t)\right\}(\omega) + \tfrac{1}{2}\mathcal{F}\left\{e^{-jat}x(t)\right\}(\omega) \\ &= \tfrac{1}{2}X(\omega-a)+\tfrac{1}{2}X(\omega+a). \end{aligned}$$

COMMENTARY. Clearly, of the above two solution methods, the second approach is simpler and much less error prone. Generally, the use of the time-domain multiplication property tends to lead to less clean solutions, as this forces a convolution to be performed in the frequency domain and convolution is often best avoided if possible. ■

6.10 Frequency Spectra of Functions

The Fourier transform representation expresses a function in terms of complex sinusoids at all frequencies. In this sense, the Fourier transform representation captures information about the frequency content of a function. For example, suppose that we have a function x with Fourier transform X. If X is nonzero at some frequency ω_0, then the function x contains some information at the frequency ω_0. On the other hand, if X is zero at the frequency ω_0, then the function x has no information at that frequency. In this way, the Fourier transform representation provides a means for measuring the frequency content of a function. This distribution of information in a function over different frequencies is referred to as the **frequency spectrum** of the function. That is, X is the frequency spectrum of x.

To gain further insight into the role played by the Fourier transform X in the context of the frequency spectrum of x, it is helpful to write the Fourier transform representation of x with X expressed in polar form as follows:

$$\begin{aligned}
x(t) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|\, e^{j\arg X(\omega)}e^{j\omega t}d\omega \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|\, e^{j[\omega t+\arg X(\omega)]}d\omega.
\end{aligned}$$

In effect, the quantity $|X(\omega)|$ is a weight that determines how much the complex sinusoid at frequency ω contributes to the integration result $x(t)$. Perhaps, this can be more easily seen if we express the above integral as the limit of a sum, derived from an approximation of the integral using the area of rectangles (i.e., $\int_{-\infty}^{\infty} f(x)dx = \lim_{\Delta x\to 0}\sum_{k=-\infty}^{\infty}\Delta x f(k\Delta x)$). Expressing x in this way, we obtain

$$\begin{aligned}
x(t) &= \lim_{\Delta\omega\to 0}\tfrac{1}{2\pi}\sum_{k=-\infty}^{\infty}\Delta\omega\, |X(k\Delta\omega)|\, e^{j[k\Delta\omega t+\arg X(k\Delta\omega)]} \\
&= \lim_{\Delta\omega\to 0}\tfrac{1}{2\pi}\sum_{k=-\infty}^{\infty}\Delta\omega\, \left|X(\omega')\right| e^{j[\omega' t+\arg X(\omega')]},
\end{aligned}$$

where $\omega' = k\Delta\omega$. From the last line of the above equation, the kth term in the summation (associated with the frequency $\omega' = k\Delta\omega$) corresponds to a complex sinusoid with fundamental frequency ω' that has had its amplitude scaled by a factor of $|X(\omega')|$ and has been time-shifted by an amount that depends on $\arg X(\omega')$. For a given $\omega' = k\Delta\omega$ (which is associated with the kth term in the summation), the larger $|X(\omega')|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\omega' t}$ will be, and therefore the larger the contribution the kth term will make to the overall summation. In this way, we can use $|X(\omega')|$ as a measure of how much information a function x has at the frequency ω'.

Note that, since the Fourier transform X is a function of a real variable, a function can, in the most general case, have information at *any arbitrary* real frequency. This is different from the case of frequency spectra in the Fourier series context (which deals only with periodic functions), where a function can only have information at certain specific frequencies (namely, at integer multiples of the fundamental frequency). There is no inconsistency here, however. As we saw in Section 6.8, in the case of periodic functions the Fourier transform will also be zero, except possibly at integer multiples of the fundamental frequency.

Since the frequency spectrum is complex (in the general case), it is usually represented using two plots, one showing the magnitude of X and one showing the argument. We refer to $|X(\omega)|$ as the **magnitude spectrum** of the function x. Similarly, we refer to $\arg X(\omega)$ as the **phase spectrum** of the function x. In the special case that X is a real (or purely imaginary) function, we usually plot the frequency spectrum directly on a single graph.

Recall, from Theorem 6.17 earlier, that the Fourier transform X of a real-valued function x must be conjugate symmetric. So, if x is real-valued, then

$$\begin{aligned}
&|X(\omega)| = |X(-\omega)| \text{ for all } \omega \quad \text{and} \\
&\arg X(\omega) = -\arg X(-\omega) \text{ for all } \omega
\end{aligned}$$

(i.e., the magnitude and argument of X are even and odd, respectively). (See (6.15a) and (6.15b).) Due to the symmetry in the frequency spectra of real-valued functions, we typically ignore negative frequencies when dealing with such

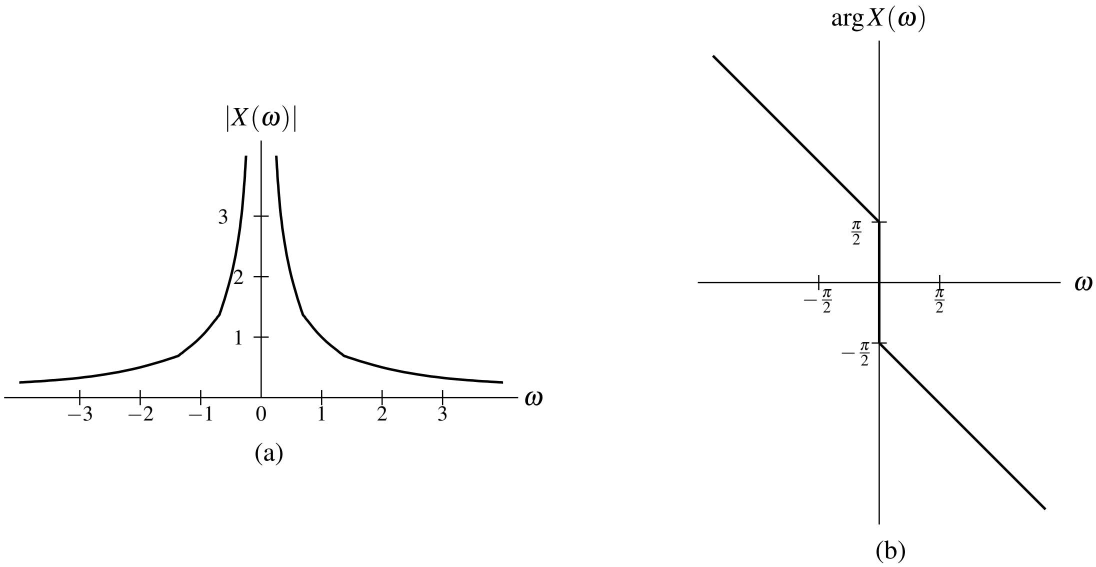


Figure 6.8: Frequency spectrum of the amplitude-scaled time-shifted signum function x. (a) Magnitude spectrum and (b) phase spectrum of x.

functions. In the case of functions that are complex-valued but not real-valued, frequency spectra do not possess the above symmetry, and negative frequencies become important.

Example 6.27 (Frequency spectrum of an amplitude-scaled time-shifted signum function)**.** Consider the function

$$x(t) = \tfrac{1}{2}\operatorname{sgn}(t-1).$$

We can show that this function has the Fourier transform

$$X(\omega) = \tfrac{1}{j\omega}e^{-j\omega}.$$

In this case, X is neither purely real nor purely imaginary, so we use two separate graphs to present the frequency spectrum X. We plot the magnitude spectrum and phase spectrum as shown in Figures 6.8(a) and (b), respectively. ■

Example 6.28 (Frequency spectrum of a time-scaled sinc function)**.** Consider the function

$$x(t) = \operatorname{sinc}\left(\tfrac{1}{2}t\right).$$

We can show that this function has the Fourier transform

$$X(\omega) = 2\pi \operatorname{rect}\omega.$$

Since, in this case, X is real, we can plot the frequency spectrum X on a single graph, as shown in Figure 6.9. ■

Example 6.29 (Frequency spectrum of a time-shifted signum function)**.** The function

$$x(t) = \operatorname{sgn}(t-1)$$

has the Fourier transform

$$X(\omega) = \tfrac{2}{j\omega}e^{-j\omega}.$$

(a) Find and plot the magnitude and phase spectra of x. (b) Determine at what frequency (or frequencies) x has the most information.

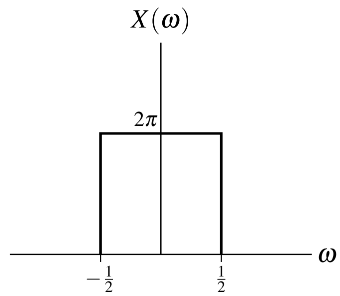


Figure 6.9: Frequency spectrum of the time-scaled sinc function x.

Solution. (a) First, we find the magnitude spectrum $|X(\omega)|$. From the expression for $X(\omega)$, we can write

$$\begin{aligned}|X(\omega)| &= \left|\tfrac{2}{j\omega}e^{-j\omega}\right| \\ &= \left|\tfrac{2}{j\omega}\right|\left|e^{-j\omega}\right| \\ &= \left|\tfrac{2}{j\omega}\right| \\ &= \tfrac{2}{|\omega|}.\end{aligned}$$

Next, we find the phase spectrum $\arg X(\omega)$. First, we observe that $\arg X(\omega)$ is not well defined if $\omega = 0$. So, we assume that $\omega \neq 0$. From the expression for $X(\omega)$, we can write (for $\omega \neq 0$)

$$\begin{aligned}\arg X(\omega) &= \arg\left(\tfrac{2}{j\omega}e^{-j\omega}\right) \\ &= \arg e^{-j\omega} + \arg\tfrac{2}{j\omega} \\ &= -\omega + \arg\tfrac{2}{j\omega} \\ &= -\omega + \arg(-\tfrac{j2}{\omega}) \\ &= \begin{cases}-\frac{\pi}{2}-\omega & \omega > 0 \\ \frac{\pi}{2}-\omega & \omega < 0\end{cases} \\ &= -\tfrac{\pi}{2}\operatorname{sgn}\omega - \omega.\end{aligned}$$

In the above simplification, we used the fact that

$$\arg\left(\tfrac{2}{j\omega}\right) = \arg(-\tfrac{j2}{\omega}) = \begin{cases}-\frac{\pi}{2} & \omega > 0 \\ \frac{\pi}{2} & \omega < 0.\end{cases}$$

Finally, using numerical calculation, we can plot the graphs of $|X(\omega)|$ and $\arg X(\omega)$ to obtain the results shown in Figures 6.10(a) and (b).

(b) Since $|X(\omega)|$ is largest for $\omega = 0$, x has the most information at the frequency 0. ■

6.11 Bandwidth of Functions

A function x with Fourier transform X is said to be **bandlimited** if, for some nonnegative real constant B,

$$X(\omega) = 0 \text{ for all } |\omega| > B.$$

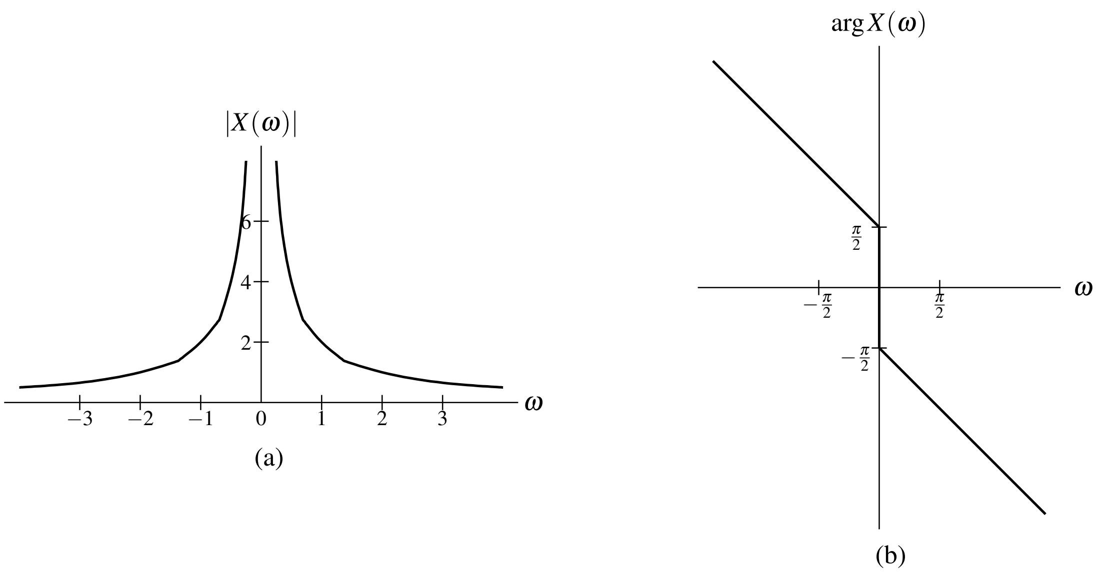


Figure 6.10: Frequency spectrum of the time-shifted signum function. (a) Magnitude spectrum and (b) phase spectrum of x.

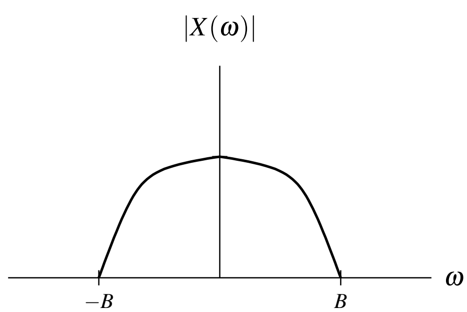


Figure 6.11: Bandwidth of a function x with the Fourier transform X.

We sometimes refer to B as the **bandwidth** of x. An illustrative example is provided in Figure 6.11.

One can show that a function cannot be both time limited and bandlimited. (A proof of this fact is considered in Exercise 6.25.) To help understand why this is so we recall the time/frequency scaling property of the Fourier transform. From this property, we know that as we compress a function x (by time scaling), its Fourier transform X will expand (by time scaling). Similarly, as we compress the Fourier transform X (by time scaling), x will expand (by time scaling). So, clearly, there is an inverse relationship between the time-extent and bandwidth of a function.

6.12 Energy-Density Spectra

Suppose that we have a function x with finite energy E and Fourier transform X. By definition, the energy contained in x is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

We can use Parseval's relation (6.12) to express E in terms of X as

$$E = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega.$$

Thus, the energy E is given by

$$E = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} E_x(\omega) d\omega,$$

where

$$E_x(\omega) = |X(\omega)|^2 .$$

We refer to E_x as the **energy-density spectrum** of the function x. The function E_x indicates how the energy in x is distributed with respect to frequency. For example, the energy contributed by frequencies in the range $[\omega_1, \omega_2]$ is simply given by

$$\tfrac{1}{2\pi} \int_{\omega_1}^{\omega_2} E_x(\omega) d\omega.$$

Example 6.30. Compute the energy-density spectrum E_x of the function

$$x(t) = \operatorname{sinc}\left(\tfrac{1}{2}t\right).$$

Determine the amount of energy contained in the frequency components in the range $\omega \in [-\frac{1}{4}, \frac{1}{4}]$. Also, determine the total amount of energy in the function.

Solution. First, we compute the Fourier transform X of x. We obtain

$$X(\omega) = 2\pi \operatorname{rect} \omega.$$

Computing the energy spectral density E_x, we have

$$\begin{aligned} E_x(\omega) &= |X(\omega)|^2 \\ &= |2\pi \operatorname{rect} \omega|^2 \\ &= 4\pi^2 \operatorname{rect}^2 \omega \\ &= 4\pi^2 \operatorname{rect} \omega. \end{aligned}$$

Let E_1 denote the energy contained in x for frequencies $|\omega| \in [-\frac{1}{4}, \frac{1}{4}]$. Then, we have

$$\begin{aligned} E_1 &= \tfrac{1}{2\pi}\int_{-1/4}^{1/4} E_x(\omega)d\omega \\ &= \tfrac{1}{2\pi}\int_{-1/4}^{1/4} 4\pi^2\operatorname{rect}(\omega)d\omega \\ &= \int_{-1/4}^{1/4} 2\pi d\omega \\ &= \pi. \end{aligned}$$

Let E denote the total amount of energy in x. Then, we have

$$\begin{aligned} E &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} E_x(\omega)d\omega \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} 4\pi^2\operatorname{rect}(\omega)d\omega \\ &= \int_{-1/2}^{1/2} 2\pi d\omega \\ &= 2\pi. \end{aligned}$$

■

6.13 Characterizing LTI Systems Using the Fourier Transform

Consider a LTI system with input x, output y, and impulse response h. Such a system is depicted in Figure 6.12. The behavior of such a system is governed by the equation

$$y(t) = x * h(t). \tag{6.32}$$

Let X, Y, and H denote the Fourier transforms of x, y, and h, respectively. Taking the Fourier transform of both sides of (6.32) and using the the time-domain convolution property of the Fourier transform, we obtain

$$Y(\omega) = X(\omega)H(\omega). \tag{6.33}$$

This result provides an alternative way of viewing the behavior of a LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output functions. In other words, we have a system resembling that in Figure 6.13. In this case, however, the convolution operation from the time domain is replaced by multiplication in the frequency domain. The frequency spectrum (i.e., Fourier transform) of the output is the product of the frequency spectrum (i.e., Fourier transform) of the input and the frequency spectrum (i.e., Fourier transform) of the impulse response. As a matter of terminology, we refer to H as the **frequency response** of the system. The system behavior is completely characterized by the frequency response H. If we know the input, we can compute its Fourier transform X, and then determine the Fourier transform Y of the output. Using the inverse Fourier transform, we can then determine the output y.

In the most general case, the frequency response H is a complex-valued function. Thus, we can represent H in terms of its magnitude and argument. We refer to the magnitude of H as the **magnitude response** of the system. Similarly, we refer to the argument of H as the **phase response** of the system.

From (6.33), we can write

$$\begin{aligned} |Y(\omega)| &= |X(\omega)H(\omega)| \\ &= |X(\omega)|\,|H(\omega)| \quad \text{and} \end{aligned} \tag{6.34a}$$

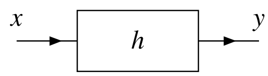


Figure 6.12: Time-domain view of a LTI system with input x, output y, and impulse response h.

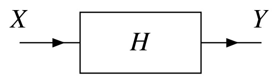


Figure 6.13: Frequency-domain view of a LTI system with input spectrum X, output spectrum Y, and frequency response H.

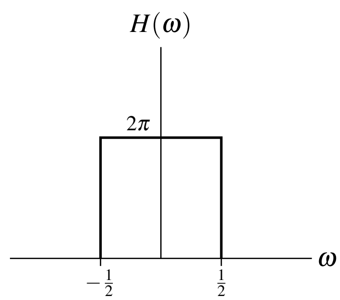


Figure 6.14: Frequency response of example system.

$$\begin{aligned}\arg Y(\omega) &= \arg[X(\omega)H(\omega)] \\ &= \arg X(\omega) + \arg H(\omega). \end{aligned} \tag{6.34b}$$

From (6.34a), we can see that the magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude spectrum of the impulse response. From (6.34b), we have that the phase spectrum of the output equals the phase spectrum of the input plus the phase spectrum of the impulse response.

Since the frequency response H is simply the frequency spectrum of the impulse response h, if h is real, then (for the reasons explained in Section 6.10)

$$|H(\omega)| = |H(-\omega)| \text{ for all } \omega \quad \text{and}$$
$$\arg H(\omega) = -\arg H(-\omega) \text{ for all } \omega$$

(i.e., the magnitude and phase responses are even and odd, respectively).

Example 6.31. Consider a LTI system with impulse response

$$h(t) = \operatorname{sinc}\left(\tfrac{1}{2}t\right).$$

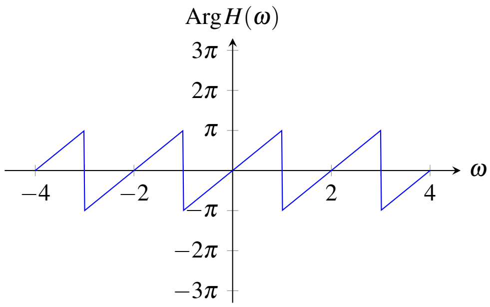
This system has the frequency response

$$H(\omega) = 2\pi \operatorname{rect} \omega.$$

In this particular case, H is real. So, we can plot the frequency response H on a single graph, as shown in Figure 6.14. ■

6.13.1 Unwrapped Phase

Since the argument of a complex number is not uniquely determined, the argument of a complex-valued function is also not uniquely determined. Consequently, we have some freedom in how we define a function that corresponds to the phase (i.e., argument) of a complex-valued function. Often, for convenience, we restrict the argument to lie in an



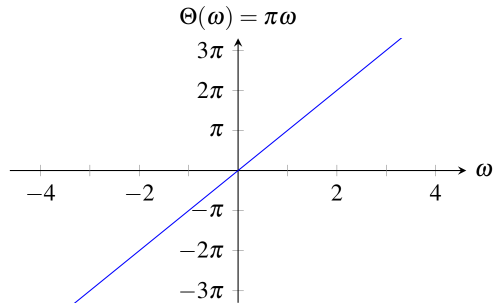


Figure 6.15: Unwrapped phase example. (a) The phase function restricted such that its range is in $(-\pi, \pi]$ and (b) the corresponding unwrapped phase.

interval of length 2π, such as the interval $(-\pi, \pi]$ which corresponds to the principal argument. Defining the phase of a complex-valued function in this way, however, can often result in a phase function with unnecessary discontinuities. This motivates the notion of unwrapped phase. The **unwrapped phase** is simply the phase defined in such a way so as not to restrict the phase to an interval of length 2π and to keep the phase function continuous to the greatest extent possible. An example illustrating the notion of unwrapped phase is given below.

Example 6.32 (Unwrapped phase)**.** Consider the phase response of a LTI system with the frequency response

$$H(\omega) = e^{j\pi\omega}.$$

We can choose to define the phase (i.e., argument) of H by simply using the principal argument (i.e., $\operatorname{Arg} H(\omega)$). This yields the phase function shown in Figure 6.15(a). Using the principal argument in this way, however, unnecessarily introduces discontinuities into the phase function. For this reason, we sometimes prefer to define the phase function in such a way as to eliminate such unnecessary discontinuities. This motivates the use of the unwrapped phase. The function H has the unwrapped phase Θ given by

$$\Theta(\omega) = \pi\omega.$$

A plot of Θ is shown in Figure 6.15(b). Unlike the function in Figure 6.15(a) (which has numerous discontinuities), the function in Figure 6.15(b) is continuous. Although the functions in these two figures are distinct, these functions are equivalent in the sense that they correspond to the same physical angular displacement (i.e., $e^{j\operatorname{Arg} H(\omega)} = e^{j\Theta(\omega)}$ for all $\omega \in \mathbb{R}$). ∎

6.13.2 Magnitude and Phase Distortion

Recall, from Corollary 5.1, that a LTI system $\mathcal{H}$ with frequency response H is such that

$$\mathcal{H}\left\{e^{j\omega t}\right\}(t) = H(\omega)e^{j\omega t}$$

(i.e., $e^{j\omega t}$ is an eigenfunction of $\mathcal{H}$ with eigenvalue $H(\omega)$). Expressing $H(\omega)$ in polar form, we have

$$\begin{aligned}\mathcal{H}\{e^{j\omega t}\}(t) &= |H(\omega)|\, e^{j\arg H(\omega)} e^{j\omega t} \\ &= |H(\omega)|\, e^{j[\omega t + \arg H(\omega)]} \\ &= |H(\omega)|\, e^{j\omega(t + \arg[H(\omega)]/\omega)}.\end{aligned}$$

This equation can be rewritten as

$$\mathcal{H}\{e^{j\omega t}\}(t) = |H(\omega)|\, e^{j\omega[t - \tau_p(\omega)]}, \tag{6.35a}$$

where

$$\tau_{\mathsf{p}}(\omega) = -\frac{\arg H(\omega)}{\omega}. \tag{6.35b}$$

Thus, the response of the system to the function $e^{j\omega t}$ is produced by applying two transformations to this function:

- (amplitude) scaling by $|H(\omega)|$; and
- translating by $\tau_{\mathsf{p}}(\omega)$.

Therefore, the magnitude response determines how different complex sinusoids are (amplitude) scaled by the system. Similarly, the phase response determines how different complex sinusoids are translated (i.e., delayed/advanced) by the system.

A system for which $|H(\omega)| = 1$ for all ω is said to be **allpass**[1]. In the case of an allpass system, the magnitude spectra of the system's input and output are identical. If a system is not allpass, it modifies the magnitude spectrum in some way. In situations where the magnitude spectrum is changed in an undesirable manner, **magnitude distortion** (i.e., distortion of the magnitude spectrum) is said to occur. If $|H(\omega)| = a$ for all ω, where a is a constant, every complex sinusoid is scaled by the same amount a when passing through the system. In practice, this type of change to the magnitude spectrum may sometimes be undesirable if $a \neq 1$. If $|H(\omega)|$ is not a constant, different complex sinusoids are scaled by different amounts. In practice, this type of change to the magnitude spectrum is usually undesirable and deemed to constitute magnitude distortion.

The function τ_{p} appearing in (6.35b) is known as the **phase delay** of the system. A system for which $\tau_{\mathsf{p}}(\omega) = 0$ for all ω is said to have **zero phase**. In the case of a system having zero phase, the phase spectra of the system's input and output are identical. In the case that the system does not have zero phase, the phase spectra of the system's input and output differ. In situations where the phase spectrum is changed in an undesirable manner, **phase distortion** (i.e., distortion of the phase spectrum) is said to occur. If $\tau_{\mathsf{p}}(\omega) = t_{\mathsf{d}}$ for all ω, where t_{d} is a constant, the system shifts all complex sinusoids by the same amount t_{d}. Note that $\tau_{\mathsf{p}}(\omega) = t_{\mathsf{d}}$ is equivalent to the (unwrapped) phase response being of the form

$$\arg H(\omega) = -t_{\mathsf{d}}\omega,$$

which is a linear function with a zero constant term. For this reason, a system with a constant phase delay is said to have **linear phase**. If $\tau_{\mathsf{p}}(\omega)$ is not a constant, different complex sinusoids are shifted by different amounts. In many practical applications, shifting different complex sinusoids by different amounts is undesirable. Therefore, systems that are not linear phase are typically deemed to introduce phase distortion. For this reason, in contexts where phase spectra are important, systems with either zero phase or linear phase are typically used.

Example 6.33 (Distortionless transmission)**.** Consider a LTI system with input x and output y given by

$$y(t) = x(t - t_0),$$

where t_0 is a real constant. That is, the output of the system is simply the input delayed by t_0. This type of system behavior is referred to as distortionless transmission, since the system allows the input to pass through to the output unmodified, except for a delay being introduced. This type of behavior is the ideal for which we strive in real-world communication systems (i.e., the received signal y equals a delayed version of the transmitted signal x). Taking the Fourier transform of the above equation, we have

$$Y(\omega) = e^{-j\omega t_0}X(\omega).$$

Thus, the system has the frequency response H given by

$$H(\omega) = e^{-j\omega t_0}.$$

[1] Some authors (e.g., [5, 7]) define an allpass system as one for which $|H(\omega)| = c$ for all ω, where c is a constant (and c is not necessarily 1).

Since $|H(\omega)| = 1$ for all ω, the system is allpass and does not introduce any magnitude distortion. The phase delay τ_p of the system is given by

$$\begin{aligned}\tau_\mathrm{p}(\omega) &= -\frac{\arg H(\omega)}{\omega} \\ &= -\left(\frac{-\omega t_0}{\omega}\right) \\ &= t_0.\end{aligned}$$

Since the phase delay is a constant, the system has linear phase and does not introduce any phase distortion (except for a trivial time shift of t_0). ■

Example 6.34 (Frequency spectra of images)**.** The human visual system is more sensitive to the phase spectrum of an image than its magnitude spectrum. This can be aptly demonstrated by separately modifying the magnitude and phase spectra of an image, and observing the effect. Below, we consider two variations on this theme.

Consider the `potatohead` and `hongkong` images shown in Figures 6.16(a) and (b), respectively. Replacing the magnitude spectrum of the `potatohead` image with the magnitude spectrum of the `hongkong` image (and leaving the phase spectrum unmodified), we obtain the new image shown in Figure 6.16(c). Although changing the magnitude spectrum has led to distortion, the basic essence of the original image has not been lost. On the other hand, replacing the phase spectrum of the `potatohead` image with the phase spectrum of the `hongkong` image (and leaving the magnitude spectrum unmodified), we obtain the image shown in Figure 6.16(d). Clearly, by changing the phase spectrum of the image, the fundamental nature of the image has been altered, with the new image more closely resembling the `hongkong` image than the original `potatohead` image.

A more extreme scenario is considered in Figure 6.17. In this case, we replace each of the magnitude and phase spectra of the `potatohead` image with random data, with this data being taken from the image consisting of random noise shown in Figure 6.17(b). When we completely replace the magnitude spectrum of the `potatohead` image with random values, we can still recognize the resulting image in Figure 6.17(c) as a very grainy version of the original `potatohead` image. On the other hand, when the phase spectrum of the `potatohead` image is replaced with random values, all visible traces of the original `potatohead` image are lost in the resulting image in Figure 6.17(d). ■

6.14 Interconnection of LTI Systems

From the properties of the Fourier transform and the definition of the frequency response, we can derive a number of equivalences involving the frequency response and series- and parallel-interconnected systems.

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with frequency responses H_1 and H_2, respectively, that are connected in a series configuration as shown in the left-hand side of Figure 6.18(a). Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(t) = h_1 * h_2(t).$$

Taking the Fourier transform of both sides of this equation yields

$$\begin{aligned}H(\omega) &= \mathcal{F}\{h_1 * h_2\}(\omega) \\ &= \mathcal{F}h_1(\omega)\mathcal{F}h_2(\omega) \\ &= H_1(\omega)H_2(\omega).\end{aligned}$$

Thus, we have the equivalence shown in Figure 6.18(a). Also, since multiplication commutes, we also have the equivalence shown in Figure 6.18(b).

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with frequency responses H_1 and H_2 that are connected in a parallel configuration as shown on the left-hand side of Figure 6.19. Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(t) = h_1(t) + h_2(t).$$

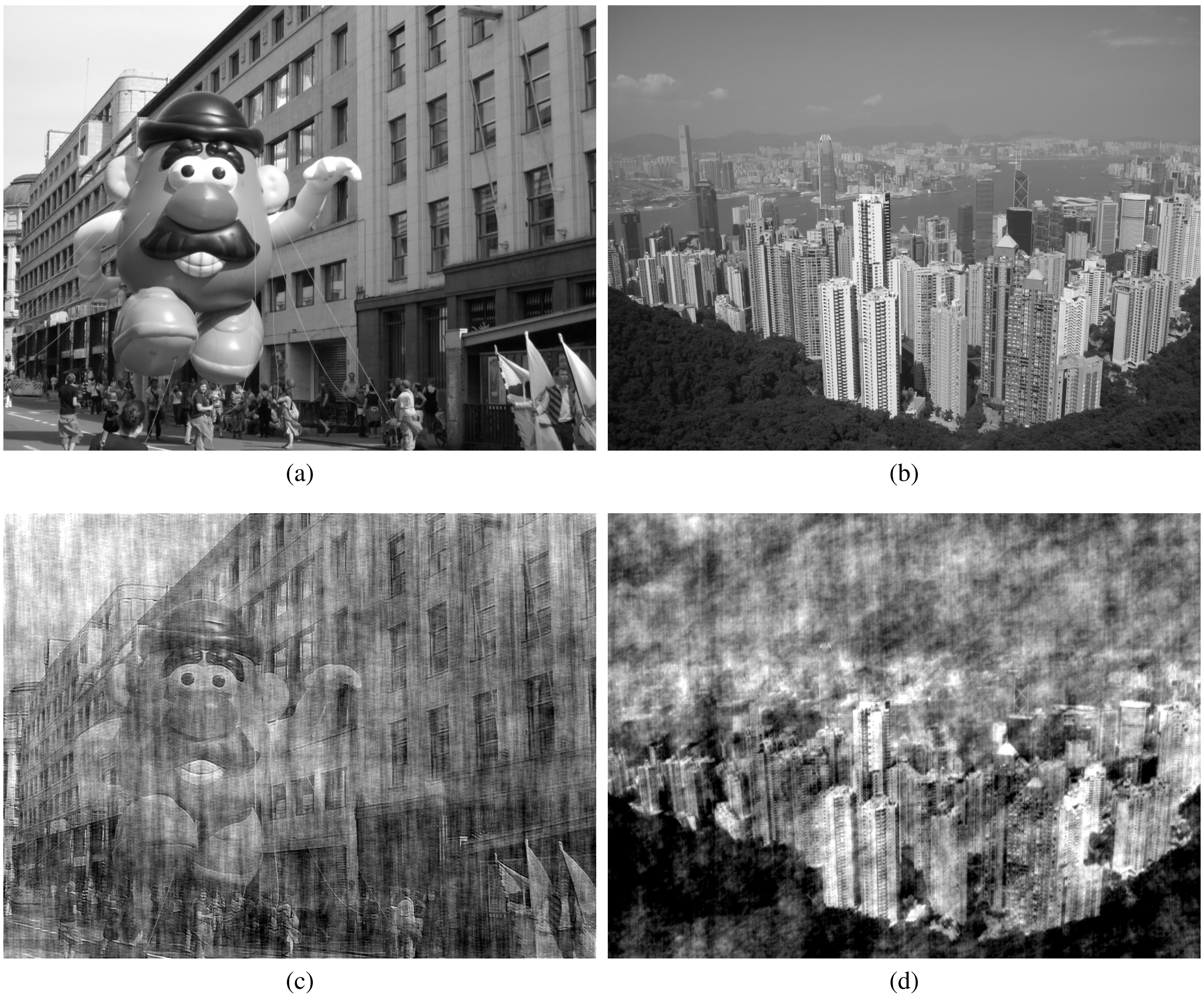


Figure 6.16: Importance of phase information in images. The (a) `potatohead` and (b) `hongkong` images. (c) The `potatohead` image after having its magnitude spectrum replaced with the magnitude spectrum of the `hongkong` image. (d) The `potatohead` image after having its phase spectrum replaced with the phase spectrum of the `hongkong` image.

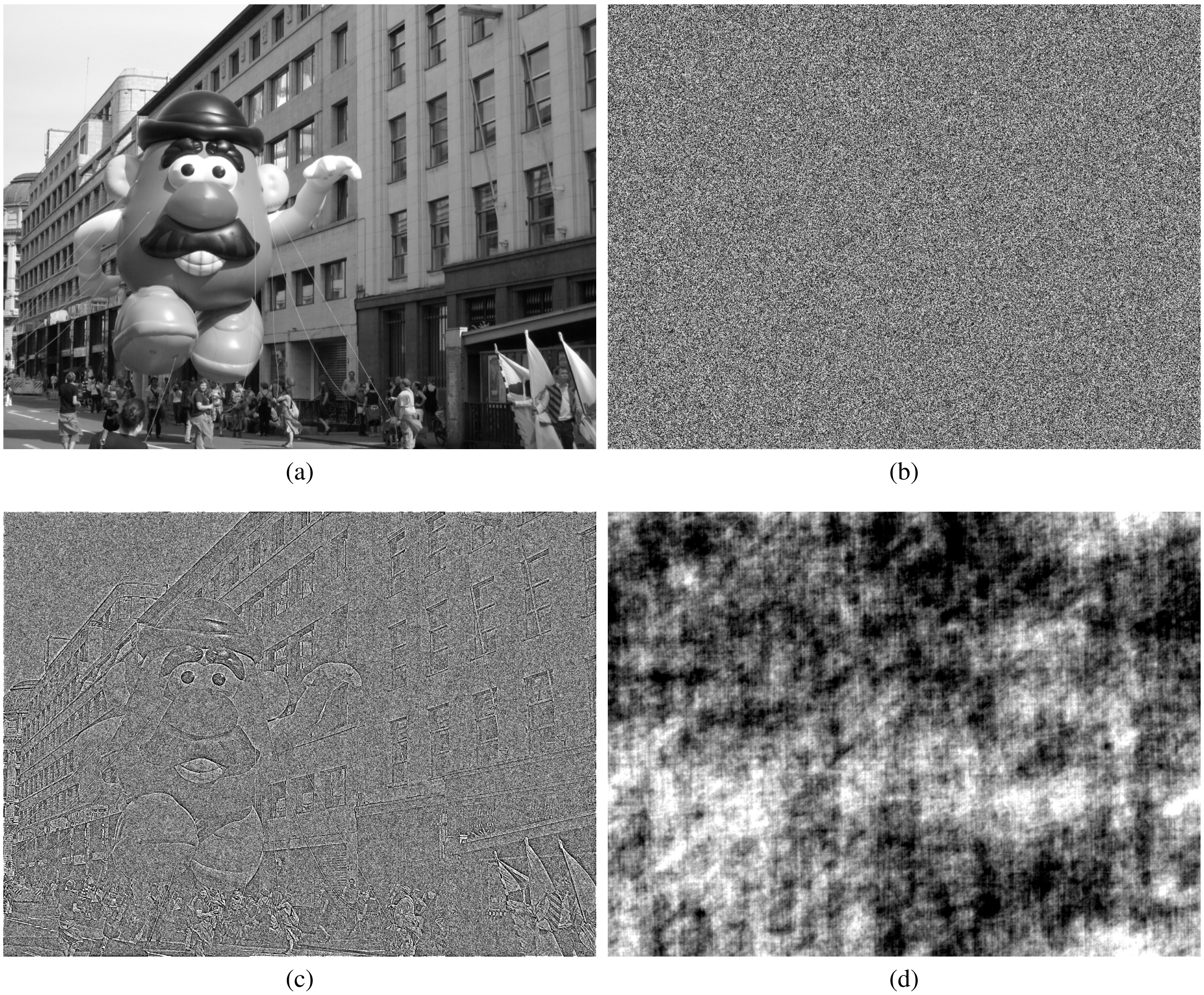


Figure 6.17: Importance of phase information in images. The (a) `potatohead` and (b) `random` images. (c) The `potatohead` image after having its magnitude spectrum replaced with the magnitude spectrum of the `random` image. (d) The `potatohead` image after having its phase spectrum replaced with the phase spectrum of the `random` image.

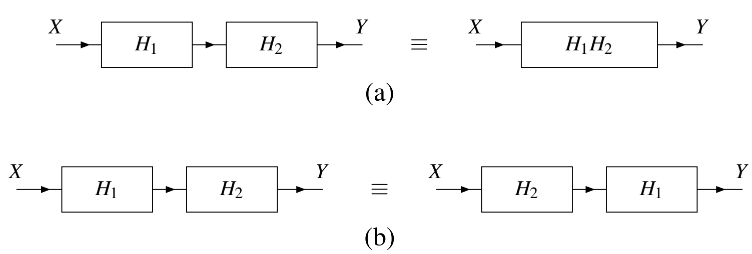


Figure 6.18: Equivalences involving frequency responses and the series interconnection of LTI systems. The (a) first and (b) second equivalences.

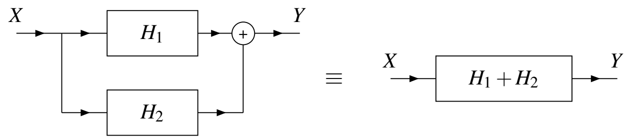


Figure 6.19: Equivalence involving frequency responses and the parallel interconnection of LTI systems.

Taking the Fourier transform of both sides of this equation yields

$$\begin{aligned}H(\omega) &= \mathcal{F}\{h_1+h_2\}(\omega)\\ &= \mathcal{F}h_1(\omega)+\mathcal{F}h_2(\omega)\\ &= H_1(\omega)+H_2(\omega).\end{aligned}$$

Thus, we have the equivalence shown in Figure 6.19.

6.15 LTI Systems and Differential Equations

Many LTI systems of practical interest can be represented using an Nth-order linear differential equation with constant coefficients. Suppose that we have such a system with input x and output y. Then, the input-output behavior of the system is given by an equation of the form

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = \sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t)$$

(where $M \leq N$). Let X and Y denote the Fourier transforms of x and y, respectively. Taking the Fourier transform of both sides of the above equation yields

$$\mathcal{F}\left\{\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t)\right\}(\omega) = \mathcal{F}\left\{\sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t)\right\}(\omega).$$

Using the linearity property of the Fourier transform, we can rewrite this as

$$\sum_{k=0}^{N} b_k \mathcal{F}\left\{\left(\tfrac{d}{dt}\right)^k y(t)\right\}(\omega) = \sum_{k=0}^{M} a_k \mathcal{F}\left\{\left(\tfrac{d}{dt}\right)^k x(t)\right\}(\omega).$$

Using the time-differentiation property of the Fourier transform, we can re-express this as

$$\sum_{k=0}^{N} b_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} a_k (j\omega)^k X(\omega).$$

Then, factoring we have

$$Y(\omega)\sum_{k=0}^{N} b_k (j\omega)^k = X(\omega)\sum_{k=0}^{M} a_k (j\omega)^k.$$

Rearranging this equation, we obtain

$$\frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} a_k (j\omega)^k}{\sum_{k=0}^{N} b_k (j\omega)^k} = \frac{\sum_{k=0}^{M} a_k j^k \omega^k}{\sum_{k=0}^{N} b_k j^k \omega^k}.$$

Since the system is LTI, $Y(\omega) = X(\omega)H(\omega)$ (or equivalently, $H(\omega) = \frac{Y(\omega)}{X(\omega)}$), and we can conclude from the above equation that the frequency response H is given by

$$H(\omega) = \frac{\sum_{k=0}^{M} a_k j^k \omega^k}{\sum_{k=0}^{N} b_k j^k \omega^k}.$$

Observe that, for a system of the form considered above, the frequency response is a rational function—hence, our interest in rational functions.

Example 6.35 (Differential equation to frequency response)**.** A LTI system with input x and output y is characterized by the differential equation

$$7y''(t)+11y'(t)+13y(t)=5x'(t)+3x(t),$$

where x', y', and y'' denote the first derivative of x, the first derivative of y, and the second derivative of y, respectively. Find the frequency response H of this system.

Solution. Taking the Fourier transform of the given differential equation, we obtain

$$7(j\omega)^2Y(\omega)+11j\omega Y(\omega)+13Y(\omega)=5j\omega X(\omega)+3X(\omega).$$

Rearranging the terms and factoring, we have

$$(-7\omega^2+11j\omega+13)Y(\omega)=(5j\omega+3)X(\omega).$$

Thus, H is given by

$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{5j\omega+3}{-7\omega^2+11j\omega+13}.$$ ■

Example 6.36 (Frequency response to differential equation)**.** A LTI system with input x and output y has the frequency response

$$H(\omega)=\frac{-7\omega^2+11j\omega+3}{-5\omega^2+2}.$$

Find the differential equation that characterizes this system.

Solution. From the given frequency response H, we have

$$\frac{Y(\omega)}{X(\omega)}=\frac{-7\omega^2+11j\omega+3}{-5\omega^2+2}.$$

Multiplying both sides by $(-5\omega^2+2)X(\omega)$, we have

$$-5\omega^2Y(\omega)+2Y(\omega)=-7\omega^2X(\omega)+11j\omega X(\omega)+3X(\omega).$$

Applying some simple algebraic manipulation yields

$$5(j\omega)^2Y(\omega)+2Y(\omega)=7(j\omega)^2X(\omega)+11(j\omega)X(\omega)+3X(\omega).$$

Taking the inverse Fourier transform of the preceding equation, we obtain

$$5y''(t)+2y(t)=7x''(t)+11x'(t)+3x(t).$$ ■

6.16 Filtering

In some applications, we want to change the relative amplitude of the frequency components of a function or possibly eliminate some frequency components altogether. This process of modifying the frequency components of a function is referred to as **filtering**. Various types of filters exist. One type is frequency-selective filters. Frequency selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies. Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

An **ideal lowpass filter** eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 6.20(a).

An **ideal highpass filter** eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \ge \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 6.20(b).

An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie between two cutoff frequencies, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

where ω_{c1} and ω_{c2} are the cutoff frequencies. A plot of this frequency response is given in Figure 6.20(c).

Example 6.37 (Ideal filters)**.** For each LTI system whose impulse response h is given below, find and plot the frequency response H of the system, and identify the type of frequency-selective filter to which the system corresponds.
(a) $h(t) = \frac{\omega_c}{\pi}\operatorname{sinc}(\omega_c t)$, where ω_c is a positive real constant;
(b) $h(t) = \delta(t) - \frac{\omega_c}{\pi}\operatorname{sinc}(\omega_c t)$, where ω_c is a positive real constant; and
(c) $h(t) = \frac{2\omega_b}{\pi}[\operatorname{sinc}(\omega_b t)]\cos(\omega_a t)$, where ω_a and ω_b are positive real constants.

Solution. In what follows, let us denote the input and output of the system as x and y, respectively. Also, let X and Y denote the Fourier transforms of x and y, respectively.

(a) The frequency response H of the system is simply the Fourier transform of the impulse response h. Thus, we have

$$\begin{aligned} H(\omega) &= \mathcal{F}\{\tfrac{\omega_c}{\pi}\operatorname{sinc}(\omega_c t)\}(\omega) \\ &= \tfrac{\omega_c}{\pi}\mathcal{F}\{\operatorname{sinc}(\omega_c t)\}(\omega) \\ &= \tfrac{\omega_c}{\pi}\left[\tfrac{\pi}{\omega_c}\operatorname{rect}\left(\tfrac{\omega}{2\omega_c}\right)\right] \\ &= \operatorname{rect}\left(\tfrac{\omega}{2\omega_c}\right) \\ &= \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The frequency response H is plotted in Figure 6.21(a). Since $Y(\omega) = H(\omega)X(\omega)$ and $H(\omega) = 0$ for $|\omega| > \omega_c$, Y will contain only those frequency components in X that lie in the frequency range $|\omega| \le \omega_c$. In other words, only the lower frequency components from X are kept. Thus, the system represents a lowpass filter.

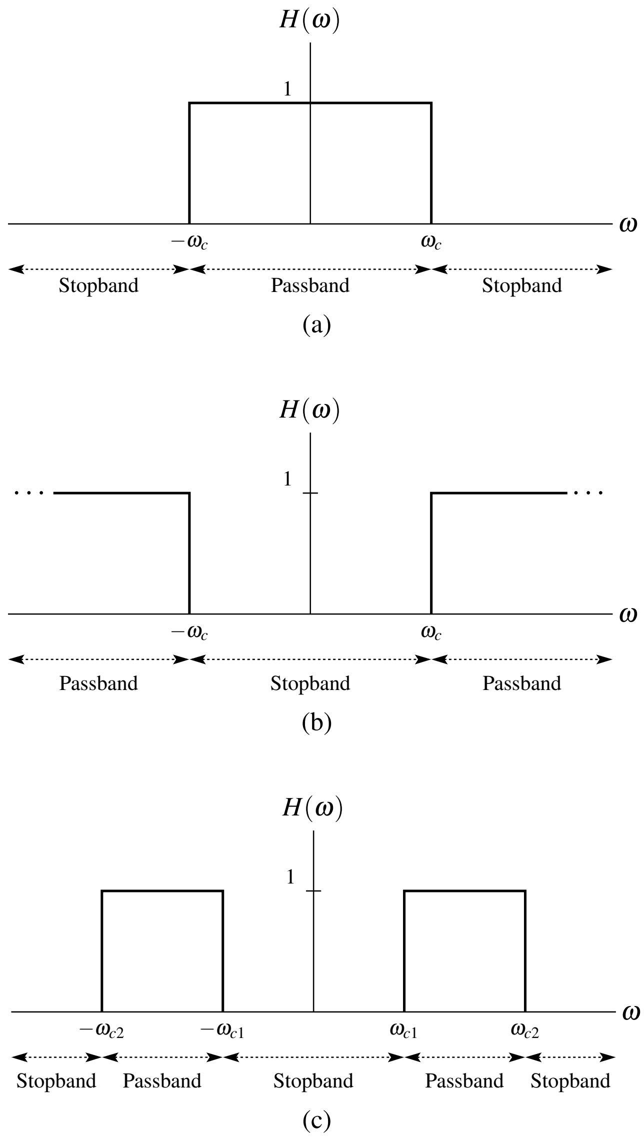


Figure 6.20: Frequency responses of (a) ideal lowpass, (b) ideal highpass, and (c) ideal bandpass filters.

(b) The frequency response H of the system is simply the Fourier transform of the impulse response h. Thus, we have

$$\begin{aligned}
H(\omega) &= \mathcal{F}\{\delta(t) - \tfrac{\omega_c}{\pi}\operatorname{sinc}(\omega_c t)\}(\omega) \\
&= \mathcal{F}\delta(\omega) - \tfrac{\omega_c}{\pi}\mathcal{F}\{\operatorname{sinc}(\omega_c t)\}(\omega) \\
&= 1 - \tfrac{\omega_c}{\pi}\left[\tfrac{\pi}{\omega_c}\operatorname{rect}\left(\tfrac{\omega}{2\omega_c}\right)\right] \\
&= 1 - \operatorname{rect}\left(\tfrac{\omega}{2\omega_c}\right) \\
&= \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The frequency response H is plotted in Figure 6.21(b). Since $Y(\omega) = H(\omega)X(\omega)$ and $H(\omega) = 0$ for $|\omega| < \omega_c$, Y will contain only those frequency components in X that lie in the frequency range $|\omega| \geq \omega_c$. In other words, only the higher frequency components from X are kept. Thus, the system represents a highpass filter.

(c) The frequency response H of the system is simply the Fourier transform of the impulse response h. Thus, we have

$$\begin{aligned}
H(\omega) &= \mathcal{F}\{\tfrac{2\omega_b}{\pi}[\operatorname{sinc}(\omega_b t)]\cos(\omega_a t)\}(\omega) \\
&= \tfrac{\omega_b}{\pi}\mathcal{F}\{[\operatorname{sinc}(\omega_b t)](2\cos[\omega_a t])\}(\omega) \\
&= \tfrac{\omega_b}{\pi}\mathcal{F}\{[\operatorname{sinc}(\omega_b t)][e^{j\omega_a t} + e^{-j\omega_a t}]\}(\omega) \\
&= \tfrac{\omega_b}{\pi}\left[\mathcal{F}\{e^{j\omega_a t}\operatorname{sinc}(\omega_b t)\}(\omega) + \mathcal{F}\{e^{-j\omega_a t}\operatorname{sinc}(\omega_b t)\}(\omega)\right] \\
&= \tfrac{\omega_b}{\pi}\left[\tfrac{\pi}{\omega_b}\operatorname{rect}\left(\tfrac{\omega-\omega_a}{2\omega_b}\right) + \tfrac{\pi}{\omega_b}\operatorname{rect}\left(\tfrac{\omega+\omega_a}{2\omega_b}\right)\right] \\
&= \operatorname{rect}\left(\tfrac{\omega-\omega_a}{2\omega_b}\right) + \operatorname{rect}\left(\tfrac{\omega+\omega_a}{2\omega_b}\right) \\
&= \begin{cases} 1 & \omega_a - \omega_b \leq |\omega| \leq \omega_a + \omega_b \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The frequency response H is plotted in Figure 6.21(c). Since $Y(\omega) = H(\omega)X(\omega)$ and $H(\omega) = 0$ for $|\omega| < \omega_a - \omega_b$ or $|\omega| > \omega_a + \omega_b$, Y will contain only those frequency components in X that lie in the frequency range $\omega_a - \omega_b \leq |\omega| \leq \omega_a + \omega_b$. In other words, only the middle frequency components of X are kept. Thus, the system represents a bandpass filter. ■

Example 6.38 (Lowpass filtering)**.** Consider a LTI system with impulse response

$$h(t) = 300\operatorname{sinc}(300\pi t).$$

Using frequency-domain methods, find the response y of the system to the input

$$x(t) = \tfrac{1}{2} + \tfrac{3}{4}\cos(200\pi t) + \tfrac{1}{2}\cos(400\pi t) - \tfrac{1}{4}\cos(600\pi t).$$

Solution. To begin, we find the Fourier transform X of x. Computing X, we have

$$\begin{aligned}
X(\omega) &= \mathcal{F}\left\{\tfrac{1}{2} + \tfrac{3}{4}\cos(200\pi t) + \tfrac{1}{2}\cos(400\pi t) - \tfrac{1}{4}\cos(600\pi t)\right\}(\omega) \\
&= \tfrac{1}{2}\mathcal{F}\{1\}(\omega) + \tfrac{3}{4}\mathcal{F}\{\cos(200\pi t)\}(\omega) + \tfrac{1}{2}\mathcal{F}\{\cos(400\pi t)\}(\omega) - \tfrac{1}{4}\mathcal{F}\{\cos(600\pi t)\}(\omega) \\
&= \tfrac{1}{2}[2\pi\delta(\omega)] + \tfrac{3\pi}{4}[\delta(\omega+200\pi) + \delta(\omega-200\pi)] + \tfrac{\pi}{2}[\delta(\omega+400\pi) + \delta(\omega-400\pi)] \\
&\quad - \tfrac{\pi}{4}[\delta(\omega+600\pi) + \delta(\omega-600\pi)] \\
&= -\tfrac{\pi}{4}\delta(\omega+600\pi) + \tfrac{\pi}{2}\delta(\omega+400\pi) + \tfrac{3\pi}{4}\delta(\omega+200\pi) + \pi\delta(\omega) + \tfrac{3\pi}{4}\delta(\omega-200\pi) \\
&\quad + \tfrac{\pi}{2}\delta(\omega-400\pi) - \tfrac{\pi}{4}\delta(\omega-600\pi).
\end{aligned}$$

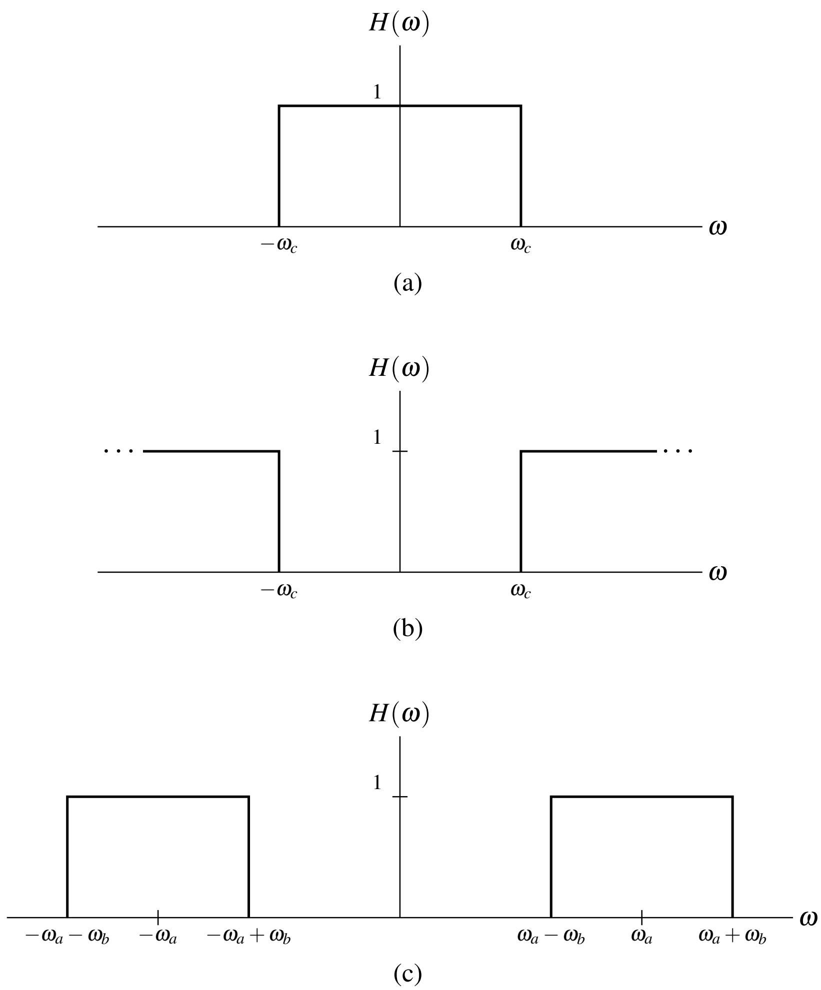


Figure 6.21: Frequency responses of each of the (a) first, (b) second, and (c) third systems from the example.

A plot of the frequency spectrum X is shown in Figure 6.22(a). Using the results of Example 6.37, we can determine the frequency response H of the system to be

$$\begin{aligned} H(\omega) &= \mathcal{F}\{300\,\mathrm{sinc}(300\pi t)\}(\omega) \\ &= \mathrm{rect}\left(\tfrac{\omega}{2[300\pi]}\right) \\ &= \begin{cases} 1 & |\omega| \leq 300\pi \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The frequency response H is shown in Figure 6.22(b). The frequency spectrum Y of the output can be computed as

$$\begin{aligned} Y(\omega) &= H(\omega)X(\omega) \\ &= \tfrac{3\pi}{4}\delta(\omega+200\pi) + \pi\delta(\omega) + \tfrac{3\pi}{4}\delta(\omega-200\pi). \end{aligned}$$

The frequency spectrum Y is shown in Figure 6.22(c). Taking the inverse Fourier transform of Y yields

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\left\{\tfrac{3\pi}{4}\delta(\omega+200\pi) + \pi\delta(\omega) + \tfrac{3\pi}{4}\delta(\omega-200\pi)\right\}(t) \\ &= \pi\mathcal{F}^{-1}\delta(t) + \tfrac{3}{4}\mathcal{F}^{-1}\{\pi[\delta(\omega+200\pi)+\delta(\omega-200\pi)]\}(t) \\ &= \pi(\tfrac{1}{2\pi}) + \tfrac{3}{4}\cos(200\pi t) \\ &= \tfrac{1}{2} + \tfrac{3}{4}\cos(200\pi t). \end{aligned}$$

■

Example 6.39 (Bandpass filtering)**.** Consider a LTI system with the impulse response

$$h(t) = \tfrac{2}{\pi}\,\mathrm{sinc}(t)\cos(4t).$$

Using frequency-domain methods, find the response y of the system to the input

$$x(t) = -1 + 2\cos(2t) + \cos(4t) - \cos(6t).$$

Solution. Taking the Fourier transform of x, we have

$$\begin{aligned} X(\omega) &= -1\mathcal{F}\{1\}(\omega) + 2\mathcal{F}\{\cos(2t)\}(\omega) + \mathcal{F}\{\cos(4t)\}(\omega) - \mathcal{F}\{\cos(6t)\}(\omega) \\ &= -2\pi\delta(\omega) + 2(\pi[\delta(\omega-2)+\delta(\omega+2)]) + \pi[\delta(\omega-4)+\delta(\omega+4)] - \pi[\delta(\omega-6)+\delta(\omega+6)] \\ &= -\pi\delta(\omega+6) + \pi\delta(\omega+4) + 2\pi\delta(\omega+2) - 2\pi\delta(\omega) + 2\pi\delta(\omega-2) + \pi\delta(\omega-4) - \pi\delta(\omega-6). \end{aligned}$$

The frequency spectrum X is shown in Figure 6.23(a). Now, we compute the frequency response H of the system. Using the results of Example 6.37, we can determine H to be

$$\begin{aligned} H(\omega) &= \mathcal{F}\{\tfrac{2}{\pi}\,\mathrm{sinc}(t)\cos(4t)\}(\omega) \\ &= \mathrm{rect}\left(\tfrac{\omega-4}{2}\right) + \mathrm{rect}\left(\tfrac{\omega+4}{2}\right) \\ &= \begin{cases} 1 & 3 \leq |\omega| \leq 5 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The frequency response H is shown in Figure 6.23(b). The frequency spectrum Y of the output is given by

$$\begin{aligned} Y(\omega) &= H(\omega)X(\omega) \\ &= \pi\delta(\omega+4) + \pi\delta(\omega-4). \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\{\pi\delta(\omega+4) + \pi\delta(\omega-4)\}(t) \\ &= \mathcal{F}^{-1}\{\pi[\delta(\omega+4)+\delta(\omega-4)]\}(t) \\ &= \cos(4t). \end{aligned}$$

■

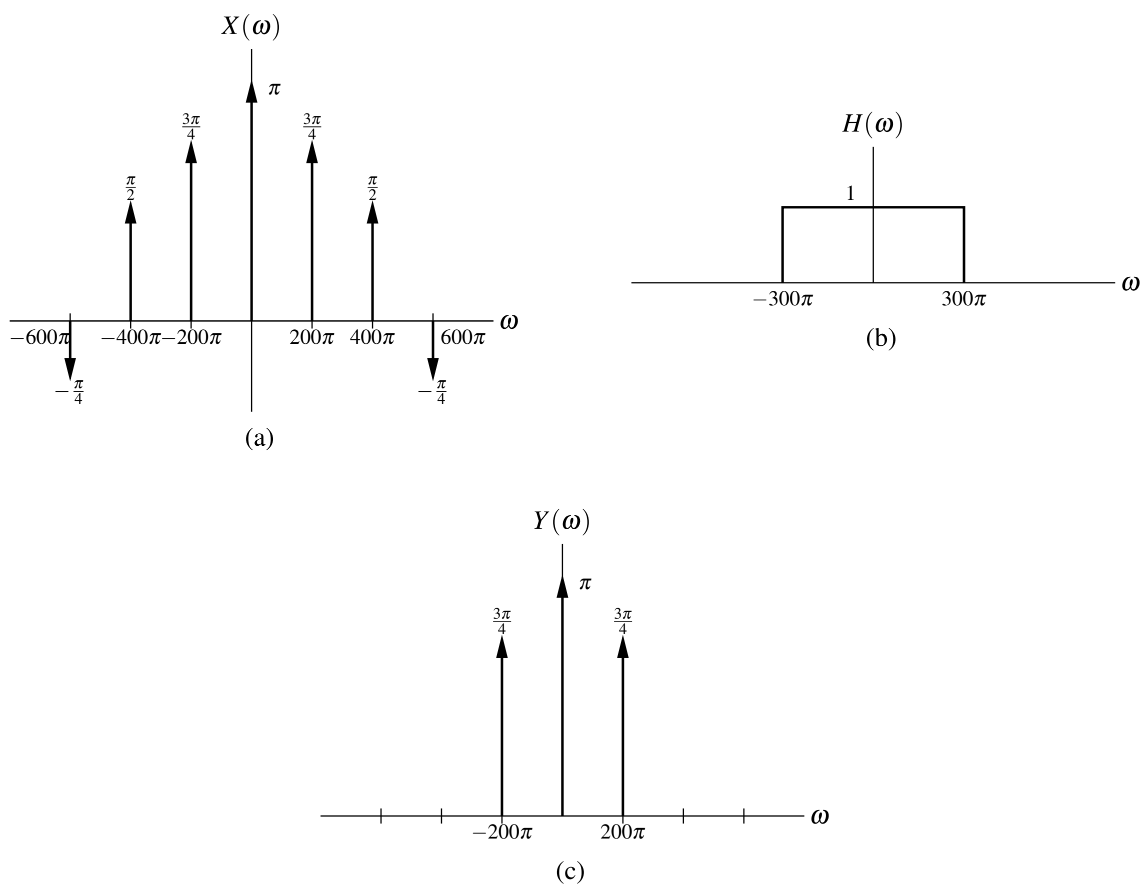


Figure 6.22: Frequency spectra for the lowpass filtering example. (a) Frequency spectrum of the input x. (b) Frequency response of the system. (c) Frequency spectrum of the output y.

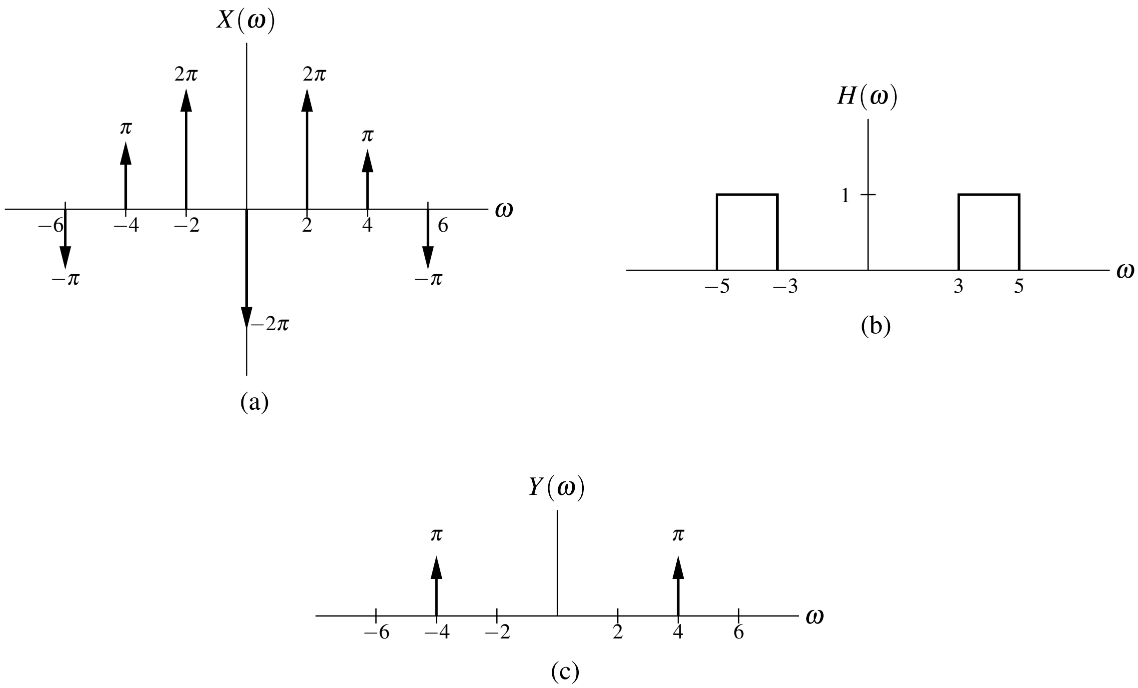


Figure 6.23: Frequency spectra for bandpass filtering example. (a) Frequency spectrum of the input x. (b) Frequency response of the system. (c) Frequency spectrum of the output y.

6.17 Equalization

Often, we find ourselves faced with a situation where we have a system with a particular frequency response that is undesirable for the application at hand. As a result, we would like to change the frequency response of the system to be something more desirable. This process of modifying the frequency response in this way is referred to as **equalization**. Essentially, equalization is just a filtering operation, where the filtering is applied with the specific goal of obtaining a more desirable frequency response.

Let us now examine the mathematics behind equalization. Consider the LTI system with impulse response h_{orig} as shown in Figure 6.24(a). Let H_{orig} denote the Fourier transform of h_{orig}. Suppose that the frequency response H_{orig} is undesirable for some reason (i.e., the system does not behave in a way that is good for the application at hand). Consequently, we would instead like to have a system with frequency response H_{d}. In effect, we would like to somehow change the frequency response H_{orig} of the original system to H_{d}. This can be accomplished by using another system called an **equalizer**. More specifically, consider the new system shown in Figure 6.24(b), which consists of a LTI equalizer with impulse response h_{eq} connected in series with the original system having impulse response h_{orig}. Let H_{eq} denote the Fourier transform of h_{eq}. From the block diagram, we have

$$Y(\omega) = H(\omega)X(\omega),$$

where $H(\omega) = H_{\mathsf{orig}}(\omega)H_{\mathsf{eq}}(\omega)$. In effect, we want to force H to be equal to H_{d} so that the overall (i.e., series-interconnected) system has the frequency response desired. So, we choose the equalizer to be such that

$$H_{\mathsf{eq}}(\omega) = \frac{H_{\mathsf{d}}(\omega)}{H_{\mathsf{orig}}(\omega)}.$$

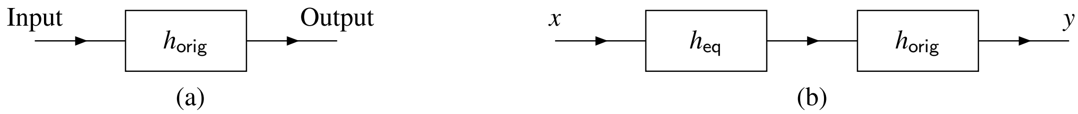


Figure 6.24: Equalization example. (a) Original system. (b) New system with equalization.

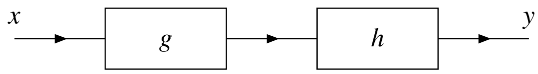


Figure 6.25: System from example that employs equalization.

Then, we have

$$\begin{aligned} H(\omega) &= H_{\text{orig}}(\omega)H_{\text{eq}}(\omega) \\ &= H_{\text{orig}}(\omega)\left[\frac{H_d(\omega)}{H_{\text{orig}}(\omega)}\right] \\ &= H_{\text{d}}(\omega). \end{aligned}$$

Thus, the system in Figure 6.24(b) has the frequency response H_{d} as desired.

Equalization is used in many applications. In real-world communication systems, equalization is used to eliminate or minimize the distortion introduced when a signal is sent over a (nonideal) communication channel. In audio applications, equalization can be employed to emphasize or de-emphasize certain ranges of frequencies. For example, often we like to boost the bass (i.e., emphasize the low frequencies) in the audio output of a stereo.

Example 6.40 (Communication channel equalization)**.** Consider a LTI communication channel with frequency response

$$H(\omega) = \tfrac{1}{3+j\omega}.$$

Unfortunately, this channel has the undesirable effect of attenuating higher frequencies. Find the frequency response G of an equalizer that when connected in series with the communication channel yields an ideal (i.e., distortionless) channel. The new system with equalization is shown in Figure 6.25, where g and h denote the inverse Fourier transforms of G and H, respectively.

Solution. An ideal communication channel has a frequency response equal to one for all frequencies. Consequently, we want $H(\omega)G(\omega) = 1$ or equivalently $G(\omega) = 1/H(\omega)$. Thus, we conclude that

$$\begin{aligned} G(\omega) &= \frac{1}{H(\omega)} \\ &= \frac{1}{\left(\frac{1}{3+j\omega}\right)} \\ &= 3 + j\omega. \end{aligned}$$ ■

6.18 Circuit Analysis

One application of the Fourier transform is circuit analysis. In this section, we consider this particular application.

The basic building blocks of many electrical networks are resistors, inductors, and capacitors. In what follows, we briefly introduce each of these circuit elements.

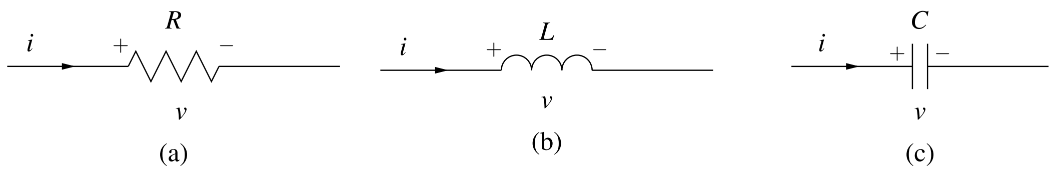


Figure 6.26: Basic electrical components. (a) Resistor, (b) inductor, and (c) capacitor.

A **resistor** is a circuit element that opposes the flow of electric current. The resistor, shown in schematic form in Figure 6.26(a), is governed by the relationship

$$v(t) = Ri(t) \quad \left(\text{or equivalently, } i(t) = \tfrac{1}{R}v(t)\right),$$

where R, v and i denote the resistance of, voltage across, and current through the resistor, respectively. Note that the resistance R is a nonnegative quantity (i.e., $R \geq 0$). In the frequency domain, the above relationship becomes

$$V(\omega) = RI(\omega) \quad \left(\text{or equivalently, } I(\omega) = \tfrac{1}{R}V(\omega)\right),$$

where V and I denote the Fourier transforms of v and i, respectively.

An **inductor** is a circuit element that converts an electric current into a magnetic field and vice versa. The inductor, shown in schematic form in Figure 6.26(b), is governed by the relationship

$$v(t) = L\tfrac{d}{dt}i(t) \quad \left(\text{or equivalently, } i(t) = \tfrac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau\right),$$

where L, v, and i denote the inductance of, voltage across, and current through the inductor, respectively. Note that the inductance L is a nonnegative quantity (i.e., $L \geq 0$). In the frequency domain, the above relationship becomes

$$V(\omega) = j\omega LI(\omega) \quad \left(\text{or equivalently, } I(\omega) = \tfrac{1}{j\omega L}V(\omega)\right),$$

where V and I denote the Fourier transforms of v and i, respectively.

A **capacitor** is a circuit element that stores electric charge. The capacitor, shown in schematic form in Figure 6.26(c), is governed by the relationship

$$v(t) = \tfrac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \left(\text{or equivalently, } i(t) = C\tfrac{d}{dt}v(t)\right),$$

where C, v, and i denote the capacitance of, voltage across, and current through the capacitor, respectively. Note that the capacitance C is a nonnegative quantity (i.e., $C \geq 0$). In the frequency domain, the above relationship becomes

$$V(\omega) = \tfrac{1}{j\omega C}I(\omega) \quad \left(\text{or equivalently, } I(\omega) = j\omega CV(\omega)\right),$$

where V and I denote the Fourier transforms of v and i, respectively.

Observe that, in the case of inductors and capacitors, the equations that characterize these circuit elements are arguably much simpler to express in the Fourier domain than in the time domain. Consequently, the use of the Fourier transform has the potential to greatly simplify the process of analyzing circuits containing inductors and capacitors.

Example 6.41 (Simple RL network)**.** Consider the resistor-inductor (RL) network shown in Figure 6.27 with input v_1 and output v_2. This system is LTI, since it can be characterized by a linear differential equation with constant coefficients. (a) Find the frequency response H of the system. (b) Find the response v_2 of the system to the input $v_1(t) = \operatorname{sgn} t$.

Solution. (a) From basic circuit analysis, we can write

$$v_1(t) = Ri(t) + L\tfrac{d}{dt}i(t) \quad \text{and} \tag{6.36}$$

$$v_2(t) = L\tfrac{d}{dt}i(t). \tag{6.37}$$

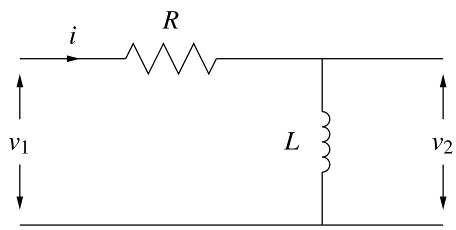


Figure 6.27: Simple RL network.

(Recall that the voltage v across an inductor L is related to the current i through the inductor as $v(t) = L\frac{d}{dt}i(t)$.) Taking the Fourier transform of (6.36) and (6.37) yields

$$\begin{aligned} V_1(\omega) &= RI(\omega) + j\omega LI(\omega) \\ &= (R + j\omega L)I(\omega) \quad \text{and} \end{aligned} \tag{6.38}$$

$$V_2(\omega) = j\omega LI(\omega). \tag{6.39}$$

From (6.38) and (6.39), we have

$$\begin{aligned} H(\omega) &= \frac{V_2(\omega)}{V_1(\omega)} \\ &= \frac{j\omega LI(\omega)}{(R + j\omega L)I(\omega)} \\ &= \frac{j\omega L}{R + j\omega L}. \end{aligned} \tag{6.40}$$

Thus, we have found the frequency response of the system.

(b) Now, suppose that $v_1(t) = \operatorname{sgn} t$ (as given). Taking the Fourier transform of the input v_1 (with the aid of Table 6.2), we have

$$V_1(\omega) = \frac{2}{j\omega}. \tag{6.41}$$

From the definition of the system, we know

$$V_2(\omega) = H(\omega)V_1(\omega). \tag{6.42}$$

Substituting (6.41) and (6.40) into (6.42), we obtain

$$\begin{aligned} V_2(\omega) &= \left(\frac{j\omega L}{R + j\omega L}\right)\left(\frac{2}{j\omega}\right) \\ &= \frac{2L}{R + j\omega L}. \end{aligned}$$

Taking the inverse Fourier transform of both sides of this equation, we obtain

$$\begin{aligned} v_2(t) &= \mathcal{F}^{-1}\left\{\frac{2L}{R + j\omega L}\right\}(t) \\ &= \mathcal{F}^{-1}\left\{\frac{2}{R/L + j\omega}\right\}(t) \\ &= 2\mathcal{F}^{-1}\left\{\frac{1}{R/L + j\omega}\right\}(t). \end{aligned}$$

Using Table 6.2, we can simplify to obtain

$$v_2(t) = 2e^{-(R/L)t}u(t).$$

Thus, we have found the response v_2 to the input $v_1(t) = \operatorname{sgn} t$. ■

6.19 Amplitude Modulation

In communication systems, we often need to transmit a signal using a frequency range that is different from that of the original signal. For example, voice/audio signals typically have information in the range of 0 to 22 kHz. Often, it is not practical to transmit such a signal using its original frequency range. Two potential problems with such an approach are: 1) interference and 2) constraints on antenna length. Since many signals are broadcast over the airwaves, we need to ensure that no two transmitters use the same frequency bands in order to avoid interference. Also, in the case of transmission via electromagnetic waves (e.g., radio waves), the length of antenna required becomes impractically large for the transmission of relatively low frequency signals. For the preceding reasons, we often need to change the frequency range associated with a signal before transmission. In what follows, we consider one possible scheme for accomplishing this. This scheme is known as amplitude modulation.

Amplitude modulation (AM) is used in many communication systems. Numerous variations on amplitude modulation are possible. Here, we consider two of the simplest variations: double-side-band/suppressed-carrier (DSB/SC) and single-side-band/suppressed-carrier (SSB/SC).

6.19.1 Modulation With a Complex Sinusoid

Consider the communication system shown in Figure 6.28. In what follows, we will analyze the behavior of this system in detail.

First, let us consider the transmitter in Figure 6.28(a). The transmitter is the system with the input x and output y that is characterized by the equation

$$y(t) = c_1(t)x(t), \tag{6.43}$$

where

$$c_1(t) = e^{j\omega_c t}.$$

Let X, Y, and C_1 denote the Fourier transforms of x, y, and c_1, respectively. Taking the Fourier transform of both sides of (6.43), we obtain

$$\begin{aligned} Y(\omega) &= \mathcal{F}\{c_1 x\}(\omega) \\ &= \mathcal{F}\left\{e^{j\omega_c t}x(t)\right\}(\omega) \\ &= X(\omega - \omega_c). \end{aligned} \tag{6.44}$$

Thus, the frequency spectrum of the (transmitter) output is simply the frequency spectrum of the (transmitter) input shifted by ω_c. The relationship between the frequency spectra of the input and output is illustrated in Figure 6.29. Clearly, the spectrum of the output has been shifted to a different frequency range as desired. Next, we need to determine whether the receiver can recover the original signal x from the transmitted signal y.

Consider the receiver shown in Figure 6.28(b). The receiver is a system with the input y and output $\hat{x}$ that is characterized by the equation

$$\hat{x}(t) = c_2(t)y(t), \tag{6.45}$$

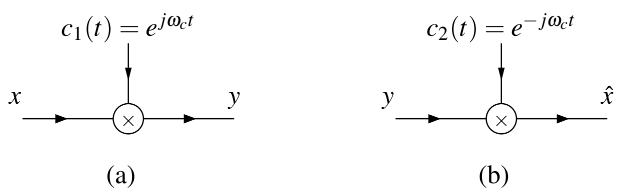


Figure 6.28: Simple communication system. (a) Transmitter and (b) receiver.

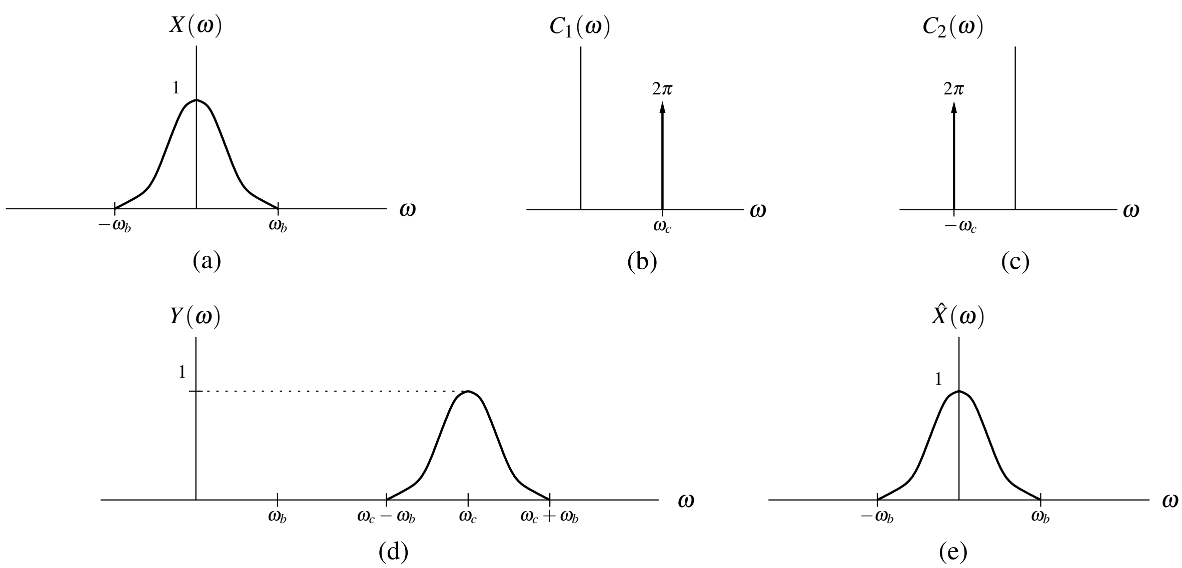


Figure 6.29: Frequency spectra for modulation with a complex sinusoid. (a) Spectrum of the transmitter input. (b) Spectrum of the complex sinusoid used in the transmitter. (c) Spectrum of the complex sinusoid used in the receiver. (d) Spectrum of the transmitted signal. (e) Spectrum of the receiver output.

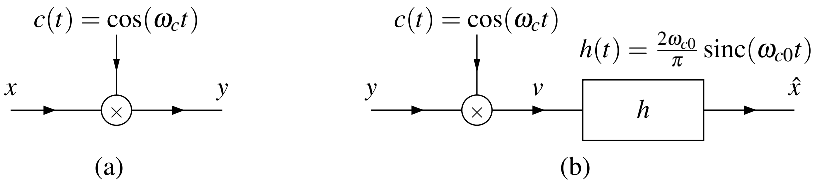


Figure 6.30: DSB/SC amplitude modulation system. (a) Transmitter and (b) receiver.

where

$$c_2(t) = e^{-j\omega_c t}.$$

In order for the communication system to be useful, we need for the received signal $\hat{x}$ to be equal to the original signal x from the transmitter. Let Y, $\hat{X}$, and C_2 denote the Fourier transform of y, $\hat{x}$, and c_2, respectively. Taking the Fourier transform of both sides of (6.45), we obtain

$$\begin{aligned}\hat{X}(\omega) &= \mathcal{F}\{c_2 y\}(\omega) \\ &= \mathcal{F}\{e^{-j\omega_c t}y(t)\}(\omega) \\ &= Y(\omega+\omega_c).\end{aligned}$$

Substituting the expression for Y in (6.44) into this equation, we obtain

$$\begin{aligned}\hat{X}(\omega) &= X([\omega+\omega_c]-\omega_c) \\ &= X(\omega).\end{aligned}$$

Since $\hat{X} = X$, we have that the received signal $\hat{x}$ is equal to the original signal x from the transmitter. Thus, the communication system has the desired behavior. The relationship between the frequency spectra of the various signals in the AM system is illustrated in Figure 6.29.

Although the above result is quite interesting mathematically, it does not have direct practical application. The difficulty here is that c_1, c_2, and y are complex-valued signals, and we cannot realize complex-valued signals in the physical world. This communication system is not completely without value, however, as it inspires the development of the practically useful system that we consider next.

6.19.2 DSB/SC Amplitude Modulation

Now, let us consider the communication system shown in Figure 6.30. This system is known as a double-side-band/suppressed-carrier (DSB/SC) amplitude modulation (AM) system. The receiver in Figure 6.30(b) contains a LTI subsystem that is labelled with its impulse response h. The DSB/SC AM system is very similar to the one considered earlier in Figure 6.28. In the new system, however, multiplication by a complex sinusoid has been replaced by multiplication by a real sinusoid. The new system also requires that the input signal x be bandlimited to frequencies in the interval $[-\omega_b, \omega_b]$ and that

$$\omega_b < \omega_{c0} < 2\omega_c - \omega_b. \tag{6.46}$$

The reasons for this restriction will become clear after having studied this system in more detail.

Consider the transmitter shown in Figure 6.30(a). The transmitter is a system with input x and output y that is characterized by the equation

$$y(t) = c(t)x(t),$$

where

$$c(t) = \cos(\omega_c t).$$

Taking the Fourier transform of both sides of the preceding equation, we obtain

$$\begin{aligned}
Y(\omega) &= \mathcal{F}\{cx\}(\omega) \\
&= \mathcal{F}\{\cos(\omega_c t)x(t)\}(\omega) \\
&= \mathcal{F}\left\{\tfrac{1}{2}[e^{j\omega_c t}+e^{-j\omega_c t}]x(t)\right\}(\omega) \\
&= \tfrac{1}{2}\left[\mathcal{F}\{e^{j\omega_c t}x(t)\}(\omega)+\mathcal{F}\{e^{-j\omega_c t}x(t)\}(\omega)\right] \\
&= \tfrac{1}{2}\left[X(\omega-\omega_c)+X(\omega+\omega_c)\right].
\end{aligned} \tag{6.47}$$

(Note that, above, we used the fact that $\cos(\omega_c t) = \frac{1}{2}(e^{j\omega_c t}+e^{-j\omega_c t})$.) Thus, the frequency spectrum of the (transmitter) output is the average of two shifted versions of the frequency spectrum of the (transmitter) input. The relationship between the frequency spectra of the input and output can be seen through Figures 6.31(a) and (d). Observe that we have managed to shift the frequency spectrum of the input signal into a different range of frequencies for transmission as desired. Next, we must determine whether the receiver can recover the original signal x.

Consider the receiver shown in Figure 6.30(b). The receiver is a system with input y and output $\hat{x}$ that is characterized by the equations

$$v(t) = c(t)y(t) \quad \text{and} \tag{6.48a}$$

$$\hat{x}(t) = v * h(t), \tag{6.48b}$$

where c is as defined earlier and

$$h(t) = \tfrac{2\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0}t). \tag{6.48c}$$

Let H, Y, V, and $\hat{X}$ denote the Fourier transforms of h, y, v and $\hat{x}$, respectively. Taking the Fourier transform of $\hat{X}$ (in (6.48b)), we have

$$\hat{X}(\omega) = H(\omega)V(\omega). \tag{6.49}$$

Taking the Fourier transform of h (in (6.48c)) with the assistance of Table 6.2, we have

$$\begin{aligned}
H(\omega) &= \mathcal{F}\left\{\tfrac{2\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0}t)\right\}(\omega) \\
&= 2\operatorname{rect}\left(\tfrac{\omega}{2\omega_{c0}}\right) \\
&= \begin{cases} 2 & |\omega| \le \omega_{c0} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Taking the Fourier transform of v (in (6.48a)) yields

$$\begin{aligned}
V(\omega) &= \mathcal{F}\{cy\}(\omega) \\
&= \mathcal{F}\{\cos(\omega_c t)y(t)\}(\omega) \\
&= \mathcal{F}\left\{\tfrac{1}{2}\left(e^{j\omega_c t}+e^{-j\omega_c t}\right)y(t)\right\}(\omega) \\
&= \tfrac{1}{2}\left[\mathcal{F}\left\{e^{j\omega_c t}y(t)\right\}(\omega)+\mathcal{F}\left\{e^{-j\omega_c t}y(t)\right\}(\omega)\right] \\
&= \tfrac{1}{2}[Y(\omega-\omega_c)+Y(\omega+\omega_c)].
\end{aligned}$$

Substituting the expression for Y in (6.47) into this equation, we obtain

$$\begin{aligned}
V(\omega) &= \tfrac{1}{2}\left[\tfrac{1}{2}\left[X([\omega-\omega_c]-\omega_c)+X([\omega-\omega_c]+\omega_c)\right]+\tfrac{1}{2}\left[X([\omega+\omega_c]-\omega_c)+X([\omega+\omega_c]+\omega_c)\right]\right] \\
&= \tfrac{1}{2}X(\omega)+\tfrac{1}{4}X(\omega-2\omega_c)+\tfrac{1}{4}X(\omega+2\omega_c).
\end{aligned} \tag{6.50}$$

The relationship between V and X can be seen via Figures 6.31(a) and (e). Substituting the above expression for V into (6.49) and simplifying, we obtain

$$\begin{aligned}\hat{X}(\omega) &= H(\omega)V(\omega)\\ &= H(\omega)\left[\tfrac{1}{2}X(\omega)+\tfrac{1}{4}X(\omega-2\omega_c)+\tfrac{1}{4}X(\omega+2\omega_c)\right]\\ &= \tfrac{1}{2}H(\omega)X(\omega)+\tfrac{1}{4}H(\omega)X(\omega-2\omega_c)+\tfrac{1}{4}H(\omega)X(\omega+2\omega_c)\\ &= \tfrac{1}{2}[2X(\omega)]+\tfrac{1}{4}(0)+\tfrac{1}{4}(0)\\ &= X(\omega).\end{aligned}$$

In the above simplification, since $H(\omega) = 2\operatorname{rect}\left(\frac{\omega}{2\omega_{c0}}\right)$ and condition (6.46) holds, we were able to deduce that $H(\omega)X(\omega) = 2X(\omega)$, $H(\omega)X(\omega-2\omega_c) = 0$, and $H(\omega)X(\omega+2\omega_c) = 0$. The relationship between $\hat{X}$ and X can be seen from Figures 6.31(a) and (f). Thus, we have that $\hat{X} = X$, implying $\hat{x} = x$. So, we have recovered the original signal x at the receiver. This system has managed to shift x into a different frequency range before transmission and then recover x at the receiver. This is exactly what we wanted to accomplish.

6.19.3 SSB/SC Amplitude Modulation

By making a minor modification to the DSB/SC AM system, we can reduce the bandwidth requirements of the system by half. The resulting system is referred to as a single-side-band/suppressed-carrier (SSB/SC) AM system. This modified system is illustrated in Figure 6.32. The transmitter in Figure 6.32(a) contains a LTI subsystem that is labelled with its impulse response g. Similarly, the receiver in Figure 6.32(b) contains a LTI subsystem that is labelled with its impulse response h. Let X, Y, Q, V, $\hat{X}$, C, G, and H denote the Fourier transforms of x, y, q, v, $\hat{x}$, c, g, and h, respectively.

The transmitter is a system with input x and output y that is characterized by the equations

$$q(t) = c(t)x(t) \quad \text{and}$$
$$y(t) = q * g(t),$$

where

$$c(t) = \cos(\omega_c t) \quad \text{and}$$
$$g(t) = \delta(t) - \tfrac{\omega_c}{\pi}\operatorname{sinc}(\omega_c t).$$

Taking the Fourier transform of g, we obtain

$$G(\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases} \quad \text{and}$$

The receiver is a system with input y and output $\hat{x}$ that is characterized by the equations

$$v(t) = c(t)y(t) \quad \text{and}$$
$$\hat{x}(t) = v * h(t)$$

where c is as defined earlier and

$$h(t) = \tfrac{4\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0}t).$$

Taking the Fourier transform of h, we obtain

$$H(\omega) = \begin{cases} 4 & |\omega| \leq \omega_{c0} \\ 0 & \text{otherwise.} \end{cases}$$

Figure 6.33 depicts the transformations that the signal undergoes as it passes through the system. Again, the output from the receiver is equal to the input to the transmitter. A detailed analysis of this communication system is left as an exercise for the reader.

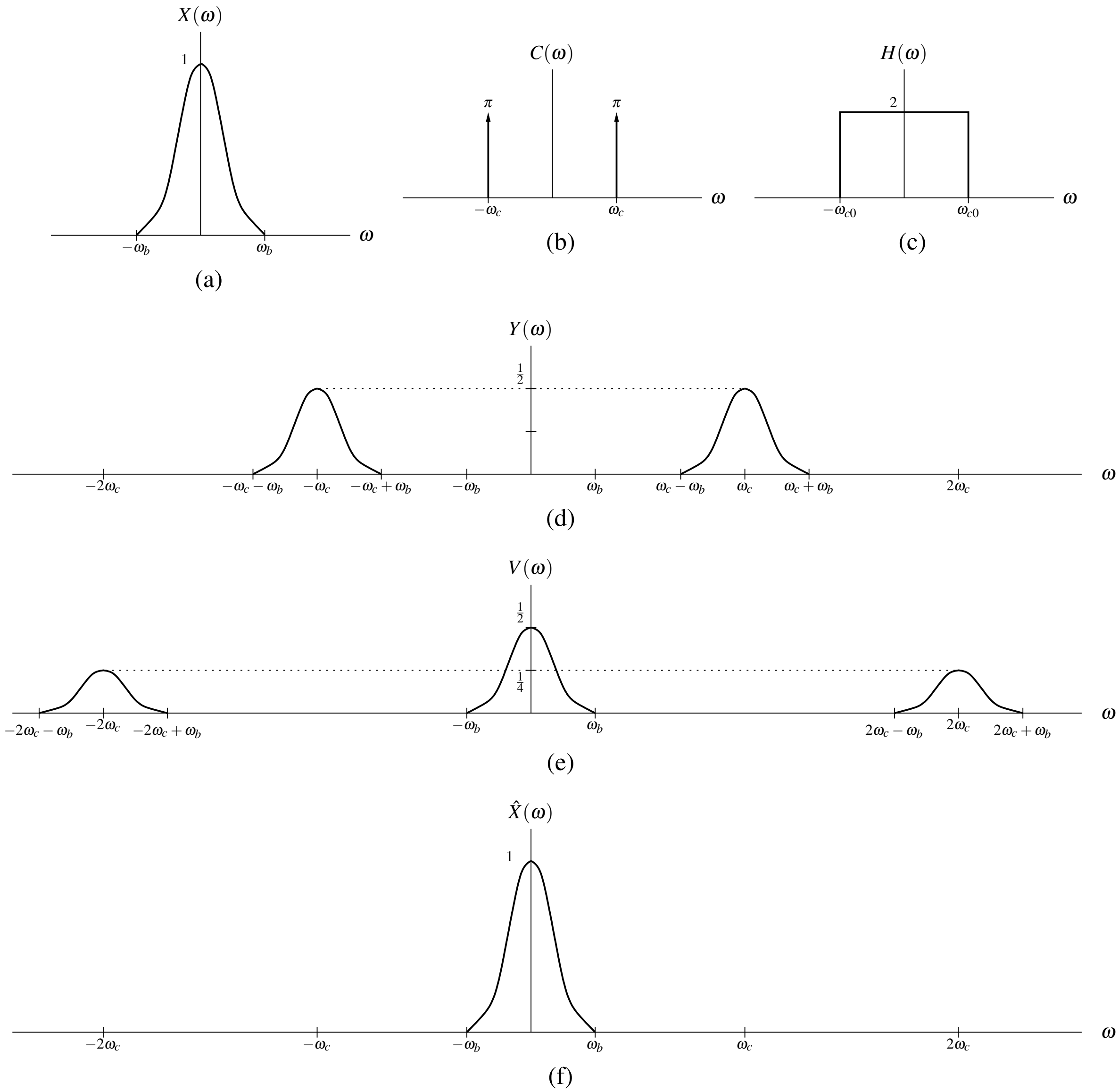


Figure 6.31: Signal spectra for DSB/SC amplitude modulation. (a) Spectrum of the transmitter input. (b) Spectrum of the sinusoidal function used in the transmitter and receiver. (c) Frequency response of the filter in the receiver. (d) Spectrum of the transmitted signal. (e) Spectrum of the multiplier output in the receiver. (f) Spectrum of the receiver output.

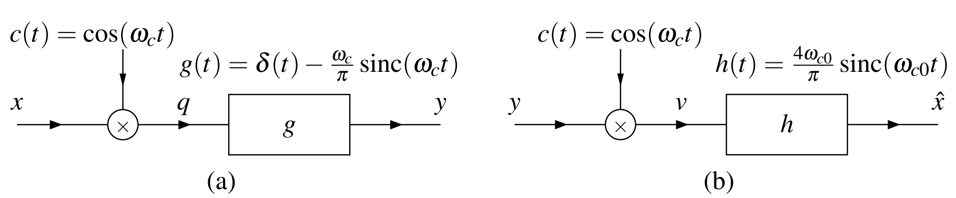


Figure 6.32: SSB/SC amplitude modulation system. (a) Transmitter and (b) receiver.

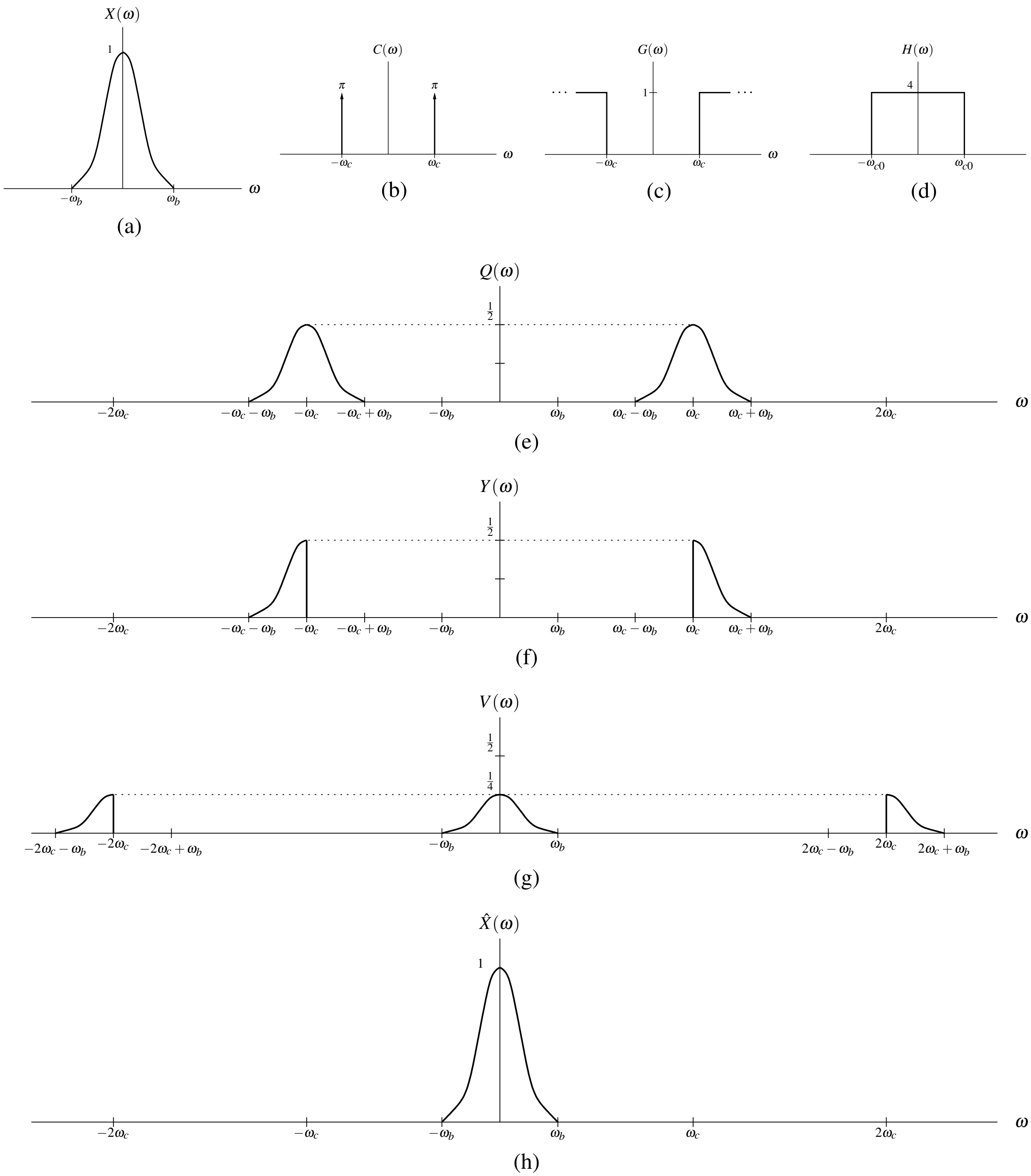


Figure 6.33: Signal spectra for SSB/SC amplitude modulation. (a) Spectrum of the transmitter input. (b) Spectrum of the sinusoid used in the transmitter and receiver. (c) Frequency response of the filter in the transmitter. (d) Frequency response of the filter in the receiver. (e) Spectrum of the multiplier output in the transmitter. (f) Spectrum of the transmitted signal. (g) Spectrum of the multiplier output in the receiver. (h) Spectrum of the receiver output.

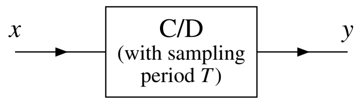


Figure 6.34: Ideal C/D converter with input function x and output sequence y.

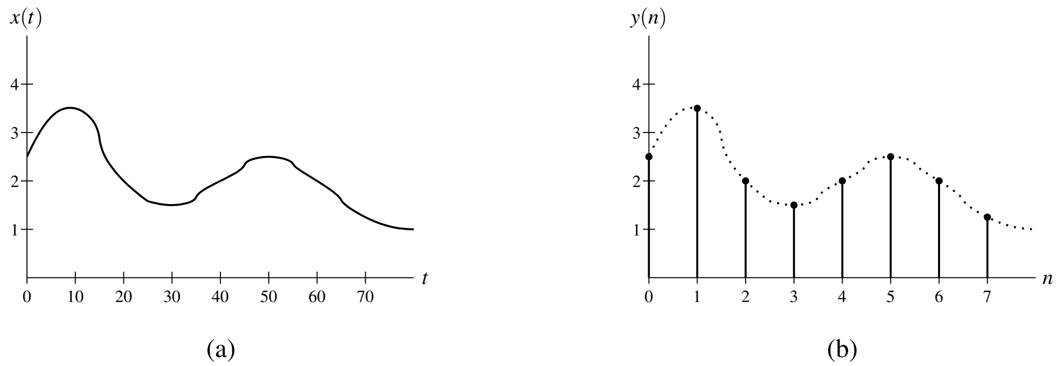


Figure 6.35: Example of periodic sampling. (a) The function x to be sampled and (b) the sequence y produced by sampling x with a sampling period of 10.

6.20 Sampling and Interpolation

Often, we encounter situations in which we would like to process a continuous-time signal in the discrete-time domain or vice versa. For example, we might have a continuous-time audio signal that we would like to process using a digital computer (which is a discrete-time system), or we might have a discrete-time audio signal that we wish to play on a loudspeaker (which is a continuous-time system). Clearly, some means is needed to link the continuous- and discrete-time domains. This connection is established through processes known as sampling and interpolation. In what follows, we will formally introduce these processes and study them in some detail.

Sampling allows us to create sequence (i.e., a discrete-time signal) from a function (i.e., a continuous-time signal). Although sampling can be performed in many different ways, the most commonly used scheme is **periodic sampling**. With this scheme, a sequence y of samples is obtained from a function x as given by

$$y(n) = x(Tn) \quad \text{for all integer } n, \tag{6.51}$$

where T is a positive real constant. As a matter of terminology, T is referred to as the **sampling period**, and $\omega_s = \frac{2\pi}{T}$ is referred to as the (angular) **sampling frequency**. A system such as that described by (6.51) is known as an **ideal continuous-to-discrete-time (C/D) converter**, and is shown diagrammatically in Figure 6.34. An example of periodic sampling is shown in Figure 6.35. Figure 6.35(a) shows a function x to be sampled, and Figure 6.35(b) shows the sequence y obtained by sampling x with the sampling period $T = 10$.

Interpolation allows us to construct a function (i.e., a continuous-time signal) from a sequence (i.e., a discrete-time signal). In effect, for a given sequence, this process constructs a function that would produce the given sequence when sampled, typically with some additional constraints imposed. More formally, for a given sequence y associated with a sampling period T, interpolation produces a function $\hat{x}$ as given by

$$\hat{x} = \mathcal{H}y,$$

subject to the constraint that

$$\hat{x}(Tn) = y(n) \quad \text{for all integer } n,$$

where $\mathcal{H}$ is an operator that maps a sequence to a function. The precise form of $\mathcal{H}$ depends on the particular interpolation scheme employed. Although there are many different ways in which to perform interpolation, we will focus

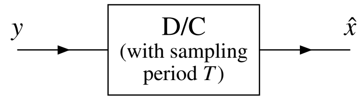


Figure 6.36: Ideal D/C converter with input sequence y and output function $\hat{x}$.

our attention in subsequent sections on one particular scheme known as bandlimited interpolation. The interpolation process is performed by a system known as an **ideal discrete-to-continuous-time (D/C) converter**, as shown in Figure 6.36.

In the absence of any constraints on the sampling process, a function cannot be uniquely determined from a sequence of its equally-spaced samples. In other words, the sampling process is not generally invertible. Consider, for example, the functions x_1 and x_2 given by

$$x_1(t) = 0 \quad \text{and} \quad x_2(t) = \sin(2\pi t).$$

If we sample each of these functions with the sampling period $T = 1$, we obtain the respective sequences

$$y_1(n) = x_1(Tn) = x_1(n) = 0 \quad \text{and} \quad y_2(n) = x_2(Tn) = \sin(2\pi n) = 0.$$

Thus, $y_1 = y_2$ in spite of the fact that $x_1 \neq x_2$. This example trivially shows that if no constraints are placed upon a function, then the function cannot be uniquely determined from its samples.

Fortunately, under certain circumstances, a function can be recovered exactly from its samples. In particular, in the case that the function being sampled is bandlimited, we can show that a sequence of its equally-spaced samples uniquely determines the function if the sampling period is sufficiently small. This result, known as the sampling theorem, is of paramount importance.

6.20.1 Sampling

In order to gain some insight into sampling, we need a way in which to mathematically model this process. To this end, we employ the simple model for the ideal C/D converter shown in Figure 6.37. In short, we may view the process of sampling as impulse train modulation followed by conversion of an impulse train to a sequence of sample values. More specifically, to sample a function x with sampling period T, we first multiply x by the periodic impulse train p to obtain

$$s(t) = x(t)p(t), \tag{6.52}$$

where

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

Then, we take the weights of successive impulses in s to form a sequence y of samples. The sampling frequency is given by $\omega_s = \frac{2\pi}{T}$. As a matter of terminology, p is referred to as a sampling function. From the diagram, we can see that the signals s and y, although very closely related, have some key differences. The impulse train s is a function (i.e., continuous-time signal) that is zero everywhere except at integer multiples of T (i.e., at sample points), while y is a sequence (i.e., discrete-time signal), defined only on the integers with its values corresponding to the weights of successive impulses in s. The various signals involved in sampling are illustrated in Figure 6.38.

In passing, we note that the above model of sampling is only a mathematical convenience. That is, the model provides us with a relatively simple way in which to study the mathematical behavior of sampling. The above model, however, is not directly useful as a means for actually realizing sampling in a real-world system. Obviously, the impulse train employed in the above model poses some insurmountable problems as far as implementation is concerned.

Now, let us consider the above model of sampling in more detail. In particular, we would like to find the relationship between the frequency spectra of the original function x and its impulse-train sampled version s. In what follows, let X, Y, P, and S denote the Fourier transforms of x, y, p, and s, respectively. Since p is T-periodic, it can be

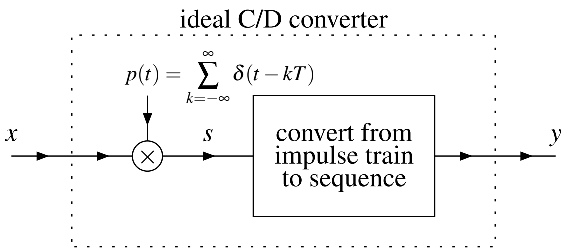


Figure 6.37: Model of ideal C/D converter with input function x and output sequence y.

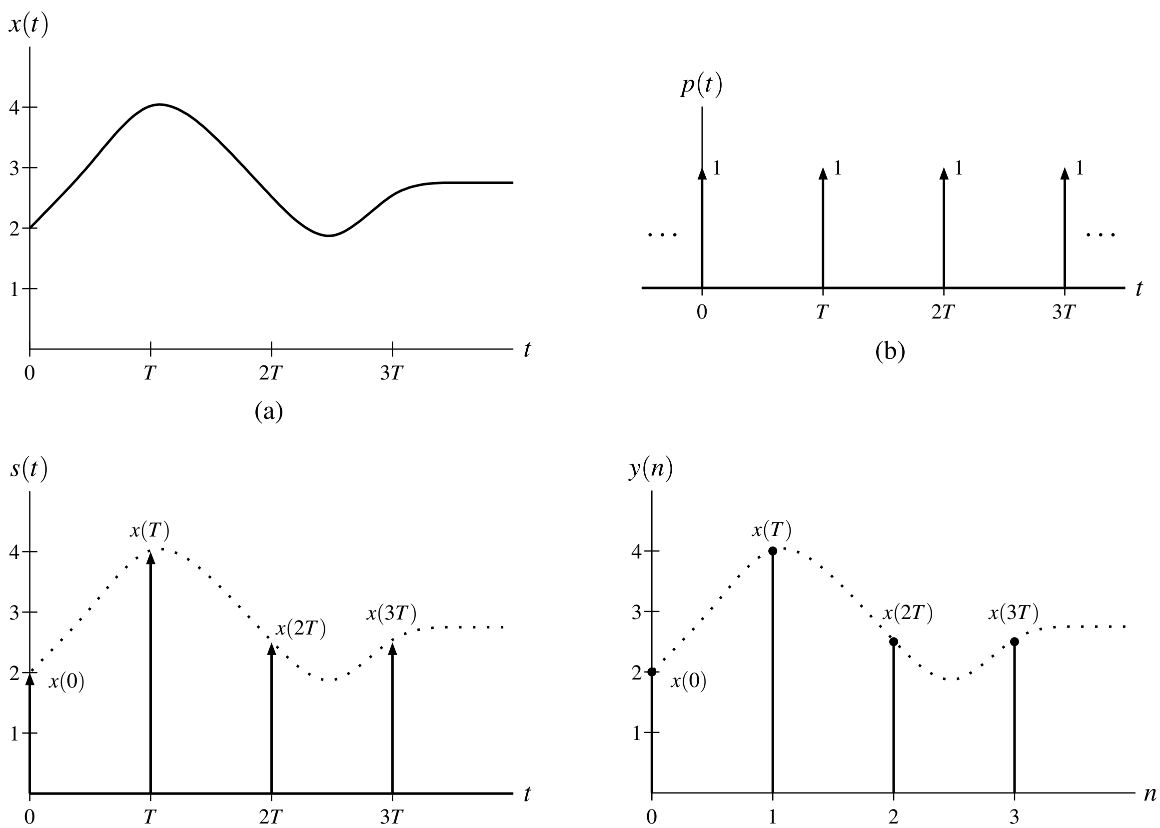


Figure 6.38: An example of the various signals involved in the sampling process for a sampling period of T. (a) The function x to be sampled. (b) The sampling function p. (c) The impulse-modulated function s. (d) The sequence y produced by sampling.

represented in terms of a Fourier series as

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}. \tag{6.53}$$

Using the Fourier series analysis equation, we calculate the coefficients c_k to be

$$\begin{aligned} c_k &= \tfrac{1}{T}\int_{-T/2}^{T/2} p(t)e^{-jk\omega_s t}dt \\ &= \tfrac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_s t}dt \\ &= \tfrac{1}{T}\int_{-\infty}^{\infty} \delta(t)e^{-jk\omega_s t}dt \\ &= \tfrac{1}{T} \\ &= \tfrac{\omega_s}{2\pi}. \end{aligned} \tag{6.54}$$

Substituting (6.53) and (6.54) into (6.52), we obtain

$$\begin{aligned} s(t) &= x(t)\sum_{k=-\infty}^{\infty} \tfrac{\omega_s}{2\pi} e^{jk\omega_s t} \\ &= \tfrac{\omega_s}{2\pi}\sum_{k=-\infty}^{\infty} x(t)e^{jk\omega_s t}. \end{aligned}$$

Taking the Fourier transform of s yields

$$S(\omega) = \tfrac{\omega_s}{2\pi}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s). \tag{6.55}$$

Thus, the spectrum of the impulse-train sampled function s is a scaled sum of an infinite number of shifted copies of the spectrum of the original function x.

Now, we consider a simple example to further illustrate the behavior of the sampling process in the frequency domain. Suppose that we have a function x with the Fourier transform X where $|X(\omega)| = 0$ for $|\omega| > \omega_m$ (i.e., x is bandlimited). To simplify the visualization process, we will assume X has the particular form shown in Figure 6.39(a). In what follows, however, we only actually rely on the bandlimited nature of x and not its specific definition. So, the results that we derive in what follows generally apply to any bandlimited function. From (6.55), we know that S is formed by the superposition of an infinite number of shifted copies of X. Upon more careful consideration, we can see that two distinct situations can arise. That is, the nonzero portions of the shifted copies of X used to form S can either: 1) overlap or, 2) not overlap. These two cases are illustrated in Figures 6.39(b) and 6.39(c), respectively. From these graphs, we can see that the nonzero portions of the shifted copies of X will not overlap if

$$\omega_m < \omega_s - \omega_m \quad \text{and} \quad -\omega_m > -\omega_s + \omega_m$$

or equivalently

$$\omega_s > 2\omega_m.$$

Consider the case in which the copies of the original spectrum X in S do not overlap, as depicted in Figure 6.39(b). In this situation, the spectrum X of the original function is clearly discernible in the spectrum S. In fact, one can see that the original spectrum X can be obtained directly from S through a lowpass filtering operation. Thus, the original function x can be exactly recovered from s.

Now, consider the case in which copies of the original spectrum X in S do overlap. In this situation, multiple frequencies in the spectrum X of the original function are mapped to the same frequency in S. This phenomenon is referred to as **aliasing**. Clearly, aliasing leads to individual periods of S having a different shape than the original spectrum X. When aliasing occurs, the shape of the original spectrum X is no longer discernible from S. Consequently, we are unable to recover the original function x from s in this case.

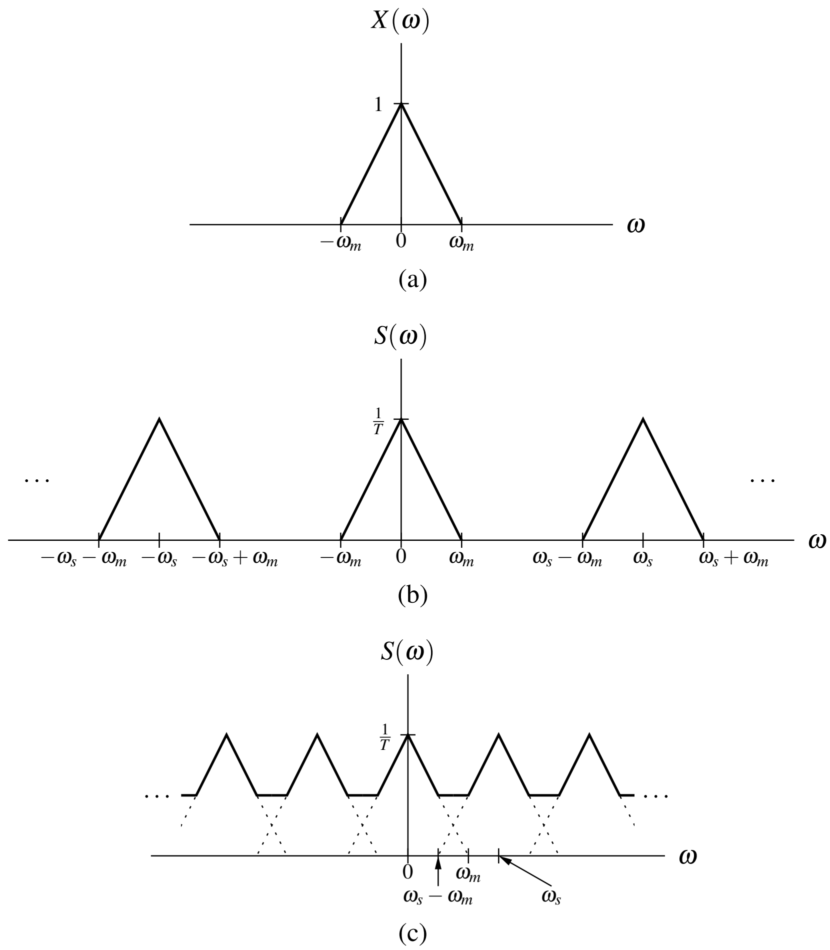


Figure 6.39: Effect of impulse-train sampling on the frequency spectrum. (a) Spectrum of the function x being sampled. (b) Spectrum of s in the absence of aliasing. (c) Spectrum of s in the presence of aliasing.

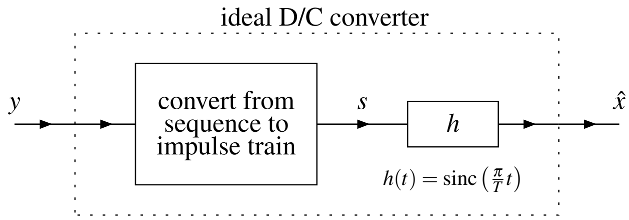


Figure 6.40: Model of ideal D/C converter with input sequence y and output function $\hat{x}$.

6.20.2 Interpolation and Reconstruction of a Function From Its Samples

Interpolation allows us to construct a function (i.e., continuous-time signal) from a sequence (i.e., discrete-time signal). This process is essentially responsible for determining the value of a function between sample points. Except in very special circumstances, it is not generally possible to exactly reproduce a function from its samples. Although many interpolation schemes exist, we shall focus our attention shortly on one particular scheme. The interpolation process can be modeled with the simple ideal D/C converter system, shown in Figure 6.40.

Recall the ideal C/D converter of Figure 6.37. Since the process of converting an impulse train to a sequence is invertible, we can reconstruct the original function x from a sequence y of its samples if we can somehow recover x from s. Let us suppose now that x is bandlimited. As we saw in the previous section, we can recover x from s provided that x is bandlimited and sampled at a sufficiently high rate so as to avoid aliasing. In the case that aliasing does not occur, we can reconstruct the original function x from y using the ideal D/C converter shown in Figure 6.40. In what follows, we will derive a formula for computing the original function $\hat{x}$ from its samples y. Consider the model of the D/C converter. We have a lowpass filter with impulse response

$$h(t) = \operatorname{sinc}\left(\tfrac{\pi t}{T}\right) = \operatorname{sinc}\left(\tfrac{\omega_s t}{2}\right)$$

and frequency response

$$H(\omega) = T\operatorname{rect}\left(\tfrac{T\omega}{2\pi}\right) = \tfrac{2\pi}{\omega_s}\operatorname{rect}\left(\tfrac{\omega}{\omega_s}\right) = \begin{cases} T & |\omega| < \frac{\omega_s}{2} \\ 0 & \text{otherwise.} \end{cases}$$

First, we convert the sequence y to the impulse train s to obtain

$$s(t) = \sum_{n=-\infty}^{\infty} y(n)\delta(t - Tn).$$

Then, we filter the resulting function s with the lowpass filter having impulse response h, yielding

$$\begin{aligned}
\hat{x}(t) &= s * h(t) \\
&= \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(t-\tau)\sum_{n=-\infty}^{\infty} y(n)\delta(\tau - Tn)d\tau \\
&= \sum_{n=-\infty}^{\infty} y(n)\int_{-\infty}^{\infty} h(t-\tau)\delta(\tau - Tn)d\tau \\
&= \sum_{n=-\infty}^{\infty} y(n)h(t - Tn) \\
&= \sum_{n=-\infty}^{\infty} y(n)\operatorname{sinc}\left[\tfrac{\pi}{T}(t - Tn)\right].
\end{aligned}$$

If x is bandlimited and aliasing is avoided, $\hat{x} = x$ and we have a formula for exactly reproducing x from its samples y.

6.20.3 Sampling Theorem

In the preceding sections, we have established the important result given by the theorem below.

Theorem 6.19 (Sampling Theorem)**.** *Let x be a function with the Fourier transform X, and let y be the sequence resulting from the periodic sampling of x with the sampling period T (i.e., $y(n) = x(Tn)$). Suppose that $|X(\omega)| = 0$ for all $|\omega| > \omega_M$ (i.e., x is bandlimited to the interval $[-\omega_M, \omega_M]$). Then, x is uniquely determined by y if*

$$\omega_s > 2\omega_M, \tag{6.56}$$

where $\omega_s = \frac{2\pi}{T}$. In particular, if (6.56) is satisfied, we have that

$$x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}\left[\tfrac{\pi}{T}(t - Tn)\right],$$

or equivalently (i.e., rewritten in terms of ω_s instead of T),

$$x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}\left(\tfrac{\omega_s}{2}t - \pi n\right).$$

As a matter of terminology, we refer to (6.56) as the **Nyquist condition** (or Nyquist criterion). Also, we call $\frac{\omega_s}{2}$ the **Nyquist frequency** and $2\omega_M$ the **Nyquist rate**. It is important to note that the Nyquist condition is a strict inequality. Therefore, to ensure aliasing does not occur in the most general case, one must choose the sampling rate larger than the Nyquist rate. One can show, however, that if the frequency spectrum does not have impulses at the Nyquist frequency, it is sufficient to sample at exactly the Nyquist rate.

Example 6.42. Let x denote a continuous-time audio signal with Fourier transform X. Suppose that $|X(\omega)| = 0$ for all $|\omega| \geq 44100\pi$. Determine the largest period T with which x can be sampled that will allow x to be exactly recovered from its samples.

Solution. The function x is bandlimited to frequencies in the range $(-\omega_m, \omega_m)$, where $\omega_m = 44100\pi$. From the sampling theorem, we know that the minimum sampling rate required is given by

$$\begin{aligned} \omega_s &= 2\omega_m \\ &= 2(44100\pi) \\ &= 88200\pi. \end{aligned}$$

Thus, the largest permissible sampling period is given by

$$\begin{aligned} T &= \tfrac{2\pi}{\omega_s} \\ &= \tfrac{2\pi}{88200\pi} \\ &= \tfrac{1}{44100}. \end{aligned}$$

■

Although the sampling theorem provides an upper bound on the sampling rate that holds in the case of arbitrary bandlimited functions, in some special cases it may be possible to employ an even smaller sampling rate. This point is further illustrated by way of the example below.

Example 6.43. Consider the function x with the Fourier transform X shown in Figure 6.41 (where $\omega_c \gg \omega_a$). (a) Using the sampling theorem directly, determine the largest permissible sampling period T that will allow x to be exactly reconstructed from its samples. (b) Explain how one can exploit the fact that $X(\omega) = 0$ for a large portion of the interval $[-\omega_c - \omega_a, \omega_c + \omega_a]$ in order to reduce the rate at which x must be sampled.

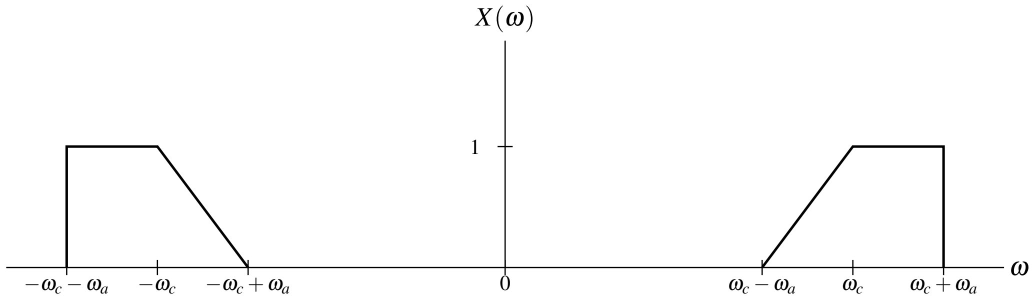


Figure 6.41: Frequency spectrum of the function x.

Solution. (a) The function x is bandlimited to $[-\omega_m, \omega_m]$, where $\omega_m = \omega_c + \omega_a$. Thus, the minimum sampling rate required is given by

$$\begin{aligned}\omega_s &> 2\omega_m \\ &= 2(\omega_c+\omega_a) \\ &= 2\omega_c + 2\omega_a.\end{aligned}$$

and the maximum sampling period is calculated as

$$\begin{aligned}T &< \tfrac{2\pi}{\omega_s} \\ &= \tfrac{2\pi}{2\omega_c+2\omega_a} \\ &= \tfrac{\pi}{\omega_c+\omega_a}.\end{aligned}$$

(b) We can modulate and lowpass filter x in order to compress all of its spectral information into the frequency range $[-2\omega_a, 2\omega_a]$, yielding the function x_1. That is, we have

$$x_1(t) = \{x(t)\cos[(\omega_c-\omega_a)t]\} * h(t)$$

where

$$h(t) = \tfrac{4\omega_a}{\pi}\operatorname{sinc}(2\omega_a t) \quad \stackrel{\text{CTFT}}{\longleftrightarrow} \quad H(\omega) = 2\operatorname{rect}\left(\tfrac{\omega}{4\omega_a}\right).$$

This process can be inverted (by modulation and filtering) to obtain x from x_1. In particular, we have that

$$x(t) = \{x_1(t)\cos[(\omega_c-\omega_a)t]\} * h_2(t)$$

where

$$h_2(t) = \delta(t) - \tfrac{2(\omega_c-\omega_a)}{\pi}\operatorname{sinc}[(\omega_c-\omega_a)t] \quad \stackrel{\text{CTFT}}{\longleftrightarrow} \quad H_2(\omega) = 2 - 2\operatorname{rect}\left[\tfrac{\omega}{4(\omega_c-\omega_a)}\right].$$

Let X_1 denote the Fourier transform of x_1. The Fourier transform X_1 is as shown in Figure 6.42. Applying the sampling theorem to x_1 we find that the minimum sampling rate is given by

$$\begin{aligned}\omega_s &> 2(2\omega_a) \\ &= 4\omega_a\end{aligned}$$

and the largest sampling period is given by

$$\begin{aligned}T &< \tfrac{2\pi}{\omega_s} \\ &= \tfrac{2\pi}{4\omega_a} \\ &= \tfrac{\pi}{2\omega_a}.\end{aligned}$$

Since $\omega_c \gg \omega_a$ (by assumption), this new sampling period is larger than the one computed in part (a) of this example. ■

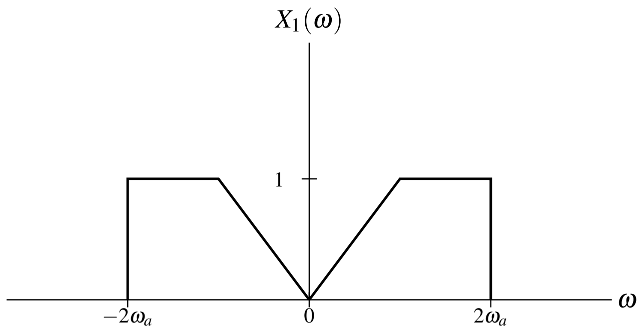


Figure 6.42: Frequency spectrum of the function x_1.

6.21 Exercises

6.21.1 Exercises Without Answer Key

6.1 Using the Fourier transform analysis equation, find the Fourier transform X of each function x below.
(a) $x(t) = A\delta(t-t_0)$, where t_0 and A are real and complex constants, respectively;
(b) $x(t) = \operatorname{rect}(t-t_0)$, where t_0 is a constant;
(c) $x(t) = e^{-4t}u(t-1)$;
(d) $x(t) = 3[u(t)-u(t-2)]$; and
(e) $x(t) = e^{-|t|}$.

6.2 Use a Fourier transform table and properties of the Fourier transform to find the Fourier transform X of each function x below.
(a) $x(t) = \cos(t-5)$;
(b) $x(t) = e^{-j5t}u(t+2)$;
(c) $x(t) = \cos(t)u(t)$;
(d) $x(t) = 6[u(t)-u(t-3)]$;
(e) $x(t) = 1/t$;
(f) $x(t) = t\operatorname{rect}(2t)$;
(g) $x(t) = e^{-j3t}\sin(5t-2)$;
(h) $x(t) = \cos(5t-2)$;
(i) $x(t) = e^{-j2t}\frac{1}{3t+1}$;
(j) $x(t) = \int_{-\infty}^{5t} e^{-\tau-1}u(\tau-1)d\tau$;
(k) $x(t) = (t+1)\sin(5t-3)$;
(l) $x(t) = \sin(2\pi t)\delta(t-\frac{\pi}{2})$;
(m) $x(t) = e^{-jt}\frac{1}{3t-2}$;
(n) $x(t) = e^{j5t}\cos(2t)u(t)$; and
(o) $x(t) = e^{-j2t}\operatorname{sgn}(-t-1)$.

6.3 Compute the Fourier transform X of the function

$$x(t) = \sum_{k=0}^{\infty} a^k \delta(t-kT),$$

where a is a constant satisfying $|a| < 1$ and T is a strictly-positive real constant. (Hint: Recall the formula for the sum of an infinite geometric series. That is, $b + br + br^2 + \ldots = \frac{b}{1-r}$ if $|r| < 1$.)

6.4 The ideal Hilbert transformer is a LTI system with the frequency response

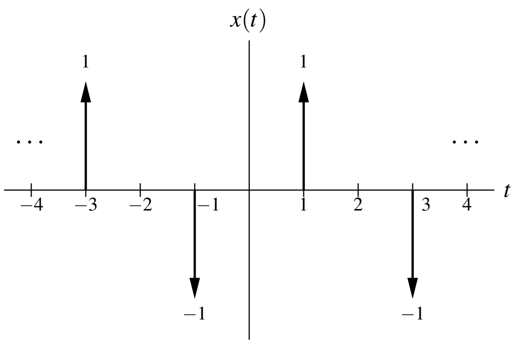
$$H(\omega) = -j\operatorname{sgn}\omega.$$

This type of system is useful in a variety of signal processing applications (e.g., SSB/SC amplitude modulation). By using the duality property of the Fourier transform, find the impulse response h of this system.

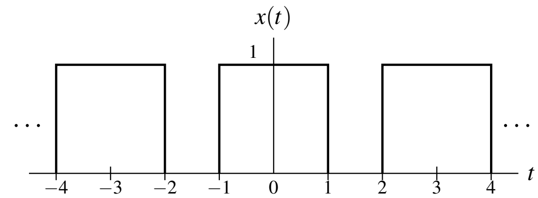
6.5 For each function y given below, find the Fourier transform Y of y in terms of the Fourier transform X of x.
(a) $y(t) = x(at-b)$, where a and b are constants and $a \neq 0$;
(b) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;
(c) $y(t) = \int_{-\infty}^{t} x^2(\tau)d\tau$;
(d) $y(t) = \mathcal{D}(x*x)(t)$, where $\mathcal{D}$ denotes the derivative operator;
(e) $y(t) = tx(2t-1)$;

(f) $y(t) = e^{j2t}x(t-1)$;
(g) $y(t) = \left(te^{-j5t}x(t)\right)^*$;
(h) $y(t) = (\mathcal{D}x) * x_1(t)$, where $x_1(t) = e^{-jt}x(t)$ and $\mathcal{D}$ denotes the derivative operator;
(i) $y(t) = \int_{-\infty}^{3t} x^*(\tau - 1)d\tau$;
(j) $y(t) = \cos(3t-1)x(t)$;
(k) $y(t) = \mathcal{D}x(t)\sin(t-2)$, where $\mathcal{D}$ denotes the derivative operator;
(l) $y(t) = tx(t)\sin(3t)$; and
(m) $y(t) = e^{j7t}\left[x * x(t-1)\right]$.

6.6 Find the Fourier transform X of each periodic function x shown below.



(a)



(b)

6.7 Using the time-domain convolution property of the Fourier transform, compute the convolution $h = h_1 * h_2$, where

$$h_1(t) = 2000\,\mathrm{sinc}(2000\pi t) \quad \text{and} \quad h_2(t) = \delta(t) - 1000\,\mathrm{sinc}(1000\pi t).$$

6.8 Compute the energy contained in the function $x(t) = 200\,\mathrm{sinc}(200\pi t)$.

6.9 For each function x given below, compute the frequency spectrum of x, and find and plot the corresponding magnitude and phase spectra.
(a) $x(t) = e^{-at}u(t)$, where a is a positive real constant; and
(b) $x(t) = \mathrm{sinc}\left(\frac{t-1}{200}\right)$.

6.10 For each differential/integral equation below that defines a LTI system with input x and output y, find the frequency response H of the system. (Note that the prime symbol denotes differentiation.)
(a) $y''(t) + 5y'(t) + y(t) + 3x'(t) - x(t) = 0$;
(b) $y'(t) + 2y(t) + \int_{-\infty}^{t} 3y(\tau)d\tau + 5x'(t) - x(t) = 0$; and
(c) $y''(t) + 5y'(t) + 6y(t) = x'(t) + 11x(t)$.

6.11 For each frequency response H given below for a LTI system with input x and output y, find the differential equation that characterizes the system.
(a) $H(\omega) = \dfrac{j\omega}{1 + j\omega}$; and
(b) $H(\omega) = \dfrac{j\omega + \frac{1}{2}}{-j\omega^3 - 6\omega^2 + 11j\omega + 6}$.

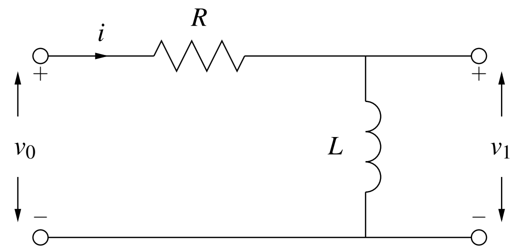
6.12 Consider the LTI system with impulse response

$$h(t) = \delta(t) - 300\,\mathrm{sinc}(300\pi t).$$

Using frequency-domain methods, find the response y of the system to the input

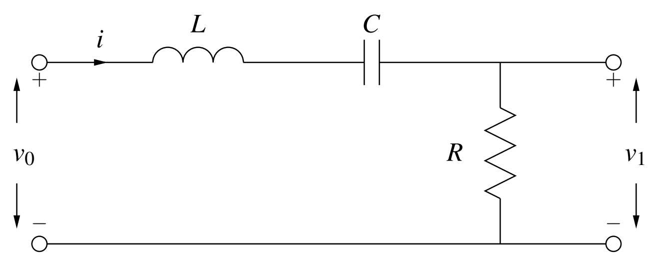
$$x(t) = \tfrac{1}{2} + \tfrac{3}{4}\cos(200\pi t) + \tfrac{1}{2}\cos(400\pi t) - \tfrac{1}{4}\cos(600\pi t).$$

6.13 Consider the LTI system with input v_0 and output v_1 as shown in the figure below, where $R = 1$ and $L = 1$.



(a) Find the frequency response H of the system.
(b) Determine the magnitude and phase responses of the system.
(c) Find the impulse response h of the system.

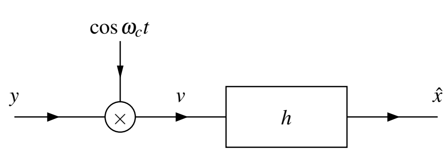
6.14 Consider the LTI system with input v_0 and output v_1 as shown in the figure below, where $R = 1$, $C = \frac{1}{1000}$, and $L = \frac{1}{1000}$.



(a) Find the frequency response H of the system.
(b) Use a computer to plot the magnitude and phase responses of the system.
(c) From the plots in part (b), identify the type of ideal filter that this system approximates.

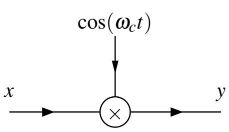
6.15 Let x be a real function with Fourier transform X satisfying $X(\omega) = 0$ for $|\omega| > \omega_b$. We use amplitude modulation to produce the function $y(t) = x(t)\sin(\omega_c t)$. Note that $\omega_c \gg \omega_b$. In order to recover the original function x, it is proposed that the system shown in the figure below be used. This system contains a LTI subsystem that is labelled with its impulse response h. Let Y, V, $\hat{X}$, and H denote the Fourier transforms of y, v, $\hat{x}$, and h, respectively. The system is such that

$$H(\omega) = \begin{cases} 2 & |\omega| < \omega_b \\ 0 & \text{otherwise.} \end{cases}$$

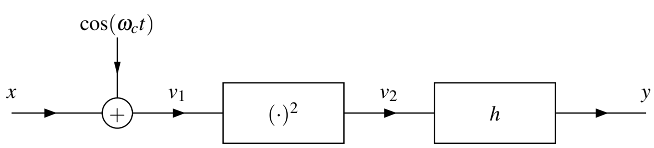


(a) Find an expression for Y in terms of X. Find an expression for $\hat{X}$ in terms of V. Find an expression for $\hat{X}$ in terms of X.
(b) Compare $\hat{x}$ and x. Comment on the utility of the proposed system.

6.16 When discussing DSB/SC amplitude modulation, we saw that a system of the form shown below in Figure A is often useful. In practice, however, the multiplier unit needed by this system is not always easy to implement. For this reason, we sometimes employ a system like that shown below in Figure B. In this second system, we sum the sinusoidal carrier and modulating signal x and then pass the result through a nonlinear squaring device (i.e., $v_2(t) = v_1^2(t)$). This system also contains a LTI subsystem with impulse response h.



(a)



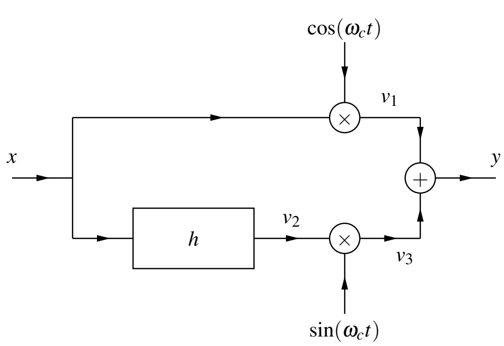
(b)

Let X, V_1, V_2, and H denote the Fourier transforms of x, v_1, v_2, and h, respectively. Suppose that $X(\omega) = 0$ for $|\omega| > \omega_b$ (i.e., x is bandlimited).
(a) Find an expression for v_1, v_2, and V_2. (Hint: If $X(\omega) = 0$ for $|\omega| > \omega_b$, then using the time-domain convolution property of the Fourier transform, we can deduce that the Fourier transform of x^2 is zero for $|\omega| > 2\omega_b$.)
(b) Determine the frequency response H required for the system shown in Figure B to be equivalent to the system in Figure A. State any assumptions made with regard to the relationship between ω_c and ω_b. (Hint: It might be helpful to sketch X and V_2 for the case of some simple X. Then, compare V_2 to X in order to deduce your answer.)

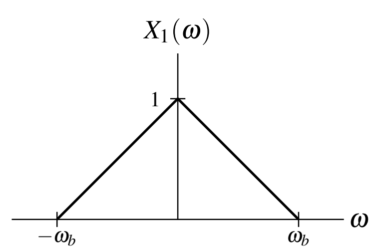
6.17 Consider the system with input x and output y as shown in Figure A below. The impulse response h is that of an ideal Hilbert transformer, whose frequency response H is given by

$$H(\omega) = -j \operatorname{sgn} \omega.$$

Let X, Y, V_1, V_2, and V_3 denote the Fourier transforms of x, y, v_1, v_2, and v_3, respectively.



(a)



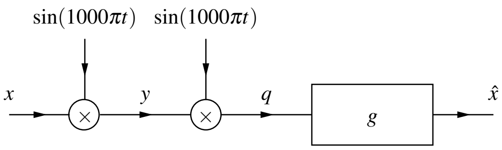
(b)

(a) Suppose that $X(\omega) = 0$ for $|\omega| > \omega_b$, where $\omega_b \ll \omega_c$. Find expressions for V_1, V_2, V_3, and Y in terms of X.
(b) Suppose that $X = X_1$, where X_1 is as shown in Figure B. Sketch V_1, V_2, V_3, and Y in this case.
(c) Draw the block diagram of a system that could be used to recover x from y.

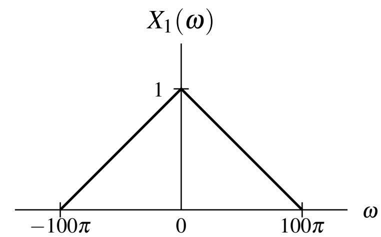
6.18 Consider the system shown below in Figure A with input x and output $\hat{x}$, where this system contains a LTI subsystem with impulse response g. The Fourier transform G of g is given by

$$G(\omega) = \begin{cases} 2 & |\omega| \leq 100\pi \\ 0 & \text{otherwise.} \end{cases}$$

Let X, $\hat{X}$, Y, and Q denote the Fourier transforms of x, $\hat{x}$, y, and q, respectively.



(a)



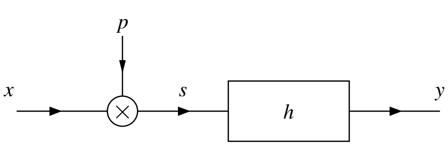
(b)

(a) Suppose that $X(\omega) = 0$ for $|\omega| > 100\pi$. Find expressions for Y, Q, and $\hat{X}$ in terms of X.
(b) If $X = X_1$ where X_1 is as shown in Figure B, sketch Y, Q, and $\hat{X}$.

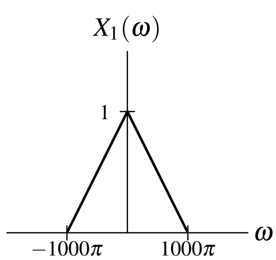
6.19 Consider the system shown below in Figure A with input x and output y. Let X, P, S, H, and Y denote the Fourier transforms of x, p, s, h, and y, respectively. Suppose that

$$p(t) = \sum_{n=-\infty}^{\infty} \delta\left(t - \tfrac{n}{1000}\right) \quad \text{and} \quad H(\omega) = \tfrac{1}{1000} \operatorname{rect}\left(\tfrac{\omega}{2000\pi}\right).$$

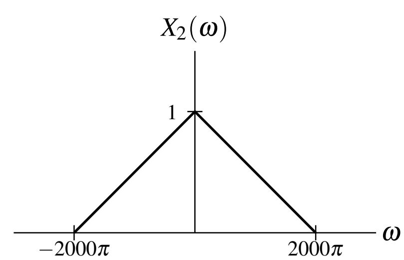
(a) Derive an expression for S in terms of X. Derive an expression for Y in terms of S and H.
(b) Suppose that $X = X_1$, where X_1 is as shown in Figure B. Using the results of part (a), plot S and Y. Indicate the relationship (if any) between the input x and output y of the system.
(c) Suppose that $X = X_2$, where X_2 is as shown in Figure C. Using the results of part (a), plot S and Y. Indicate the relationship (if any) between the input x and output y of the system.



(a)



(b)



(c)

6.20 Let X denote the Fourier transform of x. Show that:
(a) if x is even, then $X(\omega) = 2\int_0^\infty x(t)\cos(\omega t)dt$; and
(b) if x is odd, then $X(\omega) = -2j\int_0^\infty x(t)\sin(\omega t)dt$.

6.21 Let x be a real function with even part x_e and odd part x_o. Let X, X_e, and X_o denote the Fourier transforms of x, x_e, and x_o, respectively. Show that:
(a) $X_e(\omega) = \mathrm{Re}[X(\omega)]$; and
(b) $X_o(\omega) = j\,\mathrm{Im}[X(\omega)]$.

6.22 Let $y(t) = x(t+a) + x(t-a)$, where a is a real constant, and let X and Y denote the Fourier transforms of x and y, respectively. Show that $Y(\omega) = 2X(\omega)\cos(a\omega)$.

6.23 Show that, if a function x has bandwidth B, then $x^n(t)$ has bandwidth nB.

6.24 Suppose that we want to transmit a binary sequence y (where $y(n)$ is either zero or one) over a continuous-time communication channel. To do this, we choose to represent each binary sample of y with a continuous-time pulse. Using the sampling theorem, show that it is possible to transmit the bits of y at a rate of $2B$ bits per second over an ideal (i.e., noise-free) channel of bandwidth B. As it turns out, this is the theoretical upper bound on the data transmission rate, assuming that each pulse is used to represent only a single bit of data. (Hint: According to the sampling theorem, a continuous-time function of bandwidth B can be constructed from $2B$ pieces of information.)

6.25 Show that a function cannot be both timelimited and bandlimited. [Hint: Let X denote the Fourier transform of x. Suppose that x is timelimited and also bandlimited such that $X(\omega) = 0$ for $|\omega| \geq B$. Due to the bandlimited nature of x, we have that $X(\omega) = X(\omega)\,\mathrm{rect}\left(\frac{\omega}{2B'}\right)$ for $B' > B$. Then, show that the inverse Fourier transform of the preceding equation leads to a contradiction. To do this, you will need to observe that the convolution of a timelimited function with a non-timelimited function must be a non-timelimited function.]

6.26 The bandwidth of a LTI system is most simply defined as the bandwidth of the system's frequency response H. Explain why a (LTI) communication channel with (finite) bandwidth B cannot be used to (reliably) transmit a signal with bandwidth greater than B.

6.27 Let y_1 and y_2 be functions bandlimited to frequencies in the range $[-\omega_b, \omega_b]$. Suppose that these functions are sampled at a frequency ω_s satisfying the Nyquist condition to produce the sequences

$$x_1(n) = y_1(Tn) \quad \text{and} \quad x_2(n) = y_2(Tn),$$

where $T = \frac{2\pi}{\omega_s}$. Now, consider the function $y = y_1 * y_2$. Suppose that y is also sampled with period T to produce the sequence

$$x(n) = y(Tn).$$

(a) Show that y is bandlimited to frequencies in the range $[-\omega_b, \omega_b]$, meaning that it must be possible to recover y exactly from its samples.
(b) Show that the samples of y can be computed by

$$x(n) = \frac{2\pi}{\omega_s} \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k).$$

(c) Explain how we might use the above results to compute the (continuous-time) convolution of bandlimited functions using a (discrete-time) computer.

6.28 A function x is bandlimited to 22 kHz (i.e., only has spectral content for frequencies f in the range $[-22000, 22000]$). Due to excessive noise, the portion of the spectrum that corresponds to frequencies f satisfying $|f| > 20000$ has been badly corrupted and rendered useless. (a) Determine the minimum sampling rate for x that would allow the uncorrupted part of the spectrum to be recovered. (b) Suppose now that the corrupted part of the spectrum were eliminated by filtering prior to sampling. In this case, determine the minimum sampling rate for x.

6.29 A function x is bandlimited for frequencies in the range $[-B, B]$. Find the lowest rate at which the function $y(t) = x^2(t)$ can be sampled such that aliasing does not occur.

6.21.2 Exercises With Answer Key

6.30 Using the Fourier transform analysis equation, find the Fourier transform X of the function

$$x(t) = \operatorname{rect}\left(\tfrac{t}{T}\right),$$

where T is a nonzero real constant. (Hint: Be careful to correctly consider the case that $T < 0$.)

Short Answer. $X(\omega) = |T| \operatorname{sinc}\left(\frac{T\omega}{2}\right)$

6.31 Using the Fourier transform synthesis equation, find the inverse Fourier transform x of the function

$$X(\omega) = \operatorname{rect}\left(\tfrac{\omega}{2B}\right),$$

where B is a nonzero real constant. (Hint: Be careful to correctly consider the case that $B < 0$.)

Short Answer. $x(t) = \frac{|B|}{\pi} \operatorname{sinc}(Bt)$

6.32 Find the Fourier transform X of each function x given below.
(a) $x(t) = \frac{1}{2}\left[\delta(t) + \frac{j}{\pi t}\right]$; and
(b) $x(t) = \delta(t+a) + \delta(t-a)$, where a is a real constant.

Short Answer. (a) $X(\omega) = u(\omega)$; (b) $X(\omega) = 2\cos(a\omega)$.

6.33 Using properties of the Fourier transform, the Fourier transform pair $\operatorname{rect}\left(\frac{t}{T}\right) \overset{\text{CTFT}}{\longleftrightarrow} |T| \operatorname{sinc}\left(\frac{T\omega}{2}\right)$, and the fact that $\operatorname{tri}\left(\frac{t}{2}\right) = \operatorname{rect} * \operatorname{rect}(t)$, find the Fourier transform X of the function $x(t) = \operatorname{tri}\left(\frac{t}{T}\right)$, where T is a nonzero real constant.

Short Answer. $\frac{|T|}{2} \operatorname{sinc}^2\left(\frac{T\omega}{4}\right)$

6.34 Given that $e^{-a|t|} \overset{\text{CTFT}}{\longleftrightarrow} \frac{2a}{\omega^2 + a^2}$, find the Fourier transform X of the function $x(t) = \frac{2a}{t^2 + a^2}$.

Short Answer. $X(\omega) = 2\pi e^{-a|\omega|}$

6.35 Using a Fourier transform table and properties of the Fourier transform, find the Fourier transform Y of each function y given below in terms of the Fourier transform X of the function x.
(a) $y(t) = t\cos(3t)x(t)$; and
(b) $y(t) = (t-1)^{100} x^*(t-1)$.

Short Answer. (a) $Y(\omega) = \frac{j}{2}X'(\omega-3) + \frac{j}{2}X'(\omega+3)$ (where the prime symbol denotes the first derivative); (b) $Y(\omega) = e^{-j\omega}(\frac{d}{d\omega})^{100}[X^*(-\omega)]$.

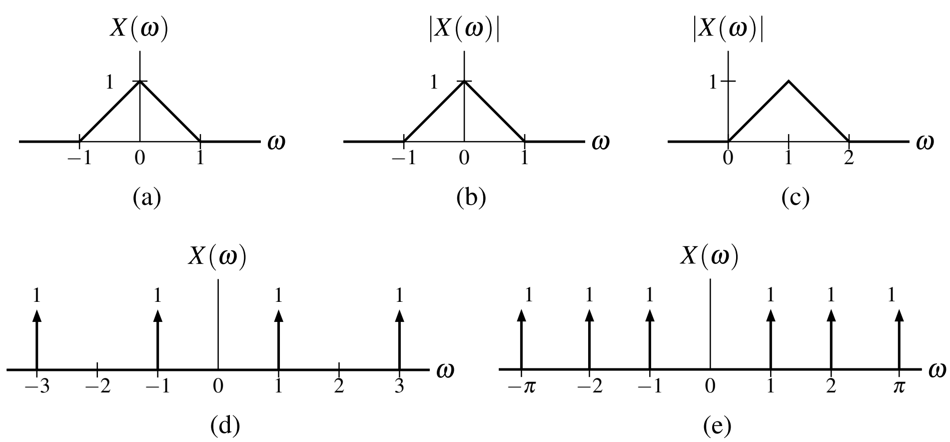
6.36 For each pair of functions M and P given below, find the function x having magnitude spectrum M and phase spectrum P.
(a) $M(\omega) = 1$ and $P(\omega) = \omega$; and
(b) $M(\omega) = \operatorname{rect}\left(\frac{\omega}{3}\right)$ and $P(\omega) = 5\omega$.

Short Answer. (a) $x(t) = \delta(t+1)$; (b) $x(t) = \frac{3}{2\pi}\operatorname{sinc}\left[\frac{3}{2}(t+5)\right]$.

6.37 Using Parseval's relation, evaluate the integral $\int_{-\infty}^{\infty}\operatorname{sinc}^2(kt)dt$, where k is a nonzero real constant.

Short Answer. π/k

6.38 Let x denote a function with the Fourier transform X. For each frequency/magnitude spectrum given below, determine (if possible) whether x has each of the following properties: real, even/odd, periodic, finite duration, and finite energy.



Short Answer. (a) real, even, aperiodic, not finite duration, finite energy; (b) aperiodic, finite energy, not finite duration; (c) not real, aperiodic, not finite duration, finite energy, not even/odd; (d) real, even, periodic (fundamental period $\frac{2\pi}{1} = 2\pi$), not finite duration, not finite energy; (e) real, even, aperiodic, not finite duration, not finite energy

6.39 Using properties of the Fourier transform and a Fourier transform table, compute the convolution $y = x_1 * x_2$, where

$$x_1(t) = \operatorname{sinc}(at - b_1) \quad \text{and} \quad x_2(t) = \operatorname{sinc}(at - b_2)$$

and a, b_1, and b_2 are real constants with $a \neq 0$.

Short Answer. $y(t) = \frac{\pi}{|a|}\operatorname{sinc}(at - b_1 - b_2)$

6.40 Let $x(t) = \operatorname{sinc}(5t-3)$. By direct application of the Nyquist sampling theorem, determine the lowest sampling rate ω_s at which x can be sampled such that it can be exactly reconstructed from its samples.

Short Answer. 10 rad/s

6.41 An audio signal x consists of two perfect sinusoidal tones at 440 Hz and 880 Hz. The signal x is sampled at a rate of f_s Hz and then played back on a loudspeaker. Determine how many tones will be heard on the loudspeaker and their frequencies, if f_s equals (a) 500 Hz and (b) 2000 Hz. (You should assume that a human can hear frequencies from 20 Hz to 20 kHz.)

Short Answer. (a) Two tones are heard with frequencies 60 Hz and 120 Hz; (b) Two tones are heard at 440 Hz and 880 Hz

6.42 Consider a bandlimited function x that has been sampled at a frequency ω_s (in radians) satisfying the Nyquist condition to produce the sequence y. Find an expression for the Fourier transform X of the function x in terms of the sequence y of samples and ω_s. [Hint: Recall that, from the sampling theorem, $x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}(\frac{\omega_s}{2}t - \pi n)$.]

Short Answer. $X(\omega) = \frac{2\pi}{\omega_s} \sum_{n=-\infty}^{\infty} y(n) e^{-j2\pi n\omega/\omega_s} \operatorname{rect}\left(\frac{\omega}{\omega_s}\right)$

6.22 MATLAB Exercises

6.101 (a) Consider a frequency response H of the form

$$H(\omega) = \frac{\sum_{k=0}^{M-1} a_k \omega^k}{\sum_{k=0}^{N-1} b_k \omega^k},$$

where a_k and b_k are complex constants. Write a MATLAB function called `freqw` that evaluates a function of the above form at an arbitrary number of specified points. The function should take three input arguments:

1) a vector containing the a_k coefficients;
2) a vector containing the b_k coefficients; and
3) a vector containing the values of ω for which to evaluate $H(\omega)$.

The function should generate two return values:

1) a vector of function values; and
2) a vector of points at which the function was evaluated.

If the function is called with no output arguments (i.e., the `nargout` variable is zero), then the function should plot the magnitude and phase responses before returning. (Hint: The `polyval` function may be helpful.)
(b) Use the function developed in part (a) to plot the magnitude and phase responses of the system with the frequency response

$$H(\omega) = \frac{16.0000}{1.0000\omega^4 - j5.2263\omega^3 - 13.6569\omega^2 + j20.9050\omega + 16.0000}.$$

For each of the plots, use the frequency range $[-5, 5]$.
(c) What type of ideal frequency-selective filter does this system approximate?

6.102 Consider the filter associated with each of the frequency responses given below. In each case, plot the magnitude and phase responses of the filter, and indicate what type of ideal frequency-selective filter it best approximates.

(a) $H(\omega) = \dfrac{\omega_b^3}{(j\omega)^3 + 2\omega_b (j\omega)^2 + 2\omega_b (j\omega) + \omega_b^3}$ where $\omega_b = 1$;

(b) $H(\omega) = \dfrac{(j\omega)^5}{(j\omega)^5 + 17.527635(j\omega)^4 + 146.32995(j\omega)^3 + 845.73205(j\omega)^2 + 2661.6442(j\omega) + 7631.0209}$; and

(c) $H(\omega) = \dfrac{13.104406(j\omega)^3}{(j\omega)^6 + 3.8776228(j\omega)^5 + 34.517979(j\omega)^4 + 75.146371(j\omega)^3 + 276.14383(j\omega)^2 + 248.16786(j\omega) + 512}$.

(Hint: Use the `freqs` function with $s = j\omega$ to compute the frequency response. The `abs`, `angle`, `linspace`, `plot`, `xlabel`, `ylabel`, and `print` functions may also prove useful for this problem.)

6.103 (a) Use the `butter` and `besself` functions to design a tenth-order Butterworth lowpass filter and tenth-order Bessel lowpass filter, each with a cutoff frequency of 10 rad/s.
(b) For each of the filters designed in part (a), plot the magnitude and phase responses using a linear scale for the frequency axis. In the case of the phase response, plot the unwrapped phase (as this will be helpful later in part (d) of this problem). (Hint: The `freqs` and `unwrap` functions may be helpful.)
(c) Consider the magnitude responses for each of the filters. Recall that an ideal lowpass filter has a magnitude response that is constant in the passband. Which of the two filters more closely approximates this ideal behavior?
(d) Consider the phase responses for each of the filters. An ideal lowpass filter has a phase response that is a linear function. Which of the two filters has a phase response that best approximates a linear (i.e., straight line) function in the passband?

Chapter 7

Laplace Transform

7.1 Introduction

In this chapter, we introduce another important mathematical tool in the study of signals and systems known as the Laplace transform. The Laplace transform can be viewed as a generalization of the (classical) Fourier transform. Due to its more general nature, the Laplace transform has a number of advantages over the Fourier transform. First, the Laplace transform representation exists for some functions that do not have a (classical) Fourier transform representation. So, we can handle some functions with the Laplace transform that cannot be handled with the Fourier transform. Second, since the Laplace transform is a more general tool, it can provide additional insights beyond those facilitated by the Fourier transform.

7.2 Motivation Behind the Laplace Transform

In Section 4.10, we showed that complex exponentials are eigenfunctions of LTI systems. Suppose that we have a LTI system with impulse response h. This eigenfunction property leads to the result that the response y of the system to the complex exponential input $x(t) = e^{st}$ (where s is a complex constant) is

$$y(t) = H(s)e^{st},$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt. \tag{7.1}$$

Previously, we referred to H as the system function. In this chapter, we will learn that H is, in fact, the Laplace transform of h. That is, the integral in (7.1) is simply the definition of the Laplace transform. In the case that $s = j\omega$ where ω is real (i.e., s is purely imaginary), (7.1) becomes the Fourier transform integral (studied in Chapter 6). From our earlier reading, we know that $H(j\omega)$ is the frequency response of the LTI system (i.e., the Fourier transform of h). Since (7.1) includes the Fourier transform as a special case, the Laplace transform can be thought of as a generalization of the (classical) Fourier transform.

7.3 Definition of the Laplace Transform

The (bilateral) **Laplace transform** of the function x is denoted as $\mathcal{L}x$ or X and is defined as

$$\mathcal{L}x(s) = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt. \tag{7.2}$$

Similarly, the **inverse Laplace transform** of X is denoted $\mathcal{L}^{-1}X$ or x and is given by

$$\mathcal{L}^{-1}X(t) = x(t) = \tfrac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds, \tag{7.3}$$

where $\sigma = \operatorname{Re}(s)$. We refer to x and X as a **Laplace transform pair** and denote this relationship as

$$x(t) \overset{\text{LT}}{\longleftrightarrow} X(s).$$

As we can see from (7.3), the calculation of the inverse Laplace transform requires a contour integration (since s is a complex variable). In particular, we must integrate along the vertical line $s = \sigma$ in the complex plane. Such a contour integration is often not so easy to compute. Therefore, in practice, we do not usually compute the inverse Laplace transform using (7.3) directly. Instead, we resort to other means (to be discussed later).

Two different versions of the Laplace transform are commonly used. The first is the bilateral version, as introduced above. The second is the unilateral version. The unilateral Laplace transform is most frequently used to solve systems of linear differential equations with nonzero initial conditions. As it turns out, the only difference between the definitions of the bilateral and unilateral Laplace transforms is in the lower limit of integration. In the bilateral case, the lower limit is $-\infty$, whereas in the unilateral case, the lower limit is 0. In the remainder of this chapter, we will focus our attention primarily on the bilateral Laplace transform. We will, however, briefly introduce the unilateral Laplace transform as a tool for solving differential equations. Unless otherwise noted, all subsequent references to the Laplace transform should be understood to mean bilateral Laplace transform.

7.4 Remarks on Notation Involving the Laplace Transform

Each of the Laplace transform operator $\mathcal{L}$ and inverse Laplace transform operator $\mathcal{L}^{-1}$ map a function to a function. Consequently, the operand for each of these operators must be a function (not a number). Consider the unnamed function that maps t to $e^{-|t|}$ as shown in Figure 7.1. Suppose that we would like to write an expression that denotes the Laplace transform of this function. At first, we might be inclined to write "$\mathcal{L}\{e^{-|t|}\}$". Strictly speaking, however, this notation is not correct, since the Laplace transform operator requires a function as an operand and "$e^{-|t|}$" (strictly speaking) denotes a number (i.e., the value of the function in the figure evaluated at t). Essentially, the cause of our problems here is that the function in question does not have a name (such as "x") by which it can be referred. To resolve this problem, we could define a function x using the equation $x(t) = e^{-|t|}$ and then write the Laplace transform as "$\mathcal{L}x$". Unfortunately, introducing a new function name just for the sake of strictly correct notation is often undesirable as it frequently leads to highly verbose writing.

One way to avoid overly verbose writing when referring to functions without names is offered by dot notation, introduced earlier in Section 2.1. Again, consider the function from Figure 7.1 that maps t to $e^{-|t|}$. Using strictly correct notation, we could write the Laplace transform of this function as "$\mathcal{L}\left\{e^{-|\cdot|}\right\}$". In other words, we can indicate that an expression refers to a function (as opposed to the value of function) by using the interpunct symbol (as discussed in Section 2.1). Some examples of the use of dot notation can be found below in Example 7.1. Dot notation is often extremely beneficial when one wants to employ precise (i.e., strictly correct) notation without being overly verbose.

Example 7.1 (Dot notation)**.** Several examples of the use of dot notation are as follows:

1. To denote the Laplace transform of the function x defined by the equation $x(t) = t^2e^{-3t}u(t)$ (without the need to introduce the named function x), we can write: $\mathcal{L}\left\{(\cdot)^2e^{-3(\cdot)}u(\cdot)\right\}$.
2. To denote the Laplace transform of the function x defined by the equation $x(t) = t^2e^{-3t}u(t)$ evaluated at $s-5$ (without the need to introduce the named function x), we can write: $\mathcal{L}\left\{(\cdot)^2e^{-3(\cdot)}u(\cdot)\right\}(s-5)$.
3. To denote the inverse Laplace transform of the function X defined by the equation $X(s) = s^{-1}$ (without the need to introduce the named function X), we can write: $\mathcal{L}^{-1}\left\{(\cdot)^{-1}\right\}$.

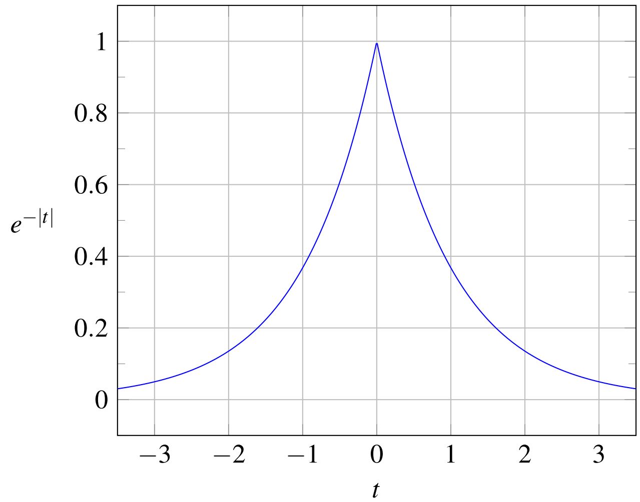


Figure 7.1: A plot of $e^{-|t|}$ versus t.

4. To denote the inverse Laplace transform of the function X defined by the equation $X(s) = s^{-1}$ evaluated at $t-3$ (without the need to introduce the named function X), we can write: $\mathcal{L}^{-1}\left\{(\cdot)^{-1}\right\}(t-3)$. ■

If the reader is comfortable with dot notation, the author would encourage the reader to use it when appropriate. Since some readers may find the dot notation to be confusing, however, this book (for the most part) attempts to minimize the use of dot notation. Instead, as a compromise solution, this book adopts the following notational conventions in order to achieve conciseness and a reasonable level of clarity without the need to use dot notation pervasively:

- unless indicated otherwise, in an expression for the operand of the Laplace transform operator $\mathcal{L}$, the variable "t" is assumed to be the independent variable for the function to which the Laplace transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "t" were a "$\cdot$");
- unless indicated otherwise, in an expression for the operand of the inverse Laplace transform operator $\mathcal{L}^{-1}$, the variable "s" is assumed to be the independent variable for the function to which the inverse Laplace transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "s" were a "$\cdot$").

Some examples of using these book-sanctioned notational conventions can be found below in Example 7.2. Admittedly, these book-sanctioned conventions are not ideal, as they abuse mathematical notation somewhat, but they seem to be the best compromise in order to accommodate those who may prefer not to use dot notation.

Example 7.2 (Book-sanctioned notation)**.** Several examples of using the notational conventions that are employed throughout most of this book (as described above) are as follows:

1. To denote the Laplace transform of the function x defined by the equation $x(t) = t^2e^{-3t}u(t)$ (without the need to introduce the named function x), we can write: $\mathcal{L}\left\{t^2e^{-3t}u(t)\right\}$.
2. To denote the Laplace transform of the function x defined by the equation $x(t) = t^2e^{-3t}u(t)$ evaluated at $s-5$ (without the need to introduce the named function x), we can write: $\mathcal{L}\left\{t^2e^{-3t}u(t)\right\}(s-5)$.
3. To denote the inverse Laplace transform of the function X defined by the equation $X(s) = s^{-1}$ (without the need to introduce the named function X), we can write: $\mathcal{L}^{-1}\left\{s^{-1}\right\}$.
4. To denote the inverse Laplace transform of the function X defined by the equation $X(s) = s^{-1}$ evaluated at $t-3$ (without the need to introduce the named function X), we can write: $\mathcal{L}^{-1}\left\{s^{-1}\right\}(t-3)$. ■

Since applying the Laplace transform operator or inverse Laplace transform operator to a function yields another function, we can evaluate this other function at some value. Again, consider the function from Figure 7.1 that maps t to $e^{-|t|}$. To denote the value of the Laplace transform of this function evaluated at $s-1$, we would write "$\mathcal{L}\{e^{-|\cdot|}\}(s-1)$" using dot notation or "$\mathcal{L}\{e^{-|t|}\}(s-1)$" using the book-sanctioned notational conventions described above.

7.5 Relationship Between Laplace Transform and Continuous-Time Fourier Transform

In Section 7.3 of this chapter, we introduced the Laplace transform, and in the previous chapter, we studied the (CT) Fourier transform. As it turns out, the Laplace transform and (CT) Fourier transform are very closely related. Recall the definition of the Laplace transform in (7.2). Consider now the special case of (7.2) where $s = j\omega$ and ω is real (i.e., $\operatorname{Re}(s) = 0$). In this case, (7.2) becomes

$$\begin{aligned}
X(j\omega) &= \left[\int_{-\infty}^{\infty} x(t)e^{-st}dt\right]\bigg|_{s=j\omega} \\
&= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \mathcal{F}x(\omega).
\end{aligned}$$

Thus, the Fourier transform is simply the Laplace transform evaluated at $s = j\omega$, assuming that this quantity is well defined (i.e., converges). In other words,

$$X(j\omega) = \mathcal{F}x(\omega). \tag{7.4}$$

Incidentally, it is due to the preceding relationship that the Fourier transform of x is sometimes written as $X(j\omega)$. When this notation is used, the function X actually corresponds to the Laplace transform of x rather than its Fourier transform (i.e., the expression $X(j\omega)$ corresponds to the Laplace transform evaluated at points on the imaginary axis).

Now, consider the general case of an arbitrary complex value for s in (7.2). Let us express s in Cartesian form as $s = \sigma + j\omega$ where σ and ω are real. Substituting $s = \sigma + j\omega$ into (7.2), we obtain

$$\begin{aligned}
X(\sigma + j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt \\
&= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt \\
&= \mathcal{F}\{e^{-\sigma t}x(t)\}(\omega).
\end{aligned}$$

Thus, we have shown

$$X(\sigma + j\omega) = \mathcal{F}\{e^{-\sigma t}x(t)\}(\omega). \tag{7.5}$$

Therefore, the Laplace transform of x can be viewed as the (CT) Fourier transform of $x'(t) = e^{-\sigma t}x(t)$ (i.e., x weighted by a real exponential function). As a consequence of multiplying by the real exponential $e^{-\sigma t}$, the Laplace transform of a function may exist when the Fourier transform of the same function does not.

By using the above relationship, we can derive the formula for the inverse Laplace transform given in (7.3). Let X denote the Laplace transform of x, and let $s = \sigma + j\omega$, where σ and ω are real. From the relationship between the Fourier and Laplace transforms in (7.5), we have

$$X(\sigma + j\omega) = \mathcal{F}\{e^{-\sigma t}x(t)\}(\omega),$$

where σ is chosen so that $X(s)$ converges for $s = \sigma + j\omega$. Taking the inverse Fourier transform of both sides of the preceding equation yields

$$\mathcal{F}^{-1}\{X(\sigma + j\omega)\}(t) = e^{-\sigma t}x(t).$$

Multiplying both sides by $e^{\sigma t}$, we obtain

$$x(t) = e^{\sigma t}\mathcal{F}^{-1}\{X(\sigma + j\omega)\}(t).$$

From the definition of the inverse Fourier transform, we have

$$\begin{aligned} x(t) &= e^{\sigma t}\left[\tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega\right] \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega. \end{aligned}$$

Since $s = \sigma + j\omega$, we have that $ds = jd\omega$, and consequently,

$$\begin{aligned} x(t) &= \tfrac{1}{2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\left(\tfrac{1}{j}\right)ds \\ &= \tfrac{1}{j2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds. \end{aligned}$$

Thus, we have just derived the inverse Laplace transform formula in (7.3).

7.6 Laplace Transform Examples

In this section, we calculate the Laplace transform of several relatively simple functions. In the process, we gain some important insights into the Laplace transform.

Example 7.3. Find the Laplace transform X of the function

$$x(t) = e^{-at}u(t),$$

where a is a real constant.

Solution. Let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform, we have

$$\begin{aligned} X(s) &= \mathcal{L}\{e^{-at}u(t)\}(s) \\ &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \\ &= \int_{0}^{\infty} e^{-(s+a)t}dt \\ &= \left[\left(-\tfrac{1}{s+a}\right)e^{-(s+a)t}\right]\Big|_0^{\infty}. \end{aligned}$$

At this point, we substitute $s = \sigma + j\omega$ in order to more easily determine when the above expression converges to a finite value. This yields

$$\begin{aligned} X(s) &= \left[\left(-\tfrac{1}{\sigma + a + j\omega}\right)e^{-(\sigma + a + j\omega)t}\right]\Big|_0^{\infty} \\ &= \left(\tfrac{-1}{\sigma + a + j\omega}\right)\left[e^{-(\sigma + a)t}e^{-j\omega t}\right]\Big|_0^{\infty} \\ &= \left(\tfrac{-1}{\sigma + a + j\omega}\right)\left[e^{-(\sigma + a)\infty}e^{-j\omega\infty} - 1\right]. \end{aligned}$$

Thus, we can see that the above expression only converges for $\sigma + a > 0$ (i.e., $\text{Re}(s) > -a$). In this case, we have that

$$\begin{aligned} X(s) &= \left(\tfrac{-1}{\sigma + a + j\omega}\right)[0 - 1] \\ &= \left(\tfrac{-1}{s+a}\right)(-1) \\ &= \frac{1}{s+a}. \end{aligned}$$

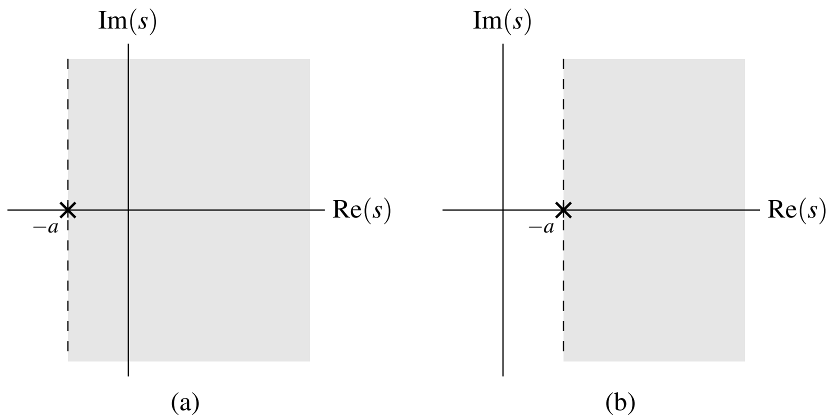


Figure 7.2: Region of convergence for the case that (a) $a > 0$ and (b) $a < 0$.

Thus, we have that

$$e^{-at}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+a} \quad \text{for } \operatorname{Re}(s) > -a.$$

The region of convergence for X is illustrated in Figures 7.2(a) and (b) for the cases of $a > 0$ and $a < 0$, respectively. ■

Example 7.4. Find the Laplace transform X of the function

$$x(t) = -e^{-at}u(-t),$$

where a is a real constant.

Solution. Let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform, we can write

$$\begin{aligned}
X(s) &= \mathcal{L}\{-e^{-at}u(-t)\}(s) \\
&= \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt \\
&= \int_{-\infty}^{0} -e^{-at}e^{-st}\,dt \\
&= \int_{-\infty}^{0} -e^{-(s+a)t}\,dt \\
&= \left[\left(\tfrac{1}{s+a}\right)e^{-(s+a)t}\right]\Big|_{-\infty}^{0}.
\end{aligned}$$

In order to more easily determine when the above expression converges to a finite value, we substitute $s = \sigma + j\omega$. This yields

$$\begin{aligned}
X(s) &= \left[\left(\tfrac{1}{\sigma+a+j\omega}\right)e^{-(\sigma+a+j\omega)t}\right]\Big|_{-\infty}^{0} \\
&= \left(\tfrac{1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)t}e^{-j\omega t}\right]\Big|_{-\infty}^{0} \\
&= \left(\tfrac{1}{\sigma+a+j\omega}\right)\left[1 - e^{(\sigma+a)\infty}e^{j\omega\infty}\right].
\end{aligned}$$

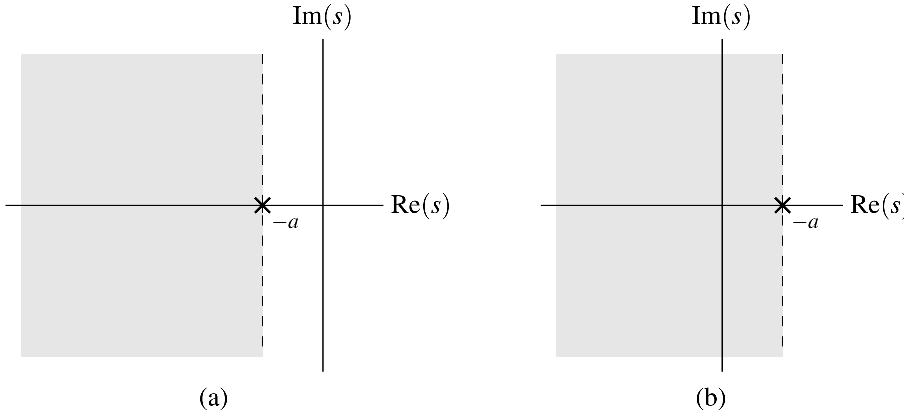


Figure 7.3: Region of convergence for the case that (a) $a > 0$ and (b) $a < 0$.

Thus, we can see that the above expression only converges for $\sigma + a < 0$ (i.e., $\mathrm{Re}(s) < -a$). In this case, we have

$$\begin{aligned} X(s) &= \left(\tfrac{1}{\sigma+a+j\omega}\right)[1-0] \\ &= \frac{1}{s+a}. \end{aligned}$$

Thus, we have that

$$-e^{-at}u(-t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+a} \quad \text{for } \mathrm{Re}(s) < -a.$$

The region of convergence for X is illustrated in Figures 7.3(a) and (b) for the cases of $a > 0$ and $a < 0$, respectively. ■

At this point, we compare the results of Examples 7.3 and 7.4, and make an important observation. Notice that the same algebraic expression for X was obtained in both of these examples (i.e., $X(s) = \frac{1}{s+a}$). The only difference is in the convergence properties of X. In one case, $X(s)$ converges for $\mathrm{Re}(s) > -a$ while in the other it converges for $\mathrm{Re}(s) < -a$. As it turns out, one must specify both the algebraic expression for X and its region of convergence in order to uniquely determine $x = \mathcal{L}^{-1}X$ from X.

Example 7.5 (Laplace transform of the unit-step function)**.** Find the Laplace transform X of the function

$$x(t) = u(t).$$

Solution. Let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform, we have

$$\begin{aligned} X(s) &= \mathcal{L}u(s) \\ &= \int_{-\infty}^{\infty} u(t)e^{-st}dt \\ &= \int_{0}^{\infty} e^{-st}dt \\ &= \left[\left(-\tfrac{1}{s}\right)e^{-st}\right]\Big|_0^{\infty}. \end{aligned}$$

At this point, we substitute $s = \sigma + j\omega$ in order to more clearly see the region of convergence for this expression. This yields

$$\begin{aligned} X(s) &= \left[\left(-\tfrac{1}{\sigma+j\omega}\right)e^{-(\sigma+j\omega)t}\right]\Big|_0^{\infty} \\ &= \left[\left(-\tfrac{1}{\sigma+j\omega}\right)e^{-\sigma t}e^{-j\omega t}\right]\Big|_0^{\infty}. \end{aligned}$$

Thus, we can see that the above expression converges only for $\sigma > 0$ (i.e., $\operatorname{Re}(s) > 0$). In this case, we have

$$\begin{aligned} X(s) &= \left(-\tfrac{1}{\sigma+j\omega}\right)[0-1] \\ &= \left(-\tfrac{1}{s}\right)(-1) \\ &= \frac{1}{s}. \end{aligned}$$

Thus, we have that

$$u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s} \quad \text{for } \operatorname{Re}(s) > 0. \qquad \blacksquare$$

Example 7.6 (Laplace transform of the delta function)**.** Find the Laplace transform X of the function

$$x(t) = A\delta(t-t_0),$$

where A and t_0 are arbitrary real constants.

Solution. From the definition of the Laplace transform, we can write

$$\begin{aligned} X(s) &= \mathcal{L}\{A\delta(t-t_0)\}(s) \\ &= \int_{-\infty}^{\infty} A\delta(t-t_0)e^{-st}dt \\ &= A\int_{-\infty}^{\infty} \delta(t-t_0)e^{-st}dt. \end{aligned}$$

Using the sifting property of the delta function, we can simplify this result to obtain

$$X(s) = Ae^{-st_0}.$$

Thus, we have shown that

$$A\delta(t-t_0) \overset{\text{LT}}{\longleftrightarrow} Ae^{-st_0} \quad \text{for all } s. \qquad \blacksquare$$

7.7 Region of Convergence for the Laplace Transform

Before discussing the region of convergence (ROC) of the Laplace transform in detail, we need to introduce some terminology involving sets in the complex plane. Let R denote a set in the complex plane. A set R comprised of all complex numbers s such that

$$\operatorname{Re}(s) < a,$$

for some real constant a, is said to be a **left-half plane** (LHP). A set R comprised of all complex numbers s such that

$$\operatorname{Re}(s) > a,$$

for some real constant a, is said to be a **right-half plane** (RHP). Examples of LHPs and RHPs are given in Figure 7.4.

Since the ROC is a set (of points in the complex plane), we often need to employ some basic set operations when dealing with ROCs. For two sets A and B, the **intersection** of A and B, denoted $A \cap B$, is the set of all points that are in both A and B. An illustrative example of set intersection is shown in Figure 7.5.

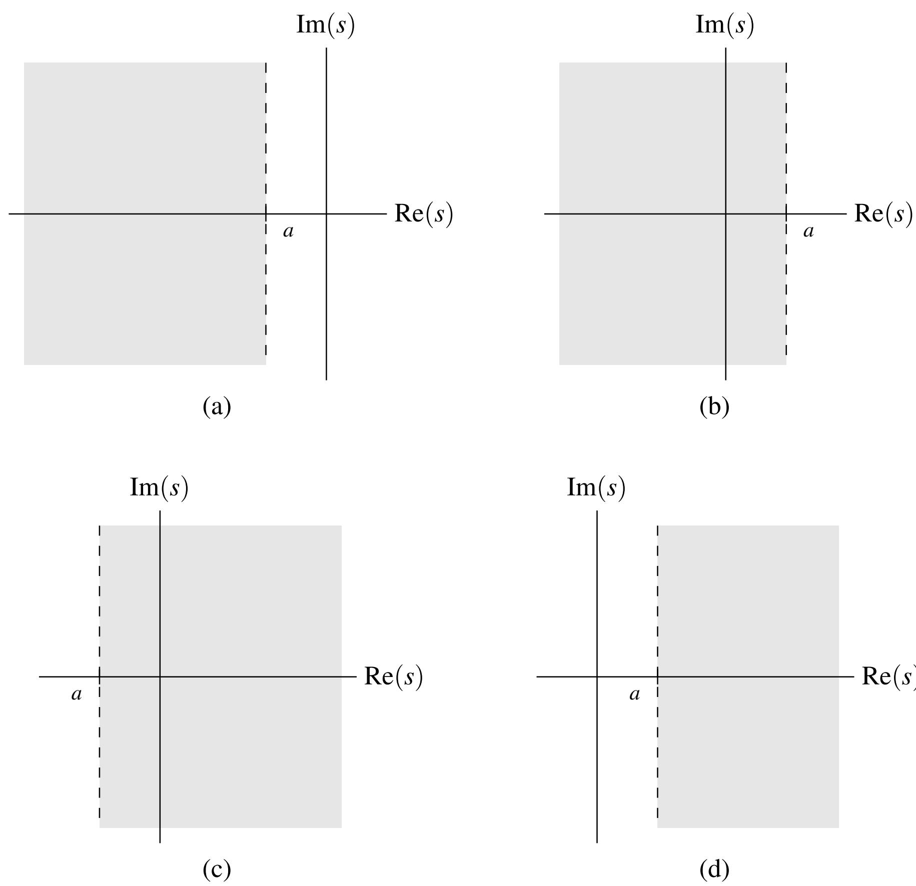


Figure 7.4: Examples of LHPs and RHPs. An example of a LHP in the case that (a) $a < 0$ and (b) $a > 0$. An example of a RHP in the case that (c) $a < 0$ and (d) $a > 0$.

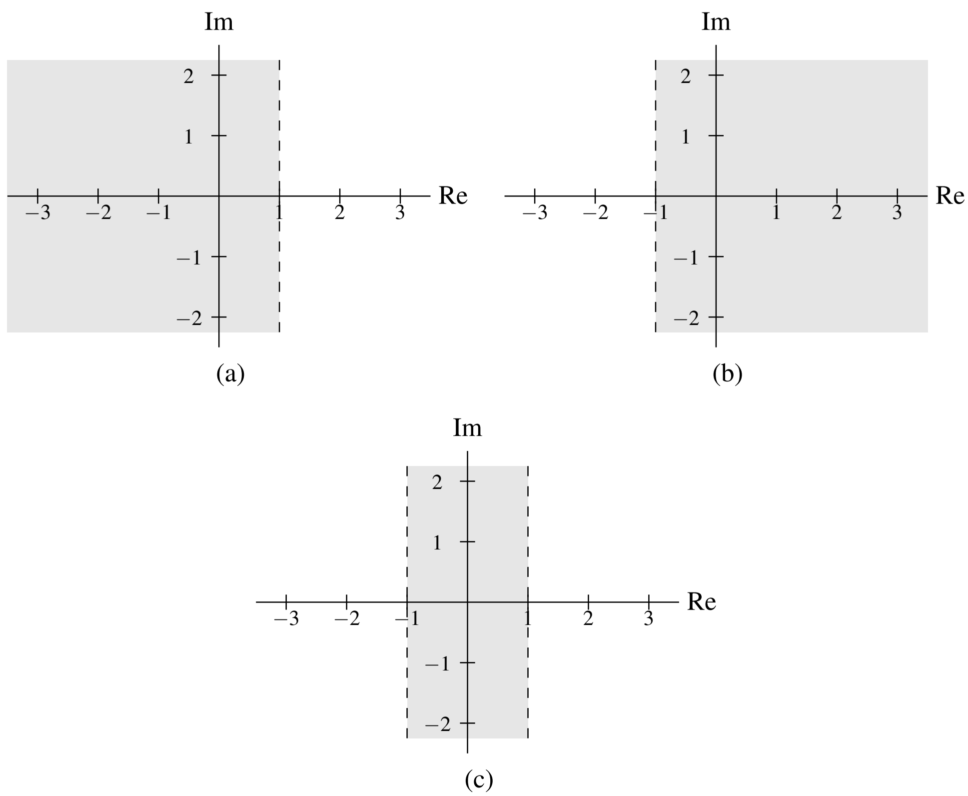


Figure 7.5: Example of set intersection. The sets (a) R_1 and (b) R_2; and (c) their intersection $R_1 \cap R_2$.

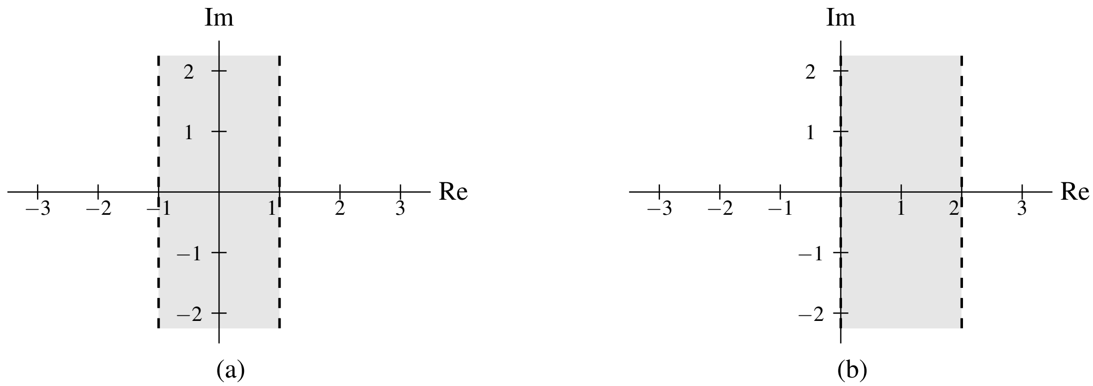


Figure 7.6: Example of adding a scalar to a set. (a) The set R. (b) The set $R+1$.

For a set S and a scalar constant a, $S+a$ denotes the set given by

$$S+a = \{z+a : z \in S\}.$$

That is, $S+a$ denotes the set formed by adding a to each element of S. For example, suppose that R is the set of complex numbers s satisfying

$$-1 < \operatorname{Re}(s) < 1,$$

as shown in Figure 7.6(a). Then, $R+1$ is the set of complex numbers s satisfying

$$0 < \operatorname{Re}(s) < 2,$$

as shown in Figure 7.6(b).

For a set S and a scalar constant a, aS denotes the set given by

$$aS = \{az : z \in S\}.$$

That is, aS denotes the set formed by multiplying each element of S by a. For example, suppose that R is the set of complex numbers s satisfying

$$-1 < \operatorname{Re}(s) < 2,$$

as shown in Figure 7.7(a). Then, $2R$ is the set of complex numbers s satisfying

$$-2 < \operatorname{Re}(s) < 4,$$

as shown in Figure 7.7(b); and $-2R$ is the set of complex numbers s satisfying

$$-4 < \operatorname{Re}(s) < 2,$$

as shown in Figure 7.7(c).

As we saw earlier, for a function x, the complete specification of its Laplace transform X requires not only an algebraic expression for X, but also the ROC associated with X. Two distinct functions can have the same algebraic expression for their Laplace transform. In what follows, we examine some of the constraints on the ROC (of the Laplace transform) for various classes of functions.

One can show that the ROC of the Laplace transform has the following properties:

1. The ROC of the Laplace transform X consists of strips parallel to the imaginary axis in the complex plane. That is, if s is in the ROC, then $s+j\omega$ is in the ROC for all real ω. Some examples of sets that would be either valid or invalid as ROCs are shown in Figure 7.8.

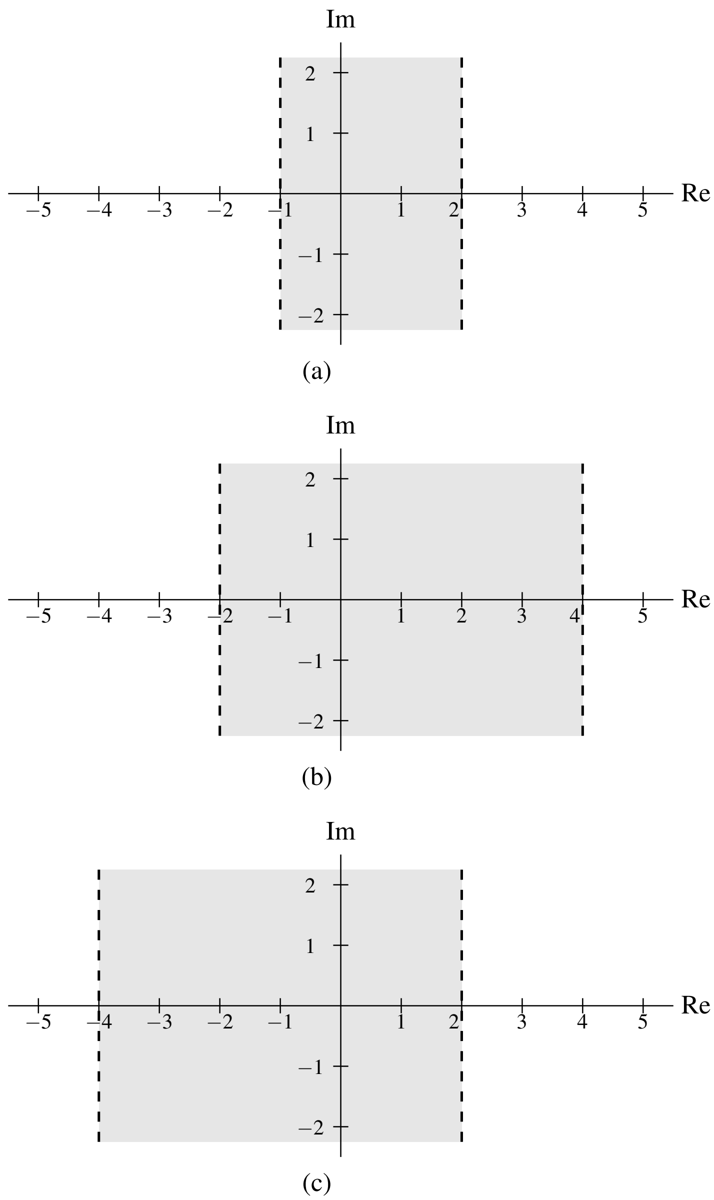


Figure 7.7: Example of multiplying a set by a scalar. (a) The set R. The sets (b) $2R$ and (c) $-2R$.

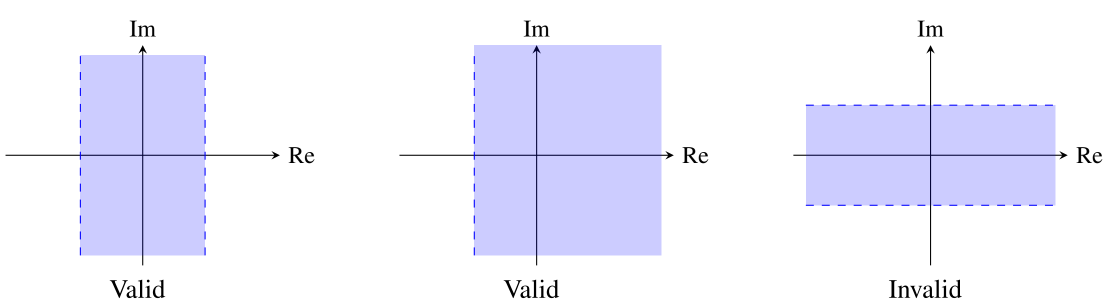


Figure 7.8: Examples of sets that would be either valid or invalid as the ROC of a Laplace transform.

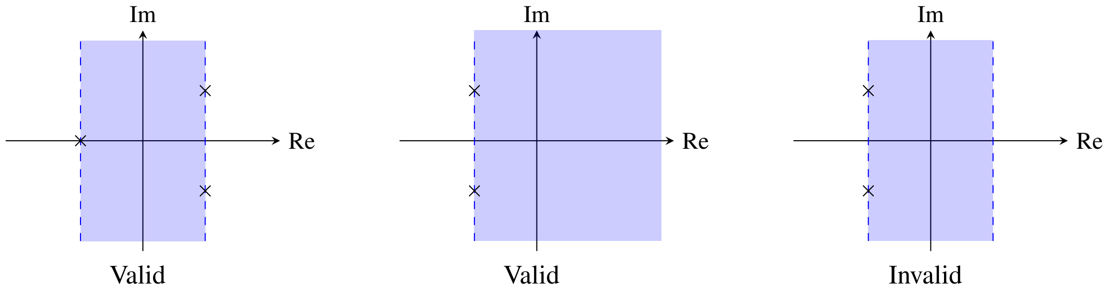


Figure 7.9: Examples of sets that would be either valid or invalid as the ROC of a rational Laplace transform.

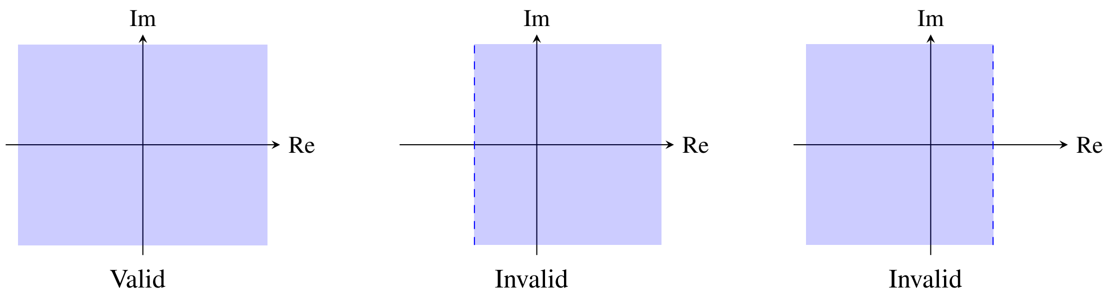


Figure 7.10: Examples of sets that would be either valid or invalid as the ROC of a Laplace transform of a finite-duration function.

Justification: The Laplace transform X of the function x is simply the (CT) Fourier transform of $x'(t) = x(t)e^{-\operatorname{Re}(s)t}$. Thus, X converges whenever this Fourier transform converges. Since the convergence of the Fourier transform only depends on $\operatorname{Re}(s)$, the convergence of the Laplace transform only depends on $\operatorname{Re}(s)$.

2. If the Laplace transform X is a rational function, the ROC does not contain any poles, and the ROC is bounded by poles or extends to infinity. Some examples of sets that would be either valid or invalid as ROCs of rational Laplace transforms are shown in Figure 7.9.

   Partial justification: Since X is rational, its value becomes infinite at a pole. So obviously, X does not converge at a pole. Therefore, it follows that the ROC cannot contain a pole.

3. If a function x is finite duration and its Laplace transform X converges for some value of s, then X converges for all values of s (i.e., the ROC is the entire complex plane). Some examples of sets that would be either valid or invalid as ROCs for X, if x is finite duration, are shown in Figure 7.10.

4. If a function x is right sided and the (vertical) line $\operatorname{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform X of x, then all values of s for which $\operatorname{Re}(s) > \sigma_0$ must also be in the ROC (i.e., the ROC includes a right-half plane containing $\operatorname{Re}(s) = \sigma_0$). Moreover, if x is right sided but not left sided, the ROC of X is a right-half plane. Some examples of sets that would be either valid or invalid as ROCs for X, if x is right sided but not left sided, are shown in Figure 7.11.

5. If a function x is left sided and the (vertical) line $\operatorname{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform X of x, then all values of s for which $\operatorname{Re}(s) < \sigma_0$ must also be in the ROC (i.e., the ROC includes a left-half plane containing $\operatorname{Re}(s) = \sigma_0$). Moreover, if x is left sided but not right sided, the ROC of X is a left-half plane. Some examples of sets that would be either valid or invalid as ROCs for X, if x is left sided but not right sided, are shown in Figure 7.12.

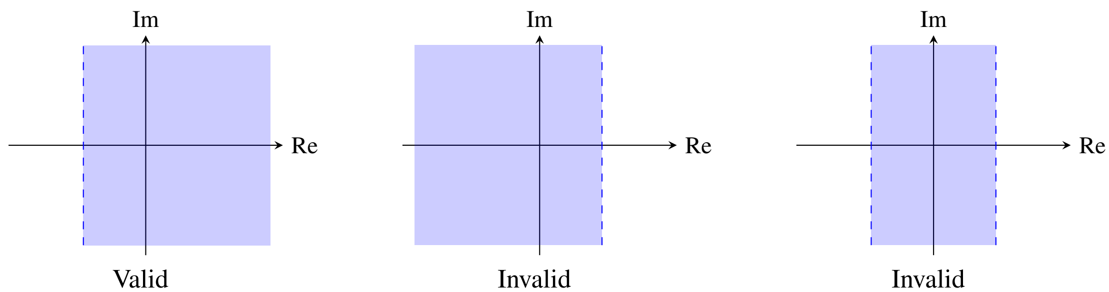


Figure 7.11: Examples of sets that would be either valid or invalid as the ROC of the Laplace transform of a function that is right sided but not left sided.

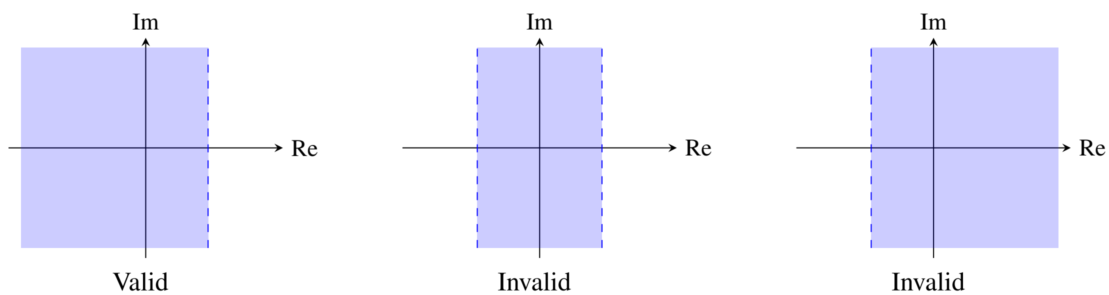


Figure 7.12: Examples of sets that would be either valid or invalid as the ROC of the Laplace transform of a function that is left sided but not right sided.

6. If a function x is two sided and the (vertical) line $\operatorname{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform X of x, then the ROC will consist of a strip in the complex plane that includes the line $\operatorname{Re}(s) = \sigma_0$. Some examples of sets that would be either valid or invalid as ROCs for X, if x is two sided, are shown in Figure 7.13.

7. If the Laplace transform X of a function x is rational, then:

   (a) If x is right sided, the ROC of X is to the right of the rightmost pole of X (i.e., the right-half plane to the right of the rightmost pole).

   (b) If x is left sided, the ROC of X is to the left of the leftmost pole of X (i.e., the left-half plane to the left of the leftmost pole).

   Some examples of sets that would be either valid or invalid as ROCs for X, if X is rational and x is left/right sided, are given in Figure 7.14.

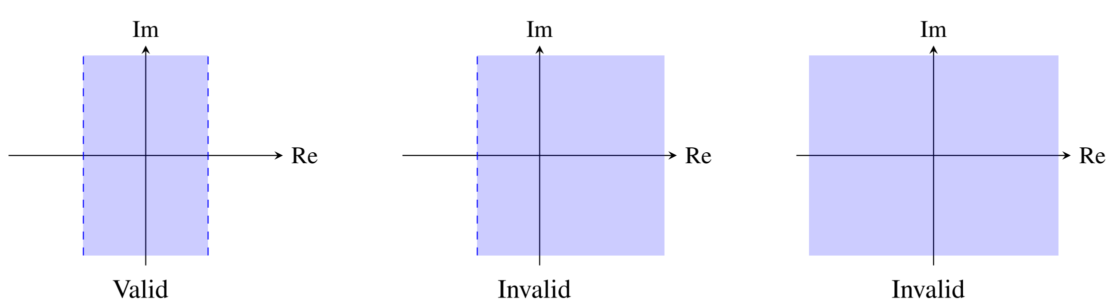


Figure 7.13: Examples of sets that would be either valid or invalid as the ROC of the Laplace transform of a two-sided function.

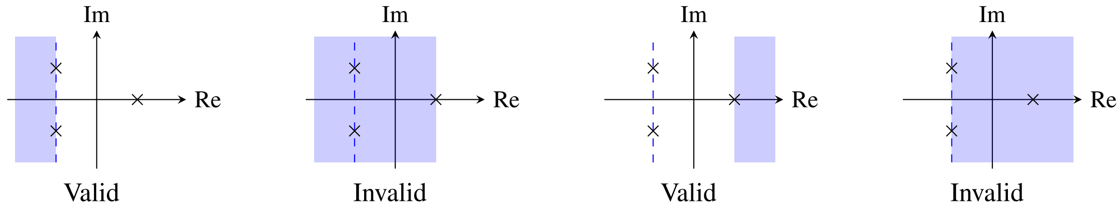


Figure 7.14: Examples of sets that would be either valid or invalid as the ROC of a rational Laplace transform of a left/right-sided function.

Figure 7.15: Relationship between the sidedness properties of x and the ROC of $X = \mathcal{L}x$

x		
left sided	right sided	ROC of X
no	no	strip
no	yes	RHP
yes	no	LHP
yes	yes	everywhere

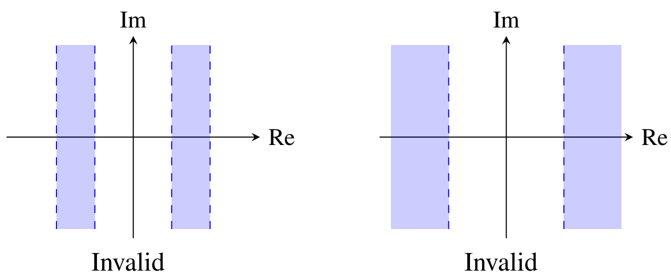


Figure 7.16: Examples of sets that would not be a valid ROC of a Laplace transform.

Note that some of the above properties are redundant. For example, properties 1, 2, and 4 imply property 7(a). Also, properties 1, 2, and 5 imply property 7(b). Moreover, since every function can be classified as one of left sided but not right sided, right sided but not left sided, two sided (i.e., neither left nor right sided), or finite duration (i.e., both left and right sided), we can infer from properties 3, 4, 5, and 6 that the ROC can only be of the form of a left-half plane, a right-half plane, a (single) vertical strip, the entire complex plane, or the empty set. In particular, the ROC of X depends on the left- and right-sidedness of x as shown in Table 7.15. Thus, the ROC must be a connected region. (A set S is said to be connected, if for every two elements a and b in S, there exists a path from a to b that is contained in S.) That is, the ROC cannot consist of multiple (unconnected) vertical strips. For example, the sets shown in Figure 7.16 would not be valid as ROCs.

Example 7.7. The Laplace transform X of the function x has the algebraic expression

$$X(s) = \frac{s+\frac{1}{2}}{(s^2+2s+2)(s^2+s-2)}.$$

Identify all of the possible ROCs of X. For each ROC, indicate whether the corresponding function x is left sided but not right sided, right sided but not left sided, two sided, or finite duration.

Solution. The possible ROCs associated with X are determined by the poles of this function. So, we must find the poles of X. Factoring the denominator of X, we obtain

$$X(s) = \frac{s+\frac{1}{2}}{(s+1-j)(s+1+j)(s+2)(s-1)}.$$

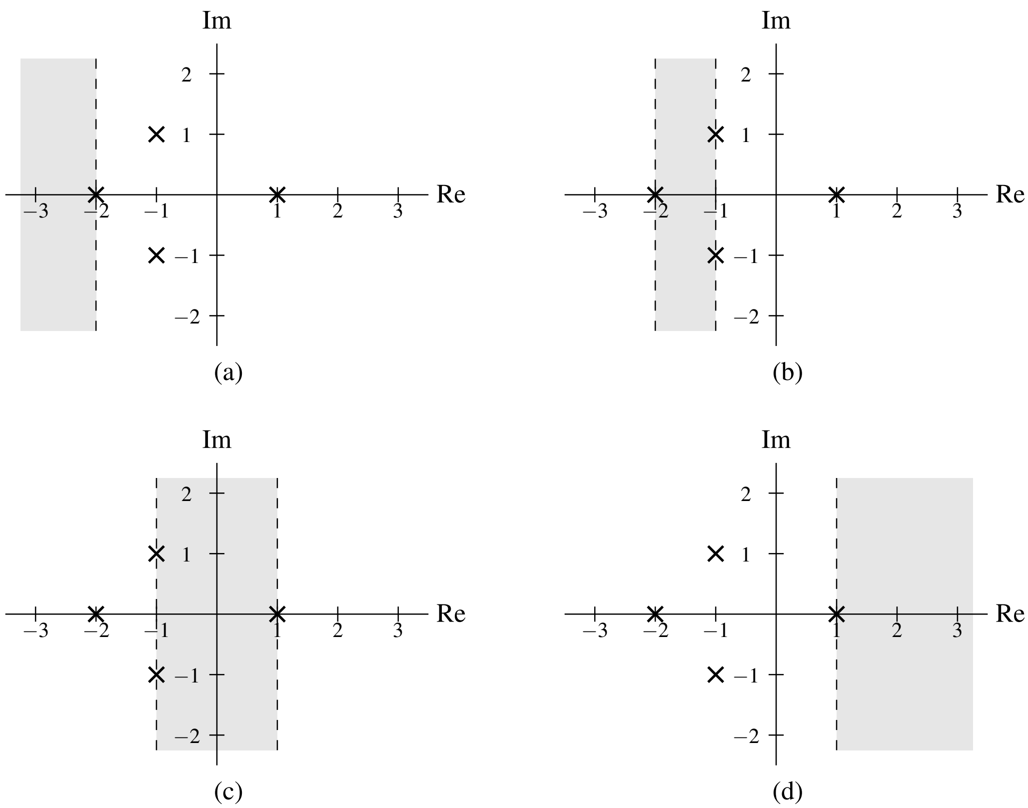


Figure 7.17: ROCs for example. The (a) first, (b) second, (c) third, and (d) fourth possible ROCs for X.

Thus, X has poles at -2, $-1-j$, $-1+j$, and 1. Since these poles only have three distinct real parts (namely, -2, -1, and 1), there are four possible ROCs:

i) $\operatorname{Re}(s) < -2$,
ii) $-2 < \operatorname{Re}(s) < -1$,
iii) $-1 < \operatorname{Re}(s) < 1$, and
iv) $\operatorname{Re}(s) > 1$.

These ROCs are plotted in Figures 7.17(a), (b), (c), and (d), respectively. The first ROC is a left-half plane, so the corresponding x must be left sided but not right sided. The second ROC is a vertical strip, so the corresponding x must be two sided. The third ROC is a vertical strip, so the corresponding x must be two sided. The fourth ROC is a right-half plane, so the corresponding x must be right sided but not left sided. ■

7.8 Properties of the Laplace Transform

The Laplace transform has a number of important properties. In the sections that follow, we introduce several of these properties. For the convenience of the reader, the properties described in the subsequent sections are also summarized in Table 7.1 (on page 259). Also, for convenience, several Laplace-transform pairs are given later in Table 7.2 (on page 260). In what follows, we will sometimes refer to transform pairs in this table.

7.8.1 Linearity

Arguably, the most important property of the Laplace transform is linearity, as introduced below.

Theorem 7.1 (Linearity)**.** *If $x_1(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)$ with ROC R_1 and $x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_2(s)$ with ROC R_2, then*

$$a_1x_1(t)+a_2x_2(t) \overset{\text{LT}}{\longleftrightarrow} a_1X_1(s)+a_2X_2(s) \;\; \text{with ROC } R \text{ containing } R_1 \cap R_2,$$

where a_1 and a_2 are arbitrary complex constants.

Proof. Let $y(t) = a_1x_1(t)+a_2x_2(t)$, and let Y denote the Laplace transform of y. Using the definition of the Laplace transform and straightforward algebraic manipulation, we have

$$\begin{aligned}
Y(s) &= \int_{-\infty}^{\infty} [a_1x_1(t)+a_2x_2(t)]e^{-st}dt \\
&= \int_{-\infty}^{\infty} a_1x_1(t)e^{-st}dt + \int_{-\infty}^{\infty} a_2x_2(t)e^{-st}dt \\
&= a_1\int_{-\infty}^{\infty} x_1(t)e^{-st}dt + a_2\int_{-\infty}^{\infty} x_2(t)e^{-st}dt \\
&= a_1X_1(s)+a_2X_2(s).
\end{aligned}$$

The ROC R can be deduced as follows. If X_1 and X_2 both converge at some point s, say $s = \lambda$, then any linear combination of these functions must also converge at $s = \lambda$. Therefore, the ROC R must contain the intersection of R_1 and R_2. Thus, we have shown that the linearity property holds. ■

In the preceding theorem, note that the ROC of the result can be larger than $R_1 \cap R_2$. When X_1 and X_2 are rational functions, this can only happen if pole-zero cancellation occurs in the expression $a_1X_1(s)+a_2X_2(s)$.

Example 7.8 (Linearity property of the Laplace transform)**.** Find the Laplace transform X of the function

$$x = x_1 + x_2,$$

where

$$x_1(t) = e^{-t}u(t) \quad \text{and} \quad x_2(t) = e^{-t}u(t) - e^{-2t}u(t).$$

Solution. Using Laplace transform pairs from Table 7.2, we have

$$\begin{aligned}
X_1(s) &= \mathcal{L}\{e^{-t}u(t)\}(s) \\
&= \frac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1 \quad \text{and} \\
X_2(s) &= \mathcal{L}\{e^{-t}u(t) - e^{-2t}u(t)\}(s) \\
&= \mathcal{L}\{e^{-t}u(t)\}(s) - \mathcal{L}\{e^{-2t}u(t)\}(s) \\
&= \frac{1}{s+1} - \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -1 \\
&= \frac{1}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1.
\end{aligned}$$

So, from the definition of X, we can write

$$\begin{aligned}
X(s) &= \mathcal{L}\{x_1+x_2\}(s) \\
&= X_1(s)+X_2(s) \\
&= \frac{1}{s+1} + \frac{1}{(s+1)(s+2)} \\
&= \frac{s+2+1}{(s+1)(s+2)} \\
&= \frac{s+3}{(s+1)(s+2)}.
\end{aligned}$$

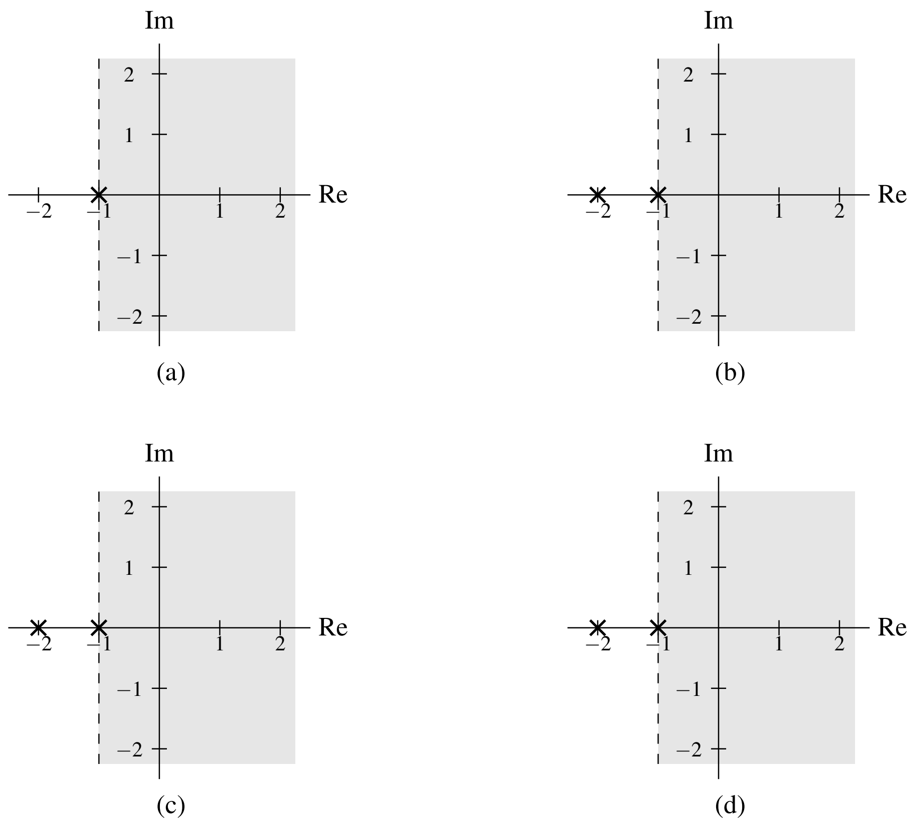


Figure 7.18: ROCs for the linearity example. The (a) ROC of X_1, (b) ROC of X_2, (c) ROC associated with the intersection of the ROCs of X_1 and X_2, and (d) ROC of X.

Now, we must determine the ROC of X. We know that the ROC of X must contain the intersection of the ROCs of X_1 and X_2. So, the ROC must contain $\mathrm{Re}(s) > -1$. Furthermore, the ROC cannot be larger than this intersection, since X has a pole at -1. Therefore, the ROC of X is $\mathrm{Re}(s) > -1$. The various ROCs are illustrated in Figure 7.18. So, in conclusion, we have

$$X(s) = \frac{s+3}{(s+1)(s+2)} \quad \text{for } \mathrm{Re}(s) > -1. \qquad \blacksquare$$

Example 7.9 (Linearity property of the Laplace transform and pole-zero cancellation)**.** Find the Laplace transform X of the function

$$x = x_1 - x_2,$$

where x_1 and x_2 are as defined in the previous example.

Solution. From the previous example, we know that

$$X_1(s) = \frac{1}{s+1} \quad \text{for } \mathrm{Re}(s) > -1 \quad \text{and}$$
$$X_2(s) = \frac{1}{(s+1)(s+2)} \quad \text{for } \mathrm{Re}(s) > -1.$$

From the definition of X, we have

$$\begin{aligned} X(s) &= \mathcal{L}\{x_1 - x_2\}(s) \\ &= X_1(s) - X_2(s) \\ &= \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} \\ &= \frac{s+2-1}{(s+1)(s+2)} \\ &= \frac{s+1}{(s+1)(s+2)} \\ &= \frac{1}{s+2}. \end{aligned}$$

Now, we must determine the ROC of X. We know that the ROC of X must at least contain the intersection of the ROCs of X_1 and X_2. Therefore, the ROC must contain $\operatorname{Re}(s) > -1$. Since X is rational, we also know that the ROC must be bounded by poles or extend to infinity. Since X has only one pole and this pole is at -2, the ROC must also include $-2 < \operatorname{Re}(s) < -1$. Therefore, the ROC of X is $\operatorname{Re}(s) > -2$. In effect, the pole at -1 has been cancelled by a zero at the same location. As a result, the ROC of X is larger than the intersection of the ROCs of X_1 and X_2. The various ROCs are illustrated in Figure 7.19. So, in conclusion, we have

$$X(s) = \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2. \qquad \blacksquare$$

7.8.2 Time-Domain Shifting

The next property of the Laplace transform to be introduced is the time-domain shifting property, as given below.

Theorem 7.2 (Time-domain shifting)**.** *If $x(t) \overset{LT}{\longleftrightarrow} X(s)$ with ROC R, then*

$$x(t-t_0) \overset{LT}{\longleftrightarrow} e^{-st_0}X(s) \;\; \textit{with ROC } R,$$

where t_0 is an arbitrary real constant.

Proof. To prove the above property, we proceed as follows. Let $y(t) = x(t-t_0)$, and let Y denote the Laplace transform of y. From the definition of the Laplace transform, we have

$$Y(s) = \int_{-\infty}^{\infty} x(t-t_0)e^{-st}dt.$$

Now, we perform a change of variable. Let $\tau = t - t_0$ so that $t = \tau + t_0$ and $d\tau = dt$. Applying this change of variable, we obtain

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau+t_0)}d\tau \\ &= e^{-st_0}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau}d\tau \\ &= e^{-st_0}X(s). \end{aligned}$$

The ROC of Y is the same as the ROC of X, since Y and X differ only by a finite constant factor (i.e., e^{-st_0}). Thus, we have proven that the time-domain shifting property holds. $\blacksquare$

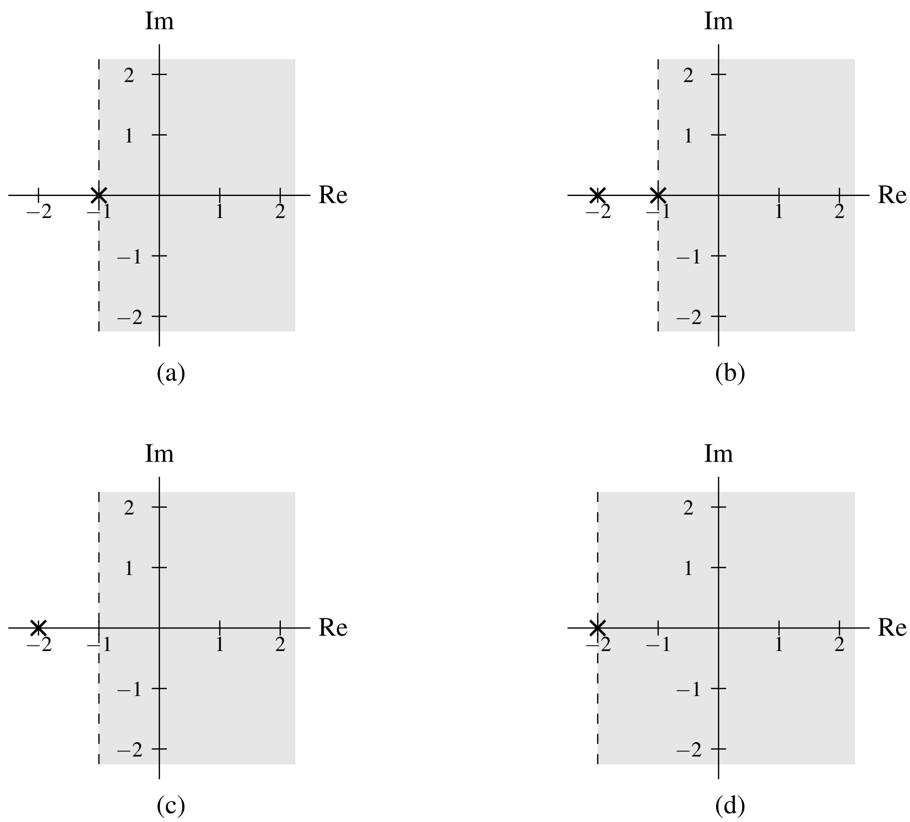


Figure 7.19: ROCs for the linearity example. The (a) ROC of X_1, (b) ROC of X_2, (c) ROC associated with the intersection of the ROCs of X_1 and X_2, and (d) ROC of X.

Example 7.10 (Time-domain shifting property)**.** Find the Laplace transform X of

$$x(t) = u(t-1).$$

Solution. From Table 7.2, we know that

$$u(t) \overset{\text{LT}}{\longleftrightarrow} 1/s \text{ for } \operatorname{Re}(s) > 0.$$

Using the time-domain shifting property, we can deduce

$$x(t) = u(t-1) \overset{\text{LT}}{\longleftrightarrow} X(s) = e^{-s}\left(\tfrac{1}{s}\right) \text{ for } \operatorname{Re}(s) > 0.$$

Therefore, we have

$$X(s) = \frac{e^{-s}}{s} \text{ for } \operatorname{Re}(s) > 0.$$

■

7.8.3 Laplace-Domain Shifting

The next property of the Laplace transform to be introduced is the Laplace-domain shifting property, as given below.

Theorem 7.3 (Laplace-domain shifting)**.** *If $x(t) \overset{LT}{\longleftrightarrow} X(s)$ with ROC R, then*

$$e^{s_0 t}x(t) \overset{LT}{\longleftrightarrow} X(s-s_0) \text{ with ROC } R + \operatorname{Re}(s_0),$$

where s_0 is an arbitrary complex constant. The ROCs are illustrated in Figure 7.20.

Proof. To prove the above property, we proceed as follows. Let $y(t) = e^{s_0 t}x(t)$, and let Y denote the Laplace transform of y. Using the definition of the Laplace transform and straightforward manipulation, we obtain

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} e^{s_0 t}x(t)e^{-st}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-(s-s_0)t}dt \\ &= X(s-s_0). \end{aligned}$$

Since $Y(s+s_0) = X(s)$, Y converges at $\lambda + s_0$ if and only if X converges at λ. Since the convergence properties of a Laplace transform only depend on the real part of the s parameter, Y converges at $\lambda + \operatorname{Re}(s_0)$ if and only if X converges at λ. Consequently, the ROC of Y is simply the ROC of X shifted by $\operatorname{Re}(s_0)$. Thus, we have shown that the Laplace-domain shifting property holds. ■

Example 7.11 (Laplace-domain shifting property)**.** Using only the properties of the Laplace transform and the transform pair

$$e^{-|t|} \overset{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \quad \text{for } -1 < \operatorname{Re}(s) < 1,$$

find the Laplace transform X of

$$x(t) = e^{5t}e^{-|t|}.$$

Solution. We are given

$$e^{-|t|} \overset{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \text{ for } -1 < \operatorname{Re}(s) < 1.$$

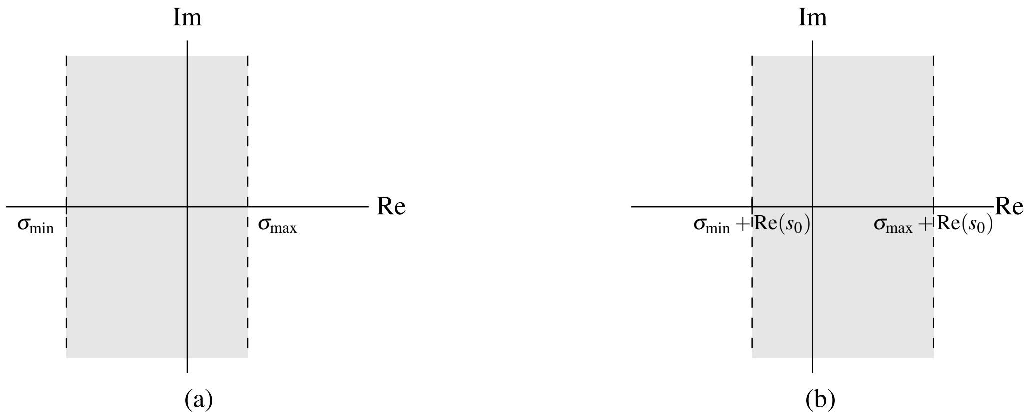


Figure 7.20: Regions of convergence for Laplace-domain shifting. (a) Before shift. (b) After shift.

Using the Laplace-domain shifting property, we can deduce

$$x(t) = e^{5t}e^{-|t|} \overset{\text{LT}}{\longleftrightarrow} X(s) = \frac{2}{1-(s-5)^2} \text{ for } -1+5 < \mathrm{Re}(s) < 1+5,$$

Thus, we have

$$X(s) = \frac{2}{1-(s-5)^2} \text{ for } 4 < \mathrm{Re}(s) < 6.$$

Rewriting X in factored form, we have

$$X(s) = \frac{2}{1-(s-5)^2} = \frac{2}{1-(s^2-10s+25)} = \frac{2}{-s^2+10s-24} = \frac{-2}{s^2-10s+24} = \frac{-2}{(s-6)(s-4)}.$$

Therefore, we have

$$X(s) = \frac{-2}{(s-4)(s-6)} \text{ for } 4 < \mathrm{Re}(s) < 6. \qquad \blacksquare$$

7.8.4 Time-Domain/Laplace-Domain Scaling

The next property of the Laplace transform to be introduced is the time-domain/Laplace-domain scaling property, as given below.

Theorem 7.4 (Time-domain/Laplace-domain scaling)**.** *If $x(t) \overset{LT}{\longleftrightarrow} X(s)$ with ROC R, then*

$$x(at) \overset{LT}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{s}{a}\right) \text{ with ROC } R_1 = aR,$$

where a is a nonzero real constant.

Proof. To prove the above property, we proceed as below. Let $y(t) = x(at)$, and let Y denote that Laplace transform of y. From the definition of the Laplace transform, we have

$$Y(s) = \int_{-\infty}^{\infty} x(at)e^{-st}dt.$$

Now, we perform a change of variable. Let $\tau = at$ so that $t = \tau/a$ and $d\tau = adt$. Performing the change of variable (and being mindful of the change in the limits of integration), we obtain

$$Y(s) = \begin{cases} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau/a} \left(\frac{1}{a}\right) d\tau & a > 0 \\ \int_{\infty}^{-\infty} x(\tau) e^{-s\tau/a} \left(\frac{1}{a}\right) d\tau & a < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau/a} d\tau & a > 0 \\ -\frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau/a} d\tau & a < 0. \end{cases}$$

Combining the two cases for a (i.e., $a > 0$ and $a < 0$), we obtain

$$Y(s) = \tfrac{1}{|a|} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau/a} d\tau$$
$$= \tfrac{1}{|a|} X\left(\tfrac{s}{a}\right).$$

Since $|a|Y(as) = X(s)$, Y converges at $a\lambda$ if and only if X converges at λ. Thus, the ROC of Y is aR. Thus, we have proven that the scaling property holds. ■

The effect of time-domain scaling on the ROC of the Laplace transform is illustrated in Figure 7.21. Suppose that the ROC of the Laplace transform of a function x is as shown in Figure 7.21(a). Then, the ROC of the Laplace transform of the function $y(t) = x(at)$ is as shown in Figure 7.21(b) for the case that $a > 0$ and Figure 7.21(c) for the case that $a < 0$.

Example 7.12 (Time-domain scaling property)**.** Using only properties of the Laplace transform and the transform pair

$$e^{-|t|} \stackrel{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \quad \text{for } -1 < \operatorname{Re}(s) < 1,$$

find the Laplace transform X of the function

$$x(t) = e^{-|3t|}.$$

Solution. We are given

$$e^{-|t|} \stackrel{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \quad \text{for } -1 < \operatorname{Re}(s) < 1.$$

Using the time-domain scaling property, we can deduce

$$x(t) = e^{-|3t|} \stackrel{\text{LT}}{\longleftrightarrow} X(s) = \frac{1}{|3|} \frac{2}{1-(\frac{s}{3})^2} \quad \text{for } 3(-1) < \operatorname{Re}(s) < 3(1).$$

Thus, we have

$$X(s) = \frac{2}{3\left[1-(\frac{s}{3})^2\right]} \quad \text{for } -3 < \operatorname{Re}(s) < 3.$$

Simplifying, we have

$$X(s) = \frac{2}{3(1-\frac{s^2}{9})} = \frac{2}{3(\frac{9-s^2}{9})} = \frac{2(9)}{3(9-s^2)} = \frac{6}{9-s^2} = \frac{-6}{(s+3)(s-3)}.$$

Therefore, we have

$$X(s) = \frac{-6}{(s+3)(s-3)} \quad \text{for } -3 < \operatorname{Re}(s) < 3.$$ ■

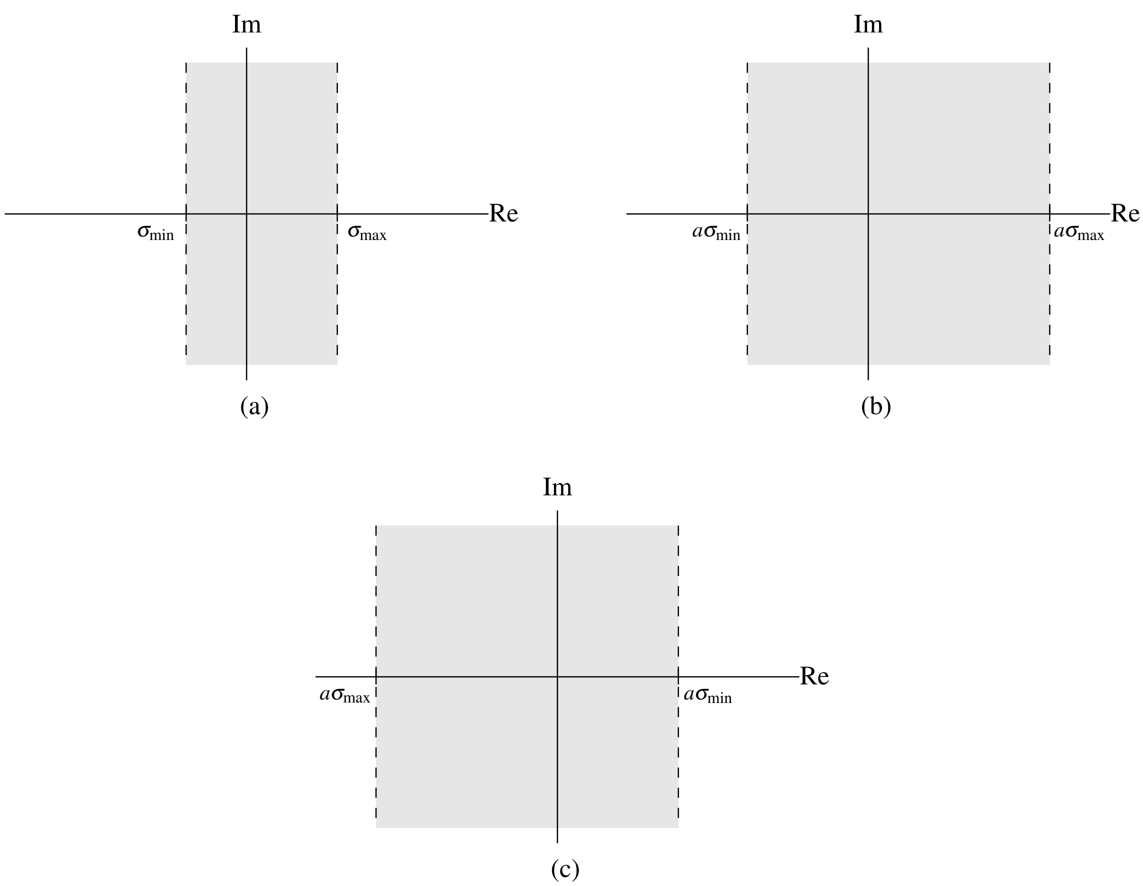


Figure 7.21: Regions of convergence for time-domain/Laplace-domain scaling. (a) Before scaling. After scaling for (b) $a > 0$ and (c) $a < 0$.

7.8.5 Conjugation

The next property of the Laplace transform to be introduced is the conjugation property, as given below.

Theorem 7.5 (Conjugation)**.** *If $x(t) \stackrel{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then*

$$x^*(t) \stackrel{\text{LT}}{\longleftrightarrow} X^*(s^*) \;\; \textit{with ROC } R.$$

Proof. To prove the above property, we proceed as follows. Let $y(t) = x^*(t)$, let Y denote the Laplace transform of y, and let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform and the properties of conjugation, we can write

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} x^*(t)e^{-st}dt \\ &= \left[\left(\int_{-\infty}^{\infty} x^*(t)e^{-st}dt\right)^*\right]^* \\ &= \left[\int_{-\infty}^{\infty} [x^*(t)]^*(e^{-st})^*dt\right]^* \\ &= \left[\int_{-\infty}^{\infty} x(t)(e^{-st})^*dt\right]^*. \end{aligned}$$

Now, we observe that $(e^{-st})^* = e^{-s^*t}$. Thus, we can write

$$\begin{aligned} Y(s) &= \left[\int_{-\infty}^{\infty} x(t)e^{-s^*t}dt\right]^* \\ &= X^*(s^*). \end{aligned}$$

We determine the ROC of Y as follows. First, we observe that $X(s) = Y^*(s^*)$. Since $Y^*(s^*) = X(s)$, Y converges at λ if and only if X converges at λ^*. We know, however, that convergence only depends on the real part of λ. So, Y converges at λ if and only if X converges at λ. From these results, we have that the ROC of Y must be identical to the ROC of X. Thus, we have shown that the conjugation property holds. ■

Example 7.13 (Conjugation property)**.** Using only properties of the Laplace transform and the transform pair

$$e^{(-1-j)t}u(t) \;\; \stackrel{\text{LT}}{\longleftrightarrow} \;\; \frac{1}{s+1+j} \;\; \text{for } \operatorname{Re}(s) > -1,$$

find the Laplace transform X of

$$x(t) = e^{(-1+j)t}u(t).$$

Solution. To begin, let $v(t) = e^{(-1-j)t}u(t)$ (i.e., v is the function whose Laplace transform is given in the Laplace-transform pair above) and let V denote the Laplace transform of v. First, we determine the relationship between x and v. We have

$$\begin{aligned} x(t) &= \left(\left(e^{(-1+j)t}u(t)\right)^*\right)^* \\ &= \left(\left(e^{(-1+j)t}\right)^* u^*(t)\right)^* \\ &= \left[e^{(-1-j)t}u(t)\right]^* \\ &= v^*(t). \end{aligned}$$

Thus, $x = v^*$. Next, we find the Laplace transform of x. We are given

$$v(t) = e^{(-1-j)t}u(t) \;\; \stackrel{\text{LT}}{\longleftrightarrow} \;\; V(s) = \frac{1}{s+1+j} \;\; \text{for } \operatorname{Re}(s) > -1.$$

Using the conjugation property, we can deduce

$$x(t) = e^{(-1+j)t}u(t) \overset{\text{LT}}{\longleftrightarrow} X(s) = \left(\frac{1}{s^* + 1 + j}\right)^* \text{ for } \operatorname{Re}(s) > -1.$$

Simplifying the algebraic expression for X, we have

$$X(s) = \left(\frac{1}{s^* + 1 + j}\right)^* = \frac{1^*}{(s^* + 1 + j)^*} = \frac{1}{s + 1 - j}.$$

Therefore, we can conclude

$$X(s) = \frac{1}{s + 1 - j} \text{ for } \operatorname{Re}(s) > -1.$$

■

7.8.6 Time-Domain Convolution

The next property of the Laplace transform to be introduced is the time-domain convolution property, as given below.

Theorem 7.6 (Time-domain convolution)**.** *If* $x_1(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)$ *with ROC* R_1 *and* $x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_2(s)$ *with ROC* R_2*, then*

$$x_1 * x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)X_2(s) \text{ with ROC } R \text{ containing } R_1 \cap R_2.$$

Proof. To prove the above property, we proceed as below. Let $y(t) = x_1 * x_2(t)$, and let Y denote the Laplace transform of y. From the definition of the Laplace transform and convolution, we have

$$\begin{aligned}
Y(s) &= \mathcal{L}\left\{\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right\}(s) \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right]e^{-st}dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)e^{-st}d\tau dt.
\end{aligned}$$

Changing the order of integration, we have

$$\begin{aligned}
Y(s) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)e^{-st}dt d\tau \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_2(t-\tau)e^{-st}dt\right]x_1(\tau)d\tau.
\end{aligned}$$

Now, we perform a change of variable. Let $v = t - \tau$ so that $t = v + \tau$ and $dv = dt$. Applying the change of variable and simplifying, we obtain

$$\begin{aligned}
Y(s) &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_2(v)e^{-s(v+\tau)}dv\right]x_1(\tau)d\tau \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_2(v)e^{-sv}dv\right]e^{-s\tau}x_1(\tau)d\tau \\
&= \left[\int_{-\infty}^{\infty} x_2(v)e^{-sv}dv\right]\left[\int_{-\infty}^{\infty} e^{-s\tau}x_1(\tau)d\tau\right] \\
&= X_1(s)X_2(s).
\end{aligned}$$

Now, we consider the ROC R of Y. If X_1 and X_2 both converge at some λ, then $Y(s) = X_1(s)X_2(s)$ must also converge at λ. Therefore, R must contain the intersection of R_1 and R_2. Thus, we have shown that the time-domain convolution property holds. ■

In the preceding theorem, note that the ROC R can be larger than $R_1 \cap R_2$. When X_1 and X_2 are rational functions, this can only happen if pole-zero cancellation occurs in the expression $X_1(s)X_2(s)$.

The time-domain convolution property of the Laplace transform has important practical implications. Since the Laplace transform effectively converts a convolution into a multiplication, the Laplace transform can be used as a means to avoid directly dealing with convolution operations. This is often extremely helpful when working with (CT) LTI systems, for example, since such systems fundamentally involve convolution.

Example 7.14 (Time-domain convolution property)**.** Find the Laplace transform X of the function

$$x(t) = x_1 * x_2(t),$$

where

$$x_1(t) = \sin(3t)u(t) \quad \text{and} \quad x_2(t) = tu(t).$$

Solution. From Table 7.2, we have that

$$x_1(t) = \sin(3t)u(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s) = \frac{3}{s^2+9} \text{ for } \operatorname{Re}(s) > 0 \quad \text{and}$$
$$x_2(t) = tu(t) \overset{\text{LT}}{\longleftrightarrow} X_2(s) = \frac{1}{s^2} \text{ for } \operatorname{Re}(s) > 0.$$

Using the time-domain convolution property, we have

$$x(t) \overset{\text{LT}}{\longleftrightarrow} X(s) = \left(\frac{3}{s^2+9}\right)\left(\frac{1}{s^2}\right) \text{ for } \{\operatorname{Re}(s) > 0\} \cap \{\operatorname{Re}(s) > 0\}.$$

The ROC of X is $\{\operatorname{Re}(s) > 0\} \cap \{\operatorname{Re}(s) > 0\}$ (as opposed to a superset thereof), since no pole-zero cancellation occurs. Simplifying the expression for X, we conclude

$$X(s) = \frac{3}{s^2(s^2+9)} \text{ for } \operatorname{Re}(s) > 0.$$ ■

7.8.7 Time-Domain Differentiation

The next property of the Laplace transform to be introduced is the time-domain differentiation property, as given below.

Theorem 7.7 (Time-domain differentiation)**.** *If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then*

$$\frac{dx(t)}{dt} \overset{\text{LT}}{\longleftrightarrow} sX(s) \text{ with ROC } R' \text{ containing } R.$$

Proof. To prove the above property, we proceed as follows. Let $\mathcal{D}$ denote that derivative operator, let $y = \mathcal{D}x$, and let Y denote the Laplace transform of y. From the definition of the inverse Laplace transform, we have

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds.$$

Differentiating both sides of this equation with respect to t, we have

$$y(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds.$$

Observing that the right-hand side of the above equation is simply the inverse Laplace transform of $sX(s)$, we can write

$$y(t) = \mathcal{L}^{-1}\{sX(s)\}(t).$$

Taking the Laplace transform of both sides yields

$$Y(s) = sX(s).$$

Now, we consider the ROC R' of $Y(s) = sX(s)$. Clearly, Y must converge at λ if X converges at λ. Since multiplication by s has the potential to cancel a pole in X, it is possible that R' may be larger than R (i.e., the ROC of X). Consequently, R' must at least contain R. Thus, we have shown that the time-domain differentiation property holds. ■

In the preceding theorem, note that the ROC R' can be larger than R. When X is a rational function, this can only happen if pole-zero cancellation occurs in the expression $sX(s)$.

The time-domain differentiation property of the Laplace transform has important practical implications. Since the Laplace transform effectively converts differentiation into multiplication (by s), the Laplace transform can be used as a means to avoid directly dealing with differentiation operations. This can often be beneficial when working with differential and integro-differential equations, for example.

Example 7.15 (Time-domain differentiation property)**.** Find the Laplace transform X of the function

$$x(t) = \tfrac{d}{dt}\delta(t).$$

Solution. From Table 7.2, we have that

$$\delta(t) \stackrel{\text{LT}}{\longleftrightarrow} 1 \text{ for all } s.$$

Using the time-domain differentiation property, we can deduce

$$x(t) = \tfrac{d}{dt}\delta(t) \stackrel{\text{LT}}{\longleftrightarrow} X(s) = s(1) \text{ for all } s.$$

Therefore, we have

$$X(s) = s \quad \text{for all } s.$$

■

7.8.8 Laplace-Domain Differentiation

The next property of the Laplace transform to be introduced is the Laplace-domain differentiation property, as given below.

Theorem 7.8 (Laplace-domain differentiation)**.** *If $x(t) \stackrel{LT}{\longleftrightarrow} X(s)$ with ROC R, then*

$$-tx(t) \stackrel{LT}{\longleftrightarrow} \tfrac{d}{ds}X(s) \text{ with ROC } R.$$

Proof. To prove the above property, we proceed as follows. Let $y(t) = -tx(t)$ and let Y denote the Laplace transform of y. From the definition of the Laplace transform, we have

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt.$$

Differentiating both sides of the above equation with respect to s, we obtain

$$\begin{aligned}\tfrac{d}{ds}X(s) &= \int_{-\infty}^{\infty} -tx(t)e^{-st}dt \\ &= Y(s).\end{aligned}$$

Thus, we have shown that the Laplace-domain differentiation property holds. ■

Example 7.16 (Laplace-domain differentiation property)**.** Using only the properties of the Laplace transform and the transform pair

$$e^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2,$$

find the Laplace transform X of the function

$$x(t) = te^{-2t}u(t).$$

Solution. We are given

$$e^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.$$

Using the Laplace-domain differentiation and linearity properties, we can deduce

$$x(t) = te^{-2t}u(t) \quad \overset{\text{LT}}{\longleftrightarrow} \quad X(s) = -\tfrac{d}{ds}\left(\frac{1}{s+2}\right) \quad \text{for } \operatorname{Re}(s) > -2.$$

Simplifying the algebraic expression for X, we have

$$X(s) = -\tfrac{d}{ds}\left(\frac{1}{s+2}\right) = -\tfrac{d}{ds}(s+2)^{-1} = (-1)(-1)(s+2)^{-2} = \frac{1}{(s+2)^2}.$$

Therefore, we conclude

$$X(s) = \frac{1}{(s+2)^2} \quad \text{for } \operatorname{Re}(s) > -2. \qquad \blacksquare$$

7.8.9 Time-Domain Integration

The next property of the Laplace transform to be introduced is the time-domain integration property, as given below.

Theorem 7.9 (Time-domain integration)**.** *If* $x(t) \overset{LT}{\longleftrightarrow} X(s)$ *with ROC R, then*

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{LT}{\longleftrightarrow} \frac{1}{s}X(s) \ \textit{ with ROC } R' \textit{ containing } R \cap \{\operatorname{Re}(s) > 0\}.$$

Proof. To prove the above property, we proceed as follows. Let $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$, and let Y and U denote the Laplace transforms of y and u, respectively. First, we observe that

$$y(t) = x * u(t).$$

Taking the Laplace transform of both sides of this equation, we have

$$Y(s) = \mathcal{L}\{x * u\}(s).$$

From the time-domain convolution property of the Laplace transform, we have

$$\begin{aligned} Y(s) &= \mathcal{L}x(s)\mathcal{L}u(s) \\ &= X(s)U(s). \end{aligned}$$

From Example 7.5, we know that $U(s) = \frac{1}{s}$ for $\operatorname{Re}(s) > 0$. So,

$$Y(s) = \frac{1}{s}X(s).$$

Now, we need to consider the ROC R' of Y. Clearly, Y must converge at λ if X and U both converge at λ. Consequently, R' must contain the intersection of the ROCs of X and U. Since U converges for $\operatorname{Re}(s) > 0$, R' must contain $R \cap (\operatorname{Re}(s) > 0)$. Thus, we have shown that the time-domain integration property holds. $\blacksquare$

In the preceding theorem, note that the ROC R' can be larger than $R \cap \{\mathrm{Re}(s) > 0\}$. When X is a rational function, this can only happen if pole-zero cancellation occurs in the expression $\frac{1}{s}X(s)$.

The time-domain integration property of the Laplace transform has important practical implications. Since the Laplace transform effectively converts integration into division (by s), the Laplace transform can be used as a means to avoid directly dealing with integration operations. This can often be beneficial when working with integral and integro-differential equations, for example.

Example 7.17 (Time-domain integration property)**.** Find the Laplace transform X of the function

$$x(t) = \int_{-\infty}^{t} e^{-2\tau}\sin(\tau)u(\tau)d\tau.$$

Solution. From Table 7.2, we have that

$$e^{-2t}\sin(t)u(t) \stackrel{\text{LT}}{\longleftrightarrow} \frac{1}{(s+2)^2+1} \quad \text{for } \mathrm{Re}(s) > -2.$$

Using the time-domain integration property, we can deduce

$$x(t) = \int_{-\infty}^{t} e^{-2\tau}\sin(\tau)u(\tau)d\tau \stackrel{\text{LT}}{\longleftrightarrow} X(s) = \frac{1}{s}\left[\frac{1}{(s+2)^2+1}\right] \quad \text{for } \{\mathrm{Re}(s) > -2\} \cap \{\mathrm{Re}(s) > 0\}.$$

The ROC of X is $\{\mathrm{Re}(s) > -2\} \cap \{\mathrm{Re}(s) > 0\}$ (as opposed to a superset thereof), since no pole-zero cancellation takes place. Simplifying the algebraic expression for X, we have

$$X(s) = \frac{1}{s}\left[\frac{1}{(s+2)^2+1}\right] = \frac{1}{s}\left(\frac{1}{s^2+4s+4+1}\right) = \frac{1}{s}\left(\frac{1}{s^2+4s+5}\right).$$

Therefore, we have

$$X(s) = \frac{1}{s(s^2+4s+5)} \quad \text{for } \mathrm{Re}(s) > 0.$$

[Note: $s^2+4s+5 = (s+2-j)(s+2+j)$.] ■

7.8.10 Initial and Final Value Theorems

The next properties of the Laplace transform to be introduced are known as the initial and final value theorems, as given below.

Theorem 7.10 (Initial value theorem)**.** *Let x be a function with the Laplace transform X. If x is causal and contains no impulses or higher order singularities at the origin, then*

$$x(0^+) = \lim_{s\to\infty} sX(s),$$

where $x(0^+)$ denotes the limit of $x(t)$ as t approaches zero from positive values of t.

Proof. To prove the above property, we proceed as below. First, we expand x as a Taylor series at 0^+.

$$x(t) = \left[x(0^+) + x^{(1)}(0^+)t + \ldots + x^{(n)}(0^+)\frac{t^n}{n!} + \ldots\right]u(t),$$

where $x^{(n)}$ denotes the nth derivative of x. Taking the Laplace transform of the above equation, and using the fact that $\mathcal{L}\{t^n u(t)\}(s) = \frac{n!}{s^{n+1}}$, we can write

$$X(s) = x(0^+)\frac{1}{s} + x^{(1)}(0^+)\frac{1}{s^2} + \ldots + x^{(n)}(0^+)\frac{1}{s^{n+1}} + \ldots.$$

Multiplying both sides of this equation by s, we obtain

$$sX(s) = x(0^+) + x^{(1)}(0^+)\frac{1}{s} + \ldots + x^{(n)}(0^+)\frac{1}{s^n} + \ldots.$$

Taking the limit as $s \to \infty$, we have

$$\lim_{s\to\infty} sX(s) = x(0^+).$$

Thus, the initial value theorem holds. ∎

Theorem 7.11 (Final value theorem)**.** *Let x be a function with the Laplace transform X. If x is causal and $x(t)$ has a finite limit as $t \to \infty$, then*

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$

Proof. Let x' denote the derivative of x. The differentiation property of the unilateral Laplace transform (introduced later in Section 7.17) states that $s\mathcal{L}_\mathrm{u}x(s) - x(0^-) = \mathcal{L}_\mathrm{u}x'(s)$. Furthermore, since x is causal, this property can be rewritten as $s\mathcal{L}x(s) - x(0^-) = \mathcal{L}_\mathrm{u}x'(s)$. Thus, we have that

$$sX(s) - x(0^-) = \int_{0^-}^{\infty} x'(t)e^{-st}dt.$$

Taking the limit of both sides of this equation as $s \to 0$, we obtain

$$\begin{aligned}
\lim_{s\to 0}\left[sX(s) - x(0^-)\right] &= \lim_{s\to 0}\int_{0^-}^{\infty} x'(t)e^{-st}dt \\
&= \int_{0^-}^{\infty} x'(t)dt \\
&= x(t)|_{0^-}^{\infty} \\
&= \lim_{t\to\infty} x(t) - x(0^-).
\end{aligned}$$

Thus, we have

$$\lim_{s\to 0} sX(s) - x(0^-) = \lim_{t\to\infty} x(t) - x(0^-).$$

Adding $x(0^-)$ to both sides of the equation yields

$$\lim_{s\to 0} sX(s) = \lim_{t\to\infty} x(t).$$ ∎

Example 7.18 (Initial and final value theorems)**.** A bounded causal function x with a (finite) limit at infinity has the Laplace transform

$$X(s) = \frac{2s^2+3s+2}{s^3+2s^2+2s} \quad \text{for } \operatorname{Re}(s) > 0.$$

Determine $x(0^+)$ and $\lim_{t\to\infty} x(t)$.

Solution. Since x is causal and does not have any singularities at the origin, the initial value theorem can be applied. From this theorem, we have

$$\begin{aligned}
x(0^+) &= \lim_{s\to\infty} sX(s) \\
&= \lim_{s\to\infty} s\left[\frac{2s^2+3s+2}{s^3+2s^2+2s}\right] \\
&= \lim_{s\to\infty} \frac{2s^2+3s+2}{s^2+2s+2} \\
&= 2.
\end{aligned}$$

Since x is bounded and causal and has well-defined limit at infinity, we can apply the final value theorem. From this theorem, we have

$$\begin{aligned}\lim_{t\to\infty} x(t) &= \lim_{s\to 0} sX(s) \\ &= \lim_{s\to 0} s\left(\frac{2s^2+3s+2}{s^3+2s^2+2s}\right) \\ &= \left.\frac{2s^2+3s+2}{s^2+2s+2}\right|_{s=0} \\ &= 1.\end{aligned}$$

In passing, we note that the inverse Laplace transform x of X can be shown to be

$$x(t) = [1+e^{-t}\cos t]u(t).$$

As we would expect, the values calculated above for $x(0^+)$ and $\lim_{t\to\infty} x(t)$ are consistent with this formula for x. ■

Amongst other things, the initial and final value theorems can be quite useful in checking for errors in Laplace transform calculations. For example, suppose that we are asked to compute the Laplace transform X of the function x. If we were to make a mistake in this computation, the values obtained for $x(0)$ and $\lim_{t\to\infty} x(t)$ using X with the initial and final value theorems and using x directly would most likely disagree. In this manner, we can relatively easily detect some types of errors in Laplace transform calculations.

7.9 More Laplace Transform Examples

Earlier in this chapter, we derived a number of Laplace transform pairs. Some of these and other important transform pairs are listed in Table 7.2. Using the various Laplace transform properties listed in Table 7.1 and the Laplace transform pairs listed in Table 7.2, we can more easily determine the Laplace transform of more complicated functions.

Example 7.19. Using properties of the Laplace transform and the Laplace transform pair

$$e^{-a|t|} \overset{\text{LT}}{\longleftrightarrow} \frac{-2a}{(s+a)(s-a)} \text{ for } -a < \operatorname{Re}(s) < a,$$

find the Laplace transform X of the function

$$x(t) = e^{-5|3t-7|}.$$

Solution. We begin by re-expressing x in terms of the following equations:

$$\begin{aligned}v_1(t) &= e^{-5|t|}, \\ v_2(t) &= v_1(t-7), \quad \text{and} \\ x(t) &= v_2(3t).\end{aligned}$$

In what follows, let R_{V_1}, R_{V_2}, and R_X denote the ROCs of V_1, V_2, and X, respectively. Taking the Laplace transform of the above three equations, we obtain

$$\begin{aligned}V_1(s) &= \frac{-10}{(s+5)(s-5)}, \quad R_{V_1} = (-5 < \operatorname{Re}(s) < 5), \\ V_2(s) &= e^{-7s}V_1(s), \quad R_{V_2} = R_{V_1}, \\ X(s) &= \tfrac{1}{3}V_2(s/3), \quad \text{and} \quad R_X = 3R_{V_2}.\end{aligned}$$

Table 7.1: Properties of the (bilateral) Laplace transform

Property	Time Domain	Laplace Domain	ROC
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$	$R+\mathrm{Re}(s_0)$
Time/Laplace-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	aR
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Time-Domain Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathrm{Re}(s)>0\}$

Property	
Initial Value Theorem	$x(0^+) = \lim_{s\to\infty} sX(s)$
Final Value Theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Table 7.2: Transform pairs for the (bilateral) Laplace transform

Pair	$x(t)$	$X(s)$	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\operatorname{Re}(s) > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\operatorname{Re}(s) < 0$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\operatorname{Re}(s) > 0$
5	$-t^n u(-t)$	$\frac{n!}{s^{n+1}}$	$\operatorname{Re}(s) < 0$
6	$e^{-at}u(t)$	$\frac{1}{s+a}$	$\operatorname{Re}(s) > -a$
7	$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\operatorname{Re}(s) < -a$
8	$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\operatorname{Re}(s) > -a$
9	$-t^n e^{-at}u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\operatorname{Re}(s) < -a$
10	$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\operatorname{Re}(s) > 0$
11	$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\operatorname{Re}(s) > 0$
12	$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\operatorname{Re}(s) > -a$
13	$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\operatorname{Re}(s) > -a$

Combining the above equations, we have

$$\begin{aligned}X(s) &= \tfrac{1}{3}V_2(s/3)\\ &= \tfrac{1}{3}e^{-7(s/3)}V_1(s/3)\\ &= \tfrac{1}{3}e^{-7s/3}V_1(s/3)\\ &= \tfrac{1}{3}e^{-7s/3}\frac{-10}{(s/3+5)(s/3-5)} \quad \text{and}\end{aligned}$$

$$\begin{aligned}R_X &= 3R_{V_2}\\ &= 3R_{V_1}\\ &= 3\{-5 < \operatorname{Re}(s) < 5\}\\ &= \{-15 < \operatorname{Re}(s) < 15\}.\end{aligned}$$

Thus, we have shown that

$$X(s) = \tfrac{1}{3}e^{-7s/3}\frac{-10}{(s/3+5)(s/3-5)} \text{ for } -15 < \operatorname{Re}(s) < 15. \qquad \blacksquare$$

Example 7.20. Find the Laplace transform X of the function

$$x(t) = [e^{-t} + e^{-2t}]u(t).$$

Solution. We can calculate X as

$$\begin{aligned}X(s) &= \mathcal{L}\{[e^{-t}+e^{-2t}]u(t)\}(s)\\ &= \mathcal{L}\{e^{-t}u(t)\}(s) + \mathcal{L}\{e^{-2t}u(t)\}(s)\\ &= \frac{1}{s+1} + \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -1 \cap \operatorname{Re}(s) > -2\\ &= \frac{s+2+s+1}{(s+1)(s+2)}\\ &= \frac{2s+3}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1.\end{aligned}$$

Thus, we have that

$$X(s) = \frac{2s+3}{(s+1)(s+2)} \text{ for } \operatorname{Re}(s) > -1. \qquad \blacksquare$$

Example 7.21. Find the Laplace transform X of the function

$$x(t) = [e^{-2t} + e^{-3t}]u(t-1).$$

Solution. To begin, we rewrite x as

$$x(t) = [e^{-2t} + e^{-3t}]v_1(t)$$

where

$$v_1(t) = u(t-1).$$

Taking the Laplace transform of the preceding equations, we have

$$\begin{aligned} V_1(s) &= \mathcal{L}\{u(t-1)\}(s) \\ &= e^{-s}\left(\tfrac{1}{s}\right) \quad \text{for } \operatorname{Re}(s) > 0. \end{aligned}$$

and

$$\begin{aligned} X(s) &= \mathcal{L}\{[e^{-2t}+e^{-3t}]v_1(t)\}(s) \\ &= \mathcal{L}\{e^{-2t}v_1(t)\}(s) + \mathcal{L}\{e^{-3t}v_1(t)\}(s). \end{aligned}$$

Now, we focus on simplifying the preceding expression for X. Let R_{V_1} denote the ROC of V_1. Then, we have

$$\begin{aligned} \mathcal{L}\{e^{-2t}v_1(t)\}(s) &= V_1(s+2) \quad \text{for } s \in R_{V_1} - 2 \quad \text{and} \\ \mathcal{L}\{e^{-3t}v_1(t)\}(s) &= V_1(s+3) \quad \text{for } s \in R_{V_1} - 3. \end{aligned}$$

Thus, we have

$$\begin{aligned} X(s) &= \mathcal{L}\{e^{-2t}v_1(t)\}(s) + \mathcal{L}\{e^{-3t}v_1(t)\}(s) \\ &= V_1(s+2) + V_1(s+3) \quad \text{for } (R_{V_1}-2) \cap (R_{V_1}-3). \end{aligned}$$

Substituting the earlier expression for V_1 into the above equation yields

$$\begin{aligned} X(s) &= e^{-(s+2)}\frac{1}{s+2} + e^{-(s+3)}\frac{1}{s+3} \quad \text{for } (\operatorname{Re}(s) > -2) \cap (\operatorname{Re}(s) > -3) \\ &= e^{-(s+2)}\frac{1}{s+2} + e^{-(s+3)}\frac{1}{s+3} \quad \text{for } \operatorname{Re}(s) > -2. \end{aligned}$$

Thus, we have that

$$X(s) = e^{-(s+2)}\frac{1}{s+2} + e^{-(s+3)}\frac{1}{s+3} \quad \text{for } \operatorname{Re}(s) > -2. \qquad \blacksquare$$

Example 7.22. Find the Laplace transform X of the function

$$x(t) = \delta(t) + u(t).$$

Solution. Taking the Laplace transform of x, we have

$$\begin{aligned} X(s) &= \mathcal{L}\{\delta + u\}(s) \\ &= \mathcal{L}\delta(s) + \mathcal{L}u(s). \end{aligned}$$

From Table 7.2, we have

$$\begin{aligned} \delta(t) &\xleftrightarrow{\text{LT}} 1 \quad \text{for all } s \quad \text{and} \\ u(t) &\xleftrightarrow{\text{LT}} \tfrac{1}{s} \quad \text{for } \operatorname{Re}(s) > 0. \end{aligned}$$

Substituting the above results, we obtain

$$\begin{aligned} X(s) &= 1 + \tfrac{1}{s} \quad \text{for } \operatorname{Re}(s) > 0 \\ &= \frac{s+1}{s}. \end{aligned}$$

Thus, we have that

$$X(s) = \frac{s+1}{s} \quad \text{for } \operatorname{Re}(s) > 0. \qquad \blacksquare$$

Example 7.23. Find the Laplace transform X of the function

$$x(t) = te^{-3|t|}.$$

Solution. To begin, we rewrite x as

$$x(t) = te^{3t}u(-t) + te^{-3t}u(t).$$

Taking the Laplace transform of x, we have

$$\begin{aligned}
X(s) &= \mathcal{L}\{te^{3t}u(-t) + te^{-3t}u(t)\}(s) \\
&= \mathcal{L}\{te^{3t}u(-t)\}(s) + \mathcal{L}\{te^{-3t}u(t)\}(s) \\
&= \frac{-(1!)}{(s-3)^2} + \frac{1!}{(s+3)^2} \quad \text{for } \operatorname{Re}(s) > -3 \cap \operatorname{Re}(s) < 3 \\
&= \frac{-(s+3)^2 + (s-3)^2}{(s+3)^2(s-3)^2} \\
&= \frac{-(s^2+6s+9) + s^2 - 6s + 9}{(s+3)^2(s-3)^2} \\
&= \frac{-12s}{(s+3)^2(s-3)^2}.
\end{aligned}$$

Thus, we have that

$$X(s) = \frac{-12s}{(s+3)^2(s-3)^2} \quad \text{for } -3 < \operatorname{Re}(s) < 3. \qquad \blacksquare$$

Example 7.24. Consider the function

$$y(t) = e^{-t}[x * x(t)].$$

Let X and Y denote the Laplace transforms of x and y, respectively. Find an expression for Y in terms of X.

Solution. To begin, we rewrite y as

$$y(t) = e^{-t}v_1(t)$$

where

$$v_1(t) = x * x(t).$$

Let V_1 denote the Laplace transform of v. Let R_X, R_Y, and R_V denote the ROCs of X, Y, and V, respectively. Taking the Laplace transform of the above equations yields

$$\begin{aligned}
V_1(s) &= \mathcal{L}v_1(s) \\
&= \mathcal{L}\{x * x\}(s) \\
&= \mathcal{L}x(s)\mathcal{L}x(s) \\
&= X^2(s) \quad \text{for } s \in R_X \quad \text{and} \\
Y(s) &= \mathcal{L}\{e^{-t}v_1(t)\}(s) \\
&= V_1(s+1) \quad \text{for } s \in R_{V_1} - 1.
\end{aligned}$$

Substituting the above expression for V_1 into the above formula for Y, we have

$$\begin{aligned}
Y(s) &= V_1(s+1) \\
&= X^2(s+1) \quad \text{for } s \in R_X - 1. \qquad \blacksquare
\end{aligned}$$

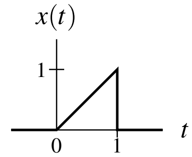


Figure 7.22: Function for the Laplace transform example.

Example 7.25. Using a Laplace transform table and properties of the Laplace transform, find the Laplace transform X of the function x shown in Figure 7.22.

First solution (which incurs more work due to differentiation). First, we express x using unit-step functions. We have

$$\begin{aligned} x(t) &= t[u(t) - u(t-1)] \\ &= tu(t) - tu(t-1). \end{aligned}$$

Taking the Laplace transform of both sides of the preceding equation, we obtain

$$X(s) = \mathcal{L}\{tu(t)\}(s) - \mathcal{L}\{tu(t-1)\}(s).$$

We have

$$\mathcal{L}\{tu(t)\}(s) = \frac{1}{s^2} \quad \text{and}$$

$$\begin{aligned} \mathcal{L}\{tu(t-1)\}(s) = -\mathcal{L}\{-tu(t-1)\}(s) &= -\tfrac{d}{ds}\left(\frac{e^{-s}}{s}\right) \\ &= -\left(\tfrac{d}{ds}e^{-s}s^{-1}\right) \\ &= -(-e^{-s}s^{-1} - s^{-2}e^{-s}) \\ &= \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}. \end{aligned}$$

Combining the above results, we have

$$\begin{aligned} X(s) &= \frac{1}{s^2} - \left[\frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}\right] \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \\ &= \frac{1 - se^{-s} - e^{-s}}{s^2}. \end{aligned}$$

Since x is finite duration, the ROC of X is the entire complex plane.

Second solution (which incurs less work by avoiding differentiation). First, we express x using unit-step functions to yield

$$\begin{aligned} x(t) &= t[u(t) - u(t-1)] \\ &= tu(t) - tu(t-1). \end{aligned}$$

To simplify the subsequent Laplace transform calculation, we choose to rewrite x as

$$\begin{aligned} x(t) &= tu(t) - tu(t-1) + u(t-1) - u(t-1) \\ &= tu(t) - (t-1)u(t-1) - u(t-1). \end{aligned}$$

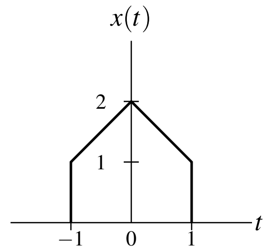


Figure 7.23: Function for the Laplace transform example.

(This is motivated by a preference to compute the Laplace transform of $(t-1)u(t-1)$ instead of $tu(t-1)$.) Taking the Laplace transform of both sides of the preceding equation, we obtain

$$X(s) = \mathcal{L}\{tu(t)\}(s) - \mathcal{L}\{(t-1)u(t-1)\}(s) - \mathcal{L}\{u(t-1)\}(s).$$

We have

$$\begin{aligned}
\mathcal{L}\{tu(t)\}(s) &= \frac{1}{s^2}, \\
\mathcal{L}\{(t-1)u(t-1)\}(s) &= e^{-s}\mathcal{L}\{tu(t)\}(s) \\
&= e^{-s}\left(\frac{1}{s^2}\right) \\
&= \frac{e^{-s}}{s^2}, \quad \text{and} \\
\mathcal{L}\{u(t-1)\}(s) &= e^{-s}\mathcal{L}\{u(t)\}(s) \\
&= e^{-s}\left(\frac{1}{s}\right) \\
&= \frac{e^{-s}}{s}.
\end{aligned}$$

Combining the above results, we have

$$\begin{aligned}
X(s) &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \\
&= \frac{1 - e^{-s} - se^{-s}}{s^2}.
\end{aligned}$$

Since x is finite duration, the ROC of X is the entire complex plane. ■

Example 7.26. Find the Laplace transform X of the function x shown in Figure 7.23.

First solution (which incurs more work due to differentiation). First, we express x using unit-step functions. We have

$$\begin{aligned}
x(t) &= (t+2)[u(t+1) - u(t)] + (-t+2)[u(t) - u(t-1)] \\
&= tu(t+1) - tu(t) + 2u(t+1) - 2u(t) - tu(t) + tu(t-1) + 2u(t) - 2u(t-1) \\
&= tu(t+1) + 2u(t+1) - 2tu(t) + tu(t-1) - 2u(t-1).
\end{aligned}$$

Taking the Laplace transform of both sides of the preceding equation, we obtain

$$\begin{aligned}
X(s) &= -\mathcal{L}\{-tu(t+1)\}(s) + 2\mathcal{L}\{u(t+1)\}(s) + 2\mathcal{L}\{-tu(t)\}(s) - \mathcal{L}\{-tu(t-1)\}(s) - 2\mathcal{L}\{u(t-1)\}(s) \\
&= -\tfrac{d}{ds}\left(\frac{e^s}{s}\right) + 2\left(\frac{e^s}{s}\right) + 2\tfrac{d}{ds}\left(\frac{1}{s}\right) - \tfrac{d}{ds}\left(\frac{e^{-s}}{s}\right) - 2\left(\frac{e^{-s}}{s}\right) \\
&= -\left(\frac{se^s - e^s}{s^2}\right) + 2\left(\frac{e^s}{s}\right) + 2\left(-\frac{1}{s^2}\right) - \left(\frac{s(-e^{-s}) - e^{-s}}{s^2}\right) - 2\left(\frac{e^{-s}}{s}\right) \\
&= \frac{-se^s + e^s + se^{-s} + e^{-s} - 2 + 2se^s - 2e^{-s}}{s^2} \\
&= \frac{se^s + e^s - 2 - se^{-s} + e^{-s}}{s^2}.
\end{aligned}$$

Since x is (bounded and) finite duration, the ROC of X must be the entire complex plane. Thus, we have that

$$X(s) = \frac{se^s + e^s - 2 - se^{-s} + e^{-s}}{s^2} \quad \text{for all } s.$$

Second solution (which incurs less work by avoiding differentiation). Alternatively, we can rearrange our above expression for x to obtain

$$\begin{aligned}
x(t) &= [tu(t+1) + u(t+1) - u(t+1)] + 2u(t+1) - 2tu(t) + [tu(t-1) - u(t-1) + u(t-1)] - 2u(t-1) \\
&= (t+1)u(t+1) + u(t+1) - 2tu(t) + (t-1)u(t-1) - u(t-1).
\end{aligned}$$

Taking the Laplace transform of both sides of the preceding equation, we have

$$\begin{aligned}
X(s) &= e^s\mathcal{L}\{tu(t)\}(s) + \mathcal{L}\{u(t+1)\}(s) - 2\mathcal{L}\{tu(t)\}(s) + e^{-s}\mathcal{L}\{tu(t)\}(s) - \mathcal{L}\{u(t-1)\}(s) \\
&= e^s\tfrac{1}{s^2} + e^s\tfrac{1}{s} - 2\tfrac{1}{s^2} + e^{-s}\tfrac{1}{s^2} - e^{-s}\tfrac{1}{s} \\
&= \frac{e^s + se^s - 2 + e^{-s} - se^{-s}}{s^2}.
\end{aligned}$$

In the case of this alternate solution, the expression for X is considerably easier to simplify. ■

7.10 Determination of the Inverse Laplace Transform

As suggested earlier, in practice, we rarely use (7.3) directly in order to compute the inverse Laplace transform. This formula requires a contour integration, which is not usually very easy to compute. Instead, we employ a partial fraction expansion of the function. In so doing, we obtain a number of simpler functions for which we can usually find the inverse Laplace transform in a table (e.g., such as Table 7.2). In what follows, we assume that the reader is already familiar with partial fraction expansions. A tutorial on partial fraction expansions is provided in Appendix B for those who might not be acquainted with such expansions.

Example 7.27. Find the inverse Laplace transform x of

$$X(s) = \frac{2}{s^2 - s - 2} \quad \text{for } -1 < \operatorname{Re}(s) < 2.$$

Solution. We begin by rewriting X in the factored form

$$X(s) = \frac{2}{(s+1)(s-2)}.$$

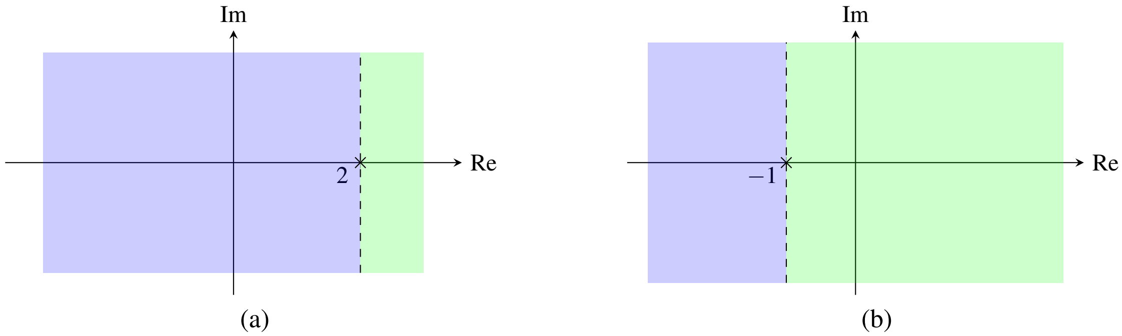


Figure 7.24: The poles and possible ROCs for the rational expressions (a) $\frac{1}{s-2}$; and (b) $\frac{1}{s+1}$.

Then, we find a partial fraction expansion of X. We know that X has an expansion of the form

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s-2}.$$

Calculating the coefficients of the expansion, we obtain

$$A_1 = (s+1)X(s)|_{s=-1} = \left.\frac{2}{s-2}\right|_{s=-1} = -\tfrac{2}{3} \quad \text{and}$$
$$A_2 = (s-2)X(s)|_{s=2} = \left.\frac{2}{s+1}\right|_{s=2} = \tfrac{2}{3}.$$

So, X has the expansion

$$X(s) = \frac{2}{3}\left(\frac{1}{s-2}\right) - \frac{2}{3}\left(\frac{1}{s+1}\right).$$

Taking the inverse Laplace transform of both sides of this equation, we have

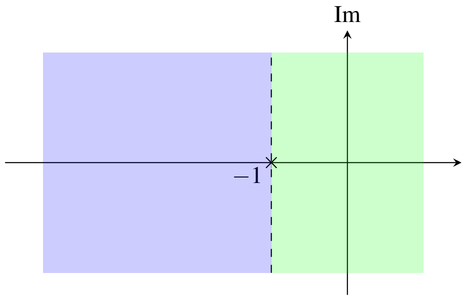
$$x(t) = \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t) - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t). \tag{7.6}$$

At this point, it is important to remember that every Laplace transform has an associated ROC, which is an essential component of the Laplace transform. So, when computing the inverse Laplace transform of a function, we must be careful to use the correct ROC for the function. Thus, in order to compute the two inverse Laplace transforms appearing in (7.6), we must associate a ROC with each of the two expressions $\frac{1}{s-2}$ and $\frac{1}{s+1}$. Some care must be exercised in doing so, since each of these expressions has more than one possible ROC and only one is correct. The possible ROCs for each of these expressions is shown in Figure 7.24. In the case of each of these expressions, the correct ROC to use is the one that contains the ROC of X (i.e., $-1 < \operatorname{Re}(s) < 2$). Using Table 7.2, we have

$$-e^{2t}u(-t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s-2} \quad \text{for } \operatorname{Re}(s) < 2 \quad \text{and}$$
$$e^{-t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1.$$

Substituting these results into (7.6), we obtain

$$\begin{aligned} x(t) &= \tfrac{2}{3}[-e^{2t}u(-t)] - \tfrac{2}{3}[e^{-t}u(t)] \\ &= -\tfrac{2}{3}e^{2t}u(-t) - \tfrac{2}{3}e^{-t}u(t). \end{aligned}$$ ■



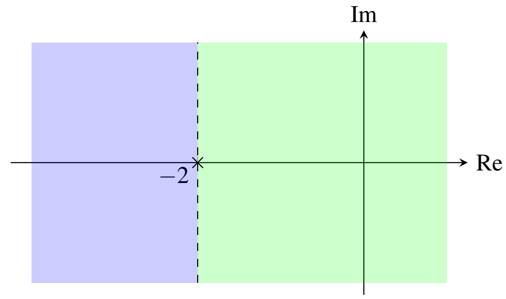


Figure 7.25: The poles and possible ROCs for the rational expressions (a) $\frac{1}{s+1}$ and $\frac{1}{(s+1)^2}$; and (b) $\frac{1}{s+2}$.

Example 7.28 (Rational function with a repeated pole)**.** Find the inverse Laplace transform x of

$$X(s) = \frac{2s+1}{(s+1)^2(s+2)} \quad \text{for } \operatorname{Re}(s) > -1.$$

Solution. To begin, we find a partial fraction expansion of X. We know that X has an expansion of the form

$$X(s) = \frac{A_{1,1}}{s+1} + \frac{A_{1,2}}{(s+1)^2} + \frac{A_{2,1}}{s+2}.$$

Calculating the coefficients of the expansion, we obtain

$$\begin{aligned}
A_{1,1} &= \tfrac{1}{(2-1)!}\left[\left(\tfrac{d}{ds}\right)^{2-1}[(s+1)^2X(s)]\right]\Big|_{s=-1} = \tfrac{1}{1!}\left[\tfrac{d}{ds}[(s+1)^2X(s)]\right]\big|_{s=-1} = \left[\tfrac{d}{ds}\left(\frac{2s+1}{s+2}\right)\right]\Big|_{s=-1} \\
&= \left[\frac{(s+2)(2)-(2s+1)(1)}{(s+2)^2}\right]\Big|_{s=-1} = \left[\frac{2s+4-2s-1}{(s+2)^2}\right]\Big|_{s=-1} = \left[\frac{3}{(s+2)^2}\right]\Big|_{s=-1} = 3, \\
A_{1,2} &= \tfrac{1}{(2-2)!}\left[\left(\tfrac{d}{ds}\right)^{2-2}\left[(s+1)^2X(s)\right]\right]\Big|_{s=-1} = \tfrac{1}{0!}\left[(s+1)^2X(s)\right]\big|_{s=-1} = \frac{2s+1}{s+2}\Big|_{s=-1} = \tfrac{-1}{1} = -1, \quad \text{and} \\
A_{2,1} &= (s+2)X(s)|_{s=-2} = \frac{2s+1}{(s+1)^2}\Big|_{s=-2} = \tfrac{-3}{1} = -3.
\end{aligned}$$

Thus, X has the expansion

$$X(s) = \frac{3}{s+1} - \frac{1}{(s+1)^2} - \frac{3}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation yields

$$x(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t) - 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t). \tag{7.7}$$

At this point, it is important to remember that every Laplace transform has an associated ROC, which is an essential component of the Laplace transform. So, when computing the inverse Laplace transform of a function, we must be careful to use the correct ROC for the function. Thus, in order to compute the three inverse Laplace transforms appearing in (7.7), we must associate a ROC with each of the three expressions $\frac{1}{s+1}$, $\frac{1}{(s+1)^2}$, and $\frac{1}{s+2}$. Some care must be exercised in doing so, since each of these expressions has more than one possible ROC and only one is correct. The possible ROCs for each of these expressions is shown in Figure 7.25. In the case of each of these expressions, the correct ROC to use is the one that contains the ROC of X (i.e., $\operatorname{Re}(s) > -1$). From Table 7.2, we have

$$e^{-t}u(t) \overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1,$$
$$te^{-t}u(t) \overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{(s+1)^2} \quad \text{for } \operatorname{Re}(s) > -1, \quad \text{and}$$
$$e^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.$$

Substituting these results into (7.7), we obtain

$$\begin{aligned} x(t) &= 3e^{-t}u(t) - te^{-t}u(t) - 3e^{-2t}u(t) \\ &= \left(3e^{-t} - te^{-t} - 3e^{-2t}\right)u(t). \end{aligned}$$

■

Example 7.29 (Inverse Laplace transform of improper rational function)**.** Find the inverse Laplace transform x of

$$X(s) = \frac{2s^2+4s+5}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1.$$

Solution. We begin by observing that, although X is rational, it is not strictly proper. So, we need to perform polynomial long division in order to express X as the sum of a polynomial and a strictly-proper rational function. By long division, we have

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad 2 \\ \hline s^2+3s+2 & 2s^2 + 4s + 5 \\ & 2s^2 + 6s + 4 \\ \hline & \quad\ -2s + 1. \end{array}$$

In other words, we have

$$X(s) = 2 + \frac{-2s+1}{s^2+3s+2}.$$

For convenience, we define

$$V(s) = \frac{-2s+1}{(s+1)(s+2)}$$

so that

$$X(s) = 2 + V(s).$$

Observe that V is a strictly-proper rational function. So, we can find a partial fraction expansion of V. Now, we find this expansion. We know that such an expansion is of the form

$$V(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}.$$

Calculating the expansion coefficients, we have

$$\begin{aligned} A_1 &= (s+1)V(s)|_{s=-1} \\ &= \left.\frac{-2s+1}{s+2}\right|_{s=-1} \\ &= 3 \quad \text{and} \\ A_2 &= (s+2)V(s)|_{s=-2} \\ &= \left.\frac{-2s+1}{s+1}\right|_{s=-2} \\ &= -5 \end{aligned}$$

So, we have

$$\begin{aligned} X(s) &= 2 + V(s) \\ &= 2 + \frac{3}{s+1} - \frac{5}{s+2}. \end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}X(t) \\ &= 2\mathcal{L}^{-1}\{1\}(t) + 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - 5\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t). \end{aligned}$$

Considering the ROC of X, we can obtain the following from Table 7.2:

$$\begin{aligned} \delta(t) &\overset{\text{LT}}{\longleftrightarrow} 1, \\ e^{-t}u(t) &\overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1, \quad \text{and} \\ e^{-2t}u(t) &\overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2. \end{aligned}$$

Finally, we can write

$$\begin{aligned} x(t) &= 2\delta(t) + 3e^{-t}u(t) - 5e^{-2t}u(t) \\ &= 2\delta(t) + (3e^{-t} - 5e^{-2t})u(t). \end{aligned}$$

■

Example 7.30. Find all possible inverse Laplace transforms of

$$X(s) = \frac{1}{s^2+3s+2}. \tag{7.8}$$

Solution. We begin by rewriting X in factored form as

$$X(s) = \frac{1}{(s+1)(s+2)}.$$

Then, we find the partial fraction expansion of X. We know that such an expansion has the form

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}.$$

Calculating the coefficients of the expansion, we have

$$\begin{aligned} A_1 &= (s+1)X(s)|_{s=-1} = \left.\frac{1}{s+2}\right|_{s=-1} = 1 \quad \text{and} \\ A_2 &= (s+2)X(s)|_{s=-2} = \left.\frac{1}{s+1}\right|_{s=-2} = -1. \end{aligned}$$

So, we have

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation yields

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t). \tag{7.9}$$

For the Laplace transform X, three possible ROCs exist:

i) $\operatorname{Re}(s) < -2$,
ii) $-2 < \operatorname{Re}(s) < -1$, and
iii) $\operatorname{Re}(s) > -1$.

Thus, three possible inverse Laplace transforms exist for X, depending on the choice of ROC.

First, let us consider the case of the ROC $\operatorname{Re}(s) < -2$. From Table 7.2, we have

$$-e^{-t}u(-t) \xleftrightarrow{\text{LT}} \tfrac{1}{s+1} \quad \text{for } \operatorname{Re}(s) < -1 \quad \text{and}$$
$$-e^{-2t}u(-t) \xleftrightarrow{\text{LT}} \tfrac{1}{s+2} \quad \text{for } \operatorname{Re}(s) < -2.$$

Substituting these results into (7.9), we have

$$\begin{aligned} x(t) &= -e^{-t}u(-t) + e^{-2t}u(-t) \\ &= \left(-e^{-t} + e^{-2t}\right)u(-t). \end{aligned}$$

Second, let us consider the case of the ROC $-2 < \operatorname{Re}(s) < -1$. From Table 7.2, we have

$$-e^{-t}u(-t) \xleftrightarrow{\text{LT}} \tfrac{1}{s+1} \quad \text{for } \operatorname{Re}(s) < -1 \quad \text{and}$$
$$e^{-2t}u(t) \xleftrightarrow{\text{LT}} \tfrac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.$$

Substituting these results into (7.9), we have

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t).$$

Third, let us consider the case of the ROC $\operatorname{Re}(s) > -1$. From Table 7.2, we have

$$e^{-t}u(t) \xleftrightarrow{\text{LT}} \tfrac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1 \quad \text{and}$$
$$e^{-2t}u(t) \xleftrightarrow{\text{LT}} \tfrac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.$$

Substituting these results into (7.9), we have

$$\begin{aligned} x(t) &= e^{-t}u(t) - e^{-2t}u(t) \\ &= \left(e^{-t} - e^{-2t}\right)u(t). \end{aligned}$$

■

7.11 Characterizing LTI Systems Using the Laplace Transform

Consider a LTI system with input x, output y, and impulse response h, as depicted in Figure 7.26. Such a system is characterized by the equation

$$y(t) = x * h(t).$$

Let X, Y, and H denote the Laplace transforms of x, y, and h, respectively. Taking the Laplace transform of both sides of the above equation and using the time-domain convolution property of the Laplace transform, we have

$$Y(s) = H(s)X(s).$$

The quantity H is known as the **system function** or **transfer function** of the system. If the ROC of H includes the imaginary axis, then $H(j\omega)$ is the frequency response of the system. The system can be represented with a block diagram labelled in the Laplace domain as shown in Figure 7.27, where the system is labelled by its system function H.

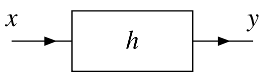


Figure 7.26: Time-domain view of a LTI system with input x, output y, and impulse response h.

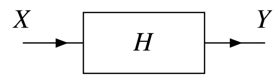


Figure 7.27: Laplace-domain view of a LTI system with input Laplace transform X, output Laplace transform Y, and system function H.

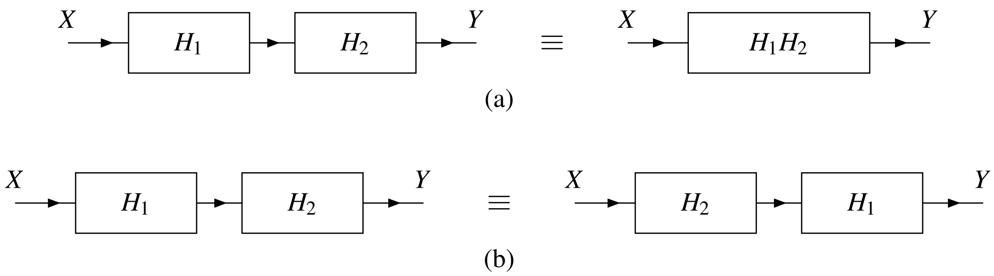


Figure 7.28: Equivalences involving system functions and the series interconnection of LTI systems. The (a) first and (b) second equivalences.

7.12 Interconnection of LTI Systems

From the properties of the Laplace transform and the definition of the system function, we can derive a number of equivalences involving the system function and series- and parallel-interconnected systems.

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with system functions H_1 and H_2, respectively, that are connected in a series configuration as shown in the left-hand side of Figure 7.28(a). Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(t) = h_1 * h_2(t).$$

Taking the Laplace transform of both sides of this equation yields

$$\begin{aligned}H(s) &= \mathcal{L}\{h_1 * h_2\}(s) \\ &= \mathcal{L}h_1(s)\mathcal{L}h_2(s) \\ &= H_1(s)H_2(s).\end{aligned}$$

Thus, we have the equivalence shown in Figure 7.28(a). Also, since multiplication commutes, we also have the equivalence shown in Figure 7.28(b).

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with system functions H_1 and H_2 that are connected in a parallel configuration as shown on the left-hand side of Figure 7.29. Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(t) = h_1(t) + h_2(t).$$

Taking the Laplace transform of both sides of the equation yields

$$\begin{aligned}H(s) &= \mathcal{L}\{h_1 + h_2\}(s) \\ &= \mathcal{L}h_1(s) + \mathcal{L}h_2(s) \\ &= H_1(s) + H_2(s).\end{aligned}$$

Thus, we have the equivalence shown in Figure 7.29.

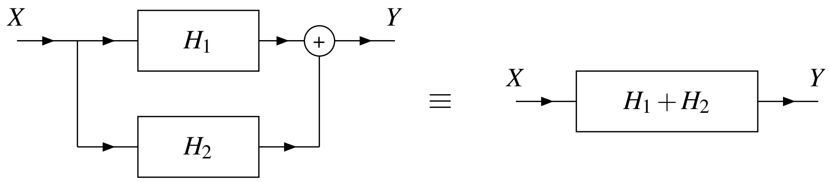


Figure 7.29: Equivalence involving system functions and the parallel interconnection of LTI systems.

7.13 System Function and System Properties

Many properties of a system can be readily determined from the characteristics of its system function, as we shall elaborate upon in the sections that follow.

7.13.1 Causality

From Theorem 4.8, we know that, a LTI system is causal if its impulse response is causal. A causal function, however, is inherently right sided. Consequently, from the properties of the ROC of the Laplace transform discussed in Section 7.7, we have the theorem below.

Theorem 7.12. *The ROC associated with the system function of a causal LTI system is a right-half plane or the entire complex plane.*

Proof. Let h and H denote the impulse response and system function of a causal LTI system $\mathcal{H}$. The system $\mathcal{H}$ is causal if and only if h is causal. So, $\mathcal{H}$ being causal implies that h is causal, which in turn implies (by definition) that h is right sided. From the properties of the ROC, h being right sided implies that the ROC of H is either a right-half plane (in the case that h is not left sided) or the entire complex plane (in the case that h is left sided). Thus, the ROC of H has the stated form. ∎

In general, the converse of the above theorem is not necessarily true. This is, it is not always true that a system function H with a ROC that is either a right-half plane or the entire complex plane is associated with a causal system. If H is rational, however, we have that the converse does hold, as indicated by the theorem below.

Theorem 7.13. *For a LTI system with a rational system function H, causality of the system is equivalent to the ROC of H being the right-half plane to the right of the rightmost pole or, if H has no poles, the entire complex plane.*

Proof. The proof is left as an exercise for the reader. ∎

Example 7.31. For the LTI system with each system function H below, determine whether the system is causal.

(a) $H(s) = \dfrac{1}{s+1}$ for $\operatorname{Re}(s) > -1$;

(b) $H(s) = \dfrac{1}{s^2-1}$ for $-1 < \operatorname{Re}(s) < 1$;

(c) $H(s) = \dfrac{e^s}{s+1}$ for $\operatorname{Re}(s) < -1$; and

(d) $H(s) = \dfrac{e^s}{s+1}$ for $\operatorname{Re}(s) > -1$.

Solution. (a) The poles of H are plotted in Figure 7.30(a) and the ROC is indicated by the shaded area. The system function H is rational and the ROC is the right-half plane to the right of the rightmost pole. Therefore, the system is causal.

(b) The poles of H are plotted in Figure 7.30(b) and the ROC is indicated by the shaded area. The system function is rational but the ROC is not a right-half plane or the entire complex plane. Therefore, the system is not causal.

(c) The system function H has a left-half plane ROC. Therefore, h is left sided but not right sided. Thus, the system is not causal.

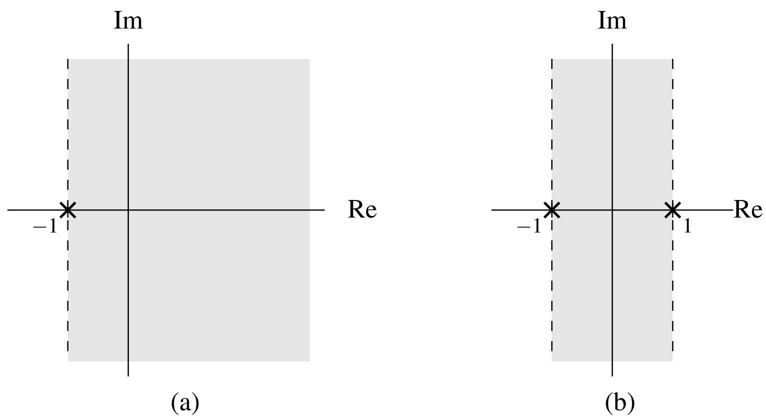


Figure 7.30: Pole and ROCs of the rational system functions in the causality example. The cases of the (a) first (b) second system functions.

(d) The system function H has a right-half plane ROC but is not rational. Thus, we cannot make any conclusion directly from (the ROC of) the system function. Instead, we draw our conclusion from the impulse response h. Taking the inverse Laplace transform of H, we obtain

$$h(t) = e^{-(t+1)}u(t+1).$$

Thus, the impulse response h is not causal. Therefore, the system is not causal. ■

7.13.2 BIBO Stability

In this section, we consider the relationship between the system function and BIBO stability. The first important result is given by the theorem below.

Theorem 7.14. *A LTI system is BIBO stable if and only if the ROC of its system function H contains the imaginary axis (i.e.,* $\operatorname{Re}(s) = 0$*).*

Proof. To begin, we observe that the ROC of H containing the imaginary axis is equivalent to the condition

$$H(j\omega) \text{ converges for all } \omega. \tag{7.10}$$

Let h denote the inverse Laplace transform of H (i.e., h is the impulse response of the system). From earlier in Theorem 4.11, we know that the system is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty \tag{7.11}$$

(i.e., h is absolutely integrable). Thus, the proof of the theorem is equivalent to showing that (7.11) holds if and only if (7.10) holds.

First, we show that (7.11) being satisfied implies that (7.10) holds. So, we assume that (7.11) holds. From the definition of H (evaluated on the imaginary axis), we have

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt.$$

Now, we note the fact that an integral is convergent if it is absolutely convergent. In particular, for any function f,

$$\int_{-\infty}^{\infty} f(t)\,dt \text{ converges} \quad \text{if} \int_{-\infty}^{\infty} |f(t)|\,dt \text{ converges.}$$

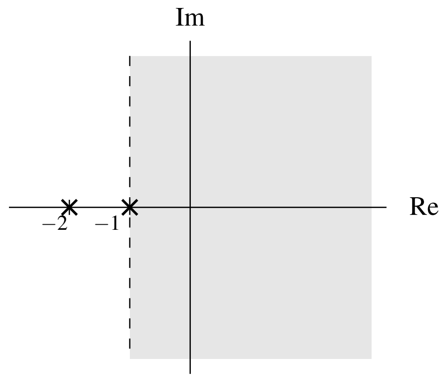


Figure 7.31: ROC for example.

Thus, $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$ converges if $\int_{-\infty}^{\infty} \left|h(t)e^{-j\omega t}\right| dt$ converges. We have, however, that

$$\int_{-\infty}^{\infty} \left|h(t)e^{-j\omega t}\right| dt = \int_{-\infty}^{\infty} |h(t)| \left|e^{-j\omega t}\right| dt = \int_{-\infty}^{\infty} |h(t)| dt.$$

So, $H(j\omega)$ converges if (7.11) holds. Thus, (7.11) being satisfied implies that (7.10) holds. The proof that (7.10) being satisfied implies that (7.11) holds is left as an exercise for the reader. ■

In the case that the system is causal, a more specific result can be derived. This result is given by the theorem below.

Theorem 7.15. *A causal LTI system with a (proper) rational system function H is BIBO stable if and only if all of the poles of H lie in the left half of the plane (i.e., all of the poles have negative real parts).*

Proof. From Theorem 7.14, we know that the system is stable if and only if the ROC of H contains the imaginary axis. Since the system is causal and H is rational, the ROC of H is to the right of its rightmost pole. Therefore, the ROC contains the imaginary axis if and only if the rightmost pole is to the left of the imaginary axis. All of the poles of H lying to the left of the imaginary axis is equivalent to all of these poles having negative real parts. ■

Observe from the preceding two theorems (i.e., Theorems 7.14 and 7.15) that, in the case of a LTI system, the characterization of the BIBO stability property is much simpler in the Laplace domain (via the system function) than the time domain (via the impulse response). For this reason, analyzing the stability of LTI systems is typically performed using the Laplace transform.

Example 7.32. A LTI system has the system function

$$H(s) = \frac{1}{(s+1)(s+2)}.$$

Given that the system is BIBO stable, determine the ROC of H.

Solution. Clearly, the system function H is rational with poles at -1 and -2. Therefore, only three possibilities exist for the ROC:
i) $\mathrm{Re}(s) < -2$,
ii) $-2 < \mathrm{Re}(s) < -1$, and
iii) $\mathrm{Re}(s) > -1$.
In order for the system to be BIBO stable, however, the ROC of H must include the entire imaginary axis. Therefore, the ROC must be $\mathrm{Re}(s) > -1$. This ROC is illustrated in Figure 7.31. ■

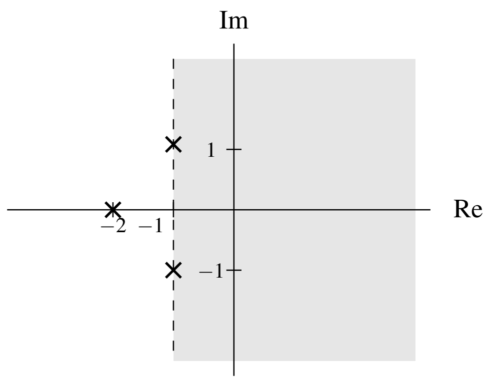


Figure 7.32: Poles of the system function.

Example 7.33. A LTI system is causal and has the system function

$$H(s) = \frac{1}{(s+2)(s^2+2s+2)}.$$

Determine whether this system is BIBO stable.

Solution. We begin by factoring H to obtain

$$H(s) = \frac{1}{(s+2)(s+1-j)(s+1+j)}.$$

(Using the quadratic formula, one can confirm that $s^2+2s+2=0$ has roots at $s=-1\pm j$.) Thus, H has poles at -2, $-1+j$, and $-1-j$. The poles are plotted in Figure 7.32. Since the system is causal and all of the poles of H are in the left half of the plane, the system is BIBO stable. ■

Example 7.34. For each LTI system with system function H given below, determine the ROC of H that corresponds to a BIBO stable system.

(a) $H(s) = \dfrac{s(s-1)}{(s+2)(s+1+j)(s+1-j)}$;

(b) $H(s) = \dfrac{s}{(s+1)(s-1)(s-1-j)(s-1+j)}$;

(c) $H(s) = \dfrac{(s+j)(s-j)}{(s+2-j)(s+2+j)}$; and

(d) $H(s) = \dfrac{s-1}{s}$.

Solution. (a) The function H has poles at -2, $-1+j$, and $-1-j$. The poles are shown in Figure 7.33(a). Since H is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only three distinct ROCs are possible:

i) $\operatorname{Re}(s) < -2$,
ii) $-2 < \operatorname{Re}(s) < -1$, and
iii) $\operatorname{Re}(s) > -1$.

Since we want a BIBO stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be $\operatorname{Re}(s) > -1$. This is the shaded region in the Figure 7.33(a).

(b) The function H has poles at -1, 1, $1+j$, and $1-j$. The poles are shown in Figure 7.33(b). Since H is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only three distinct ROCs are possible:

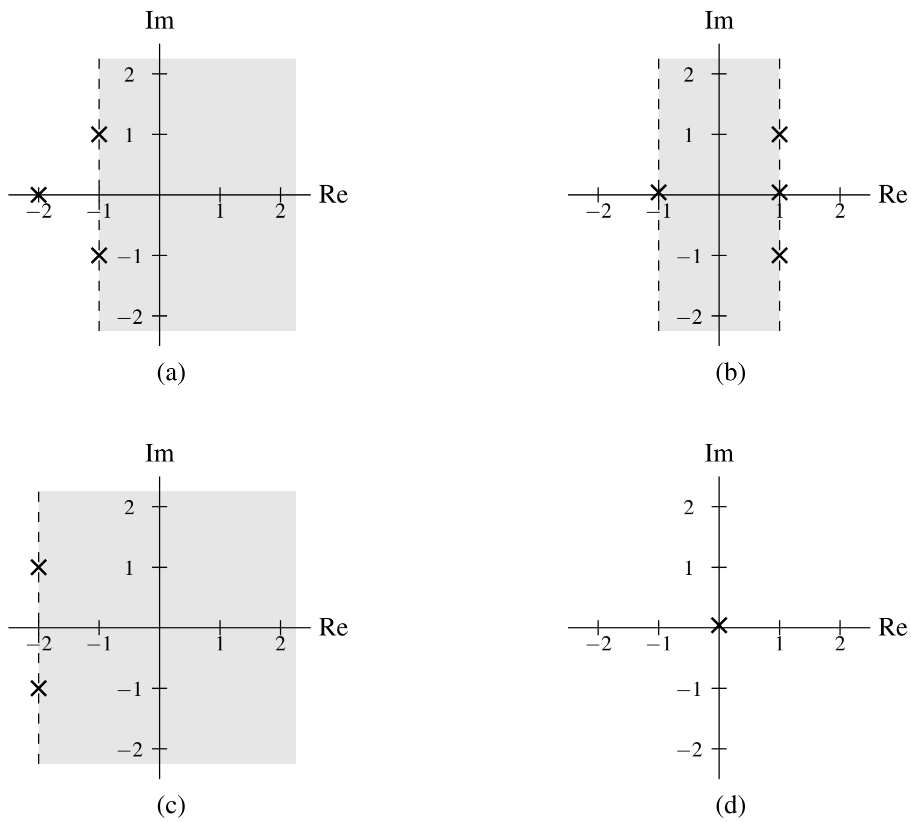


Figure 7.33: Poles and ROCs of the system function H in the (a) first, (b) second, (c) third, and (d) fourth parts of the example.

i) $\operatorname{Re}(s) < -1$,
ii) $-1 < \operatorname{Re}(s) < 1$, and
iii) $\operatorname{Re}(s) > 1$.

Since we want a BIBO stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be $-1 < \operatorname{Re}(s) < 1$. This is the shaded region in Figure 7.33(b).

(c) The function H has poles at $-2+j$ and $-2-j$. The poles are shown in Figure 7.33(c). Since H is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only two distinct ROCs are possible:

i) $\operatorname{Re}(s) < -2$ and
ii) $\operatorname{Re}(s) > -2$.

Since we want a BIBO stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be $\operatorname{Re}(s) > -2$. This is the shaded region in Figure 7.33(c).

(d) The function H has a pole at 0. The pole is shown in Figure 7.33(d). Since H is rational, it cannot converge at 0 (which is a pole of H). Consequently, the ROC can never include the entire imaginary axis. Therefore, the system function H can never be associated with a BIBO stable system. ■

7.13.3 Invertibility

In this section, we consider the relationship between the system function and invertibility. The first important result is given by the theorem below.

Theorem 7.16 (Inverse of LTI system)**.** *Let $\mathcal{H}$ be a LTI system with system function H. If the inverse $\mathcal{H}^{-1}$ of $\mathcal{H}$ exists, $\mathcal{H}^{-1}$ is LTI and has a system function H_{inv} that satisfies*

$$H(s)H_{\mathsf{inv}}(s) = 1. \tag{7.12}$$

Proof. Let h denote the inverse Laplace transform of H. From Theorem 4.9, we know that the system $\mathcal{H}$ is invertible if and only if there exists another LTI system with impulse response h_{inv} satisfying

$$h * h_{\mathsf{inv}} = \delta.$$

Let H_{inv} denote the Laplace transform of h_{inv}. Taking the Laplace transform of both sides of the above equation, we have

$$\mathcal{L}\{h * h_{\mathsf{inv}}\} = \mathcal{L}\delta.$$

From the time-domain convolution property of the Laplace transform and Table 7.2 (i.e., $\mathcal{L}\delta(s) = 1$), we obtain

$$H(s)H_{\mathsf{inv}}(s) = 1. \qquad \blacksquare$$

From the preceding theorem, we have the result below.

Theorem 7.17 (Invertibility of LTI system)**.** *A LTI system $\mathcal{H}$ with system function H is invertible if and only if there exists a function H_{inv} satisfying*

$$H(s)H_{\mathsf{inv}}(s) = 1.$$

Proof. The proof follows immediately from the result of Theorem 7.16 by simply observing that $\mathcal{H}$ being invertible is equivalent to the existence of $\mathcal{H}^{-1}$. $\blacksquare$

From the above theorems, we have that a LTI system $\mathcal{H}$ with system function H has an inverse if and only if a solution for H^{inv} exists in (7.12). Furthermore, if an inverse system exists, its system function is given by

$$H_{\mathsf{inv}}(s) = \frac{1}{H(s)}.$$

Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is not necessarily unique. In practice, however, we often desire a stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in one specific choice of inverse system (due to these additional constraints of stability and/or causality).

Example 7.35. Consider the LTI system with system function

$$H(s) = \frac{s+1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.$$

Determine all possible inverses of this system. Comment on the BIBO stability of each of these inverse systems.

Solution. The system function H_{inv} of the inverse system is given by

$$H_{\mathsf{inv}}(s) = \frac{1}{H(s)} = \frac{s+2}{s+1}.$$

Two ROCs are possible for H_{inv}:
i) $\operatorname{Re}(s) < -1$ and
ii) $\operatorname{Re}(s) > -1$.
Each ROC is associated with a distinct inverse system. The first ROC is associated with a system that is not BIBO stable, since this ROC does not contain the imaginary axis. The second ROC is associated with a BIBO stable system, since this ROC contains the imaginary axis. $\blacksquare$

7.14 LTI Systems and Differential Equations

Many LTI systems of practical interest can be described by Nth-order linear differential equations with constant coefficients. Such a system with input x and output y can be characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = \sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t), \tag{7.13}$$

where $M \leq N$. Let X and Y denote the Laplace transforms of x and y, respectively. Let H denote the system function of the system. Taking the Laplace transform of both sides of the above equation, we obtain

$$\mathcal{L}\left\{\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t)\right\}(s) = \mathcal{L}\left\{\sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t)\right\}(s).$$

Using the linearity property of the Laplace transform, we can rewrite this equation as

$$\sum_{k=0}^{N} b_k \mathcal{L}\left\{\left(\tfrac{d}{dt}\right)^k y(t)\right\}(s) = \sum_{k=0}^{M} a_k \mathcal{L}\left\{\left(\tfrac{d}{dt}\right)^k x(t)\right\}(s).$$

Using the time differentiation property of the Laplace transform, we have

$$\sum_{k=0}^{N} b_k s^k Y(s) = \sum_{k=0}^{M} a_k s^k X(s).$$

Factoring, we have

$$Y(s) \sum_{k=0}^{N} b_k s^k = X(s) \sum_{k=0}^{M} a_k s^k.$$

Dividing both sides of this equation by $X(s)\sum_{k=0}^{N} b_k s^k$, we obtain

$$\frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} a_k s^k}{\sum_{k=0}^{N} b_k s^k}.$$

Since $H(s) = \frac{Y(s)}{X(s)}$, we have that H is given by

$$H(s) = \frac{\sum_{k=0}^{M} a_k s^k}{\sum_{k=0}^{N} b_k s^k}.$$

Observe that, for a system of the form considered above (i.e., a system characterized by an equation of the form of (7.13)), the system function is always rational. It is for this reason that rational functions are of particular interest.

Example 7.36 (Differential equation to system function)**.** A LTI system with input x and output y is characterized by the differential equation

$$y''(t) + \tfrac{D}{M} y'(t) + \tfrac{K}{M} y(t) = x(t),$$

where D, K, and M are positive real constants, and the prime symbol is used to denote derivative. Find the system function H of this system.

Solution. Taking the Laplace transform of the given differential equation, we obtain

$$s^2 Y(s) + \tfrac{D}{M} s Y(s) + \tfrac{K}{M} Y(s) = X(s).$$

Rearranging the terms and factoring, we have

$$\left(s^2 + \tfrac{D}{M}s + \tfrac{K}{M}\right)Y(s) = X(s).$$

Dividing both sides by $\left(s^2 + \tfrac{D}{M}s + \tfrac{K}{M}\right)X(s)$, we obtain

$$\frac{Y(s)}{X(s)} = \frac{1}{s^2 + \frac{D}{M}s + \frac{K}{M}}.$$

Thus, H is given by

$$H(s) = \frac{1}{s^2 + \frac{D}{M}s + \frac{K}{M}}. \qquad \blacksquare$$

Example 7.37 (System function to differential equation)**.** A LTI system with input x and output y has the system function

$$H(s) = \frac{s}{s + R/L},$$

where L and R are positive real constants. Find the differential equation that characterizes this system.

Solution. Let X and Y denote the Laplace transforms of x and y, respectively. To begin, we have

$$\begin{aligned} Y(s) &= H(s)X(s) \\ &= \left(\frac{s}{s + R/L}\right)X(s). \end{aligned}$$

Rearranging this equation, we obtain

$$\begin{aligned} & (s + \tfrac{R}{L})Y(s) = sX(s) \\ \Rightarrow \quad & sY(s) + \tfrac{R}{L}Y(s) = sX(s). \end{aligned}$$

Taking the inverse Laplace transform of both sides of this equation (by using the linearity and time-differentiation properties of the Laplace transform), we have

$$\begin{aligned} & \mathcal{L}^{-1}\{sY(s)\}(t) + \tfrac{R}{L}\mathcal{L}^{-1}Y(t) = \mathcal{L}^{-1}\{sX(s)\}(t) \\ \Rightarrow \quad & \tfrac{d}{dt}y(t) + \tfrac{R}{L}y(t) = \tfrac{d}{dt}x(t). \end{aligned}$$

Therefore, the system is characterized by the differential equation

$$\tfrac{d}{dt}y(t) + \tfrac{R}{L}y(t) = \tfrac{d}{dt}x(t). \qquad \blacksquare$$

7.15 Circuit Analysis

One application of the Laplace transform is circuit analysis. In this section, we consider this particular application. The basic building blocks of many electrical networks are resistors, inductors, and capacitors. In what follows, we briefly introduce each of these circuit elements.

A **resistor** is a circuit element that opposes the flow of electric current. The resistor, shown in schematic form in Figure 7.34(a), is governed by the relationship

$$v(t) = Ri(t) \quad \left(\text{or equivalently, } i(t) = \tfrac{1}{R}v(t)\right),$$

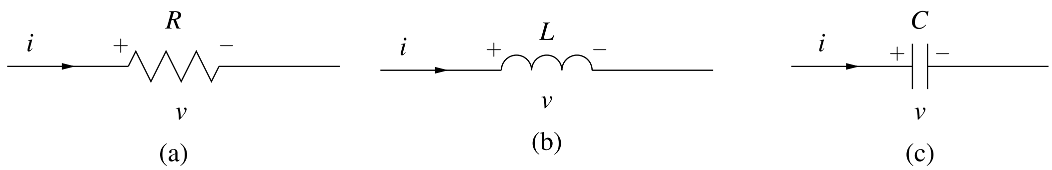


Figure 7.34: Basic electrical components. (a) Resistor, (b) inductor, and (c) capacitor.

where R, v and i denote the resistance of, voltage across, and current through the resistor, respectively. Note that the resistance R is a nonnegative quantity (i.e., $R \geq 0$). In the Laplace domain, the above relationship becomes

$$V(s) = RI(s) \quad \left(\text{or equivalently, } I(s) = \tfrac{1}{R}V(s)\right),$$

where V and I denote the Laplace transforms of v and i, respectively.

An **inductor** is a circuit element that converts an electric current into a magnetic field and vice versa. The inductor, shown in schematic form in Figure 7.34(b), is governed by the relationship

$$v(t) = L\tfrac{d}{dt}i(t) \quad \left(\text{or equivalently, } i(t) = \tfrac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau\right),$$

where L, v, and i denote the inductance of, voltage across, and current through the inductor, respectively. Note that the inductance L is a nonnegative quantity (i.e., $L \geq 0$). In the Laplace domain, the above relationship becomes

$$V(s) = sLI(s) \quad \left(\text{or equivalently, } I(s) = \tfrac{1}{sL}V(s)\right),$$

where V and I denote the Laplace transforms of v and i, respectively.

A **capacitor** is a circuit element that stores electric charge. The capacitor, shown in schematic form in Figure 7.34(c), is governed by the relationship

$$v(t) = \tfrac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \left(\text{or equivalently, } i(t) = C\tfrac{d}{dt}v(t)\right),$$

where C, v, and i denote the capacitance of, voltage across, and current through the capacitor, respectively. Note that the capacitance C is a nonnegative quantity (i.e., $C \geq 0$). In the Laplace domain, the above relationship becomes

$$V(s) = \tfrac{1}{sC}I(s) \quad \left(\text{or equivalently, } I(s) = sCV(s)\right),$$

where V and I denote the Laplace transforms of v and i, respectively.

Observe that the equations that characterize inductors and capacitors are arguably much simpler to express in the Laplace domain than in the time domain. Consequently, the use of the Laplace transform can greatly simplify the process of analyzing circuits containing inductors and capacitors.

Example 7.38 (Simple RC network)**.** Consider the resistor-capacitor (RC) network shown in Figure 7.35 with input v_1 and output v_2. This system is LTI and can be characterized by a linear differential equation with constant coefficients. (a) Find the system function H of this system. (b) Determine whether the system is BIBO stable. (c) Determine the step response of the system.

Solution. (a) From basic circuit analysis, we have

$$v_1(t) = Ri(t) + v_2(t) \quad \text{and} \tag{7.14a}$$
$$i(t) = C\tfrac{d}{dt}v_2(t). \tag{7.14b}$$

Taking the Laplace transform of (7.14) yields

$$V_1(s) = RI(s) + V_2(s) \quad \text{and} \tag{7.15a}$$
$$I(s) = CsV_2(s). \tag{7.15b}$$

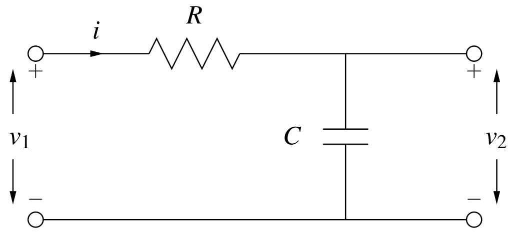


Figure 7.35: Simple RC network.

Substituting (7.15b) into (7.15a) and rearranging, we obtain

$$\begin{aligned} & V_1(s) = R[CsV_2(s)] + V_2(s) \\ \Rightarrow\quad & V_1(s) = RCsV_2(s) + V_2(s) \\ \Rightarrow\quad & V_1(s) = [1+RCs]V_2(s) \\ \Rightarrow\quad & \frac{V_2(s)}{V_1(s)} = \frac{1}{1+RCs}. \end{aligned}$$

Thus, we have that the system function H is given by

$$\begin{aligned} H(s) &= \frac{1}{1+RCs} \\ &= \frac{\frac{1}{RC}}{s+\frac{1}{RC}} \\ &= \frac{\frac{1}{RC}}{s-(-\frac{1}{RC})}. \end{aligned}$$

Since the system can be physically realized, it must be causal. Therefore, the ROC of H must be a right-half plane. Thus, we may infer that the ROC of H is $\operatorname{Re}(s) > -\frac{1}{RC}$. So, we have

$$H(s) = \frac{1}{1+RCs} \quad \text{for } \operatorname{Re}(s) > -\tfrac{1}{RC}.$$

(b) Since resistance and capacitance are (strictly) positive quantities, $R > 0$ and $C > 0$. Thus, $-\frac{1}{RC} < 0$. Consequently, the ROC contains the imaginary axis and the system is BIBO stable.

(c) Now, let us calculate the step response of the system. We know that the system input-output behavior is characterized by the equation

$$\begin{aligned} V_2(s) &= H(s)V_1(s) \\ &= \left(\frac{1}{1+RCs}\right)V_1(s). \end{aligned}$$

To compute the step response, we need to consider an input equal to the unit-step function. So, $v_1 = u$, implying that $V_1(s) = \frac{1}{s}$. Substituting this expression for V_1 into the above expression for V_2, we have

$$\begin{aligned} V_2(s) &= \left(\frac{1}{1+RCs}\right)\left(\frac{1}{s}\right) \\ &= \frac{\frac{1}{RC}}{s(s+\frac{1}{RC})}. \end{aligned}$$

Now, we need to compute the inverse Laplace transform of V_2 in order to determine v_2. To simplify this task, we find the partial fraction expansion for V_2. We know that this expansion is of the form

$$V_2(s) = \frac{A_1}{s} + \frac{A_2}{s+\frac{1}{RC}}.$$

Solving for the coefficients of the expansion, we obtain

$$\begin{aligned} A_1 &= sV_2(s)|_{s=0} \\ &= 1 \quad \text{and} \\ A_2 &= (s+\tfrac{1}{RC})V_2(s)\big|_{s=-\frac{1}{RC}} \\ &= \frac{\frac{1}{RC}}{-\frac{1}{RC}} \\ &= -1. \end{aligned}$$

Thus, we have that V_2 has the partial fraction expansion given by

$$V_2(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{RC}}.$$

Taking the inverse Laplace transform of both sides of the equation, we obtain

$$v_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\right\}(t).$$

Using Table 7.2 and the fact that the system is causal (which implies the necessary ROC), we obtain

$$\begin{aligned} v_2(t) &= u(t) - e^{-t/(RC)}u(t) \\ &= \left(1-e^{-t/(RC)}\right)u(t). \end{aligned}$$

■

7.16 Stability Analysis

As mentioned earlier, since BIBO stability is more easily characterized for LTI systems in the Laplace domain than the time domain, the Laplace domain is often used to analyze system stability. In what follows, we will consider this application of the Laplace transform in more detail.

Example 7.39 (Feedback system)**.** Consider the system shown in Figure 7.36 that has input Laplace transform X and output Laplace transform Y, and is formed by the interconnection of two causal LTI systems labelled with their system functions H_1 and H_2. The system functions H_1 and H_2 are given by

$$H_1(s) = \frac{1}{s^2+as+(a-2)} \quad \text{and} \quad H_2(s) = -1,$$

where a is a real constant. (a) Find the system function H of the (overall) system. (b) Determine the values of the parameter a for which the system is BIBO stable.

Solution. (a) From the system diagram, we can write

$$\begin{aligned} V(s) &= X(s) + H_2(s)Y(s) \quad \text{and} \\ Y(s) &= H_1(s)V(s). \end{aligned}$$

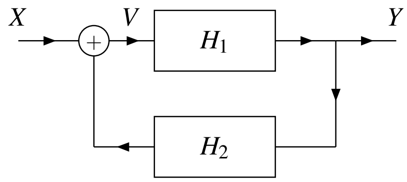


Figure 7.36: Feedback system.

Combining these two equations and simplifying, we obtain

$$\begin{aligned}
& Y(s) = H_1(s)[X(s) + H_2(s)Y(s)] \\
\Rightarrow\ & Y(s) = H_1(s)X(s) + H_1(s)H_2(s)Y(s) \\
\Rightarrow\ & Y(s)[1 - H_1(s)H_2(s)] = H_1(s)X(s) \\
\Rightarrow\ & \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 - H_1(s)H_2(s)}.
\end{aligned}$$

Since $H(s) = \frac{Y(s)}{X(s)}$, we have

$$H(s) = \frac{H_1(s)}{1 - H_1(s)H_2(s)}.$$

Substituting the given expressions for H_1 and H_2, and simplifying, we can write

$$\begin{aligned}
H(s) &= \frac{\left(\frac{1}{s^2+as+(a-2)}\right)}{1 + \left(\frac{1}{s^2+as+(a-2)}\right)} \\
&= \frac{1}{s^2 + as + (a-2) + 1} \\
&= \frac{1}{s^2 + as + (a-1)}.
\end{aligned}$$

(b) In order to assess the BIBO stability of the system, we need to know the poles of the system function H. So, we use the quadratic formula in order to factor the denominator of H. Solving for the roots s of the denominator of H, we obtain

$$\begin{aligned}
s &= \frac{-a \pm \sqrt{a^2 - 4(a-1)}}{2} \\
&= \frac{-a \pm \sqrt{a^2 - 4a + 4}}{2} \\
&= \frac{-a \pm \sqrt{(a-2)^2}}{2} \\
&= \frac{-a \pm (a-2)}{2} \\
&= \{-1, 1-a\}.
\end{aligned}$$

So, $s^2 + as + (a-1) = (s+1)(s+a-1)$. Thus, we have

$$H(s) = \frac{1}{(s+1)(s+a-1)}.$$

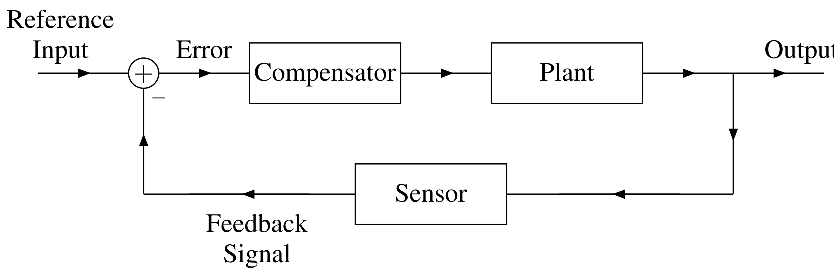


Figure 7.37: Feedback control system.

Since the system is causal, the system is BIBO stable if and only if all of the poles are strictly to the left of the imaginary axis. The system has two poles, one at -1 and one at $1-a$. Thus, we know that

$$1-a<0 \Rightarrow a>1.$$

Therefore, the system is BIBO stable if and only if $a>1$. ■

7.16.1 Feedback Control Systems

In control systems applications, we wish to have some physical quantity (such as a position or force) track a desired value over time. The input to the control system is the desired (i.e., reference) value for the quantity being controlled, and the output is the actual value of the quantity. The difference between the actual value and reference value constitutes an error. The goal of the control system is to force this error to be as close to zero as possible. When the error is zero, this corresponds to the actual value being equal to the desired value. For example, in a simple heating/cooling thermostat system, the reference input would be temperature setting on the thermostat wall control and the output would be the actual room temperature. In a flight control system, the reference input would be the desired values for the position and orientation of the plane specified by the pilot's flight controls and the output would be the plane's actual position and orientation.

A very commonly occurring configuration used for control system applications is known as a feedback control system. A feedback control system consists of three interconnected subsystems:

1. a plant, which is the system whose output corresponds to the quantity being controlled;
2. a sensor, which is used to measure the actual value of the quantity being controlled; and
3. a compensator (also called a controller), which is used to ensure that the overall system behaves in a manner that the output closely tracks the reference input in addition to possibly satisfying other criteria such as being stable.

The general structure of a feedback control system is shown in Figure 7.37. The reference input corresponds to the desired value for the quantity being controlled. The output corresponds to the actual value of the quantity being controlled (which is measured by a sensor). The adder output corresponds to error (i.e., the difference between the desired and actual values of the controlled quantity). Again, in control applications, the objective is to have the output track the reference input as closely as possible over time. In other words, we wish for the error to be as close to zero as possible. It is the responsibility of the compensator to ensure that this happens. If the compensator is well designed, the error will remain close to zero over time.

Consider a simple control system application corresponding to a robot arm. In this system, the reference input corresponds to the desired position and orientation of the robot-arm end effector (i.e., the device at the end of the robot arm), while the output corresponds to the actual position and orientation of the end effector. If the compensator is well designed, then the actual position and orientation of the end effector should track the desired value over time.

Having introduced the notion of feedback control systems, we now consider the application of stabilizing an unstable plant with a feedback control system.

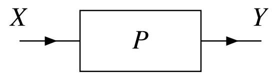


Figure 7.38: Plant.

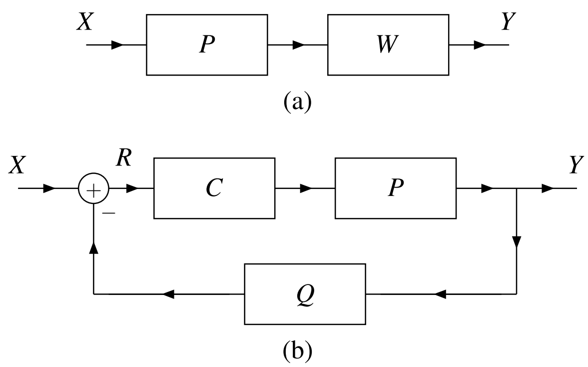


Figure 7.39: Two configurations for stabilizing the unstable plant. (a) Simple cascade system and (b) feedback control system.

Example 7.40 (Stabilization of unstable plant)**.** Consider the causal LTI system with input Laplace transform X, output Laplace transform Y, and system function

$$P(s) = \frac{10}{s-1},$$

as depicted in Figure 7.38. One can easily confirm that this system is not BIBO stable, due to the pole of P at 1. (Since the system is causal, the ROC of P is the RHP given by $\operatorname{Re}(s) > 1$. Clearly, this ROC does not include the imaginary axis. Therefore, the system is not BIBO stable.) In what follows, we consider two different strategies for stabilizing this unstable system as well as their suitability for use in practice.

(a) STABILIZATION OF UNSTABLE PLANT VIA POLE-ZERO CANCELLATION. Suppose that the system in Figure 7.38 is connected in series with another causal LTI system with system function

$$W(s) = \frac{s-1}{10(s+1)},$$

in order to yield a new system with input Laplace transform X and output Laplace transform Y, as shown in Figure 7.39(a). Show that this new system is BIBO stable.

(b) STABILIZATION OF UNSTABLE PLANT VIA FEEDBACK. Suppose now that the system in Figure 7.38 is interconnected with two other causal LTI systems with system functions C and Q, as shown in Figure 7.39(b), in order to yield a new system with input Laplace transform X, output Laplace transform Y, and system function H. Moreover, suppose that

$$C(s) = \beta \quad \text{and} \quad Q(s) = 1,$$

where β is a real constant. Show that, with an appropriate choice of β, the resulting system is BIBO stable.

(c) PRACTICAL ISSUES. Parts (a) and (b) of this example consider two different schemes for stabilizing the unstable system in Figure 7.38. As it turns out, a scheme like the one in part (a) is not useful in practice. Identify the practical problems associated with this approach. Indicate whether the scheme in part (b) suffers from the same shortcomings.

Solution. (a) From the block diagram in Figure 7.39(a), the system function H of the overall system is

$$\begin{aligned} H(s) &= P(s)W(s) \\ &= \left(\frac{10}{s-1}\right)\left(\frac{s-1}{10(s+1)}\right) \\ &= \frac{1}{s+1}. \end{aligned}$$

Since the system is causal and H is rational, the ROC of H is $\mathrm{Re}(s) > -1$. Since the ROC includes the imaginary axis, the system is BIBO stable.

Although our only objective in this example is to stabilize the unstable plant, we note that, as it turns out, the system also has a somewhat reasonable step response. Recall that, for a control system, the output should track the input. Since, in the case of the step response, the input is u, we would like the output to at least approximate u. The step response s is given by

$$\begin{aligned} s(t) &= \mathcal{L}^{-1}\{U(s)H(s)\}(t) \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}(t) \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}(t) \\ &= (1-e^{-t})u(t). \end{aligned}$$

Evidently, s is a somewhat crude approximation of the desired response u.

(b) From the block diagram in Figure 7.39(b), we can write

$$\begin{aligned} R(s) &= X(s) - Q(s)Y(s) \quad \text{and} \\ Y(s) &= C(s)P(s)R(s). \end{aligned}$$

Combining these equations (by substituting the expression for R in the first equation into the second equation), we obtain

$$\begin{aligned} & Y(s) = C(s)P(s)[X(s) - Q(s)Y(s)] \\ \Rightarrow\quad & Y(s) = C(s)P(s)X(s) - C(s)P(s)Q(s)Y(s) \\ \Rightarrow\quad & [1 + C(s)P(s)Q(s)]Y(s) = C(s)P(s)X(s) \\ \Rightarrow\quad & \frac{Y(s)}{X(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)}. \end{aligned}$$

Since $H(s) = \frac{Y(s)}{X(s)}$, we have

$$H(s) = \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)}.$$

Substituting the given expressions for P, C, and Q, we have

$$\begin{aligned} H(s) &= \frac{\beta(\frac{10}{s-1})}{1 + \beta(\frac{10}{s-1})(1)} \\ &= \frac{10\beta}{s - 1 + 10\beta} \\ &= \frac{10\beta}{s - (1 - 10\beta)}. \end{aligned}$$

The system function H is rational and has a single pole at $1-10\beta$. Since the system is causal, the ROC must be the RHP given by $\operatorname{Re}(s) > 1-10\beta$. For the system to be BIBO stable, we require that the ROC includes the imaginary axis. Thus, the system is BIBO stable if $1-10\beta < 0$ which implies $10\beta > 1$, or equivalently $\beta > \frac{1}{10}$.

Although our only objective in this example is to stabilize the unstable plant, we note that, as it turns out, the system also has a reasonable step response. (This is not by chance, however. Some care had to be exercised in the choice of the form of the compensator system function C. The process involved in making this choice requires knowledge of control systems beyond the scope of this book, however.) Recall that, for a control system, the output should track the input. Since, in the case of the step response, the input is u, we would like the output to at least approximate u. The step response s is given by

$$\begin{aligned}
s(t) &= \mathcal{L}^{-1}\{U(s)H(s)\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{10\beta}{s(s-[1-10\beta])}\right\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{10\beta}{10\beta-1}\left(\frac{1}{s}-\frac{1}{s-(1-10\beta)}\right)\right\}(t) \\
&= \frac{10\beta}{10\beta-1}\left(1-e^{-(10\beta-1)t}\right)u(t) \\
&\approx u(t) \quad \text{for large } \beta.
\end{aligned}$$

Clearly, as β increases, s better approximates the desired response u.

(c) The scheme in part (a) for stabilizing the unstable plant relies on pole-zero cancellation. Unfortunately, in practice, it is not possible to achieve pole-zero cancellation. In short, the issue is one of approximation. Our analysis of systems is based on theoretical models specified in terms of equations. These theoretical models, however, are only approximations of real-world systems. This approximate nature is due to many factors, including (but not limited to) the following:

1. We cannot determine the system function of a system exactly, since this involves measurement, which always has some error.

2. We cannot build a system with such precision that it will have exactly some prescribed system function. The system function will only be close to the desired one.

3. The system function of most systems will vary at least slightly with changes in the physical environment (e.g., changes in temperature and pressure, or changes in gravitational forces due to changes in the phase of the moon, and so on).

4. Although a system may be represented by a LTI model, the likely reality is that the system is not exactly LTI, which introduces error.

For reasons such as these, the effective poles and zeros of the system function will only be approximately where we expect them to be. Pole-zero cancellation, however, requires a pole and zero to be placed at exactly the same location. So, any error will prevent the pole-zero cancellation from occurring. Since at least some small error is unavoidable in practice, the desired pole-zero cancellation will not be achieved.

The scheme in part (b) for stabilizing the unstable plant is based on feedback. With the feedback approach, the poles of the system function are not cancelled with zeros. Instead, the poles are completely changed/relocated. For this reason, we can place the poles such that, even if the poles are displaced slightly (due to approximation error), the stability of the system will not be compromised. Therefore, this second scheme does not suffer from the same practical problem that the first one does. ■

7.17 Unilateral Laplace Transform

As mentioned earlier, two different versions of the Laplace transform are commonly employed, namely, the bilateral and unilateral versions. So far, we have considered only the bilateral Laplace transform. Now, we turn our attention to the unilateral Laplace transform. The **unilateral Laplace transform** of the function x is denoted as $\mathcal{L}_{\mathrm{u}}x$ or X and is defined as

$$\mathcal{L}_{\mathrm{u}}x(s) = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt. \tag{7.16}$$

The **inverse unilateral Laplace transform** has the same definition as in the case of the bilateral transform, namely (7.3).

Comparing the definitions of the unilateral and bilateral Laplace transforms given by (7.16) and (7.2), respectively, we can see that these definitions only differ in the lower limit of integration. Due to the similarity in these definitions, an important relationship exists between these two transforms, as we shall now demonstrate. Consider the bilateral Laplace transform of the function xu for an arbitrary function x. We have

$$\begin{aligned}\mathcal{L}\{xu\}(s) &= \int_{-\infty}^{\infty} x(t)u(t)e^{-st}dt \\ &= \int_{0^-}^{\infty} x(t)e^{-st}dt \\ &= \mathcal{L}_{\mathrm{u}}x(s).\end{aligned}$$

In other words, the unilateral Laplace transform of the function x is simply the bilateral Laplace transform of the function xu. Since $\mathcal{L}_{\mathrm{u}}x = \mathcal{L}\{xu\}$ and xu is always right sided, the ROC associated with $\mathcal{L}_{\mathrm{u}}x$ is always a right-half plane (or the entire complex plane). For this reason, we often do not explicitly indicate the ROC when working with the unilateral Laplace transform.

From earlier in this chapter, we know that the bilateral Laplace transform is invertible. That is, if the function x has the bilateral Laplace transform $X = \mathcal{L}x$, then $\mathcal{L}^{-1}X = x$. Now, let us consider the invertibility of the unilateral Laplace transform. To do this, we must consider the quantity $\mathcal{L}_{\mathrm{u}}^{-1}\mathcal{L}_{\mathrm{u}}x$. Since $\mathcal{L}_{\mathrm{u}}x = \mathcal{L}\{xu\}$ and the inverse equations for the unilateral and bilateral Laplace transforms are identical, we can write

$$\begin{aligned}\mathcal{L}_{\mathrm{u}}^{-1}\mathcal{L}_{\mathrm{u}}x(t) &= \mathcal{L}_{\mathrm{u}}^{-1}\{\mathcal{L}\{xu\}\}(t) \\ &= \mathcal{L}^{-1}\{\mathcal{L}\{xu\}\}(t) \\ &= x(t)u(t) \\ &= \begin{cases} x(t) & t \geq 0 \\ 0 & t < 0. \end{cases}\end{aligned}$$

Thus, we have that $\mathcal{L}_{\mathrm{u}}^{-1}\mathcal{L}_{\mathrm{u}}x = x$ only if x is causal. In other words, the unilateral Laplace transform is invertible only for causal functions. For noncausal functions, we can only recover $x(t)$ for $t \geq 0$. In essence, the unilateral Laplace transform discards all information about the value of the function x at t for $t < 0$. Since this information is discarded, it cannot be recovered by an inverse unilateral Laplace transform operation.

Due to the close relationship between the unilateral and bilateral Laplace transforms, these two transforms have some similarities in their properties. Since these two transforms are not identical, however, their properties differ in some cases, often in subtle ways. The properties of the unilateral Laplace transform are summarized in Table 7.3.

By comparing the properties of the unilateral and bilateral Laplace transform listed in Tables 7.3 and 7.1, respectively, we can see that the unilateral Laplace transform has some of the same properties as its bilateral counterpart, namely, the linearity, Laplace-domain shifting, conjugation, and Laplace-domain differentiation properties. The initial-value and final-value theorems also apply in the case of the unilateral Laplace transform.

Since the unilateral and bilateral Laplace transforms are defined differently, their properties also differ in some cases. These differences can be seen by comparing the bilateral Laplace transform properties listed in Table 7.1 with the unilateral Laplace transform properties listed in Table 7.3. In the unilateral case, we can see that:

1. the time-domain convolution property has the additional requirement that the functions being convolved must be causal;

Table 7.3: Properties of the unilateral Laplace transform

Property	Time Domain	Laplace Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$
Time/Laplace-Domain Scaling	$x(at), a>0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$X^*(s^*)$
Time-Domain Convolution	$x_1 * x_2(t), x_1$ and x_2 are causal	$X_1(s)X_2(s)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)-x(0^-)$
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$
Time-Domain Integration	$\int_{0^-}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$

Property	
Initial Value Theorem	$x(0^+) = \lim_{s\to\infty} sX(s)$
Final Value Theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

2. the time/Laplace-domain scaling property has the additional constraint that the scaling factor must be positive;
3. the time-domain differentiation property has an extra term in the expression for $\mathcal{L}_\mathsf{u}\{\mathcal{D}x\}(t)$, where $\mathcal{D}$ denotes the derivative operator (namely, $-x(0^-)$); and
4. the time-domain integration property has a different lower limit in the time-domain integral (namely, 0^- instead of $-\infty$).

Also, in the unilateral case, the time-domain shifting property does not hold (except in special circumstances).

Since $\mathcal{L}_\mathsf{u}x = \mathcal{L}\{xu\}$, we can easily generate a table of unilateral Laplace transform pairs from a table of bilateral transform pairs. Using the bilateral Laplace transform pairs from Table 7.2, and the preceding relationship between the unilateral and bilateral Laplace transforms, we can trivially deduce the unilateral Laplace transform pairs in Table 7.4. Since, in the unilateral case, the ROC always corresponds to a right-sided function, we do not explicitly indicate the ROC in the table. That is, the ROC is implicitly assumed to be the one that corresponds to a right-sided function (i.e., a right-half plane or the entire complex plane).

The inverse unilateral Laplace transform is computed through the same means used in the bilateral case (e.g., partial fraction expansions). The only difference is that the ROC is always assumed to correspond to a right-sided function.

7.18 Solving Differential Equations Using the Unilateral Laplace Transform

Many systems of interest in engineering applications can be characterized by constant-coefficient linear differential equations. As it turns out, a system that is described by such an equation need not be linear. In particular, the

Table 7.4: Transform pairs for the unilateral Laplace transform

Pair	$x(t),\ t \geq 0$	$X(s)$
1	$\delta(t)$	1
2	1	$\frac{1}{s}$
3	t^n	$\frac{n!}{s^{n+1}}$
4	e^{-at}	$\frac{1}{s+a}$
5	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
6	$\cos(\omega_0 t)$	$\frac{s}{s^2+\omega_0^2}$
7	$\sin(\omega_0 t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
8	$e^{-at}\cos(\omega_0 t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
9	$e^{-at}\sin(\omega_0 t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

system will be linear only if the initial conditions for the differential equation are all zero. If one or more of the initial conditions is nonzero, then the system is what we refer to as **incrementally linear**. For our purposes here, incrementally linear systems can be thought of as a generalization of linear systems. The unilateral Laplace transform is sometimes quite useful due to its ability to easily handle nonzero initial conditions. For example, one common use of the unilateral Laplace transform is in solving constant-coefficient linear differential equations with nonzero initial conditions. In what follows, we consider some examples that exploit the unilateral Laplace transform to this end.

Example 7.41. Consider the causal incrementally-linear TI system with input x and output y characterized by the differential equation

$$y'(t) + 3y(t) = x(t),$$

where the prime symbol denotes derivative. If $x(t) = e^{-t}u(t)$ and $y(0^-) = 1$, find y.

Solution. We begin by taking the unilateral Laplace transform of both sides of the given differential equation. This yields

$$\begin{aligned}
&\mathcal{L}_{\mathsf{u}}\left\{y'\right\}(s) + 3\mathcal{L}_{\mathsf{u}}y(s) = \mathcal{L}_{\mathsf{u}}x(s) \\
\Rightarrow\quad & sY(s) - y(0^-) + 3Y(s) = X(s) \\
\Rightarrow\quad & (s+3)Y(s) = X(s) + y(0^-) \\
\Rightarrow\quad & Y(s) = \frac{X(s)+y(0^-)}{s+3}.
\end{aligned}$$

Since $x(t) = e^{-t}u(t)$, we have

$$X(s) = \mathcal{L}_{\mathsf{u}}\{e^{-t}\}(s) = \frac{1}{s+1}.$$

Substituting this expression for X and the given initial conditions (i.e., $y(0^-) = 1$) into the above equation for Y, we obtain

$$Y(s) = \frac{\left(\frac{1}{s+1}\right)+1}{s+3} = \frac{\left(\frac{s+2}{s+1}\right)}{s+3} = \frac{s+2}{(s+1)(s+3)}.$$

Now, we find a partial fraction expansion for Y. Such an expansion has the form

$$Y(s) = \frac{A_1}{s+1} + \frac{A_2}{s+3}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned} A_1 &= (s+1)Y(s)|_{s=-1} \\ &= \left.\frac{s+2}{s+3}\right|_{s=-1} \\ &= \tfrac{1}{2} \quad \text{and} \\ A_2 &= (s+3)Y(s)|_{s=-3} \\ &= \left.\frac{s+2}{s+1}\right|_{s=-3} \\ &= \tfrac{1}{2}. \end{aligned}$$

So, we can rewrite Y as

$$Y(s) = \tfrac{1}{2}\left(\frac{1}{s+1}\right) + \tfrac{1}{2}\left(\frac{1}{s+3}\right).$$

Taking the inverse unilateral Laplace transform of Y, we obtain

$$\begin{aligned} y(t) &= \mathcal{L}_{\mathrm{u}}^{-1}Y(t) \\ &= \tfrac{1}{2}\mathcal{L}_{\mathrm{u}}^{-1}\left\{\frac{1}{s+1}\right\}(t) + \tfrac{1}{2}\mathcal{L}_{\mathrm{u}}^{-1}\left\{\frac{1}{s+3}\right\}(t) \\ &= \tfrac{1}{2}e^{-t} + \tfrac{1}{2}e^{-3t} \quad \text{for } t \geq 0. \end{aligned}$$

■

Example 7.42 (Unilateral Laplace transform of second-order derivative)**.** Find the unilateral Laplace transform Y of y in terms of the unilateral Laplace transform X of x, where

$$y(t) = x''(t)$$

and the prime symbol denotes derivative (e.g., x'' is the second derivative of x)

Solution. Define the function

$$v(t) = x'(t) \tag{7.17}$$

so that

$$y(t) = v'(t). \tag{7.18}$$

Let V denote the unilateral Laplace transform of v. Taking the unilateral Laplace transform of (7.17) (using the time-domain differentiation property), we have

$$\begin{aligned} V(s) &= \mathcal{L}_{\mathrm{u}}\left\{x'\right\}(s) \\ &= sX(s) - x(0^-). \end{aligned} \tag{7.19}$$

Taking the unilateral Laplace transform of (7.18) (using the time-domain differentiation property), we have

$$\begin{aligned} Y(s) &= \mathcal{L}_{\mathrm{u}}\left\{v'\right\}(s) \\ &= sV(s) - v(0^-). \end{aligned} \tag{7.20}$$

Substituting (7.19) into (7.20), we have

$$\begin{aligned} Y(s) &= s\left[sX(s) - x(0^-)\right] - v(0^-) \\ &= s^2X(s) - sx(0^-) - x'(0^-). \end{aligned}$$

Thus, we have that

$$Y(s) = s^2X(s) - sx(0^-) - x'(0^-). \qquad \blacksquare$$

Example 7.43. Consider the causal incrementally-linear TI system with input x and output y characterized by the differential equation

$$y''(t) + 3y'(t) + 2y(t) = x(t),$$

where the prime symbol denotes derivative. If $x(t) = 5u(t)$, $y(0^-) = 1$, and $y'(0^-) = -1$, find y.

Solution. We begin by taking the unilateral Laplace transform of both sides of the given differential equation. This yields

$$\begin{aligned} & \mathcal{L}_{\mathrm{u}}\left\{y'' + 3y' + 2y\right\}(s) = \mathcal{L}_{\mathrm{u}}x(s) \\ \Rightarrow\quad & \mathcal{L}_{\mathrm{u}}\left\{y''\right\}(s) + 3\mathcal{L}_{\mathrm{u}}\left\{y'\right\}(s) + 2\mathcal{L}_{\mathrm{u}}y(s) = \mathcal{L}_{\mathrm{u}}x(s) \\ \Rightarrow\quad & \left[s^2Y(s) - sy(0^-) - y'(0^-)\right] + 3\left[sY(s) - y(0^-)\right] + 2Y(s) = X(s) \\ \Rightarrow\quad & s^2Y(s) - sy(0^-) - y'(0^-) + 3sY(s) - 3y(0^-) + 2Y(s) = X(s) \\ \Rightarrow\quad & \left[s^2 + 3s + 2\right]Y(s) = X(s) + sy(0^-) + y'(0^-) + 3y(0^-) \\ \Rightarrow\quad & Y(s) = \frac{X(s) + sy(0^-) + y'(0^-) + 3y(0^-)}{s^2 + 3s + 2}. \end{aligned}$$

Since $x(t) = 5u(t)$, we have

$$X(s) = \mathcal{L}_{\mathrm{u}}\{5u(t)\}(s) = \tfrac{5}{s}.$$

Substituting this expression for X and the given initial conditions into the above equation yields

$$Y(s) = \frac{\left(\frac{5}{s}\right) + s - 1 + 3}{s^2 + 3s + 2} = \frac{s^2 + 2s + 5}{s(s+1)(s+2)}.$$

Now, we must find a partial fraction expansion of Y. Such an expansion is of the form

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{s+2}.$$

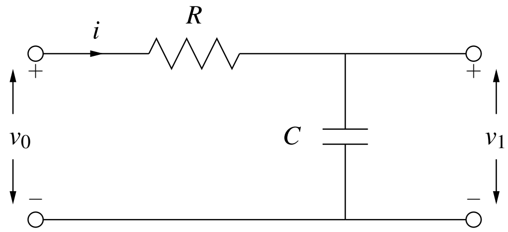


Figure 7.40: RC network.

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1 &= sY(s)\big|_{s=0} \\
&= \left.\frac{s^2+2s+5}{(s+1)(s+2)}\right|_{s=0} \\
&= \tfrac{5}{2}, \\
A_2 &= (s+1)Y(s)\big|_{s=-1} \\
&= \left.\frac{s^2+2s+5}{s(s+2)}\right|_{s=-1} \\
&= -4, \quad \text{and} \\
A_3 &= (s+2)Y(s)\big|_{s=-2} \\
&= \left.\frac{s^2+2s+5}{s(s+1)}\right|_{s=-2} \\
&= \tfrac{5}{2}.
\end{aligned}$$

So, we can rewrite Y as

$$Y(s) = \frac{5/2}{s} - \frac{4}{s+1} + \frac{5/2}{s+2}.$$

Taking the inverse unilateral Laplace transform of Y yields

$$\begin{aligned}
y(t) &= \mathcal{L}_u^{-1}Y(t) \\
&= \tfrac{5}{2}\mathcal{L}_u^{-1}\left\{\frac{1}{s}\right\}(t) - 4\mathcal{L}_u^{-1}\left\{\frac{1}{s+1}\right\}(t) + \tfrac{5}{2}\mathcal{L}_u^{-1}\left\{\frac{1}{s+2}\right\}(t) \\
&= \tfrac{5}{2} - 4e^{-t} + \tfrac{5}{2}e^{-2t} \quad \text{for } t \geq 0.
\end{aligned}$$

■

Example 7.44 (RC network)**.** Consider the resistor-capacitor (RC) network shown in Figure 7.40 with input v_0 and output v_1. If $R = 100$, $C = \frac{1}{100}$, $v_0(t) = 3e^{-2t}u(t)$, and $v_1(0^-) = 1$, find v_1.

Solution. From basic circuit analysis, we have

$$\begin{aligned}
v_0(t) &= Ri(t) + v_1(t) \quad \text{and} \\
i(t) &= C\tfrac{d}{dt}v_1(t).
\end{aligned}$$

Combining the preceding two equations, we obtain

$$\begin{aligned}
v_0(t) &= R\left[C\tfrac{d}{dt}v_1(t)\right] + v_1(t) \\
&= RC\tfrac{d}{dt}v_1(t) + v_1(t).
\end{aligned}$$

Taking the unilateral Laplace transform of both sides of this equation yields

$$\begin{aligned}
& \mathcal{L}_{\mathsf{u}} v_0(s) = \mathcal{L}_{\mathsf{u}}\left\{RC\tfrac{d}{dt}v_1(t) + v_1(t)\right\}(s) \\
\Rightarrow\quad & \mathcal{L}_{\mathsf{u}} v_0(s) = RC\mathcal{L}_{\mathsf{u}}\left\{\tfrac{d}{dt}v_1(t)\right\}(s) + \mathcal{L}_{\mathsf{u}} v_1(s) \\
\Rightarrow\quad & V_0(s) = RC\left[sV_1(s) - v_1(0^-)\right] + V_1(s) \\
\Rightarrow\quad & V_0(s) = RCsV_1(s) - RCv_1(0^-) + V_1(s) \\
\Rightarrow\quad & V_0(s) + RCv_1(0^-) = RCsV_1(s) + V_1(s) \\
\Rightarrow\quad & V_1(s) = \frac{V_0(s) + RCv_1(0^-)}{RCs+1}.
\end{aligned}$$

Since $v_0(t) = 3e^{-2t}u(t)$, we have

$$V_0(s) = \frac{3}{s+2}.$$

Substituting this expression for V_0 into the above equation for V_1, we obtain

$$\begin{aligned}
V_1(s) &= \frac{\left(\frac{3}{s+2}\right)+1}{s+1} \\
&= \frac{3}{(s+1)(s+2)} + \frac{1}{s+1} \\
&= \frac{s+5}{(s+1)(s+2)}.
\end{aligned}$$

Now, we find a partial fraction expansion of V_1. Such an expansion is of the form

$$V_1(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}.$$

Calculating the expansion coefficients yields

$$\begin{aligned}
A_1 &= (s+1)V_1(s)|_{s=-1} \\
&= \left.\frac{s+5}{s+2}\right|_{s=-1} \\
&= 4 \quad \text{and} \\
A_2 &= (s+2)V_1(s)|_{s=-2} \\
&= \left.\frac{s+5}{s+1}\right|_{s=-2} \\
&= -3.
\end{aligned}$$

Thus, we can rewrite V_1 as

$$V_1(s) = \frac{4}{s+1} - \frac{3}{s+2}.$$

Taking the inverse unilateral Laplace transform of V_1, we obtain

$$\begin{aligned}
v_1(t) &= \mathcal{L}_{\mathsf{u}}^{-1} V_1(t) \\
&= \mathcal{L}_{\mathsf{u}}^{-1}\left\{\frac{4}{s+1} - \frac{3}{s+2}\right\}(t) \\
&= 4\mathcal{L}_{\mathsf{u}}^{-1}\left\{\frac{1}{s+1}\right\}(t) - 3\mathcal{L}_{\mathsf{u}}^{-1}\left\{\frac{1}{s+2}\right\}(t) \\
&= 4e^{-t} - 3e^{-2t} \quad \text{for } t \geq 0.
\end{aligned}$$

■

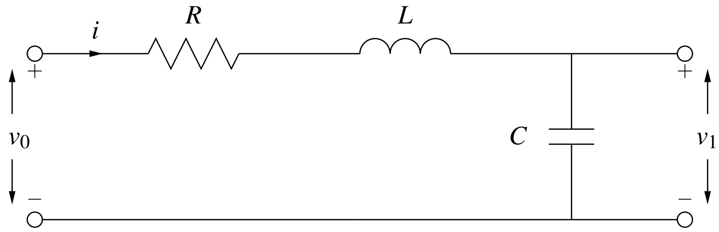


Figure 7.41: RLC network.

Example 7.45 (RLC network)**.** Consider the resistor-inductor-capacitor (RLC) network shown in Figure 7.41 with input v_0 and output v_1. If $R = 2$, $L = 1$, $C = 1$, $v_0(t) = u(t)$, $v_1(0^-) = 0$, and $v_1'(0^-) = 1$, find v_1.

Solution. From basic circuit analysis, we can write

$$v_0(t) = Ri(t) + L\frac{d}{dt}i(t) + v_1(t) \quad \text{and}$$
$$v_1(t) = \tfrac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \Rightarrow \quad i(t) = C\frac{d}{dt}v_1(t).$$

Combining the above equations, we obtain

$$\begin{aligned} v_0(t) &= R\left[C\tfrac{d}{dt}v_1(t)\right] + L\tfrac{d}{dt}\left[C\tfrac{d}{dt}v_1(t)\right] + v_1(t) \\ &= RC\tfrac{d}{dt}v_1(t) + LC\left(\tfrac{d}{dt}\right)^2 v_1(t) + v_1(t). \end{aligned}$$

Taking the unilateral Laplace transform of both sides of the preceding equation yields

$$\begin{aligned} & \mathcal{L}_{\mathrm{u}}v_0(s) = RC\mathcal{L}_{\mathrm{u}}\left\{\tfrac{d}{dt}v_1(t)\right\}(s) + LC\mathcal{L}_{\mathrm{u}}\left\{\left(\tfrac{d}{dt}\right)^2 v_1(t)\right\}(s) + \mathcal{L}_{\mathrm{u}}v_1(s) \\ \Rightarrow\quad & V_0(s) = RC\left[sV_1(s) - v_1(0^-)\right] + LC\left[s^2V_1(s) - sv_1(0^-) - v_1'(0^-)\right] + V_1(s) \\ \Rightarrow\quad & V_0(s) = RCsV_1(s) - RCv(0^-) + LCs^2V_1(s) - LCsv_1(0^-) - LCv_1'(0^-) + V_1(s) \\ \Rightarrow\quad & \left[LCs^2 + RCs + 1\right]V_1(s) = V_0(s) + RCv(0^-) + LCsv_1(0^-) + LCv_1'(0^-) \\ \Rightarrow\quad & V_1(s) = \frac{V_0(s) + [RC + LCs]\,v_1(0^-) + LCv_1'(0^-)}{LCs^2 + RCs + 1}. \end{aligned}$$

Since $v_0 = u$, we have $V_0(s) = \frac{1}{s}$. Substituting this expression for V_0 into the preceding equation for V_1, we obtain

$$\begin{aligned} V_1(s) &= \frac{\left(\frac{1}{s}\right) + 1}{s^2 + 2s + 1} \\ &= \frac{\left(\frac{s+1}{s}\right)}{(s+1)^2} \\ &= \frac{1}{s(s+1)}. \end{aligned}$$

Now, we find a partial fraction expansion of V_1. Such an expansion is of the form

$$V_1(s) = \frac{A_1}{s} + \frac{A_2}{s+1}.$$

Solving for the expansion coefficients, we obtain

$$
\begin{aligned}
A_1 &= sV_1(s)|_{s=0} \\
&= \left.\frac{1}{s+1}\right|_{s=0} \\
&= 1 \quad \text{and} \\
A_2 &= (s+1)V_1(s)|_{s=-1} \\
&= \left.\frac{1}{s}\right|_{s=-1} \\
&= -1.
\end{aligned}
$$

Thus, we can rewrite V_1 as

$$V_1(s) = \frac{1}{s} - \frac{1}{s+1}.$$

Taking the inverse unilateral Laplace transform of V_1, we obtain

$$
\begin{aligned}
v_1(t) &= \mathcal{L}_\mathrm{u}^{-1}\left\{\frac{1}{s}\right\}(t) - \mathcal{L}_\mathrm{u}^{-1}\left\{\frac{1}{s+1}\right\}(t) \\
&= 1 - e^{-t} \quad \text{for } t \geq 0.
\end{aligned}
$$

■

7.19 Exercises

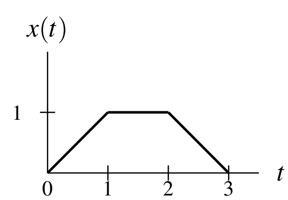
7.19.1 Exercises Without Answer Key

7.1 Using the definition of the Laplace transform, find the Laplace transform X of each of function x below.
(a) $x(t) = e^{-at}u(t)$;
(b) $x(t) = e^{-a|t|}$; and
(c) $x(t) = \cos(\omega_0 t)u(t)$. [Note: $\int e^{ax}\cos(bx)dx = \frac{1}{a^2+b^2}(e^{ax}[a\cos(bx)+b\sin(bx)])+C$.]

7.2 Using properties of the Laplace transform and a table of Laplace transform pairs, find the Laplace transform X of each function x below.
(a) $x(t) = e^{-2t}u(t)$;
(b) $x(t) = 3e^{-2t}u(t) + 2e^{5t}u(-t)$;
(c) $x(t) = e^{-2t}u(t+4)$;
(d) $x(t) = \int_{-\infty}^{t} e^{-2\tau}u(\tau)d\tau$;
(e) $x(t) = -e^{at}u(-t+b)$, where a and b are real constants and $a > 0$;
(f) $x(t) = te^{-3t}u(t+1)$; and
(g) $x(t) = tu(t+2)$.

7.3 Using properties of the Laplace transform and a Laplace transform table, find the Laplace transform X of each function x below.
(a) $x(t) = \begin{cases} t & 0 \leq t < 1 \\ t-2 & 1 \leq t < 2 \\ 0 & \text{otherwise;} \end{cases}$
(b) $x(t) = \begin{cases} 1+t & -1 \leq t < 0 \\ 1-t & 0 \leq t < 1 \\ 0 & \text{otherwise;} \end{cases}$ and
(c) $x(t) = \begin{cases} t & 0 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ t-1 & 2 \leq t < 3 \\ 0 & \text{otherwise.} \end{cases}$

7.4 Using properties of the Laplace transform and a Laplace transform table, find the Laplace transform X of the function x shown in the figure below.



7.5 For each case below, find the Laplace transform Y of the function y in terms of the Laplace transform X of the function x.
(a) $y(t) = x(at-b)$ where a and b are real constants and $a \neq 0$;
(b) $y(t) = e^{-3t}[x * x(t-1)]$;
(c) $y(t) = tx(3t-2)$;
(d) $y(t) = \mathcal{D}x_1(t)$, where $x_1(t) = x^*(t-3)$ and $\mathcal{D}$ denotes the derivative operator;
(e) $y(t) = e^{-5t}x(3t+7)$; and
(f) $y(t) = e^{-j5t}x(t+3)$.

7.6 The function x has the Laplace transform

$$X(s) = \frac{(s+\frac{1}{2})(s-\frac{1}{2})}{s(s+1)(s-1)}.$$

Determine whether x is left sided but not right sided, right sided but not left sided, two sided, or finite duration for each ROC of X below.
(a) $\operatorname{Re}(s) < -1$;
(b) $-1 < \operatorname{Re}(s) < 0$;
(c) $0 < \operatorname{Re}(s) < 1$; and
(d) $\operatorname{Re}(s) > 1$.

7.7 A function x has the Laplace transform

$$X(s) = \frac{s+\frac{1}{2}}{(s+1-j)(s+1+j)(s+2)}.$$

Plot the ROC of X if x is (a) left sided; (b) right sided; (c) two sided; (d) causal.

7.8 A function x has the Laplace transform

$$X(s) = \frac{s+\frac{1}{2}}{(s-1)(s+1-j)(s+1+j)(s+2)}.$$

Determine how many distinct possibilities exist for x. (It is not necessary to explicitly find all of them.)

7.9 A causal function x has the Laplace transform

$$X(s) = \frac{-2s}{s^2+3s+2}.$$

(a) Assuming that x has no singularities at 0, find $x(0^+)$.
(b) Assuming that $\lim_{t\to\infty} x(t)$ exists, find this limit.

7.10 Find the inverse Laplace transform x of each function X below.
(a) $X(s) = \dfrac{s-5}{s^2-1}$ for $-1 < \operatorname{Re}(s) < 1$;
(b) $X(s) = \dfrac{2s^2+4s+5}{(s+1)(s+2)}$ for $\operatorname{Re}(s) > -1$; and
(c) $X(s) = \dfrac{3s+1}{s^2+3s+2}$ for $-2 < \operatorname{Re}(s) < -1$.

7.11 Find the causal inverse Laplace transform x of each function X below.
(a) $X(s) = \dfrac{s^2+4s+5}{s^2+2s+1}$; and
(b) $X(s) = \dfrac{-3s^2-6s-2}{(s+1)^2(s+2)}$.

7.12 Find all possible inverse Laplace transforms of

$$H(s) = \frac{7s-1}{s^2-1} = \frac{4}{s+1} + \frac{3}{s-1}.$$

7.13 Consider a LTI system with input x, output y, and system function H, where

$$H(s) = \frac{s+1}{s^2+2s+2}.$$

Find the differential equation that characterizes the behavior of the system.

7.14 A causal LTI system with input x and output y is characterized by the differential equation

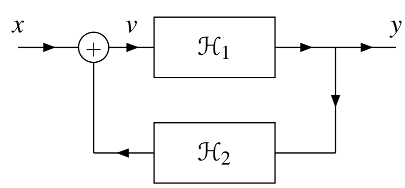
$$y''(t) + 4y'(t) + 3y(t) = 2x'(t) + x(t),$$

where the prime symbol denotes derivative. Find the system function H of the system.

7.15 Consider the LTI system with input x, output y, and system function H, as shown in the figure below. Suppose that the systems $\mathcal{H}_1$ and $\mathcal{H}_2$ are causal and LTI with the respective system functions

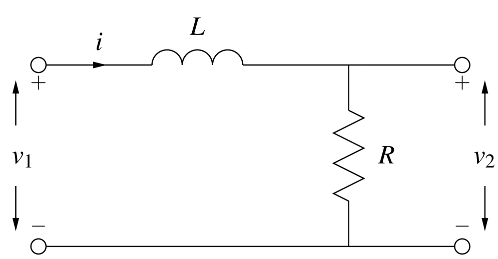
$$H_1(s) = \frac{1}{s-1} \quad \text{and} \quad H_2(s) = A,$$

where A is a real constant.



(a) Find an expression for H in terms of H_1 and H_2.
(b) Determine for what values of A the system is BIBO stable.

7.16 Consider the LTI resistor-inductor (RL) network with input v_1 and output v_2 shown in the figure below.



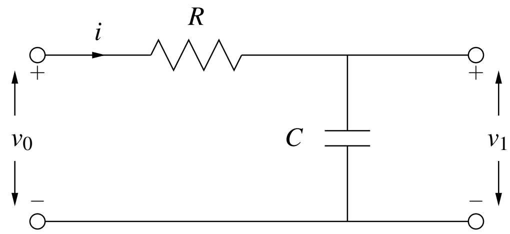
(a) Find the system function H of the system.
(b) Determine whether the system is BIBO stable.
(c) Find the step response s of the system.

7.17 Consider the causal (incrementally-linear TI) system with input x and output y that is characterized by the differential equation

$$y''(t) + 7y'(t) + 12y(t) = x(t),$$

where the prime symbol denotes derivative. If $y(0^-) = -1$, $y'(0^-) = 0$, and $x(t) = u(t)$, find y.

7.18 Consider the resistor-capacitor (RC) network shown in the figure below, where $R = 1000$ and $C = \frac{1}{1000}$.



(a) Find the differential equation that characterizes the relationship between the input v_0 and output v_1.
(b) If $v_1(0^-) = 2$, and $v_0(t) = 2e^{-3t}$, find v_1.

7.19 Consider a LTI system with the system function

$$H(s) = \frac{s^2+7s+12}{s^2+3s+12}.$$

Find all possible inverses of this system. For each inverse, identify its system function and the corresponding ROC. Also, indicate whether the inverse is causal and/or stable. (Note: You do not need to find the impulse responses of these inverse systems.)

7.20 Consider the causal LTI system with input x and output y that is characterized by the differential equation

$$y''(t) - y(t) = x'(t) + ax(t),$$

where a is a real constant and the prime symbol denotes derivative. Determine for what values of a the system is BIBO stable.

7.21 In wireless communication channels, the transmitted signal is propagated simultaneously along multiple paths of varying lengths. Consequently, the signal received from the channel is the sum of numerous delayed and amplified/attenuated versions of the original transmitted signal. In this way, the channel distorts the transmitted signal. This is commonly referred to as the multipath problem. In what follows, we examine a simple instance of this problem.

Consider a LTI communication channel with input x and output y. Suppose that the transmitted signal x propagates along two paths. Along the intended direct path, the channel has a delay of T and gain of one. Along a second (unintended indirect) path, the signal experiences a delay of $T+\tau$ and gain of a. Thus, the received signal y is given by $y(t) = x(t-T) + ax(t-T-\tau)$. Find the system function H of a LTI system that can be connected in series with the output of the communication channel in order to recover the (delayed) signal $x(t-T)$ without any distortion. Determine whether this system is physically realizable.

7.19.2 Exercises With Answer Key

7.22 Using the definition of the Laplace transform, find the Laplace transform X of each of function x below.
(a) $x(t) = \sin(at)u(t)$, where a is a real constant; and
(b) $x(t) = \sinh(at)u(t)$, where $\sinh t = \frac{1}{2}(e^t - e^{-t})$ and a is a real constant.
(c) $x(t) = \cosh(at)u(t)$, where $\cosh t = \frac{1}{2}(e^t + e^{-t})$ and a is a real constant.

Short Answer. (a) $X(s) = \frac{a}{s^2+a^2}$ for $\operatorname{Re}(s) > 0$; (b) $X(s) = \frac{a}{s^2-a^2}$ for $\operatorname{Re}(s) > |a|$; (c) $X(s) = \frac{s}{s^2-a^2}$ for $\operatorname{Re}(s) > |a|$

7.23 Using properties of the Laplace transform and a table of Laplace transform pairs, find the Laplace transform X of each function x given below.
(a) $x(t) = t^2 e^{-t} u(t-1)$;
(b) $x(t) = t^2 u(t-1)$;
(c) $x(t) = (t+1)u(t-1)$; and
(d) $x(t) = u(t-1) - u(t-2)$.

Short Answer. (a) $X(s) = e^{-s-1}\left[\frac{s^2+4s+5}{(s+1)^3}\right]$ for $\operatorname{Re}(s) > -1$; (b) $X(s) = e^{-s}\left(\frac{s^2+2s+2}{s^3}\right)$ for $\operatorname{Re}(s) > 0$; (c) $X(s) = e^{-s}\left(\frac{2s+1}{s^2}\right)$ for $\operatorname{Re}(s) > 0$; (d) $X(s) = \frac{e^{-s}-e^{-2s}}{s}$ for all s.

7.24 Find the inverse Laplace transform x of each function X given below.
(a) $X(s) = e^{-7s}\dfrac{6s+13}{(s+2)(s+3)}$ for $\operatorname{Re}(s) > -2$;
(b) $X(s) = \dfrac{-3s+2}{(s+1)^2}$ for $\operatorname{Re}(s) > -1$;
(c) $X(s) = \dfrac{7s^2+19s+17}{(s+1)^2(s+2)}$ for $\operatorname{Re}(s) > -1$;
(d) $X(s) = \dfrac{s^2+s+2}{(s+1)^2}$ for $\operatorname{Re}(s) > -1$; and
(e) $X(s) = \dfrac{s^2-s+1}{(s+3)^2(s+2)}$ for $\operatorname{Re}(s) > -2$.

Short Answer. (a) $x(t) = e^{-2(t-7)}u(t-7) + 5e^{-3(t-7)}u(t-7)$; (b) $x(t) = 5te^{-t}u(t) - 3e^{-t}u(t)$; (c) $x(t) = 7e^{-2t}u(t) + 5te^{-t}u(t)$; (d) $x(t) = \delta(t) - e^{-t}u(t) + 2te^{-t}u(t)$; (e) $x(t) = 7e^{-2t}u(t) - 6e^{-3t}u(t) - 13te^{-3t}u(t)$

7.25 Find the inverse Laplace transform x of the function $X(s) = \frac{-3}{(s+2)(s-1)}$ if the ROC of X is:
(a) $-2 < \operatorname{Re}(s) < 1$;
(b) $\operatorname{Re}(s) > 1$; and
(c) $\operatorname{Re}(s) < -2$.

Short Answer. (a) $x(t) = e^{-2t}u(t) + e^{t}u(-t)$; (b) $x(t) = \left(e^{-2t} - e^{t}\right)u(t)$; (c) $x(t) = \left(-e^{-2t} + e^{t}\right)u(-t)$

7.26 Let X denote the Laplace transform of x. Suppose that x is absolutely integrable and X is rational with a pole at 1 and the other poles unknown. Determine if x can be each of the following:
(a) left sided;
(b) right sided;
(c) two sided;
(d) finite duration.

Short Answer. (a) yes; (b) no; (c) yes; (d) no

7.27 Use the Laplace transform to compute the convolution $y(t) = x_1 * x_2(t)$, where

$$x_1(t) = e^{-at}u(t), \quad x_2(t) = e^{-bt}u(t),$$

and a and b are strictly positive real constants. [Note that there are two cases to consider, namely $a = b$ and $a \neq b$.]

Short Answer. if $a \neq b$, $y(t) = \frac{1}{b-a}(e^{-at} - e^{-bt})u(t)$; if $a = b$, $y(t) = te^{-at}u(t)$

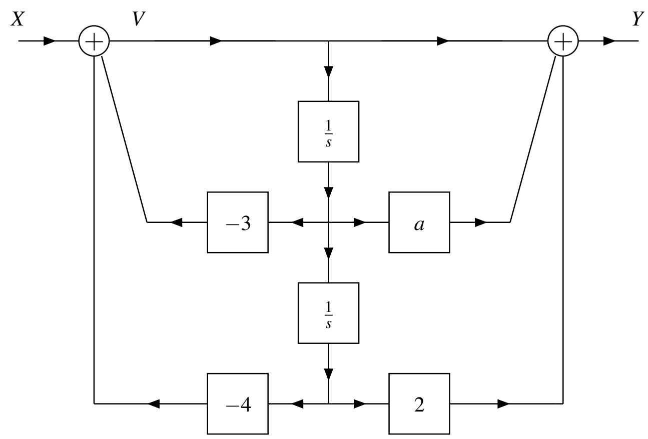
7.28 For the causal LTI system with input x and output y that is characterized by each differential equation given below, find the system function H of the system.
(a) $y''(t) + 3y'(t) + 2y(t) = 5x'(t) + 7x(t)$.

Short Answer. (a) $H(s) = \frac{5s+7}{s^2+3s+2}$ for $\operatorname{Re}(s) > -1$

7.29 For the LTI system with input x and output y and each system function H given below, find the differential equation that characterizes the system.
(a) $H(s) = \frac{7s+3}{15s^2+4s+1}$.

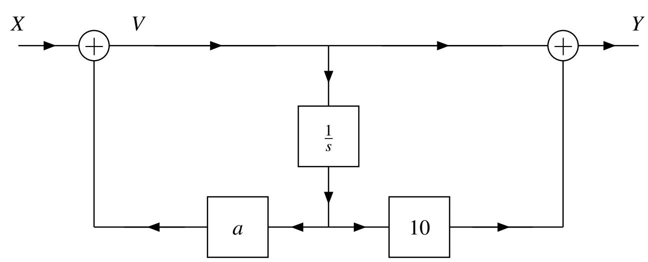
Short Answer. (a) $15y''(t) + 4y'(t) + y(t) = 7x'(t) + 3x(t)$

7.30 Consider the system $\mathcal{H}$ with input Laplace transform X and output Laplace transform Y as shown in the figure. In the figure, each subsystem is LTI and causal and labelled with its system function, and a is a real constant. (a) Find the system function H of the system $\mathcal{H}$. (b) Determine whether the system $\mathcal{H}$ is BIBO stable.



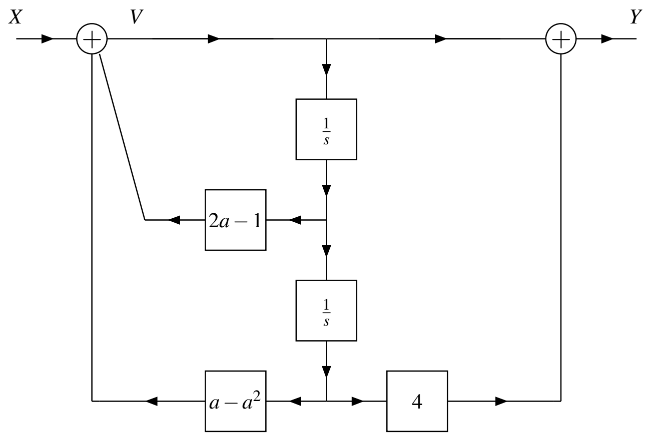
Short Answer. (a) $H(s) = \frac{s^2+as+2}{s^2+3s+4}$ for $\operatorname{Re}(s) > -\frac{3}{2}$; (b) system is BIBO stable.

7.31 Consider the system $\mathcal{H}$ with input Laplace transform X and output Laplace transform Y as shown in the figure. In the figure, each subsystem is LTI and causal and labelled with its system function, and a is a real constant. Determine for what values of a the system $\mathcal{H}$ is BIBO stable.



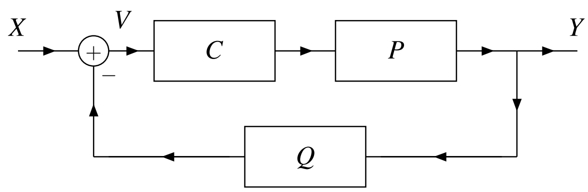
Short Answer. $H(s) = \frac{s+10}{s-a}$ for $\operatorname{Re}(s) > a$; system is BIBO stable if $a < 0$

7.32 Consider the system $\mathcal{H}$ with input Laplace transform X and output Laplace transform Y as shown in the figure. In the figure, each subsystem is LTI and causal and labelled with its system function, and a is a real constant. Determine for what values of a the system $\mathcal{H}$ is BIBO stable.



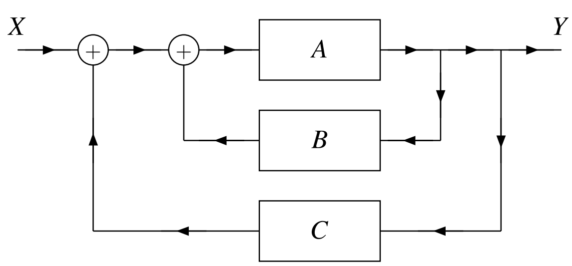
Short Answer. $H(s) = \frac{s^2+4}{s^2+(1-2a)s+a^2-a}$ for $\operatorname{Re}(s) > a$; BIBO stable if $a < 0$

7.33 Consider the system with input Laplace transform X and output Laplace transform Y as shown in the figure. In the figure, each subsystem is LTI and causal and labelled with its system function; and $P(s) = 1/s$, $C(s) = as+3$, $Q(s) = 1$, and a is a real constant. (a) Find the system function H of the system $\mathcal{H}$. (b) Determine for what values of a the system $\mathcal{H}$ is BIBO stable.



Short Answer. (a) $H(s) = \frac{as+3}{s+as+3}$ for $\operatorname{Re}(s) > \frac{-3}{a+1}$; (b) $a > -1$

7.34 Consider the LTI system $\mathcal{H}$ shown in the figure below, where the Laplace transforms of the input and output are denoted as X and Y, respectively. Each of the subsystems in the diagram are LTI and labelled with their system functions. Find the system function H of the system $\mathcal{H}$ in terms of A, B, and C.



Short Answer. $H(s) = \frac{A(s)}{1-A(s)[B(s)+C(s)]}$.

7.35 Consider the design of a thermostat system for a room in a building, where the input x is the desired room temperature and the output y is the actual room temperature. For the purposes of this design, the system will be modelled as a LTI system with system function H. Determine which of the functions H_1 and H_2 given below would be a more desirable choice for H. Explain the rationale for your decision. [Hint: Consider the unit-step response of the system.]
(a) $H_1(s) = \frac{s^2}{s^2+9}$; and
(b) $H_2(s) = 1 + \frac{1}{20}\left[\frac{s(s+10)}{(s+10)^2+1}\right]$.

Short Answer. H_2

7.20 MATLAB Exercises

7.101 Consider a causal LTI system with the system function

$$H(s) = \frac{1}{-2s^7 - s^6 - 3s^5 + 2s^3 + s - 3}.$$

(a) Use MATLAB to find and plot the poles of H.
(b) Determine whether the system is BIBO stable.

7.102 Consider a LTI system with the system function

$$H(s) = \frac{1}{1.0000s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1.0000}.$$

(This system corresponds to a fourth-order Butterworth lowpass filter with a cutoff frequency of 1 rad/s.) Plot the response y of the system to each input x given below. In each case, plot y over the interval $[0, 20]$.
(a) $x = \delta$; and
(b) $x = u$.
(Hint: The `tf`, `impulse`, and `step` functions may be helpful.)

Part II

Discrete-Time Signals and Systems

Chapter 8

Discrete-Time Signals and Systems

8.1 Overview

In this chapter, we will examine discrete-time signals and systems in more detail.

8.2 Transformations of the Independent Variable

An important concept in the study of signals and systems is the transformation of a signal. Here, we introduce several elementary signal transformations. Each of these transformations involves a simple modification of the independent variable.

8.2.1 Time Shifting (Translation)

The first type of signal transformation that we shall consider is known as time shifting. **Time shifting** (also known as **translation**) maps a sequence x to the sequence y given by

$$y(n) = x(n-b), \tag{8.1}$$

where b is an integer constant. In other words, the sequence y is formed by replacing n by $n-b$ in the expression for the $x(n)$. Geometrically, the transformation (8.1) shifts the sequence x along the time axis to yield y. If $b > 0$, y is shifted to the right relative to x (i.e., delayed in time). If $b < 0$, y is shifted to the left relative to x (i.e., advanced in time).

The effects of time shifting are illustrated in Figure 8.1. By applying a time-shifting transformation to the sequence x shown in Figure 8.1(a), each of the sequences in Figures 8.1(b) and (c) can be obtained.

8.2.2 Time Reversal (Reflection)

The next type of signal transformation that we consider is called **time reversal**. **Time reversal** (also known as **reflection**) maps a sequence x to the sequence y given by

$$y(n) = x(-n). \tag{8.2}$$

In other words, the sequence y is formed by replacing n with $-n$ in the expression for $x(n)$. Geometrically, the transformation 8.2 reflects the sequence x about the origin to yield y.

To illustrate the effects of time reversal, an example is provided in Figure 8.2. Applying a time-reversal transformation to the sequence x in Figure 8.2(a) yields the sequence in Figure 8.2(b).

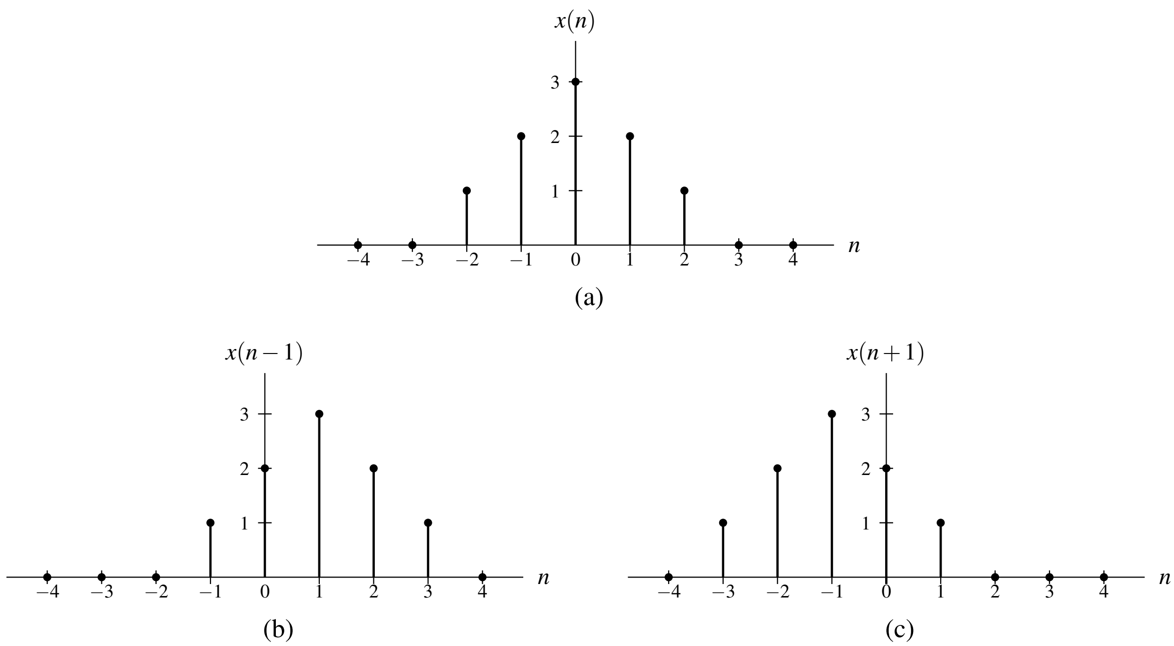


Figure 8.1: Example of time shifting.

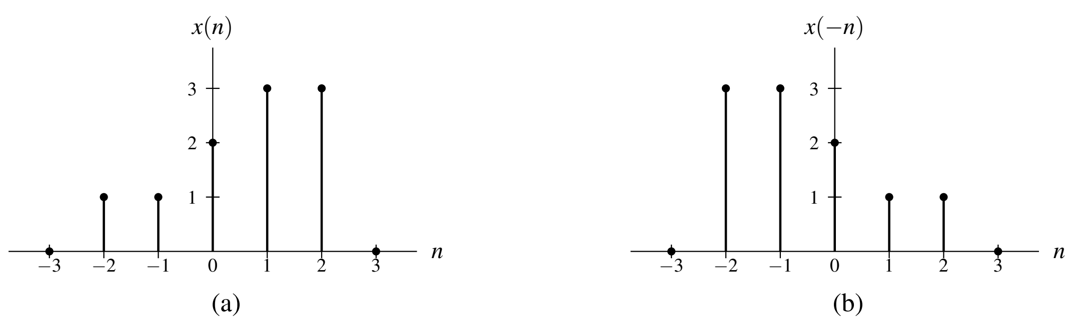


Figure 8.2: Example of time reversal.

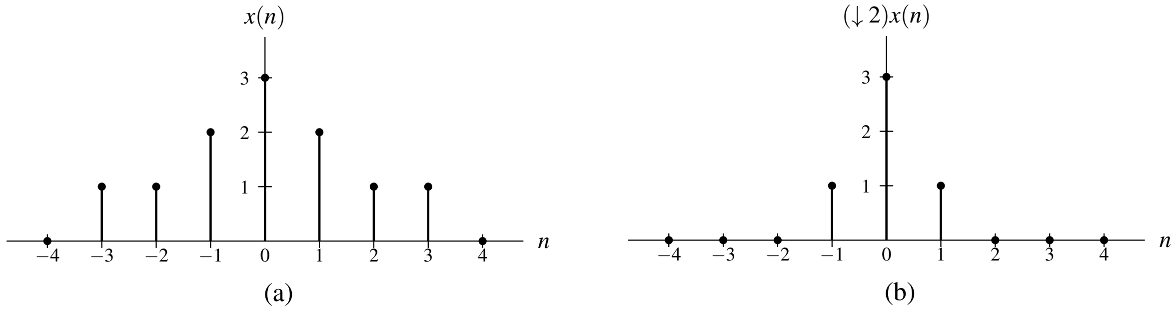


Figure 8.3: Downsampling example. (a) Original sequence x. (b) Result obtained by 2-fold downsampling of x.

8.2.3 Downsampling

The next transformation to be considered is called downsampling. **Downsampling** maps a sequence x to the sequence y given by

$$y(n) = (\downarrow a)x(n) = x(an), \tag{8.3}$$

where a is a (strictly) positive integer. The constant a is referred to as the **downsampling factor**. Downsampling with a downsampling factor of a is referred to as ***a*-fold downsampling**. In simple terms, a-fold downsampling keeps every ath sample from the original sequence and discards the others. Thus, downsampling reduces the sampling rate/density by a factor of a.

To illustrate the effects of downsampling, an example is provided in Figure 8.3. Applying 2-fold downsampling to the sequence x shown in Figure 8.3(a) yields the sequence shown in Figure 8.3(b).

8.2.4 Upsampling (Time Expansion)

The next transformation to be considered is called upsampling. **Upsampling** (also known as **time expansion**) maps a sequence x to the sequence y given by

$$y(n) = (\uparrow a)x(n) = \begin{cases} x(n/a) & n/a \text{ is an integer} \\ 0 & \text{otherwise,} \end{cases}$$

where a is a (strictly) positive integer. The constant a is referred to as the **upsampling factor**. Upsampling with a upsampling factor of a is referred to as ***a*-fold upsampling**. In simple terms, a-fold upsampling results in the insertion of $a-1$ zeros between the samples of the original input sequence. Thus, upsampling increases the sampling rate/density by a factor of a.

To illustrate the effects of upsampling, an example is provided in Figure 8.4. Applying 2-fold upsampling to the sequence x in Figure 8.4(a) yields the sequence in Figure 8.4(b).

8.2.5 Combined Independent-Variable Transformations

Some independent-variable transformations commute, while other do not. The issue of commutativity is important, for example, when trying to simplify or manipulate expressions involving combined transformations. A time reversal operation commutes with a downsampling or upsampling operation. A time shift operation (with a nonzero shift) does not commute with a time-reversal, downsampling, or upsampling operation. A downsampling operation does not commute with an upsampling operation unless the upsampling and downsampling factors are coprime (i.e., have no common factors).

Consider a transformation that maps the input sequence x to the output sequence y given by

$$y(n) = x(an - b), \tag{8.4}$$

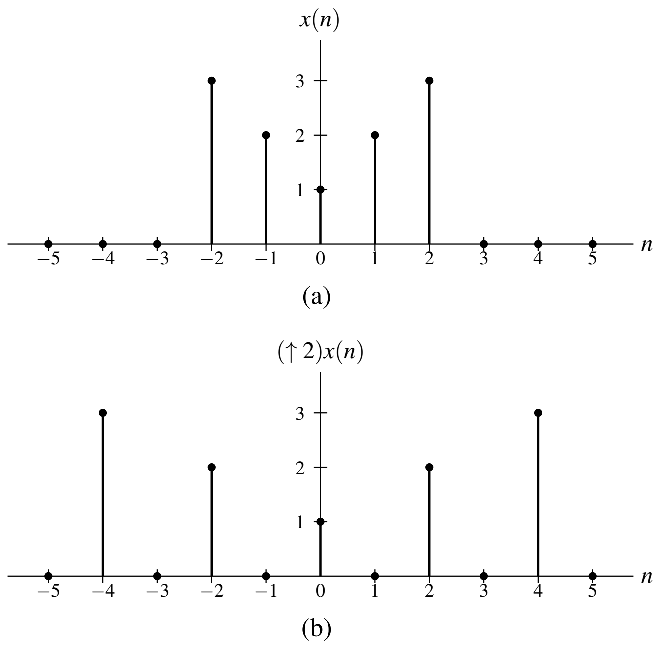


Figure 8.4: Upsampling example. (a) Original sequence x. (b) Result obtained by 2-fold upsampling of x.

where a and b are integers and $a \neq 0$. Such a transformation is a combination of time shifting, downsampling, and time-reversal operations.

The transformation (8.4) is equivalent to:

1. first, time shifting x by b;
2. then, downsampling the result by $|a|$ and, if $a < 0$, time reversing as well.

If $\frac{b}{a}$ is an integer, the transformation (8.4) is also equivalent to:

1. first, downsampling x by $|a|$ and, if $a < 0$, time reversing;
2. then, time shifting the result by $\frac{b}{a}$.

Observe that the shifting amount differs in these two interpretations (i.e., b versus $\frac{b}{a}$). This is due to the fact that a time shift operation does not commute with either a downsampling or time-reversal operation. The proof that the above two interpretations are valid is left as an exercise for the reader in Exercise 8.2.

8.2.6 Two Perspectives on Independent-Variable Transformations

A transformation of the independent variable can be viewed in terms of:

1. the effect that the transformation has on the *sequence*; or
2. the effect that the transformation has on the *horizontal axis*.

This distinction is important because such a transformation has *opposite* effects on the sequence and horizontal axis. For example, the (time-shifting) transformation that replaces n by $n-b$ (where b is an integer) in the expression for $x(n)$ can be viewed as a transformation that:

1. shifts the sequence x *right* by b units; or
2. shifts the horizontal axis *left* by b units.

In our treatment of independent-variable transformations, we are only interested in the effect that a transformation has on the *sequence*. If one is not careful to consider that we are interested in the sequence perspective (as opposed to the axis perspective), many aspects of independent-variable transformations will not make sense.

8.3 Properties of Sequences

Sequences can possess a number of interesting properties. In what follows, we consider the properties of symmetry and periodicity (introduced earlier) in more detail. Also, we present several other sequence properties. The properties considered are frequently useful in the analysis of signals and systems.

8.3.1 Remarks on Symmetry

At this point, we make some additional comments about even and odd sequences (introduced earlier). Since sequences are often summed or multiplied, one might wonder what happens to the even/odd symmetry properties of sequences under these operations. In what follows, we introduce a few results in this regard.

Sums involving even and odd sequences have the following properties:

- The sum of two even sequences is even.
- The sum of two odd sequences is odd.
- The sum of an even sequence and odd sequence is neither even nor odd, provided that the sequences are not identically zero.

Products involving even and odd sequences have the following properties:

- The product of two even sequences is even.
- The product of two odd sequences is even.
- The product of an even sequences and an odd sequences is odd.

(The proofs of the above properties involving sums and products of even and odd sequences is left as an exercise for the reader in Exercise 8.9.)

As it turns out, any arbitrary sequence can be expressed as the sum of an even and odd sequence, as elaborated upon by the theorem below.

Theorem 8.1 (Decomposition of sequence into even and odd parts)**.** *Any arbitrary sequence x can be uniquely represented as the sum of the form*

$$x(n) = x_{\mathrm{e}}(n) + x_{\mathrm{o}}(n), \tag{8.5}$$

where x_{e} and x_{o} are even and odd, respectively, and given by

$$x_{\mathrm{e}}(n) = \tfrac{1}{2}\left[x(n) + x(-n)\right] \quad \textit{and} \tag{8.6}$$

$$x_{\mathrm{o}}(n) = \tfrac{1}{2}\left[x(n) - x(-n)\right]. \tag{8.7}$$

As a matter of terminology, x_{e} is called the ***even part*** *of x and is denoted* $\mathrm{Even}\{x\}$, *and x_{o} is called the* ***odd part*** *of x and is denoted* $\mathrm{Odd}\{x\}$.

Proof. From (8.6) and (8.7), we can easily confirm that $x_{\mathrm{e}} + x_{\mathrm{o}} = x$ as follows:

$$\begin{aligned} x_{\mathrm{e}}(n) + x_{\mathrm{o}}(n) &= \tfrac{1}{2}(x(n) + x(-n)) + \tfrac{1}{2}(x(n) - x(-n)) \\ &= \tfrac{1}{2}x(n) + \tfrac{1}{2}x(-n) + \tfrac{1}{2}x(n) - \tfrac{1}{2}x(-n) \\ &= x(n). \end{aligned}$$

Furthermore, we can easily verify that x_e is even and x_o is odd. From the definition of x_e in (8.6), we have

$$\begin{aligned} x_e(-n) &= \tfrac{1}{2}(x(-n)+x(-(-n))) \\ &= \tfrac{1}{2}(x(n)+x(-n)) \\ &= x_e(n). \end{aligned}$$

Thus, x_e is even. From the definition of x_o in (8.7), we have

$$\begin{aligned} x_o(-n) &= \tfrac{1}{2}(x(-n)-x(n)) \\ &= \tfrac{1}{2}(-x(n)+x(-n)) \\ &= -x_o(n). \end{aligned}$$

Thus, x_o is odd.

Lastly, we show that the decomposition of x into the sum of an even sequence and odd sequence is unique. Suppose that x can be written as the sum of an even sequence and odd sequence in two ways as

$$x(n) = f_e(n) + f_o(n) \quad \text{and} \tag{8.8a}$$
$$x(n) = g_e(n) + g_o(n), \tag{8.8b}$$

where f_e and g_e are even and f_o and g_o are odd. Equating these two expressions for x, we have

$$f_e(n) + f_o(n) = g_e(n) + g_o(n).$$

Rearranging this equation, we have

$$f_e(n) - g_e(n) = g_o(n) - f_o(n).$$

Now, we consider the preceding equation more carefully. Since the sum of even sequences is even and the sum of odd sequences is odd, we have that the left- and right-hand sides of the preceding equation correspond to even and odd sequences, respectively. Thus, we have that the even sequence $f_e(n) - g_e(n)$ is equal to the odd sequence $g_o(n) - f_o(n)$. The only sequence, however, that is both even and odd is the zero sequence. (A proof of this fact is left as an exercise for the reader in Exercise 8.12.) Therefore, we have that

$$f_e(n) - g_e(n) = g_o(n) - f_o(n) = 0.$$

In other words, we have that

$$f_e(n) = g_e(n) \quad \text{and} \quad f_o(n) = g_o(n).$$

This implies that the two decompositions of x given by (8.8a) and (8.8b) must be the same decomposition (i.e., they cannot be distinct). Thus, the decomposition of x into the sum of an even sequence and odd sequence is unique. ∎

8.3.2 Remarks on Periodicity

Since we often add sequences, it is helpful to know if the sum of periodic sequences is also periodic. We will consider this issue next, but before doing so we first must introduce the notion of a least common multiple and greatest common divisor.

The **least common multiple (LCM)** of two nonzero integers a and b, denoted $\operatorname{lcm}(a,b)$, is the smallest positive integer that is divisible by both a and b. From this definition, it immediately follows that, if a and b are coprime, then $\operatorname{lcm}(a,b) = |ab|$. The quantity $\operatorname{lcm}(a,b)$ can be easily determined from a prime factorization of the integers a and b by taking the product of the highest power for each prime factor appearing in these factorizations.

The **greatest common divisor (GCD)** of two integers a and b, denoted $\gcd(a,b)$, is the largest positive integer that divides both a and b, where at least one of a and b is nonzero. In the case that a and b are both zero, we define $\gcd(0,0) = 0$. Since $0/a = 0 \in \mathbb{Z}$ (i.e., a divides 0) and $a/a = 1 \in \mathbb{Z}$ (i.e., a divides a), $\gcd(a,0) = \gcd(0,a) = |a|$. Since, the signs of integers do not affect divisibility, $\gcd(a,b) = \gcd(-a,b) = \gcd(a,-b) = \gcd(-a,-b)$. From the definition of the GCD, it immediately follows that, if a and b are coprime (i.e., have no common factors), then $\gcd(a,b) = 1$. The quantity $\gcd(a,b)$ can be easily determined from a prime factorization of the integers a and b by taking the product of the lowest power for each prime factor appearing in these factorizations.

Example 8.1 (Least common multiple)**.** Find the LCM of each pair of integers given below.
(a) 20 and 6;
(b) 54 and 24; and
(c) 24 and 90.

Solution. (a) First, we write the prime factorizations of 20 and 6, which yields

$$20 = 2^2 \cdot 5^1 \quad \text{and} \quad 6 = 2^1 \cdot 3^1.$$

To obtain the LCM, we take the highest power of each prime factor in these two factorizations. Thus, we have 2^2 from the factorization of 20, 3^1 from the factorization of 6, and 5^1 from the factorization of 20. Therefore, we conclude

$$\begin{aligned} \operatorname{lcm}(20,6) &= 2^2 \cdot 3^1 \cdot 5^1 \\ &= 60. \end{aligned}$$

(b) Using a similar process as above, we have

$$\begin{aligned} \operatorname{lcm}(54,24) &= \operatorname{lcm}(2^1 \cdot 3^3, 2^3 \cdot 3^1) \\ &= 2^3 \cdot 3^3 \\ &= 216. \end{aligned}$$

(c) Again, using a similar process as above, we have

$$\begin{aligned} \operatorname{lcm}(24,90) &= \operatorname{lcm}(2^3 \cdot 3^1, 2^1 \cdot 3^2 \cdot 5^1) \\ &= 2^3 \cdot 3^2 \cdot 5^1 \\ &= 360. \end{aligned}$$

■

Example 8.2 (Greatest common divisor)**.** Find the GCD of each pair of integers given below.
(a) 20 and 6;
(b) 54 and 24; and
(c) 24 and 90.

Solution. (a) First, we write the prime factorizations of 20 and 6, which yields

$$20 = 2^2 \cdot 5^1 \quad \text{and} \quad 6 = 2^1 \cdot 3^1.$$

To obtain the GCD, we take the lowest power of each prime factor in these two factorizations. Thus, we have 2^1 from the factorization of 6, 3^0 from the factorization of 20, and 5^0 from the factorization of 6. Therefore, we conclude

$$\begin{aligned} \gcd(20,6) &= 2^1 \cdot 3^0 \cdot 5^0 \\ &= 2. \end{aligned}$$

(b) Using a similar process as above, we have

$$\begin{aligned} \gcd(54,24) &= \gcd(2^1 \cdot 3^3, 2^3 \cdot 3^1) \\ &= 2^1 \cdot 3^1 \\ &= 6. \end{aligned}$$

(c) Again, using a similar process as above, we have

$$\begin{aligned} \gcd(24,90) &= \gcd(2^3 \cdot 3^1, 2^1 \cdot 3^2 \cdot 5^1) \\ &= 2^1 \cdot 3^1 \cdot 5^0 \\ &= 6. \end{aligned}$$

■

Having introduced the LCM, we know consider whether the sum of two periodic sequences is periodic. In this regard, the theorem below is enlightening.

Theorem 8.2 (Sum of periodic sequences)**.** *For any two periodic sequences x_1 and x_2 with periods N_1 and N_2, respectively, the sequence $x = x_1 + x_2$ is periodic with period $N = \operatorname{lcm}(N_1, N_2)$.*

Proof. Since N is an integer multiple of both N_1 and N_2, we can write $N = k_1 N_1$ and $N = k_2 N_2$ for some positive integers k_1 and k_2. So, we can write

$$\begin{aligned} x(n+N) &= x_1(n+N) + x_2(n+N) \\ &= x_1(n+k_1N_1) + x_2(n+k_2N_2) \\ &= x_1(n) + x_2(n) \\ &= x(n). \end{aligned}$$

Thus, x is periodic with period N. ■

Unlike in the case of the sum of periodic functions, the sum of periodic sequences is always periodic.

Example 8.3. The sequences $x_1(n) = \cos\left(\frac{\pi}{6}n\right)$ and $x_2(n) = \sin\left(\frac{2\pi}{45}n\right)$ have fundamental periods $N_1 = 12$ and $N_2 = 45$, respectively. Find the fundamental period N of the sequence $y = x_1 + x_2$.

Solution. We have

$$\begin{aligned} N &= \operatorname{lcm}(N_1, N_2) \\ &= \operatorname{lcm}(12, 45) \\ &= \operatorname{lcm}(2^2 \cdot 3, 3^2 \cdot 5) \\ &= 2^2 \cdot 3^2 \cdot 5 \\ &= 180. \end{aligned}$$

The sequences x_1, x_2, and $x_1 + x_2$ are plotted in Figure 8.5. ■

Sometimes a sequence may be both periodic and possess even or odd symmetry. In this situation, the result of the following theorem is sometimes useful.

Theorem 8.3. *Let x be an arbitrary N-periodic sequence x. Then, the following assertions hold:*

1. *if x is even, then $x(n) = x(N-n)$ for all $n \in \mathbb{Z}$;*
2. *if x is odd, then $x(n) = -x(N-n)$ for all $n \in \mathbb{Z}$; and*
3. *if x is odd, then $x(0) = 0$ for both even and odd N, and $x\left(\frac{N}{2}\right) = 0$ for even N.*

Proof. A proof of this theorem is left as an exercise for the reader in Exercise 8.5. ■

8.3.3 Support of Sequences

We can classify sequences based on the interval over which their value is nonzero. This is sometimes referred to as the support of a sequence. In what follows, we introduce some terminology related to the support of sequences.

A sequence x is said to be **left sided** if, for some finite constant n_0, the following condition holds:

$$x(n) = 0 \quad \text{for all } n > n_0.$$

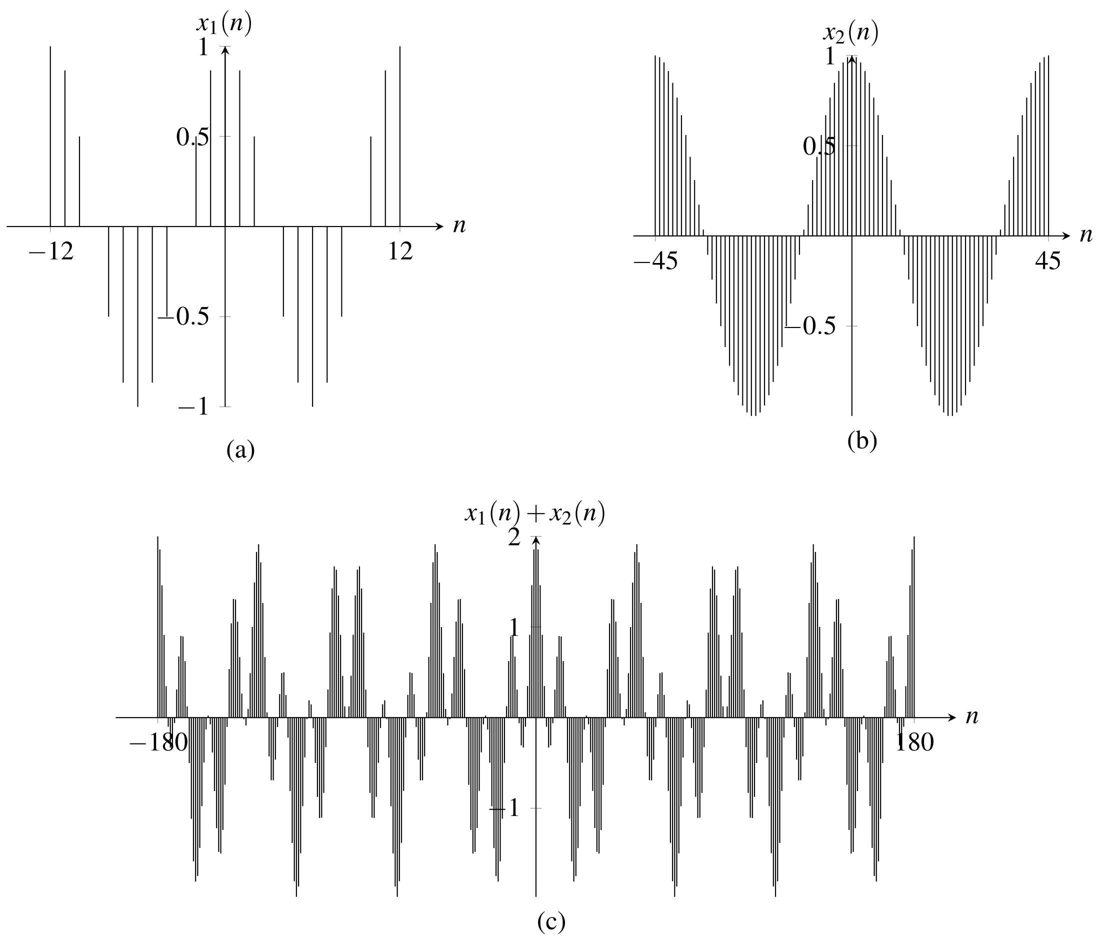


Figure 8.5: Sequences for Example 8.3.

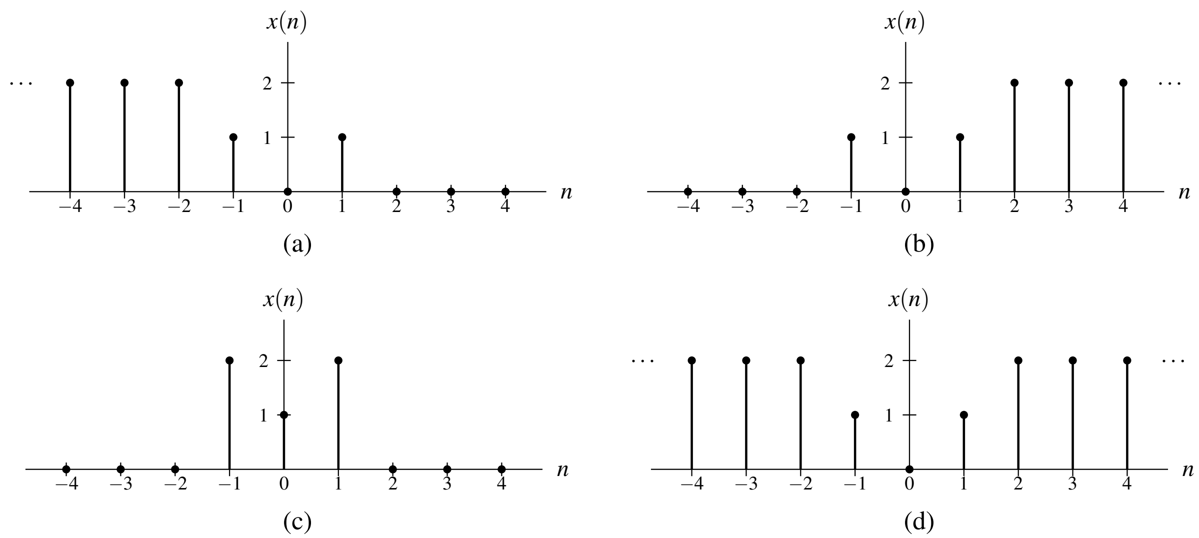


Figure 8.6: Examples of sequences with various sidedness properties. A sequence that is (a) left sided but not right sided, (b) right sided but not left sided, (c) finite duration, and (d) two sided.

In other words, the sequence is only potentially nonzero to the left of some point. A sequence x is said to be **right sided** if, for some finite constant n_0, the following condition holds:

$$x(n) = 0 \quad \text{for all } n < n_0.$$

In other words, the sequence is only potentially nonzero to the right of some point. A sequence that is both left sided and right sided is said to be **time limited** or **finite duration**. A sequence that is neither left sided nor right sided is said to be **two sided**. Note that every sequence is exactly one of: left sided but not right sided, right sided but not left sided, finite duration, or two sided. Examples of left-sided (but not right-sided), right-sided (but not left-sided), finite-duration, and two-sided sequences are shown in Figure 8.6.

A sequence x is said to be **causal** if

$$x(n) = 0 \quad \text{for all } n < 0.$$

A causal sequence is a special case of a right-sided sequence. Similarly, a sequence x is said to be **anticausal** if

$$x(n) = 0 \quad \text{for all } n > 0.$$

An anticausal sequence is a special case of a left-sided sequence. Note that the qualifiers "causal" and "anticausal", when applied to sequences, have nothing to do with cause and effect. In this sense, this choice of terminology is arguably not the best.

8.3.4 Bounded Sequences

A sequence x is said to be **bounded** if there exists some (finite) nonnegative real constant A such that

$$|x(n)| \leq A \quad \text{for all } n$$

(i.e., $x(n)$ is finite for all n). For example, the sequence $x(n) = (-1)^n$ bounded, since

$$|(-1)^n| \leq 1 \text{ for all } n.$$

In contrast, the sequence $x(n) = n$ is not bounded, since

$$\lim_{n\to\infty} |n| = \infty.$$

8.3.5 Signal Energy

The **energy** E contained in the sequence x is given by

$$E = \sum_{k=-\infty}^{\infty} |x(n)|^2 .$$

As a matter of terminology, a signal x with finite energy is said to be an **energy signal**.

8.3.6 Examples

Example 8.4. Let x be a sequence with the following properties:
- $v(n) = x(n-3)$ is causal; and
- x is odd.

Determine for what values of n the quantity $x(n)$ must be zero.

Solution. Since v is causal, we have

$$\begin{aligned} & v(n) = 0 \text{ for } n < 0 \\ \Rightarrow \quad & x(n-3) = 0 \text{ for } n < 0 \\ \Rightarrow \quad & x(n) = 0 \text{ for } n+3 < 0 \\ \Rightarrow \quad & x(n) = 0 \text{ for } n < -3. \end{aligned}$$

Since x is odd, $x(0) = 0$ and $x(n) = 0$ for $n > 3$. Therefore, $x(n) = 0$ for all $n \notin \{-3,-2,-1,1,2,3\}$. ■

Example 8.5. A sequence x has the following properties:
- $x(n) = n+2$ for $-1 \leq n \leq 1$;
- $v_1(n) = x(n-1)$ is causal; and
- $v_2(n) = x(n+1)$ is even.

Find $x(n)$ for all n.

Solution. Since $v_1(n) = x(n-1)$ is causal, we have

$$\begin{aligned} & x(n-1) = 0 \ \text{ for } n < 0 \\ \Rightarrow \quad & x([n+1]-1) = 0 \ \text{ for } (n+1) < 0 \\ \Rightarrow \quad & x(n) = 0 \ \text{ for } n < -1. \end{aligned}$$

From this and the fact that $x(n) = n+2$ for $-1 \leq n \leq 1$, we have

$$x(n) = \begin{cases} n+2 & -1 \leq n \leq 1 \\ 0 & n \leq -2. \end{cases} \tag{8.9}$$

So, we only need to determine $x(n)$ for $n \geq 2$. Since $v_2(n) = x(n+1)$ is even, we have

$$\begin{aligned} & v_2(n) = v_2(-n) \\ \Rightarrow \quad & x(n+1) = x(-n+1) \\ \Rightarrow \quad & x([n-1]+1) = x(-[n-1]+1) \\ \Rightarrow \quad & x(n) = x(-n+2) \\ \Rightarrow \quad & x(n) = x(2-n). \end{aligned}$$

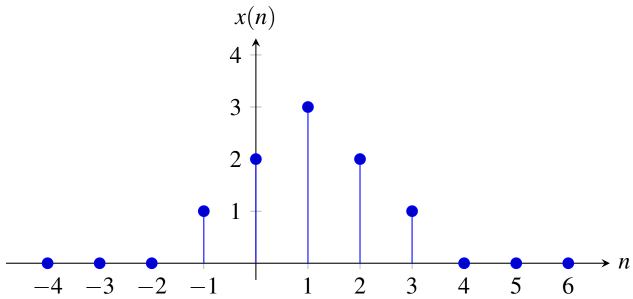


Figure 8.7: The sequence x from Example 8.5.

Using this with (8.9), we obtain

$$\begin{aligned} x(n) &= x(2-n) \\ &= \begin{cases} (2-n)+2 & -1 \le 2-n \le 1 \\ 0 & 2-n \le -2 \end{cases} \\ &= \begin{cases} 4-n & -3 \le -n \le -1 \\ 0 & -n \le -4 \end{cases} \\ &= \begin{cases} 4-n & 1 \le n \le 3 \\ 0 & n \ge 4. \end{cases} \end{aligned}$$

Therefore, we conclude

$$x(n) = \begin{cases} 0 & n \le -2 \\ 2+n & n \in \{-1,0\} \\ 4-n & n \in \{1,2,3\} \\ 0 & n \ge 4. \end{cases}$$

A plot of x is given in Figure 8.7. ■

8.4 Elementary Sequences

A number of elementary sequences are particularly useful in the study of signals and systems. In what follows, we introduce some of the more beneficial ones for our purposes.

8.4.1 Real Sinusoidal Sequences

One important class of sequences is the real sinusoids. A **real sinusoidal sequence** is a sequence of the form

$$x(n) = A\cos(\Omega n + \theta), \tag{8.10}$$

where A, Ω, and θ are real constants. For all integer k,

$$x_k(n) = A\cos[(\Omega + 2\pi k)n + \theta] \tag{8.11}$$

is the same sequence (due to the fact that the cos function is 2π-periodic).

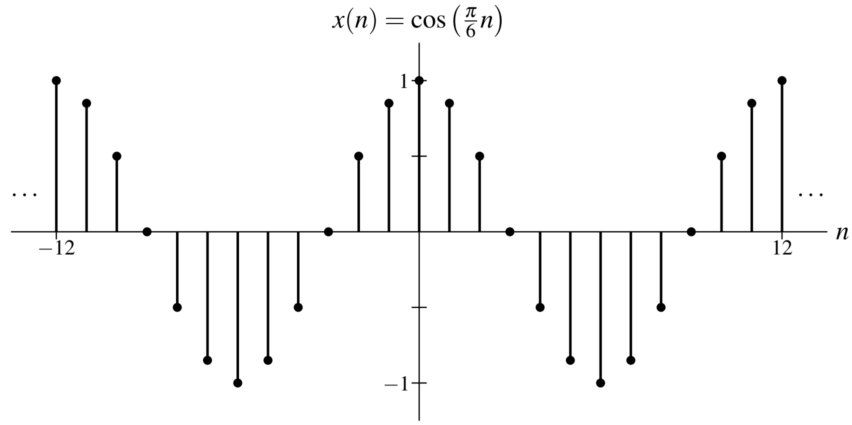


Figure 8.8: Example of a real-sinusoidal sequence.

The real sinusoidal sequence x in (8.10) is periodic if and only if $\frac{\Omega}{2\pi}$ is a rational number, in which case the fundamental period is the smallest integer of the form $\frac{2\pi k}{|\Omega|}$ where k is a (strictly) positive integer. In particular, if $\Omega = \frac{2\pi\ell}{m}$ where ℓ and m are integers, x can be shown to have the fundamental period

$$N = \frac{m}{\gcd(\ell, m)}.$$

In the case that ℓ and m are coprime (i.e., have no common factors), $N = \frac{m}{\gcd(\ell,m)} = \frac{m}{1} = m$. (A proof of this is left as an exercise for the reader in Exercise 8.6.)

An example of a periodic real sinusoid with fundamental period 12 is shown plotted in Figure 8.8.

In the case of periodic real sinusoidal sequences, the frequency of the sequence is often treated as a signed quantity. In other words, we often employ the notion of signed frequency, as discussed in Section 2.10.2. Assuming that the real sinusoidal sequence x given by (8.10) is periodic, x has a signed frequency of Ω. In most cases, we simply refer to the signed frequency as the "frequency". Normally, this does not cause any confusion, as it is usually clear from the context whether the frequency is being treated as a signed or unsigned quantity.

Unlike their continuous-time counterparts, real sinusoidal sequences have an upper bound on the rate at which they can oscillate. This is evident from (8.11) above, since every sinusoid with a frequency Ω outside the range $[0, 2\pi)$ is identical to a sinusoid whose frequency is contained in this range. Moreover, for frequencies $\Omega \in [0, 2\pi)$, the highest rate of oscillation would correspond to π, since π is the point that is equally far from both 0 and 2π (which correspond to the lowest rate of oscillation). As the frequency of a real sinusoidal sequence is increased from 0, its rate of oscillation will increase until a frequency of π is reached and then start to decrease until a frequency of 2π is hit. This behavior then repeats periodically for each subsequent interval of length 2π. Thus, more generally, a real sinusoidal sequence achieves the highest rate of oscillation when its frequency is an odd integer multiple of π (i.e., $(2k+1)\pi$ for integer k) and achieves the lowest rate of oscillation when its frequency is an even integer multiple of π (i.e., $(2k)\pi$ for integer k). Figure 8.9 shows the effect of increasing the frequency of a real sinusoidal sequence from 0 to 2π. The sequence with frequency 0 has no oscillation (i.e., is constant) while the sequence with frequency π oscillates most rapidly.

Example 8.6 (Fundamental period of real sinusoid)**.** Determine if each sequence x given below is periodic, and if it is, find its fundamental period.
(a) $x(n) = \cos(42n)$; and
(b) $x(n) = \sin\left(\frac{4\pi}{11}n\right)$.

Solution. (a) Since $\frac{2\pi}{42} = \frac{\pi}{21}$ is not rational, x is not periodic.

(b) Since

$$(2\pi)/\left(\tfrac{4\pi}{11}\right) = (2\pi)\left(\tfrac{11}{4\pi}\right) = \tfrac{11}{2}$$

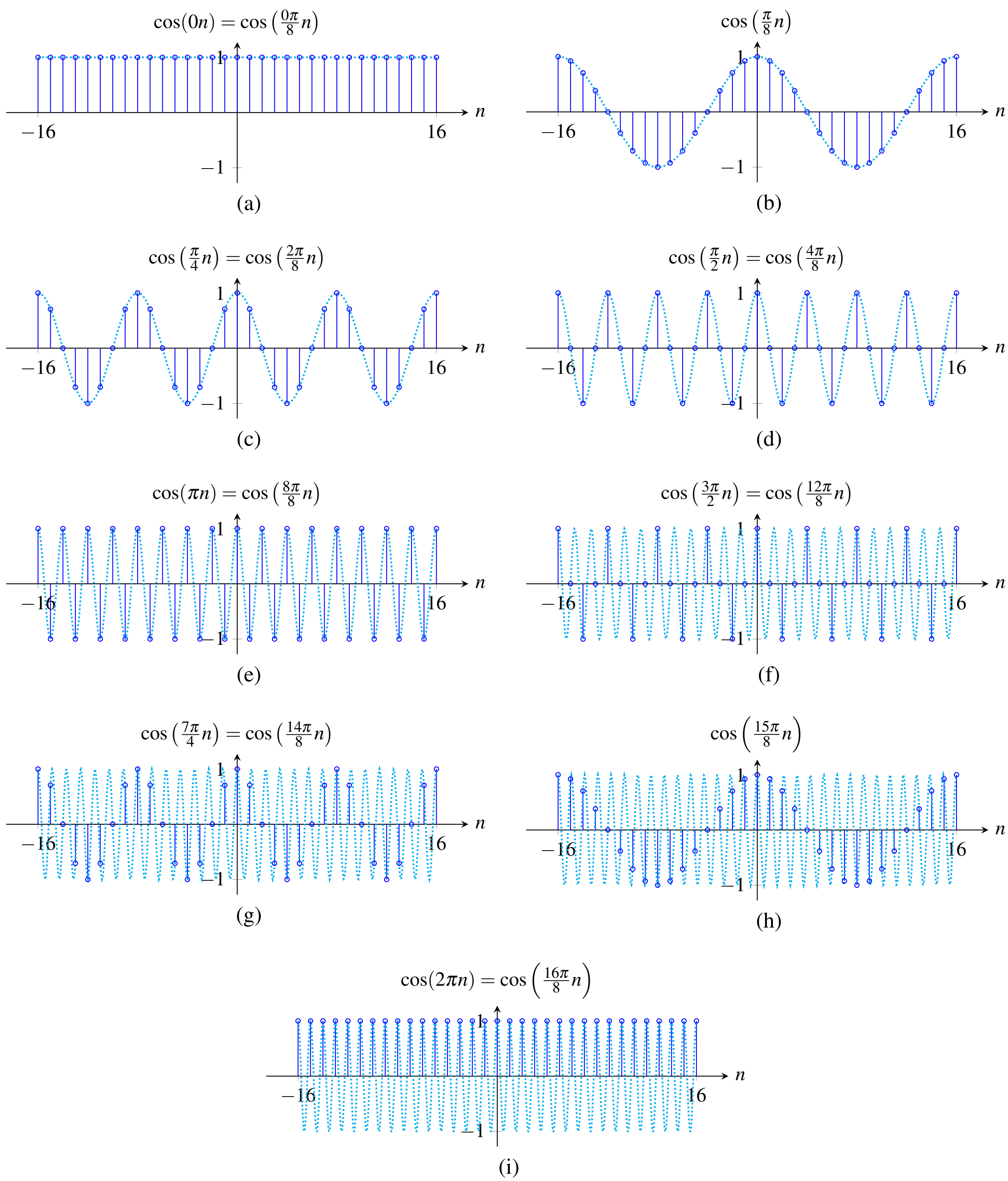


Figure 8.9: The effect of increasing the frequency of a real sinusoidal sequence. A plot of $x(n) = \cos(\Omega n)$ for Ω having each of the values (a) $\frac{0\pi}{8} = 0$, (b) $\frac{1\pi}{8} = \frac{\pi}{8}$, (c) $\frac{2\pi}{8} = \frac{\pi}{4}$, (d) $\frac{4\pi}{8} = \frac{\pi}{2}$, (e) $\frac{8\pi}{8} = \pi$, (f) $\frac{12\pi}{8} = \frac{3\pi}{2}$, (g) $\frac{14\pi}{8} = \frac{7\pi}{4}$, (h) $\frac{15\pi}{8}$, and (i) $\frac{16\pi}{8} = 2\pi$.

is rational, x is periodic. The fundamental period N is the smallest integer of the form $\frac{11}{2}k$, where k is a strictly positive integer. Thus, $N = 11$ (corresponding to $k = 2$). ■

8.4.2 Complex Exponential Sequences

Another important class of sequences is the complex exponentials. A **complex exponential sequence** is a sequence of the form

$$x(n) = ca^n, \tag{8.12}$$

where c and a are complex constants. Such a sequence can also be equivalently expressed in the form

$$x(n) = ce^{bn},$$

where b is a complex constant chosen as $b = \ln a$. (This form is more similar to that presented for complex exponential functions). A complex exponential can exhibit one of a number of distinct modes of behavior, depending on the values of the parameters c and a. For example, as special cases, complex exponentials include real exponentials and complex sinusoids. In what follows, we examine some special cases of complex exponentials, in addition to the general case.

8.4.2.1 Real Exponential Sequences

The first special case of the complex exponentials to be considered is the real exponentials. In the case of a **real exponential sequence**, we restrict c and a in (8.12) to be real. A real exponential can exhibit one of several distinct modes of behavior, depending on the magnitude and sign of a, as illustrated in Figure 8.10. If $|a| > 1$, the magnitude of $x(n)$ increases exponentially as n increases (i.e., a growing exponential). If $|a| < 1$, the magnitude of $x(n)$ decreases exponentially as n increases (i.e., a decaying exponential). If $|a| = 1$, the magnitude of $x(n)$ is a constant, independent of n. If $a > 0$, $x(n)$ has the same sign for all n. If $a < 0$, $x(n)$ alternates in sign as n increases/decreases.

8.4.2.2 Complex Sinusoidal Sequences

The second special case of the complex exponentials that we shall consider is the complex sinusoids. In the case of a **complex sinusoidal sequence**, the parameters in (8.12) are such that c and a are complex and $|a| = 1$ (i.e., a is of the form $e^{j\Omega}$ where Ω is real). That is, a **complex sinusoidal sequence** is a sequence of the form

$$x(n) = ce^{j\Omega n}, \tag{8.13}$$

where c is complex and Ω is real. Using Euler's relation, we can rewrite $x(n)$ as

$$x(n) = \underbrace{|c|\cos(\Omega n + \arg c)}_{\mathrm{Re}\{x(n)\}} + j\underbrace{|c|\sin(\Omega n + \arg c)}_{\mathrm{Im}\{x(n)\}}.$$

Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are real sinusoids. To illustrate the form of a complex sinusoid, the real and imaginary parts of a particular complex sinusoid are plotted in Figure 8.11.

The complex sinusoidal sequence x in (8.13) is periodic if and only if $\frac{\Omega}{2\pi}$ is a rational number, in which case the fundamental period is the smallest integer of the form $\frac{2\pi k}{|\Omega|}$, where k is a (strictly) positive integer. In particular, if $\Omega = \frac{2\pi\ell}{m}$ where ℓ and m are integers, x can be shown to have the fundamental period

$$N = \frac{m}{\gcd(\ell, m)}.$$

In the case that ℓ and m are coprime (i.e., have no common factors), $N = \frac{m}{\gcd(\ell,m)} = \frac{m}{1} = m$. (A proof of this is left as an exercise for the reader in Exercise 8.6.)

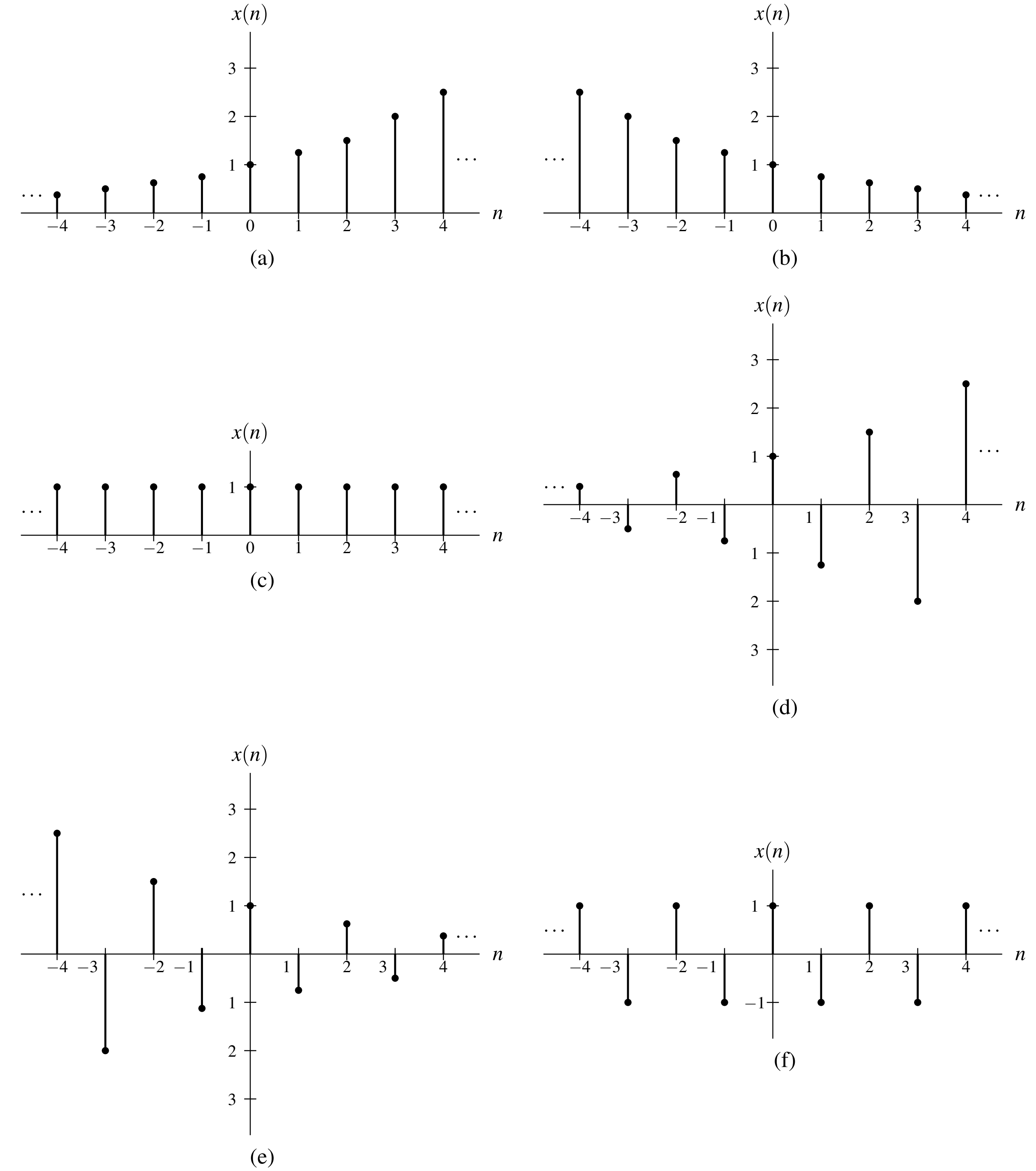


Figure 8.10: Examples of real exponential sequences. (a) $|a| > 1$, $a > 0$ [$a = \frac{5}{4}$; $c = 1$]; (b) $|a| < 1$, $a > 0$ [$a = \frac{4}{5}$; $c = 1$]; (c) $|a| = 1$, $a > 0$ [$a = 1$; $c = 1$]; (d) $|a| > 1$, $a < 0$ [$a = -\frac{5}{4}$; $c = 1$]; (e) $|a| < 1$, $a < 0$ [$a = -\frac{4}{5}$; $c = 1$]; and (f) $|a| = 1$, $a < 0$ [$a = -1$; $c = 1$].

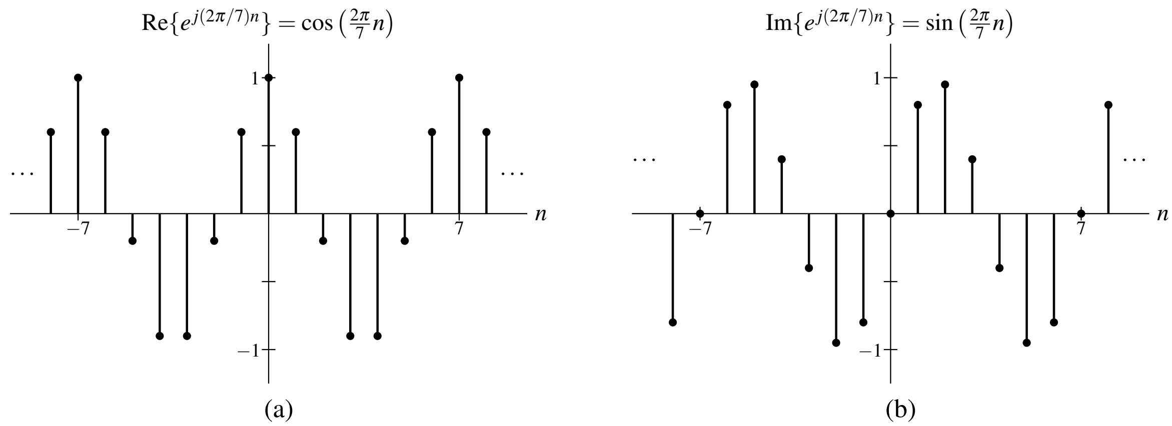


Figure 8.11: Example of complex sinusoidal sequence $x(n) = e^{j(2\pi/7)n}$. The (a) real and (b) imaginary parts of x.

In the case of periodic complex sinusoidal sequences, the frequency of the sequence is often treated as a signed quantity. In other words, we often employ the notion of signed frequency, as discussed in Section 2.10.2. Assuming that the complex sinusoidal sequence x given by (8.13) is periodic, x has a signed frequency of Ω. In most cases, we simply refer to the signed frequency as the "frequency". Normally, this does not cause any confusion, as it is usually clear from the context whether the frequency is being treated as a signed or unsigned quantity.

Unlike their continuous-time counterparts, complex sinusoidal sequences have an upper bound on the rate at which they can oscillate. This is due to similar reasoning as in the case of real sinusoidal sequences (discussed earlier in Section 8.4.1). For frequencies $\Omega \in [0, 2\pi)$, the oscillation rate of complex sinusoidal sequences is greatest at a frequency of π and lowest at a frequency of 0. As the frequency Ω increases from 0 to π, the oscillation rate increases, and as the frequency increases from π to 2π, the oscillation rate decreases.

Example 8.7 (Fundamental period of complex sinusoid)**.** Determine if each sequence x given below is periodic, and if it is, find its fundamental period.
(a) $x(n) = e^{j42n}$;
(b) $x(n) = e^{j(4\pi/11)n}$; and
(c) $x(n) = e^{j(\pi/3)n}$.

Solution. (a) Since $\frac{2\pi}{42} = \frac{\pi}{21}$ is not rational, x is not periodic.

(b) Since

$$(2\pi)/\left(\tfrac{4\pi}{11}\right) = (2\pi)\left(\tfrac{11}{4\pi}\right) = \tfrac{11}{2}$$

is rational, x is periodic. The fundamental period N is the smallest integer of the form $\frac{11}{2}k$, where k is a strictly positive integer. Thus, $N = 11$ (corresponding to $k = 2$). Alternatively, the fundamental period N of $x(n) = e^{j(2\pi[2/11])n}$ is given by

$$N = \frac{11}{\gcd(11,2)} = \frac{11}{\gcd(11^1, 2^1)} = \frac{11}{1} = 11.$$

(c) Since

$$(2\pi)/\left(\tfrac{\pi}{3}\right) = (2\pi)\left(\tfrac{3}{\pi}\right) = \tfrac{6}{1}$$

is rational, x is periodic. The fundamental period N is the smallest integer of the form $\frac{6}{1}k$, where k is a strictly positive integer. Thus, $N = 6$ (corresponding to $k = 1$). Alternatively, the fundamental period N of $x(n) = e^{j(\pi/3)n} = e^{j(2\pi[1]/6)n}$ is given by

$$N = \frac{6}{\gcd(6,1)} = \frac{6}{\gcd(2^1 \cdot 3^1, 1)} = \frac{6}{1} = 6. \quad \blacksquare$$

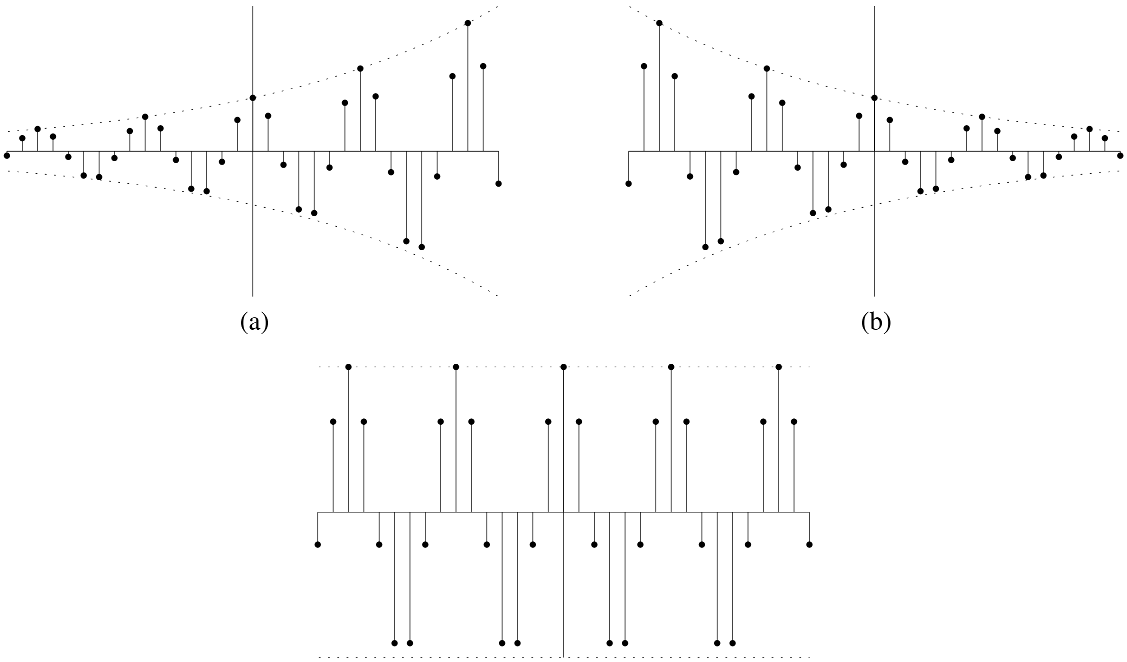


Figure 8.12: Various mode of behavior for the real and imaginary parts of a complex exponential sequence. (a) $|a| > 1$; (b) $|a| < 1$; and (c) $|a| = 1$.

8.4.2.3 General Complex Exponential Sequences

Lastly, we consider general complex exponential sequences. That is, we consider the general case of (8.12) where c and a are both complex. Letting $c = |c| e^{j\theta}$ and $a = |a| e^{j\Omega}$ where θ and Ω are real, and using Euler's relation, we can rewrite $x(n)$ as

$$x(n) = \underbrace{|c| |a|^n \cos(\Omega n + \theta)}_{\mathrm{Re}\{x(n)\}} + j \underbrace{|c| |a|^n \sin(\Omega n + \theta)}_{\mathrm{Im}\{x(n)\}}.$$

Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the product of a real exponential and real sinusoid. One of several distinct modes of behavior is exhibited by x, depending on the value of a, as illustrated in Figure 8.12. If $|a| = 1$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are real sinusoids. If $|a| > 1$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the product of a real sinusoid and a growing real exponential. If $|a| < 1$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the product of a real sinusoid and a decaying real exponential.

8.4.3 Relationship Between Complex Exponentials and Real Sinusoids

From Euler's relation, a complex sinusoid can be expressed as the sum of two real sinusoids as

$$ce^{j\Omega n} = c\cos(\Omega n) + jc\sin(\Omega n).$$

Moreover, a real sinusoid can be expressed as the sum of two complex sinusoids using the identities

$$c\cos(\Omega n + \theta) = \tfrac{c}{2}\left[e^{j(\Omega n+\theta)} + e^{-j(\Omega n+\theta)}\right] \quad \text{and}$$
$$c\sin(\Omega n + \theta) = \tfrac{c}{2j}\left[e^{j(\Omega n+\theta)} - e^{-j(\Omega n+\theta)}\right].$$

This result follows from Euler's relation and is simply a restatement of (A.7).

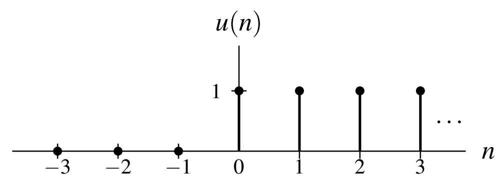


Figure 8.13: The unit-step sequence.

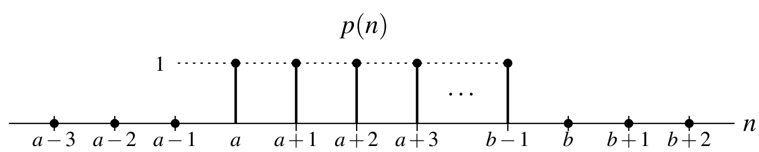


Figure 8.14: The rectangular sequence.

8.4.4 Unit-Step Sequence

Another elementary sequence often used in systems theory is the unit-step sequence. The **unit-step sequence**, denoted u, is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

A plot of this sequence is shown in Figure 8.13.

8.4.5 Unit-Rectangular Pulse

A family of sequences that is often useful is unit-rectangular pulses. A **unit rectangular pulse** is a sequence of the form

$$p(n) = \begin{cases} 1 & a \leq n < b \\ 0 & \text{otherwise,} \end{cases} \tag{8.14}$$

where a and b are integer constants satisfying $a < b$. The graph of a unit rectangular pulse has the general form shown in Figure 8.14. As is formally shown in Example 8.8 below, p can be expressed in terms of the unit-step sequence u as

$$p(n) = u(n-a) - u(n-b). \tag{8.15}$$

This particular way of expressing rectangular-pulse sequences is often extremely useful.

Example 8.8. Show that the unit rectangular pulse sequence p given by (8.14) can be written as specified in (8.15).

Solution. Recall that u is defined as

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

From this definition, we can write

$$u(n-a) = \begin{cases} 1 & n-a \geq 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & n \geq a \\ 0 & \text{otherwise.} \end{cases} \tag{8.16}$$

Similarly, we can write

$$u(n-b) = \begin{cases} 1 & n-b \geq 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & n \geq b \\ 0 & \text{otherwise.} \end{cases} \tag{8.17}$$

From (8.16) and (8.17), we can write

$$\begin{aligned} u(n-a) - u(n-b) &= \begin{cases} 0-0 & n < a \\ 1-0 & a \leq n < b \\ 1-1 & n \geq b. \end{cases} \\ &= \begin{cases} 0 & n < a \\ 1 & a \leq n < b \\ 0 & n \geq b. \end{cases} \\ &= \begin{cases} 1 & a \leq n < b \\ 0 & \text{otherwise.} \end{cases} \\ &= p(n). \end{aligned}$$

Thus, we have shown that

$$p(n) = u(n-a) - u(n-b).$$

■

8.4.6 Unit-Impulse Sequence

In systems theory, one elementary sequence of fundamental importance is the unit-impulse sequence. The **unit-impulse sequence** (also known as the **delta sequence**), denoted δ, is defined as

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{8.18}$$

(Note that the unit-impulse sequence is a very different thing from the unit-impulse function.) The first-order difference of u is δ. That is,

$$\delta(n) = u(n) - u(n-1).$$

The accumulation (i.e., running sum) of δ is u. That is,

$$u(n) = \sum_{k=-\infty}^{n} \delta(k).$$

A plot of δ is shown in Figures 8.15.

The unit-impulse sequence has two important properties that follow from its definition in (8.18). These properties are given by the theorems below.

Theorem 8.4 (Equivalence property)**.** *For any sequence x and any integer constant n_0, the following identity holds:*

$$x(n)\delta(n-n_0) = x(n_0)\delta(n-n_0). \tag{8.19}$$

Proof. The proof essentially follows immediately from the fact that the unit-impulse sequence is only nonzero at a single point. ■

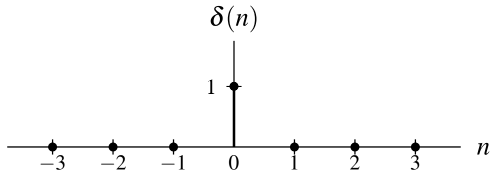


Figure 8.15: The unit-impulse sequence.

Theorem 8.5 (Sifting property). *For any sequence x and any integer constant n_0, the following identity holds:*

$$\sum_{n=-\infty}^{\infty} x(n)\delta(n-n_0) = x(n_0).$$

Proof. From the definition of the delta sequence, we have

$$\sum_{n=-\infty}^{\infty} \delta(n) = 1.$$

This implies that, for any integer constant n_0,

$$\sum_{n=-\infty}^{\infty} \delta(n-n_0) = 1.$$

Multiplying both sides of the equation by $x(n_0)$, we obtain

$$\sum_{n=-\infty}^{\infty} x(n_0)\delta(n-n_0) = x(n_0).$$

Then, using the equivalence property in (8.19), we have

$$\sum_{n=-\infty}^{\infty} x(n)\delta(n-n_0) = x(n_0).$$ ■

Trivially, the sequence δ is also even.

Example 8.9. Evaluate the summation

$$\sum_{n=-\infty}^{\infty} \sin\left(\tfrac{\pi}{2}n\right)\delta(n-1).$$

Solution. Using the sifting property of the unit impulse sequence, we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \sin\left(\tfrac{\pi}{2}n\right)\delta(n-1) &= \sin\left(\tfrac{\pi}{2}n\right)\Big|_{n=1} \\ &= \sin\left(\tfrac{\pi}{2}\right) \\ &= 1.\end{aligned}$$ ■

8.5 Representing Arbitrary Sequences Using Elementary Sequences

In the earlier sections, we introduced a number of elementary sequences. Often in signal analysis, it is convenient to represent arbitrary sequences in terms of elementary sequences. Here, we consider how the unit-step sequence can be exploited in order to obtain alternative representations of sequences.

Example 8.10. Consider the piecewise-linear sequence x given by

$$x(n) = \begin{cases} n+7 & -6 \le n \le -4 \\ 4 & -3 \le n \le 2 \\ 6-n & 3 \le n \le 5 \\ 0 & \text{otherwise.} \end{cases}$$

Find a single expression for $x(n)$ (involving unit-step sequences) that is valid for all n.

Solution. A plot of x is shown in Figure 8.16(a). We consider each segment of the piecewise-linear sequence separately. The first segment (i.e., for n) can be expressed as

$$v_1(n) = (n+7)[u(n+6) - u(n+3)].$$

This sequence is plotted in Figure 8.16(b). The second segment (i.e., for n) can be expressed as

$$v_2(n) = 4[u(n+3) - u(n-3)].$$

This sequence is plotted in Figure 8.16(c). The third segment (i.e., for n) can be expressed as

$$v_3(n) = (6-n)[u(n-3) - u(n-6)].$$

This sequence is plotted in Figure 8.16(d). Now, we observe that $x = v_1 + v_2 + v_3$. That is, we have

$$\begin{aligned} x(n) &= v_1(n) + v_2(n) + v_3(n) \\ &= (n+7)[u(n+6) - u(n+3)] + 4[u(n+3) - u(n-3)] + (6-n)[u(n-3) - u(n-6)] \\ &= (n+7)u(n+6) - (n+7)u(n+3) + 4u(n+3) - 4u(n-3) + (6-n)u(n-3) - (6-n)u(n-6) \\ &= (n+7)u(n+6) + (-n-3)u(n+3) + (2-n)u(n-3) + (n-6)u(n-6). \end{aligned}$$

Thus, we have found a single expression for $x(n)$ that is valid for all n. ■

8.6 Discrete-Time Systems

Suppose that we have a system with input x and output y. Such a system can be described mathematically by the equation

$$y = \mathcal{H}x, \tag{8.20}$$

where $\mathcal{H}$ denotes an operator (i.e., transformation). The operator $\mathcal{H}$ simply maps the input sequence x to the output sequence y. Such an operator might be associated with a system of difference equations, for example.

Alternatively, we sometimes express the relationship (8.20) using the notation

$$x \xrightarrow{\mathcal{H}} y.$$

Furthermore, if clear from the context, the operator $\mathcal{H}$ is often omitted, yielding the abbreviated notation

$$x \to y.$$

Note that the symbols "$\to$" and "=" have very different meanings. For example, the notation $x \to y$ does not in any way imply that $x = y$. The symbol "$\to$" should be read as "produces" (not as "equals"). That is, "$x \to y$" should be read as "the input x produces the output y".

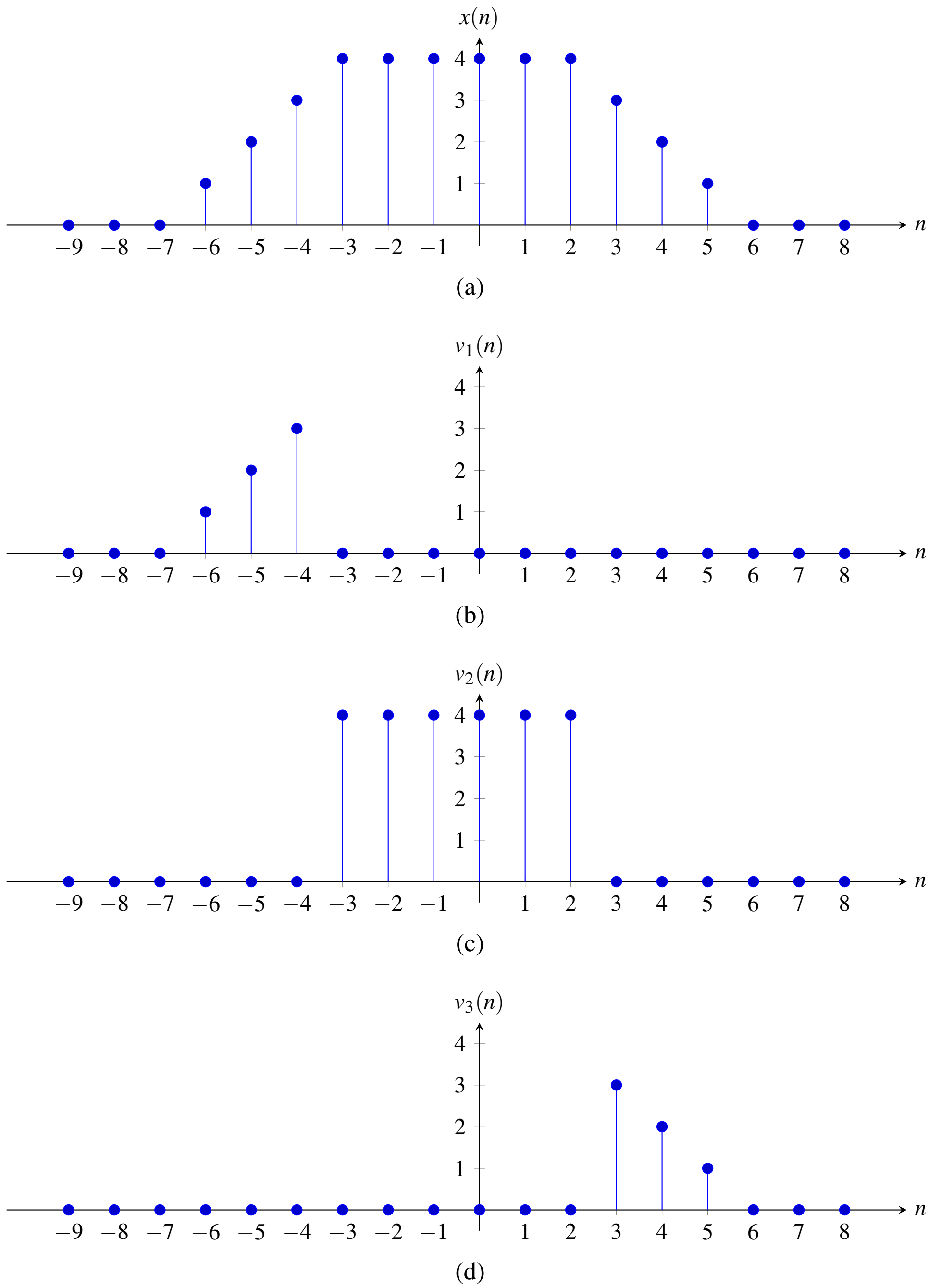


Figure 8.16: Representing a piecewise-linear sequence using unit-step sequences. (a) The sequence x. (b), (c), and (d) Three sequences whose sum is x.

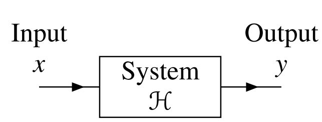
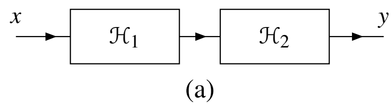


Figure 8.17: Block diagram of system.



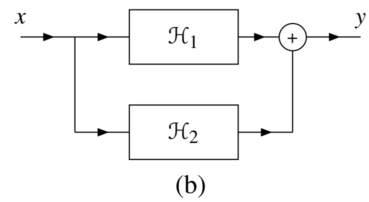


Figure 8.18: Interconnection of systems. The (a) series interconnection and (b) parallel interconnection of the systems $\mathcal{H}_1$ and $\mathcal{H}_2$.

8.6.1 Block Diagram Representation

Suppose that we have a system defined by the operator $\mathcal{H}$ and having the input x and output y. Often, we represent such a system using a block diagram as shown in Figure 8.17.

8.6.2 Interconnection of Systems

Systems may be interconnected in a number of ways. Two basic types of connections are as shown in Figure 8.18. The first type of connection, as shown in Figure 8.18(a), is known as a **series** or **cascade** connection. In this case, the overall system is defined by

$$y = \mathcal{H}_2\mathcal{H}_1 x. \tag{8.21}$$

The second type of connection, as shown in Figure 8.18(b), is known as a **parallel** connection. In this case, the overall system is defined by

$$y = \mathcal{H}_1 x + \mathcal{H}_2 x. \tag{8.22}$$

The system equations in (8.21) and (8.22) cannot be simplified further unless the definitions of the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ are known.

8.7 Properties of Systems

In what follows, we will define a number of important properties that a system may possess. These properties are useful in classifying systems, as well as characterizing their behavior.

8.7.1 Memory

A system $\mathcal{H}$ is said to be **memoryless** if, for every integer n_0, $\mathcal{H}x(n)$ does not depend on $x(n)$ for some $n \neq n_0$. In other words, a memoryless system is such that the value of its output at any given point in time can depend on the value of its input at only the *same* point in time. A system that is not memoryless is said to have **memory**. Although simple, a memoryless system is not very flexible, since its current output value cannot rely on past or future values of the input.

Example 8.11 (Ideal amplifier)**.** Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(n) = Ax(n)$$

and A is a nonzero real constant.

Solution. Consider the calculation of $\mathcal{H}x(n)$ at any arbitrary point $n = n_0$. We have

$$\mathcal{H}x(n_0) = Ax(n_0).$$

Thus, $\mathcal{H}x(n_0)$ depends on $x(n)$ only for $n = n_0$. Therefore, the system is memoryless. ■

Example 8.12 (Ideal accumulator)**.** Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(n) = \sum_{k=-\infty}^{n} x(k).$$

Solution. Consider the calculation of $\mathcal{H}x(n)$ at any arbitrary point $n = n_0$. We have

$$\mathcal{H}x(n_0) = \sum_{k=-\infty}^{n_0} x(n).$$

Thus, $\mathcal{H}x(n_0)$ depends on $x(n)$ for $-\infty < n \leq n_0$. So, $\mathcal{H}x(n_0)$ is dependent on $x(n)$ for some $n \neq n_0$ (e.g., $n_0 - 1$). Therefore, the system has memory (i.e., is not memoryless). ■

Example 8.13. Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(n) = e^{x(n)}.$$

Solution. Consider the calculation of $\mathcal{H}x(n)$ at any arbitrary point $n = n_0$. We have

$$\mathcal{H}x(n_0) = e^{x(n_0)}.$$

Thus, $\mathcal{H}x(n_0)$ depends on $x(n)$ only for $n = n_0$. Therefore, the system is memoryless. ■

Example 8.14. Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(n) = \mathrm{Odd}\{x\}(n) = \tfrac{1}{2}\left[x(n) - x(-n)\right].$$

Solution. For any x and any integer n_0, we have that $\mathcal{H}x(n_0)$ depends on $x(n)$ for $n = n_0$ and $n = -n_0$. Since $\mathcal{H}x(n_0)$ depends on $x(n)$ for $n \neq n_0$, the system has memory (i.e., the system is not memoryless). ■

8.7.2 Causality

A system $\mathcal{H}$ is said to be **causal** if, for every integer n_0, $\mathcal{H}x(n)$ does not depend on $x(n)$ for some $n > n_0$. In other words, a causal system is such that the value of its output at any given point in time can depend on the value of its input at only the *same or earlier* points in time (i.e., *not later* points in time). A memoryless system is always causal, although the converse is not necessarily true.

If the independent variable represents time, a system must be causal in order to be physically realizable. Noncausal systems can sometimes be useful in practice, however, as the independent variable need not always represent time.

Example 8.15 (Ideal accumulator)**.** Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(n) = \sum_{k=-\infty}^{n} x(n).$$

Solution. Consider the calculation of $\mathcal{H}x(n_0)$ for arbitrary n_0. We have

$$\mathcal{H}x(n_0) = \sum_{k=-\infty}^{n_0} x(n).$$

Thus, we can see that $\mathcal{H}x(n_0)$ depends only on $x(n)$ for $-\infty < n \leq n_0$. Since all of the values in this interval are less than or equal to n_0, the system is causal. ■

Example 8.16. Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(n) = \sum_{k=n-1}^{n+1} x(k).$$

Solution. Consider the calculation of $\mathcal{H}x(n_0)$ for arbitrary n_0. We have

$$\mathcal{H}x(n_0) = \sum_{k=n_0-1}^{n_0+1} x(k).$$

Thus, we can see that $\mathcal{H}x(n_0)$ only depends on $x(n)$ for $n_0 - 1 \leq n \leq n_0 + 1$. Since at least one value in this interval is greater than n_0 (i.e., $n_0 + 1$), the system is not causal. ■

Example 8.17. Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(n) = (n+1)e^{x(n-1)}.$$

Solution. Consider the calculation of $\mathcal{H}x(n_0)$ for arbitrary n_0. We have

$$\mathcal{H}x(n_0) = (n_0+1)e^{x(n_0-1)}.$$

Thus, we can see that $\mathcal{H}x(n_0)$ depends only on $x(n)$ for $n = n_0 - 1$. Since $n_0 - 1 \leq n_0$, the system is causal. ■

Example 8.18. Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(n) = \operatorname{Odd}\{x\}(n) = \tfrac{1}{2}\left[x(n) - x(-n)\right].$$

Solution. For any x and any integer constant n_0, we have that $\mathcal{H}x(n_0)$ depends only on $x(n)$ for $n = n_0$ and $n = -n_0$. Suppose that $n_0 = -1$. In this case, we have that $\mathcal{H}x(n_0)$ (i.e., $\mathcal{H}x(-1)$) depends on $x(n)$ for $n = 1$ but $n = 1 > n_0$. Therefore, the system is not causal. ■

8.7.3 Invertibility

The **inverse** of a system $\mathcal{H}$ (if such an inverse exists) is a system $\mathcal{G}$ such that, for every sequence x,

$$\mathcal{G}\mathcal{H}x = x$$

(i.e., the system formed by the cascade interconnection of $\mathcal{H}$ followed by $\mathcal{G}$ is a system whose input and output are equal). As a matter of notation, the inverse of $\mathcal{H}$ is denoted $\mathcal{H}^{-1}$. The relationship between a system and its inverse is illustrated in Figure 8.19. The two systems in this figure must be equivalent, due to the relationship between $\mathcal{H}$ and $\mathcal{H}^{-1}$ (i.e., $\mathcal{H}^{-1}$ cancels $\mathcal{H}$).

A system $\mathcal{H}$ is said to be **invertible** if it has a corresponding inverse system (i.e., its inverse exists). An invertible system must be such that its input x can always be uniquely determined from its output $\mathcal{H}x$. From this definition, it follows that an invertible system will always produce distinct outputs from any two distinct inputs.

To show that a system is invertible, we simply find the inverse system. To show that a system is not invertible, it is sufficient to find two distinct inputs to that system that result in identical outputs. In practical terms, invertible systems are nice in the sense that their effects can be undone.

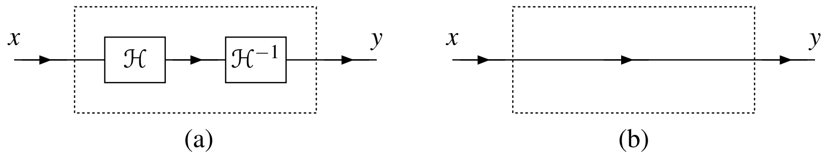


Figure 8.19: Systems that are equivalent (assuming $\mathcal{H}^{-1}$ exists). (a) First and (b) second system.

Example 8.19. Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(n) = x(n - n_0)$$

and n_0 is an integer constant.

Solution. Let $y = \mathcal{H}x$. By substituting $n + n_0$ for n in $y(n) = x(n - n_0)$, we obtain

$$\begin{aligned} y(n+n_0) &= x(n+n_0-n_0) \\ &= x(n). \end{aligned}$$

Thus, we have shown that

$$x(n) = y(n+n_0).$$

This, however, is simply the equation of the inverse system $\mathcal{H}^{-1}$. In particular, we have that

$$x(n) = \mathcal{H}^{-1}y(n)$$

where

$$\mathcal{H}^{-1}y(n) = y(n+n_0).$$

Thus, we have found $\mathcal{H}^{-1}$. Therefore, the system is invertible. ■

Example 8.20 (Ideal squarer)**.** Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(n) = x^2(n).$$

Solution. Consider the sequences $x_1(n) = -1$ and $x_2(n) = 1$. We have

$$\mathcal{H}x_1(n) = 1 \quad \text{and} \quad \mathcal{H}x_2(n) = 1.$$

Thus, we have found two distinct inputs that result in the same output. Therefore, the system is not invertible. ■

Example 8.21. Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(n) = \frac{e^{x(n)}}{n^2+1}.$$

Solution. Let $y = \mathcal{H}x$. Attempting to solve for x in terms of y, we have

$$\begin{aligned} & y(n) = \frac{e^{x(n)}}{n^2+1} \\ \Rightarrow \quad & \left(n^2+1\right) y(n) = e^{x(n)} \\ \Rightarrow \quad & \ln\left[(n^2+1)y(n)\right] = x(n). \end{aligned}$$

Thus, we have shown that

$$x(n) = \ln[(n^2+1)y(n)].$$

This, however, is simply the equation of the inverse system $\mathcal{H}^{-1}$. In particular, we have that

$$x(n) = \mathcal{H}^{-1}y(n)$$

where

$$\mathcal{H}^{-1}y(n) = \ln\left[\left(n^2+1\right)y(n)\right].$$

Thus, we have found $\mathcal{H}^{-1}$. Therefore, the system is invertible. ■

Example 8.22. Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(n) = \operatorname{Odd}\{x\}(n) = \tfrac{1}{2}\left[x(n) - x(-n)\right].$$

Solution. Consider the response $\mathcal{H}x$ of the system to an input x of the form

$$x(n) = \alpha,$$

where α is a real constant. We have that

$$\begin{aligned}\mathcal{H}x(n) &= \tfrac{1}{2}[x(n) - x(-n)]\\ &= \tfrac{1}{2}(\alpha-\alpha)\\ &= 0.\end{aligned}$$

Therefore, any constant input yields the same zero output. This, however, implies that distinct inputs can yield identical outputs. Therefore, the system is not invertible. ■

8.7.4 BIBO Stability

Although stability can be defined in numerous ways, in systems theory, we are often most interested in bounded-input bounded-output (BIBO) stability.

A system $\mathcal{H}$ is **BIBO stable** if, for every bounded sequence x, $\mathcal{H}x$ is also bounded (i.e., $|x(n)| < \infty$ for all n implies that $|\mathcal{H}x(n)| < \infty$ for all n). In other words, a BIBO stable system is such that it guarantees to always produce a bounded output as long as its input is bounded.

To prove that a system is BIBO stable, we must show that every bounded input leads to a bounded output. To show that a system is not BIBO stable, we simply need to find one counterexample (i.e., a single bounded input that leads to an unbounded output).

In practical terms, a BIBO stable system is well behaved in the sense that, as long as the system input remains finite for all time, the output will also remain finite for all time. Usually, a system that is not BIBO stable will have serious safety issues. For example, a portable music player with a battery input of 3.7 volts and headset output of ∞ volts would result in one vaporized human (and likely one big lawsuit as well).

Example 8.23 (Ideal squarer)**.** Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(n) = x^2(n).$$

Solution. Suppose that the input x is bounded such that (for all n)

$$|x(n)| \leq A,$$

where A is a (finite) real constant. Squaring both sides of the inequality, we obtain

$$|x(n)|^2 \leq A^2.$$

Interchanging the order of the squaring and magnitude operations, we have

$$\left|x^2(n)\right| \leq A^2.$$

Using the fact that $\mathcal{H}x(n) = x^2(n)$, we can write

$$|\mathcal{H}x(n)| \leq A^2.$$

Since A is finite, A^2 is also finite. Thus, we have that $\mathcal{H}x$ is bounded (i.e., $|\mathcal{H}x(n)| \leq A^2 < \infty$ for all n). Therefore, the system is BIBO stable. ■

Example 8.24 (Ideal accumulator)**.** Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(n) = \sum_{k=-\infty}^{n} x(k).$$

Solution. Suppose that we choose the input $x(n) = 1$. Clearly, x is bounded (i.e., $|x(n)| \leq 1$ for all n). We can calculate the response $\mathcal{H}x$ to this input as follows:

$$\begin{aligned}\mathcal{H}x(n) &= \sum_{k=-\infty}^{n} x(k) \\ &= \sum_{k=-\infty}^{n} 1 \\ &= \infty.\end{aligned}$$

Thus, the output $\mathcal{H}x$ is unbounded for the bounded input x. Therefore, the system is not BIBO stable. ■

Example 8.25. Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(n) = \operatorname{Odd}\{x\}(n) = \tfrac{1}{2}\left[x(n) - x(-n)\right].$$

Solution. Suppose that x is bounded. Then, $x(-n)$ is also bounded. Since the difference of two bounded sequences is bounded, $x(n) - x(-n)$ is bounded. Multiplication of a bounded sequence by a finite constant yields a bounded result. So, the sequence $\frac{1}{2}[x(n) - x(-n)]$ is bounded. Thus, $\mathcal{H}x$ is bounded. Since a bounded input must yield a bounded output, the system is BIBO stable. ■

Example 8.26. Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(n) = \frac{x(n)}{x^2(n) - 4}.$$

Solution. Consider the input $x(n) = 2$. Clearly, x is bounded (i.e., $|x(n)| \leq 2$ for all n). We can calculate the response $\mathcal{H}x$ to this input as follows:

$$\begin{aligned}\mathcal{H}x(n) &= \frac{2}{2^2 - 4} = \frac{2}{0} \\ &= \infty.\end{aligned}$$

Thus, the output $\mathcal{H}x$ is unbounded for the bounded input x. Therefore, the system is not BIBO stable. ■

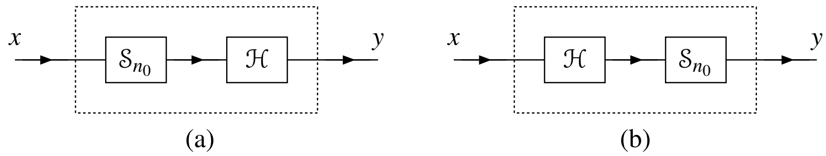


Figure 8.20: Systems that are equivalent if $\mathcal{H}$ is time invariant (i.e., $\mathcal{H}$ commutes with $\mathcal{S}_{n_0}$). (a) A system that first time shifts by n_0 and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}\mathcal{S}_{n_0}x$); and (b) a system that first applies $\mathcal{H}$ and then time shifts by n_0 (i.e., $y = \mathcal{S}_{n_0}\mathcal{H}x$).

8.7.5 Time Invariance

A system $\mathcal{H}$ is said to be **time invariant (TI)** (or **shift invariant (SI)**) if, for every sequence x and every integer n_0, the following condition holds:

$$\mathcal{H}x(n-n_0) = \mathcal{H}x'(n) \text{ for all } n \quad \text{where} \quad x'(n) = x(n-n_0)$$

(i.e., $\mathcal{H}$ *commutes with time shifts*). In other words, a system is time invariant, if a time shift (i.e., advance or delay) in the input sequence results in an identical time shift in the output sequence. A system that is not time invariant is said to be **time varying** (or **shift varying**). In effect, time invariance means that the two systems shown in Figure 8.20 are equivalent, where $\mathcal{S}_{n_0}$ denotes an operator that applies a time shift of n_0 to a sequence (i.e., $\mathcal{S}_{n_0}x(n) = x(n-n_0)$).

In simple terms, a time invariant system is a system whose behavior does not change with respect to time. Practically speaking, compared to time-varying systems, time-invariant systems are much easier to design and analyze, since their behavior does not change over time.

Example 8.27. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(n) = \sin[x(n)].$$

Solution. Let $x'(n) = x(n-n_0)$, where n_0 is an arbitrary integer constant. From the definition of $\mathcal{H}$, we can easily deduce that

$$\begin{aligned} \mathcal{H}x(n-n_0) &= \sin[x(n-n_0)] \quad \text{and} \\ \mathcal{H}x'(n) &= \sin[x'(n)] \\ &= \sin[x(n-n_0)]. \end{aligned}$$

Since $\mathcal{H}x(n-n_0) = \mathcal{H}x'(n)$ for all x and n_0, the system is time invariant. ■

Example 8.28. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(n) = nx(n).$$

Solution. Let $x'(n) = x(n-n_0)$, where n_0 is an arbitrary integer constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} \mathcal{H}x(n-n_0) &= (n-n_0)x(n-n_0) \quad \text{and} \\ \mathcal{H}x'(n) &= nx'(n) \\ &= nx(n-n_0). \end{aligned}$$

Since $\mathcal{H}x(n-n_0) = \mathcal{H}x'(n)$ does not hold for all x and n_0, the system is not time invariant (i.e., the system is time varying). ■

Example 8.29. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(n) = \sum_{k=-10}^{10} kx(n-k).$$

Solution. Let $x'(n) = x(n-n_0)$, where n_0 is an arbitrary integer constant. From the definition of $\mathcal{H}$, we can easily deduce that

$$\begin{aligned}
\mathcal{H}x(n-n_0) &= \sum_{k=-10}^{10} kx(n-n_0-k) \quad \text{and} \\
\mathcal{H}x'(n) &= \sum_{k=-10}^{10} kx'(n-k) \\
&= \sum_{k=-10}^{10} kx(n-k-n_0) \\
&= \sum_{k=-10}^{10} kx(n-n_0-k).
\end{aligned}$$

Since $\mathcal{H}x(n-n_0) = \mathcal{H}x'(n)$ for all x and n_0, the system is time invariant. ■

Example 8.30. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(n) = \operatorname{Odd}\{x\}(n) = \tfrac{1}{2}\left[x(n) - x(-n)\right].$$

Solution. Let $x'(n) = x(n-n_0)$, where n_0 is an arbitrary integer constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
\mathcal{H}x(n-n_0) &= \tfrac{1}{2}[x(n-n_0) - x(-(n-n_0))] \\
&= \tfrac{1}{2}[x(n-n_0) - x(-n+n_0)] \quad \text{and} \\
\mathcal{H}x'(n) &= \tfrac{1}{2}[x'(n) - x'(-n)] \\
&= \tfrac{1}{2}[x(n-n_0) - x(-n-n_0)].
\end{aligned}$$

Since $\mathcal{H}x(n-n_0) = \mathcal{H}x'(n)$ does not hold for all x and n_0, the system is not time invariant. ■

8.7.6 Linearity

Two of the most and frequently-occurring mathematical operations are addition and scalar multiplication. For this reason, it is often extremely helpful to know if these operations commute with the operation performed by a given system. The system properties to be introduced next relate to this particular issue.

A system $\mathcal{H}$ is said to be **additive** if, for all sequences x_1 and x_2, the following condition holds:

$$\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$$

(i.e., $\mathcal{H}$ *commutes with addition*). Essentially, a system $\mathcal{H}$ being additive means that the two systems shown in Figure 8.21 are equivalent.

A system $\mathcal{H}$ is said to be **homogeneous** if, for every sequence x and every complex constant a, the following condition holds:

$$\mathcal{H}(ax) = a\mathcal{H}x$$

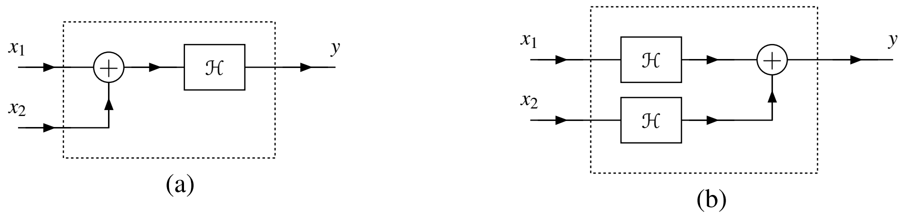


Figure 8.21: Systems that are equivalent if $\mathcal{H}$ is additive (i.e., $\mathcal{H}$ commutes with addition). (a) A system that first performs addition and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}(x_1 + x_2)$); and (b) a system that first applies $\mathcal{H}$ and then performs addition (i.e., $y = \mathcal{H}x_1 + \mathcal{H}x_2$).

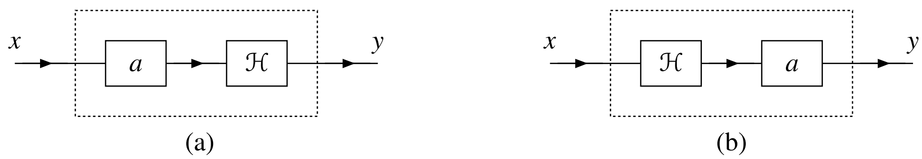


Figure 8.22: Systems that are equivalent if $\mathcal{H}$ is homogeneous (i.e., $\mathcal{H}$ commutes with scalar multiplication). (a) A system that first performs scalar multiplication and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}(ax)$); and (b) a system that first applies $\mathcal{H}$ and then performs scalar multiplication (i.e., $y = a\mathcal{H}x$).

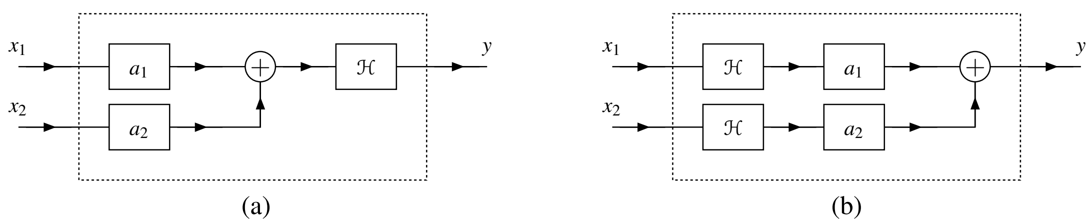


Figure 8.23: Systems that are equivalent if $\mathcal{H}$ is linear (i.e., $\mathcal{H}$ commutes with linear combinations). (a) A system that first computes a linear combination and then applies $\mathcal{H}$ (i.e., $y = \mathcal{H}(a_1x_1 + a_2x_2)$); and (b) a system that first applies $\mathcal{H}$ and then computes a linear combination (i.e., $y = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$).

(i.e., $\mathcal{H}$ *commutes with scalar multiplication*). Essentially, a system $\mathcal{H}$ being homogeneous means that the two systems shown in Figure 8.22 are equivalent.

The additivity and homogeneity properties can be combined into a single property known as superposition. In particular, a system $\mathcal{H}$ is said to have the **superposition** property, if for all sequences x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$$

(i.e., $\mathcal{H}$ *commutes with linear combinations*). A system that is both additive and homogeneous (or equivalently satisfies superposition) is said to be **linear**. Essentially, a system $\mathcal{H}$ being linear means that the two systems shown in Figure 8.23 are equivalent. To show that a system is linear, we can show that it possesses both the additivity and homogeneity properties, or we can simply show that the superposition property holds. Practically speaking, linear systems are much easier to design and analyze than nonlinear systems.

Example 8.31. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(n) = nx(n).$$

Solution. Let $x'(n) = a_1x_1(n) + a_2x_2(n)$, where x_1 and x_2 are arbitrary sequences and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we can write

$$\begin{aligned} a_1\mathcal{H}x_1(n)+a_2\mathcal{H}x_2(n) &= a_1nx_1(n)+a_2nx_2(n) \quad \text{and} \\ \mathcal{H}x'(n) &= nx'(n) \\ &= n\left[a_1x_1(n)+a_2x_2(n)\right] \\ &= a_1nx_1(n)+a_2nx_2(n). \end{aligned}$$

Since $\mathcal{H}(a_1x_1+a_2x_2) = a_1\mathcal{H}x_1+a_2\mathcal{H}x_2$ for all x_1, x_2, a_1, and a_2, the superposition property holds and the system is linear. ■

Example 8.32. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(n) = |x(n)|.$$

Solution. Let $x'(n) = a_1x_1(n)+a_2x_2(n)$, where x_1 and x_2 are arbitrary sequences and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} a_1\mathcal{H}x_1(n)+a_2\mathcal{H}x_2(n) &= a_1\left|x_1(n)\right|+a_2\left|x_2(n)\right| \quad \text{and} \\ \mathcal{H}x'(n) &= \left|x'(n)\right| \\ &= \left|a_1x_1(n)+a_2x_2(n)\right|. \end{aligned}$$

At this point, we recall the triangle inequality (i.e., for $a,b \in \mathbb{C}$, $|a+b| \leq |a|+|b|$). Thus, $\mathcal{H}(a_1x_1+a_2x_2) = a_1\mathcal{H}x_1+a_2\mathcal{H}x_2$ cannot hold for all x_1, x_2, a_1, and a_2 due, in part, to the triangle inequality. For example, this condition fails to hold for

$$a_1 = -1, \quad x_1(n) = 1, \quad a_2 = 0, \quad \text{and} \quad x_2(n) = 0,$$

in which case

$$a_1\mathcal{H}x_1(n)+a_2\mathcal{H}x_2(n) = -1 \quad \text{and} \quad \mathcal{H}x'(n) = 1.$$

Therefore, the superposition property does not hold and the system is not linear. ■

Example 8.33. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(n) = \operatorname{Odd}\{x\}(n) = \tfrac{1}{2}\left[x(n)-x(-n)\right].$$

Solution. Let $x'(n) = a_1x_1(n)+a_2x_2(n)$, where x_1 and x_2 are arbitrary sequences and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} a_1\mathcal{H}x_1(n)+a_2\mathcal{H}x_2(n) &= \tfrac{1}{2}a_1[x_1(n)-x_1(-n)]+\tfrac{1}{2}a_2[x_2(n)-x_2(-n)] \quad \text{and} \\ \mathcal{H}x'(n) &= \tfrac{1}{2}[x'(n)-x'(-n)] \\ &= \tfrac{1}{2}[a_1x_1(n)+a_2x_2(n)-[a_1x_1(-n)+a_2x_2(-n)]] \\ &= \tfrac{1}{2}[a_1x_1(n)-a_1x_1(-n)+a_2x_2(n)-a_2x_2(-n)] \\ &= \tfrac{1}{2}a_1[x_1(n)-x_1(-n)]+\tfrac{1}{2}a_2[x_2(n)-x_2(-n)]. \end{aligned}$$

Since $\mathcal{H}(a_1x_1+a_2x_2) = a_1\mathcal{H}x_1+a_2\mathcal{H}x_2$ for all x_1, x_2, a_1, and a_2, the system is linear. ■

Example 8.34. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(n) = x(n)x(n-1).$$

Solution. Let $x'(n) = a_1x_1(n) + a_2x_2(n)$, where x_1 and x_2 are arbitrary sequences and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
a_1\mathcal{H}x_1(n) + a_2\mathcal{H}x_2(n) &= a_1x_1(n)x_1(n-1) + a_2x_2(n)x_2(n-1) \quad \text{and} \\
\mathcal{H}x'(n) &= x'(n)x'(n-1) \\
&= [a_1x_1(n) + a_2x_2(n)][a_1x_1(n-1) + a_2x_2(n-1)] \\
&= a_1^2x_1(n)x_1(n-1) + a_1a_2x_1(n)x_2(n-1) + a_1a_2x_1(n-1)x_2(n) + a_2^2x_2(n)x_2(n-1).
\end{aligned}$$

Clearly, the expressions for $\mathcal{H}(a_1x_1 + a_2x_2)$ and $a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$ are quite different. Consequently, these expressions are not equal for many choices of a_1, a_2, x_1, and x_2 (e.g., $a_1 = 2$, $a_2 = 0$, $x_1(n) = 1$, and $x_2(n) = 0$). Therefore, the superposition property does not hold and the system is not linear. ■

Example 8.35 (Ideal accumulator)**.** A system $\mathcal{H}$ is defined by the equation

$$\mathcal{H}x(n) = \sum_{k=-\infty}^{n} x(k).$$

Determine whether this system is additive and/or homogeneous. Also, determine whether this system is linear.

Solution. First, we consider the additivity property. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
\mathcal{H}x_1(n) + \mathcal{H}x_2(n) &= \sum_{k=-\infty}^{n} x_1(k) + \sum_{k=-\infty}^{n} x_2(k) \quad \text{and} \\
\mathcal{H}\{x_1 + x_2\}(n) &= \sum_{k=-\infty}^{n} [x_1(k) + x_2(k)] \\
&= \sum_{k=-\infty}^{n} x_1(k) + \sum_{k=-\infty}^{n} x_2(k).
\end{aligned}$$

Since $\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$ for all x_1 and x_2, the system is additive.

Second, we consider the homogeneity property. Let a denote an arbitrary complex constant. From the definition of $\mathcal{H}$, we can write

$$\begin{aligned}
a\mathcal{H}x(n) &= a\sum_{k=-\infty}^{n} x(k) \quad \text{and} \\
\mathcal{H}\{ax\}(n) &= \sum_{k=-\infty}^{n} ax(k) \\
&= a\sum_{k=-\infty}^{n} x(k).
\end{aligned}$$

Since $\mathcal{H}(ax) = a\mathcal{H}x$ for all x and a, the system is homogeneous.

Lastly, we consider the linearity property. The system is linear since it has both the additivity and homogeneity properties. ■

Example 8.36. A system $\mathcal{H}$ is given by

$$\mathcal{H}x(n) = \operatorname{Re}[x(n)].$$

Determine whether this system is additive and/or homogeneous. Also, determine whether this system is linear.

Solution. First, we check if the additivity property is satisfied. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} \mathcal{H}x_1(n) + \mathcal{H}x_2(n) &= \operatorname{Re}[x_1(n)] + \operatorname{Re}[x_2(n)] \quad \text{and} \\ \mathcal{H}\{x_1 + x_2\}(n) &= \operatorname{Re}[x_1(n) + x_2(n)] \\ &= \operatorname{Re}[x_1(n)] + \operatorname{Re}[x_2(n)]. \end{aligned}$$

Since $\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$ for all x_1 and x_2, the system is additive.

Second, we check if the homogeneity property is satisfied. Let a denote an arbitrary complex constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} a\mathcal{H}x(n) &= a\operatorname{Re}x(n) \quad \text{and} \\ \mathcal{H}\{ax\}(n) &= \operatorname{Re}[ax(n)]. \end{aligned}$$

In order for $\mathcal{H}$ to be homogeneous, $a\mathcal{H}x = \mathcal{H}(ax)$ must hold for all x and all complex a. Suppose that $a = j$ and x is not identically zero (i.e., x is not the sequence $x(n) = 0$). In this case, we have

$$\begin{aligned} a\mathcal{H}x(n) &= j\operatorname{Re}[x(n)] \quad \text{and} \\ \mathcal{H}\{ax\}(n) &= \operatorname{Re}[jx(n)] \\ &= \operatorname{Re}[j(\operatorname{Re}[x(n)] + j\operatorname{Im}[x(n)])] \\ &= \operatorname{Re}(-\operatorname{Im}[x(n)] + j\operatorname{Re}[x(n)]) \\ &= -\operatorname{Im}[x(n)]. \end{aligned}$$

Thus, the quantities $\mathcal{H}(ax)$ and $a\mathcal{H}x$ are clearly not equal. Therefore, the system is not homogeneous.

Lastly, we consider the linearity property. Since the system does not possess both the additivity and homogeneity properties, it is not linear. ■

8.7.7 Eigensequences

An **eigensequence** of a system $\mathcal{H}$ is a sequence x that satisfies

$$\mathcal{H}x = \lambda x,$$

for some complex constant λ, which is called an **eigenvalue**. Essentially, a system behaves as an ideal amplifier (i.e., performs amplitude scaling) when presented with one of its eigensequences as input. The significance of the eigensequence property cannot be overstated. No matter how complicated a system might be, it exhibits extremely simple behavior for its eigensequences. We can often exploit this simplicity to reduce the complexity of solving many types of problems involving systems. In fact, as we will see later, eigensequences essentially form the basis for many of the mathematical tools that we use for studying systems.

Example 8.37. Consider the system $\mathcal{H}$ characterized by the equation

$$\mathcal{H}x(n) = x(n) - x(n-1).$$

For each sequence x given below, determine if x is an eigensequence of $\mathcal{H}$, and if it is, find the corresponding eigenvalue.
(a) $x(n) = 2$; and
(b) $x(n) = n$.

Solution. (a) We have

$$\begin{aligned}\mathcal{H}x(n) &= 2-2\\ &= 0\\ &= 0x(n).\end{aligned}$$

Therefore, x is an eigensequence of $\mathcal{H}$ with the eigenvalue 0.

(b) We have

$$\begin{aligned}\mathcal{H}x(n) &= n-(n-1)\\ &= 1\\ &= \left(\tfrac{1}{n}\right)n\\ &= \tfrac{1}{n}x(n).\end{aligned}$$

Therefore, x is not an eigensequence of $\mathcal{H}$. ■

Example 8.38 (Ideal amplifier)**.** Consider the system $\mathcal{H}$ given by

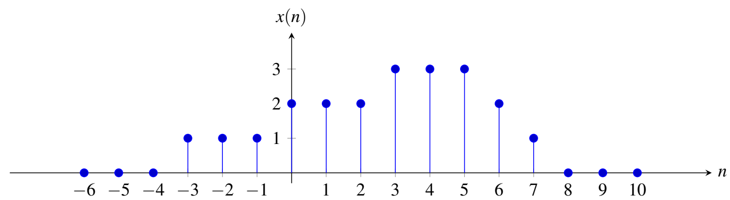
$$\mathcal{H}x(n) = ax(n),$$

where a is a complex constant. Clearly, every sequence is an eigensequence of $\mathcal{H}$ with eigenvalue a. ■

8.8 Exercises

8.8.1 Exercises Without Answer Key

8.1 Given the sequence x shown in the figure below, sketch a graph of each sequence y given below.
(a) $y(n) = x(n-2)$;
(b) $y(n) = x(n+2)$;
(c) $y(n) = x(2n)$;
(d) $y(n) = x(2n-1)$;
(e) $y(n) = x(3n+1)$; and
(f) $y(n) = x(1-2n)$.



8.2 Let x and y be sequences that are related by $y(n) = x(an-b)$, where a and b are integers and $a \geq 1$.
(a) Show that y can be obtained by time shifting x by b and then downsampling the resulting sequence by a factor of a.
(b) Show that, if $\frac{b}{a}$ is an integer, y can be obtained by downsampling x by a factor of a and then time shifting the resulting sequence by $\frac{b}{a}$.

8.3 Let x be an N-periodic sequence. Determine whether each sequence y given below is periodic.
(a) $y(n) = \operatorname{Odd}\{x\}(n)$;
(b) $y(n) = x(Nn)$; and
(c) $y(n) = x(2n)$.

8.4 Determine if each sequence x given below is periodic, and if it is, find its fundamental period N.
(a) $x(n) = 2e^{j(3\pi/10)n}$;
(b) $x(n) = 2e^{j(3/10)n}$;
(c) $x(n) = 5e^{j3n/2} + 3e^{j5n/2}$;
(d) $x(n) = e^{j7(2n+1)/3} + e^{j7(3n+2)/5}$;
(e) $x(n) = \cos(2n)$;
(f) $x(n) = \cos\left(\frac{8\pi}{31}n\right)$;
(g) $x(n) = \sin\left(\frac{6\pi}{7}n+1\right)$;
(h) $x(n) = \sin\left(\frac{1}{10}n-\pi\right)$; and
(i) $x(n) = \cos\left(\frac{\pi}{9}n\right) + \sin\left(\frac{\pi}{10}n\right) + \cos\left(\frac{\pi}{4}n\right)$.

8.5 Let x denote an arbitrary N-periodic sequence x. Show that:
(a) if x is even, then $x(n) = x(N-n)$;
(b) if x is odd, then $x(n) = -x(N-n)$; and
(c) if x is odd, then $x(0) = 0$ for both even and odd N, and $x\left(\frac{N}{2}\right) = 0$ for even N.

8.6 Show that a complex sinusoid of the form

$$x(n) = e^{j(2\pi\ell/m)n},$$

has the fundamental period N given by

$$N = \frac{m}{\gcd(\ell, m)},$$

where $\gcd(\ell, m)$ denotes the greatest common divisor (GCD) of ℓ and m.

8.7 Let y be the sequence given by

$$y(n) = \sum_{k=-\infty}^{\infty} x(n - Nk),$$

where x is any arbitrary sequence and N is a strictly positive integer constant. Show that y is N periodic.

8.8 Determine whether each sequence x given below is even, odd, or neither even nor odd.
(a) $x(n) = n^3$;
(b) $x(n) = n^3 \left|n\right|$;
(c) $x(n) = \left|n^3\right|$;
(d) $x(n) = \frac{1}{2}(e^n + e^{-n})$; and
(e) $x(n) = n + 1$.

8.9 Prove each of the assertions given below.
(a) The sum of two even sequences is even.
(b) The sum of two odd sequences is odd.
(c) The sum of an even sequence and an odd sequence, where neither of the sequences is identically zero, is neither even nor odd.
(d) The product of two even sequences is even.
(e) The product of two odd sequences is even.
(f) The product of an even sequence and an odd sequence is odd.

8.10 Show that, if a sequence x is odd, then

$$\sum_{k=-n}^{n} x(k) = 0,$$

where n is a nonnegative integer constant.

8.11 Show that, for any sequence x,

$$\sum_{k=-\infty}^{\infty} x^2(k) = \sum_{k=-\infty}^{\infty} x_{\mathrm{e}}^2(k) + \sum_{k=-\infty}^{\infty} x_{\mathrm{o}}^2(k),$$

where x_{e} and x_{o} denote the even and odd parts of x, respectively.

8.12 Show that the only sequence that is both even and odd is the zero sequence (i.e., the sequence x satisfying $x(n) = 0$ for all n).

8.13 Consider the sequence

$$x(n) = \begin{cases} n^2 & 0 \leq n \leq 2 \\ 5 & 3 \leq n \leq 5 \\ 9-n & 6 \leq n \leq 10 \\ 0 & \text{otherwise.} \end{cases}$$

Use the unit-step sequence to find a single expression for $x(n)$ that is valid for all n.

8.14 Determine whether each system $\mathcal{H}$ given below is memoryless.
(a) $\mathcal{H}x(n) = 3x(n)$;
(b) $\mathcal{H}x(n) = x(n+1) - x(n)$;
(c) $\mathcal{H}x(n) = \sum_{k=n}^{\infty} x(k)$; and
(d) $\mathcal{H}x(n) = 42$.

8.15 Determine whether each system $\mathcal{H}$ given below is causal.
(a) $\mathcal{H}x(n) = 3x(n)$;
(b) $\mathcal{H}x(n) = x(n-1) + 1$;
(c) $\mathcal{H}x(n) = x(n+1) - x(n)$;
(d) $\mathcal{H}x(n) = \sum_{k=n}^{\infty} x(k)$;
(e) $\mathcal{H}x(n) = \sum_{k=n-4}^{n} x(k)$; and
(f) $\mathcal{H}x(n) = 3x(3n+3)$.

8.16 Determine whether each system $\mathcal{H}$ given below is invertible.
(a) $\mathcal{H}x(n) = x(n-3)$;
(b) $\mathcal{H}x(n) = x(2n-1)$;
(c) $\mathcal{H}x(n) = e^{x(n)}$;
(d) $\mathcal{H}x(n) = x(n) - x(n-1)$; and
(e) $\mathcal{H}x(n) = \mathrm{Even}\{x\}(n)$.

8.17 Determine whether each system $\mathcal{H}$ given below is BIBO stable.
(a) $\mathcal{H}x(n) = 2x(n) + 1$;
(b) $\mathcal{H}x(n) = \sum_{k=n}^{n+4} x(k)$;
(c) $\mathcal{H}x(n) = \frac{1}{x(n)}$;
(d) $\mathcal{H}x(n) = \frac{1}{1+x^2(n)}$;
(e) $\mathcal{H}x(n) = e^{-|x(n)|}$;
(f) $\mathcal{H}x(n) = \sum_{k=n}^{\infty} x(k)$; and
(g) $\mathcal{H}x(n) = \frac{1}{1+|x(n)|}$.

8.18 Determine whether each system $\mathcal{H}$ given below is time invariant.
(a) $\mathcal{H}x(n) = x(n) - x(n-1)$;
(b) $\mathcal{H}x(n) = n^2 x(n)$;
(c) $\mathcal{H}x(n) = \mathrm{Even}\{x\}(n)$;
(d) $\mathcal{H}x(n) = \sum_{k=-\infty}^{\infty} x(k)x(n-k)$; and
(e) $\mathcal{H}x(n) = \sum_{k=n-n_0}^{n} x(k)$, where n_0 is a strictly positive integer constant.

8.19 Determine whether each system $\mathcal{H}$ given below is linear.
(a) $\mathcal{H}x(n) = x(n)+1$;
(b) $\mathcal{H}x(n) = \sum_{k=n-1}^{n+1} x(k)$;
(c) $\mathcal{H}x(n) = e^{x(n)}$;
(d) $\mathcal{H}x(n) = \text{Even}\{x\}(n)$;
(e) $\mathcal{H}x(n) = x^2(n)$; and
(f) $\mathcal{H}x(n) = n^2 x(n)$.

8.20 Show that, for any system $\mathcal{H}$ that is either additive or homogeneous, if a sequence x is identically zero (i.e., $x(n) = 0$ for all n), then $\mathcal{H}x$ is identically zero (i.e., $\mathcal{H}x(n) = 0$ for all n).

8.21 For each system $\mathcal{H}$ and the sequences $\{x_k\}$ given below, determine if each of the x_k is an eigensequence of $\mathcal{H}$, and if it is, also state the corresponding eigenvalue.
(a) $\mathcal{H}x(n) = x^2(n)$, $x_1(n) = a$, $x_2(n) = e^{-an}$, and $x_3(n) = \cos n$, where a is a complex constant;
(b) $\mathcal{H}x(n) = x(n+1) - x(n)$, $x_1(n) = e^{an}$, $x_2(n) = e^{an^2}$, and $x_3(n) = 42$, where a is a real constant; and
(c) $\mathcal{H}x(n) = |x(n)|$, $x_1(n) = a$, $x_2(n) = n$, $x_3(n) = n^2$, where a is a strictly positive real constant.

8.8.2 Exercises With Answer Key

8.22 Determine if each sequence x given below is periodic and, if it is, find its fundamental period.
(a) $x(n) = e^{j2\pi n/3}$;
(b) $x(n) = e^{j3\pi n/4}$;
(c) $x(n) = \cos(2\pi n)$;
(d) $x(n) = e^{j(3\pi/7)n} + e^{j(\pi/2)n} + \sin\left(\frac{5\pi}{9}n\right)$;
(e) $x(n) = \cos(\pi n) + \sin\left(\frac{\pi}{2}n\right) + e^{j(\pi/3)n}$;
(f) $x(n) = \sin\left(\frac{16\pi}{3}n + \frac{\pi}{11}\right)$; and
(g) $x(n) = \sin\left(-\frac{7\pi}{13}n + \frac{\pi}{4}\right)$.

Short Answer. (a) 3-periodic; (b) 8-periodic; (c) 1-periodic (d) 252-periodic; (e) 12-periodic; (f) 3-periodic; (g) 26-periodic

8.23 Determine whether each system $\mathcal{H}$ given below is memoryless.
(a) $\mathcal{H}x(n) = x(-n)$; and
(b) $\mathcal{H}x(n) = 3x(3n+3)$.

Short Answer. (a) has memory; (b) has memory

8.24 Determine whether each system $\mathcal{H}$ given below is invertible.
(a) $\mathcal{H}x(n) = x(3n)$.

Short Answer. (a) not invertible

8.25 Determine whether each system $\mathcal{H}$ given below is BIBO stable.
(a) $\mathcal{H}x(n) = 3x(3n+3)$.

Short Answer. (a) BIBO stable

8.26 Determine whether each system $\mathcal{H}$ given below is time invariant.
(a) $\mathcal{H}x(n) = 3x(3n+3)$.

Short Answer. (a) not time invariant

8.27 Determine whether each system $\mathcal{H}$ given below is linear.
(a) $\mathcal{H}x(n) = 3x(3n+3)$.

Short Answer. (a) linear

8.28 For each system $\mathcal{H}$ and the sequences $\{x_k\}$ given below, determine if each of the x_k is an eigensequence of $\mathcal{H}$, and if it is, also state the corresponding eigenvalue.
(a) $\mathcal{H}x(n) = \sum_{k=-\infty}^{n} x(k)$, $x_1(n) = 2^n$, $x_2(n) = \delta(n)$, and $x_3(n) = u(n)$.

Short Answer. (a) x_1 is an eigensequence with eigenvalue 2; x_2 is not an eigensequence; x_3 is not an eigensequence

Chapter 9

Discrete-Time Linear Time-Invariant Systems

9.1 Introduction

In the previous chapter, we identified a number of properties that a system may possess. Two of these properties were linearity and time invariance. In this chapter, we focus our attention exclusively on systems with both of these properties. Such systems are referred to as **linear time-invariant (LTI)** systems.

9.2 Discrete-Time Convolution

In the context of LTI systems, a mathematical operation known as convolution turns out to be particularly important. The **convolution** of the sequences x and h, denoted $x*h$, is defined as the sequence

$$x*h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k). \tag{9.1}$$

Herein, the asterisk (or star) symbol (i.e., "$*$") will be used to denote convolution, not multiplication. It is important to make a distinction between convolution and multiplication, since these two operations are quite different and do not generally yield the same result.

Notationally, $x*h$ denotes a sequence, namely the sequence that results from convolving x and h. In contrast, $x*h(n)$ denotes the sequence $x*h$ evaluated at n. Although we could equivalently write $x*h(n)$ with an extra pair of brackets as $(x*h)(n)$, we usually omit this extra pair of brackets, since doing so does not introduce any ambiguity and leads to more compact notation. That is, there is only one sensible way to group operations in the expression $x*h(n)$. The grouping $x*[h(n)]$ would not make sense since a convolution requires two sequences as operands and $h(n)$ is not a sequence, but rather the value of h evaluated at n. Thus, the only sensible way to interpret the expression $x*h(n)$ is as $(x*h)(n)$.

Since the convolution operation is used extensively in system theory, we need some practical means for evaluating a convolutional sum. Suppose that, for the given sequences x and h, we wish to compute

$$x*h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k).$$

Of course, we could naively attempt to compute $x*h$ by evaluating $x*h(n)$ as a separate summation for each possible value of n. This approach, however, is not feasible, as n can assume an infinite number of values, and therefore, an infinite number of summations would need to be evaluated. Instead, we consider a slightly different approach. Let us redefine the summation in terms of the intermediate sequence $w_n(k)$ where

$$w_n(k) = x(k)h(n-k).$$

(Note that $w_n(k)$ is implicitly a function of n.) This means that we need to compute

$$x * h(n) = \sum_{k=-\infty}^{\infty} w_n(k).$$

Now, we observe that, for most sequences x and h of practical interest, the form of $w_n(k)$ typically remains fixed over particular ranges of n. Thus, we can compute the convolution result $x * h$ by first identifying each of the distinct expressions for $w_n(k)$ and the range over which each expression is valid. Then, for each range, we evaluate a summation. In this way, we typically only need to compute a small number of summations instead of the infinite number required with the naive approach suggested above.

The above discussion leads us to propose the following general approach for computing a convolution:

1. Plot $x(k)$ and $h(n-k)$ with respect to k.

2. Initially, consider an arbitrarily large negative value for n. This will result in $h(n-k)$ being shifted very far to the left on the time axis.

3. Write the mathematical expression for $w_n(k)$.

4. Increase n gradually until the expression for $w_n(k)$ changes form. Record the interval over which the expression for $w_n(k)$ was valid.

5. Repeat steps 3 and 4 until n is an arbitrarily large positive value. This corresponds to $h(n-k)$ being shifted very far to the right on the time axis.

6. For each of the intervals identified above, sum $w_n(k)$ in order to find an expression for $x * h(n)$. This will yield an expression for $x * h(n)$ for each interval.

7. The results for the various intervals can be combined in order to obtain the convolution result $x * h(n)$ for all n.

Depending on the form of the sequences being convolved, we may employ a number of variations on the above strategy. In most cases, however, drawing graphs of the various sequences involved in the convolution computation is extremely helpful, as this typically makes the general form of the answer much easier to visualize. Sometimes, the information from the graphs can be more conveniently represented in table form. So, the use of tables can also be quite helpful. In what follows, we consider several examples of computing convolutions.

Example 9.1. Compute $x * h$, where

$$x(n) = 2^{-|n|} \quad \text{and} \quad h(n) = u(n-2).$$

Solution. From the definition of convolution, we have

$$\begin{aligned} x * h(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= \sum_{k=-\infty}^{\infty} 2^{-|k|} u(n-k-2). \end{aligned}$$

Since $u(n-k-2)=0$ for $k>n-2$, we can write

$$\begin{aligned}
x*h(n) &= \sum_{k=-\infty}^{n-2} 2^{-|k|} \\
&= \begin{cases} \sum_{k=-\infty}^{n-2} 2^k & n-2 \leq 0 \\ \sum_{k=-\infty}^{0} 2^k + \sum_{k=1}^{n-2} 2^{-k} & n-2 > 0 \end{cases} \\
&= \begin{cases} \sum_{k=-\infty}^{n-2} 2^k & n \leq 2 \\ \sum_{k=-\infty}^{0} 2^k + \sum_{k=1}^{n-2} 2^{-k} & n > 2. \end{cases}
\end{aligned}$$

Often, it can be somewhat tricky to identify the various cases that arise in the convolution computation. In this example, we have two cases: $n \leq 2$ and $n > 2$. The reason for these cases is more easily seen by examining the plots of $x(k)$ and $h(n-k)$ versus k, as shown in Figure 9.1. Now, we simplify each of the summations appearing in the above expression for $x*h$. We have

$$\begin{aligned}
\sum_{k=-\infty}^{n-2} 2^k &= \sum_{k=2-n}^{\infty} \left(\tfrac{1}{2}\right)^k = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^{k+2-n} = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^{2-n} \left(\tfrac{1}{2}\right)^k \\
&= \frac{\left(\frac{1}{2}\right)^{2-n}}{1-\frac{1}{2}} = \left(\tfrac{1}{2}\right)^{2-n} \left(\tfrac{1}{2}\right)^{-1} = \left(\tfrac{1}{2}\right)^{1-n} \\
&= 2^{n-1}, \\
\sum_{k=-\infty}^{0} 2^k &= \sum_{k=0}^{\infty} 2^{-k} = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k \\
&= \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = (1)(2) \\
&= 2, \quad \text{and} \\
\sum_{k=1}^{n-2} 2^{-k} &= \sum_{k=0}^{n-3} 2^{-(k+1)} = \sum_{k=0}^{n-3} \left(\tfrac{1}{2}\right) \left(\tfrac{1}{2}\right)^k \\
&= \left(\tfrac{1}{2}\right) \left(\frac{\left(\frac{1}{2}\right)^{n-2} - 1}{\frac{1}{2} - 1}\right) = \left(\tfrac{1}{2}\right) \left(\frac{\left(\frac{1}{2}\right)^{n-2} - 1}{-\frac{1}{2}}\right) \\
&= 1 - \left(\tfrac{1}{2}\right)^{n-2}.
\end{aligned}$$

Substituting these simplified expressions into the earlier formula for $x*h$ yields

$$x*h(n) = \begin{cases} 2^{n-1} & n \leq 2 \\ 3 - \left(\frac{1}{2}\right)^{n-2} & n > 2. \end{cases}$$

■

Example 9.2. Compute $x*h$, where

$$x(n) = \begin{cases} \left(\frac{3}{4}\right)^{-n-4} & n \leq -5 \\ 1 & -4 \leq n \leq 4 \\ \left(\frac{3}{4}\right)^{n-4} & n \geq 5. \end{cases} \quad \text{and} \quad h(n) = u(n+2) - u(n-5).$$

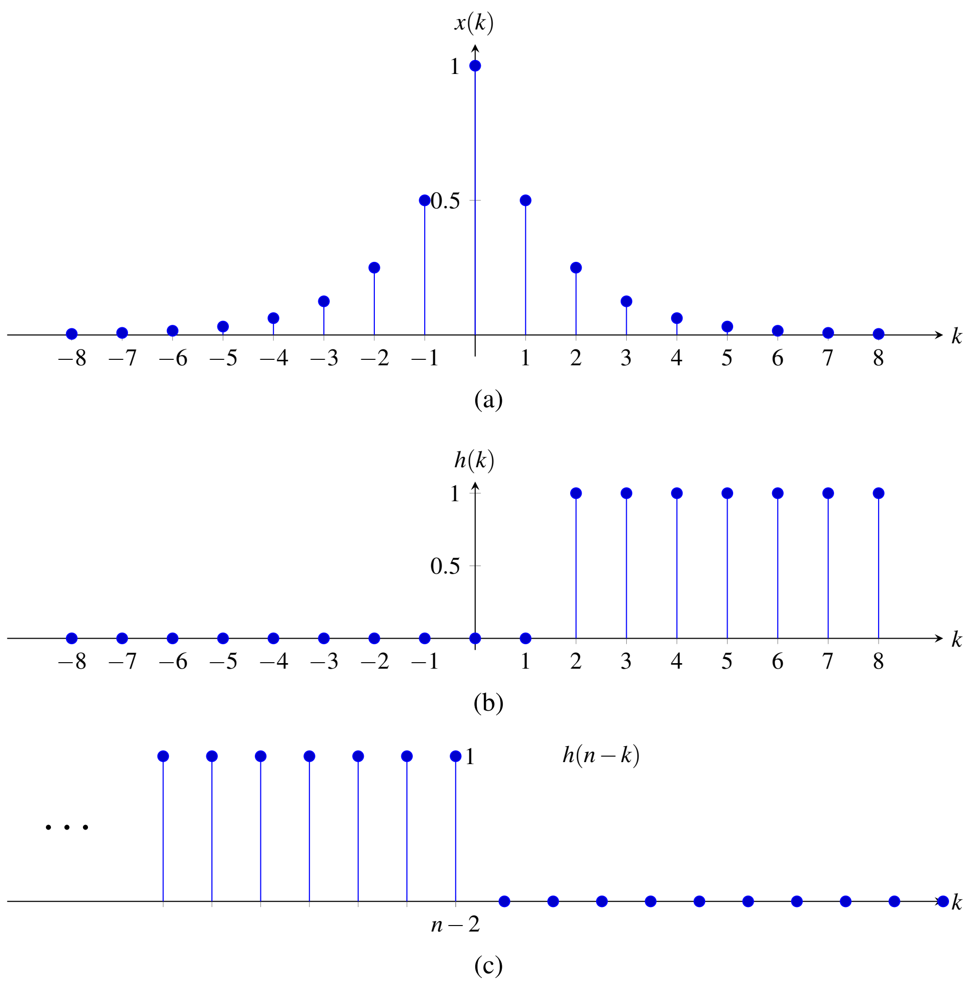


Figure 9.1: Plots for Example 9.1. Plots of (a) $x(k)$, (b) $h(k)$, and (c) $h(n-k)$ versus k.

Solution. To assist in visualizing the various cases involved in the convolution computation, we plot $x(k)$, $h(k)$, and $h(n-k)$ versus k in Figure 9.2. From these figures, we can deduce that that there are five cases in the convolution computation, namely:

1. $n \leq -7$, which comes from $n+2 \leq -5$;
2. $-6 \leq n \leq -1$, which comes from $n-4 \leq -5$ and $n+2 \geq -4$;
3. $0 \leq n \leq 2$, which comes from $n-4 \geq -4$ and $n+2 \leq 4$;
4. $3 \leq n \leq 8$, which comes from $n-4 \leq 4$ and $n+2 \geq 5$; and
5. $n \geq 9$, which comes from $n-4 \geq 5$.

We consider each of the preceding cases in turn.

In the first case, we have

$$\begin{aligned}
x * h(n) &= \sum_{k=n-4}^{n+2} \left(\tfrac{3}{4}\right)^{-k-4}(1) = \sum_{k=n-4}^{n+2} \left(\tfrac{3}{4}\right)^{-k-4} \\
&= \sum_{k=0}^{6} \left(\tfrac{3}{4}\right)^{-[k+n-4]-4} = \sum_{k=0}^{6} \left(\tfrac{3}{4}\right)^{-k-n} = \sum_{k=0}^{6} \left(\tfrac{4}{3}\right)^{n} \left(\tfrac{4}{3}\right)^{k} = \left(\tfrac{4}{3}\right)^{n} \frac{\left(\tfrac{4}{3}\right)^{7}-1}{\tfrac{4}{3}-1} \\
&= \tfrac{14197}{729} \left(\tfrac{4}{3}\right)^{n}.
\end{aligned}$$

In the second case, we have

$$\begin{aligned}
x * h(n) &= \sum_{k=n-4}^{-5} \left(\tfrac{3}{4}\right)^{-k-4}(1) + \sum_{k=-4}^{n+2} (1)(1) = \left[\sum_{k=n-4}^{-5} \left(\tfrac{4}{3}\right)^{k+4}\right] + n + 7 \\
&= n+7+\sum_{k=0}^{-n-1} \left(\tfrac{4}{3}\right)^{-(k+n-4)-4} = n+7+\sum_{k=0}^{-n-1} \left(\tfrac{4}{3}\right)^{n} \left(\tfrac{4}{3}\right)^{k} = n+7+\left(\tfrac{4}{3}\right)^{n} \frac{\left(\tfrac{4}{3}\right)^{-n}-1}{\tfrac{4}{3}-1} \\
&= n+7+3\left(\tfrac{4}{3}\right)^{n}\left[\left(\tfrac{4}{3}\right)^{-n}-1\right] = n+7+3\left[1-\left(\tfrac{4}{3}\right)^{n}\right] \\
&= n-3\left(\tfrac{4}{3}\right)^{n}+10.
\end{aligned}$$

In the third case, we have

$$\begin{aligned}
x * h(n) &= \sum_{k=n-4}^{n+2} (1)(1) \\
&= 7.
\end{aligned}$$

In the fourth case, we have

$$\begin{aligned}
x * h(n) &= \sum_{k=n-4}^{4} (1)(1) + \sum_{k=5}^{n+2} \left(\tfrac{3}{4}\right)^{k-4}(1) = 8-n+\sum_{k=5}^{n+2} \left(\tfrac{3}{4}\right)^{k-4} \\
&= 8-n+\sum_{k=0}^{n-3} \left(\tfrac{3}{4}\right)^{k+5-4} = 8-n+\sum_{k=0}^{n-3} \left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)^{k} = 8-n+\tfrac{3}{4}\,\frac{\left(\tfrac{3}{4}\right)^{n-2}-1}{\tfrac{3}{4}-1} \\
&= 8-n-4\left(\tfrac{3}{4}\right)\left[\left(\tfrac{3}{4}\right)^{n-2}-1\right] = 8-n-\left[3\left(\tfrac{3}{4}\right)^{n-2}-4\right] = 12-n-3\left(\tfrac{3}{4}\right)^{n-2} \\
&= 12-\tfrac{16}{3}\left(\tfrac{3}{4}\right)^{n}-n.
\end{aligned}$$

In the fifth case, we have

$$\begin{aligned}
x*h(n) &= \sum_{k=n-4}^{n+2} \left(\tfrac{3}{4}\right)^{k-4}(1) = \sum_{k=n-4}^{n+2} \left(\tfrac{3}{4}\right)^{k-4} \\
&= \sum_{k=0}^{6} \left(\tfrac{3}{4}\right)^{k+n-4-4} = \sum_{k=0}^{6} \left(\tfrac{3}{4}\right)^{n-8}\left(\tfrac{3}{4}\right)^{k} = \left(\tfrac{3}{4}\right)^{n-8} \frac{\left(\tfrac{3}{4}\right)^{7}-1}{\tfrac{3}{4}-1} \\
&= -4\left(\tfrac{3}{4}\right)^{n-8}\left[\left(\tfrac{3}{4}\right)^{7}-1\right] = \tfrac{14197}{4096}\left(\tfrac{3}{4}\right)^{n-8} \\
&= \tfrac{227152}{6561}\left(\tfrac{3}{4}\right)^{n}.
\end{aligned}$$

Combining the above results, we conclude

$$x*h(n) = \begin{cases} \tfrac{14197}{729}\left(\tfrac{4}{3}\right)^{n} & n \le 7 \\ n-3\left(\tfrac{4}{3}\right)^{n}+10 & -6 \le n \le -1 \\ 7 & 0 \le n \le 2 \\ 12-\tfrac{16}{3}\left(\tfrac{3}{4}\right)^{n}-n & 3 \le n \le 8 \\ \tfrac{227152}{6561}\left(\tfrac{3}{4}\right)^{n} & n \ge 9. \end{cases}$$

A plot of $x*h$ is given in Figure 9.3. ■

Example 9.3. Compute $x*h$, where

$$x(n) = \delta(n)+2\delta(n-1)+\delta(n-2) \quad \text{and} \quad h(n) = \tfrac{1}{2}\delta(n+1)+\tfrac{1}{2}\delta(n).$$

Solution. This convolution is likely most easily performed by using a graphical or tabular approach. In what follows, we elect to use a tabular approach. By constructing a table that shows x and the various time-reversed and shifted versions of h, we can easily compute the elements of $x*h$. The result of this process is shown in Table 9.1. From this table, we have

$$x*h(n) = \tfrac{1}{2}\delta(n+1)+\tfrac{3}{2}\delta(n)+\tfrac{3}{2}\delta(n-1)+\tfrac{1}{2}\delta(n-2).$$ ■

Example 9.4. Compute $x*h$, where

$$x(n) = \left(\tfrac{1}{2}\right)^{n}u(n) \quad \text{and} \quad h(n) = u(n).$$

Solution. From the definition of convolution, we have

$$\begin{aligned}
x*h(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\
&= \sum_{k=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^{k}u(k)u(n-k).
\end{aligned}$$

Since $u(k)=0$ for $k<0$ and $u(n-k)=0$ for $k>n$, we can write

$$\begin{aligned}
x*h(n) &= \begin{cases} \sum_{k=0}^{n}\left(\tfrac{1}{2}\right)^{k} & n \ge 0 \\ 0 & \text{otherwise.} \end{cases} \\
&= \left(\sum_{k=0}^{n}\left(\tfrac{1}{2}\right)^{k}\right)u(n).
\end{aligned}$$

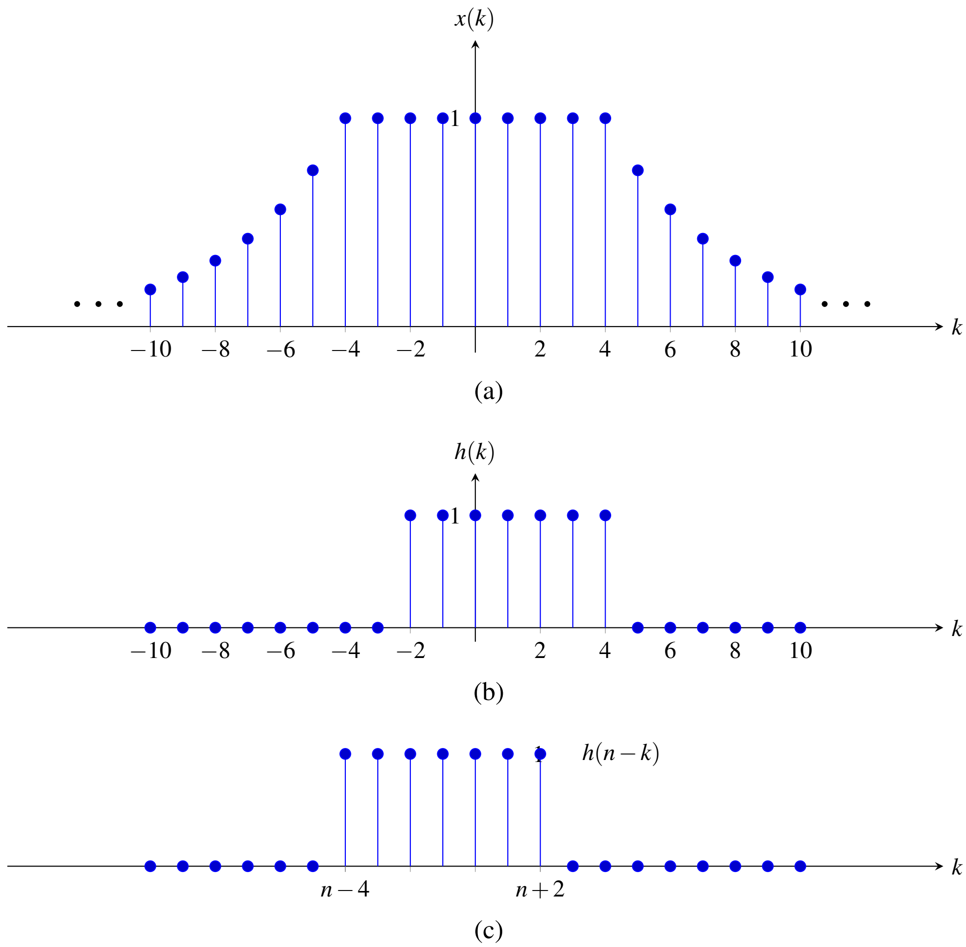


Figure 9.2: Plots for Example 9.2. Plots of (a) $x(k)$, (b) $h(k)$, and (c) $h(n-k)$ versus k.

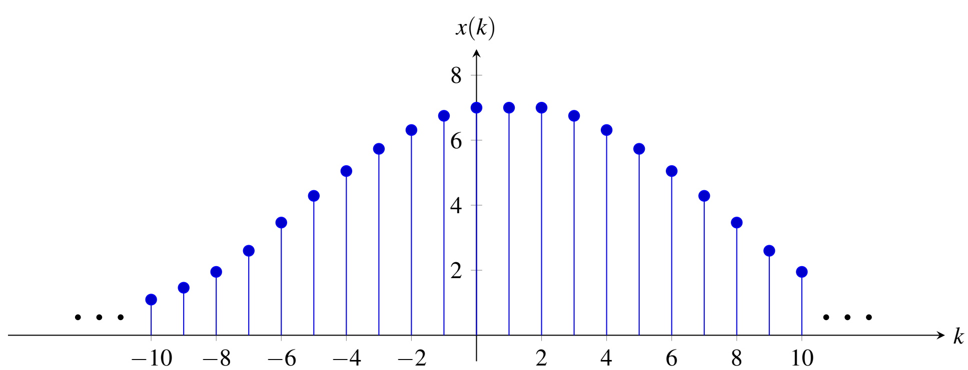


Figure 9.3: The sequence $x * h$ for Example 9.2.

Table 9.1: Convolution computation for Example 9.3

n \ k	-2	-1	0	1	2	3	4	
	$x(k)$							
			1	2	1			
	$h(k)$							
		$\frac{1}{2}$	$\frac{1}{2}$					
	$h(-k)$							
			$\frac{1}{2}$	$\frac{1}{2}$				
	$h(n-k)$							$x * h(n)$
-2	$\frac{1}{2}$	$\frac{1}{2}$						0
-1		$\frac{1}{2}$	$\frac{1}{2}$					$\frac{1}{2}(0)+\frac{1}{2}(1)=\frac{1}{2}$
0			$\frac{1}{2}$	$\frac{1}{2}$				$\frac{1}{2}(1)+\frac{1}{2}(2)=\frac{3}{2}$
1				$\frac{1}{2}$	$\frac{1}{2}$			$\frac{1}{2}(2)+\frac{1}{2}(1)=\frac{3}{2}$
2					$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}(1)+\frac{1}{2}(0)=\frac{1}{2}$
3						$\frac{1}{2}$	$\frac{1}{2}$	0

Often, it can be somewhat tricky to identify the various cases that arise in the convolution computation. In this example, we have two cases: $n \geq 0$ and $n < 0$. The reason for these cases is more easily seen by examining the plots of $x(k)$ and $h(n-k)$ versus k, as shown in Figure 9.4. Using the formula for the sum of a geometric sequence, we can write

$$\begin{aligned} x * h(n) &= \left[\frac{\left(\frac{1}{2}\right)^{n+1}-1}{\frac{1}{2}-1}\right] u(n) \\ &= -2\left[\left(\tfrac{1}{2}\right)^{n+1}-1\right] u(n) \\ &= \left[2-\left(\tfrac{1}{2}\right)^{n}\right] u(n). \end{aligned}$$

■

Example 9.5. Compute $x * h$, where

$$x(n) = \delta(n) + 3\delta(n-1) + 9\delta(n-2) + 9\delta(n-3) + 3\delta(n-4) + \delta(n-5) \quad \text{and}$$
$$h(n) = \delta(n) - \delta(n-1).$$

Solution. This convolution is likely most easily performed by using a graphical or tabular approach. In what follows, we elect to use a tabular approach. By constructing a table that shows x and the various time-reversed and shifted versions of h, we can easily compute the elements of $x * h$. The result of this process is shown in Table 9.2. From this table, we have

$$x * h(n) = \delta(n) + 2\delta(n-1) + 6\delta(n-2) - 6\delta(n-4) - 2\delta(n-5) - \delta(n-6).$$

■

9.3 Properties of Convolution

Since convolution is frequently employed in the study of LTI systems, it is important for us to know some of its basic properties. In what follows, we examine some of these properties.

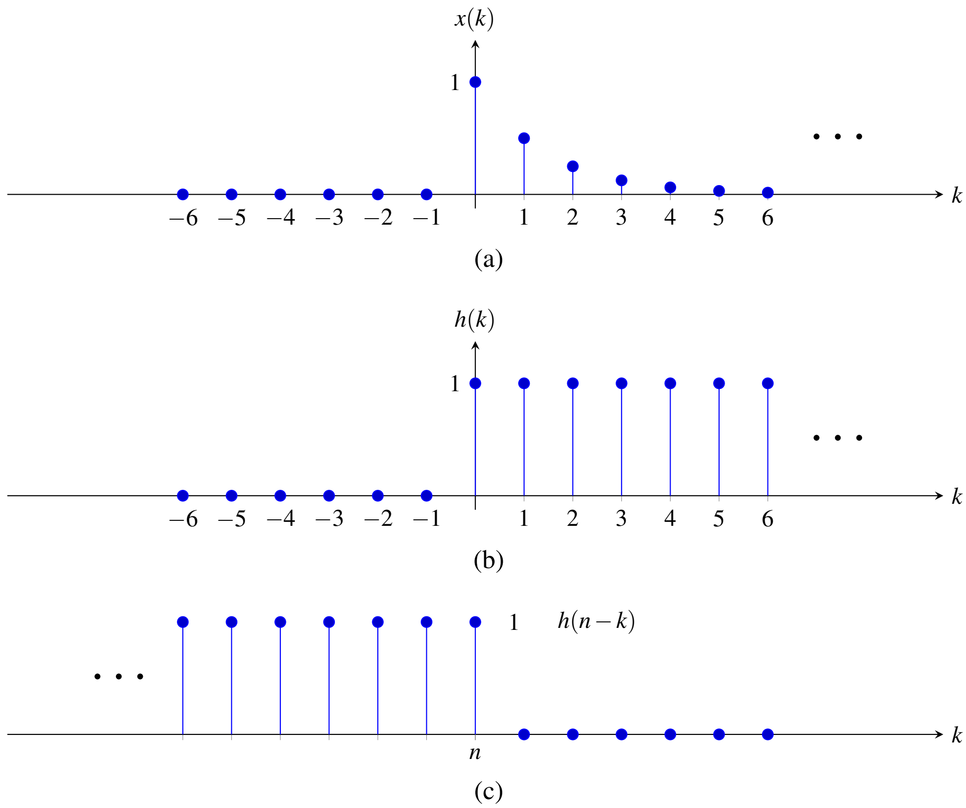


Figure 9.4: Plots for Example 9.4. Plots of (a) $x(k)$, (b) $h(k)$, and (c) $h(n-k)$ versus k.

Table 9.2: Convolution computation for Example 9.5

$n \backslash k$	-2	-1	0	1	2	3	4	5	6	7	
	$x(k)$										
			1	3	9	9	3	1			
	$h(k)$										
			1	-1							
	$h(-k)$										
		-1	1								
	$h(n-k)$										$x*h(n)$
-1	-1	1									0
0		-1	1								$1-0=1$
1			-1	1							$3-1=2$
2				-1	1						$9-3=6$
3					-1	1					$9-9=0$
4						-1	1				$3-9=-6$
5							-1	1			$1-3=-2$
6								-1	1		$0-1=-1$
7									-1	1	0

Theorem 9.1 (Commutativity of convolution)**.** *Convolution is commutative. That is, for any two sequences x and h,*

$$x * h = h * x. \tag{9.2}$$

In other words, the result of a convolution is not affected by the order of its operands.

Proof. We now provide a proof of the commutative property stated above. To begin, we expand the left-hand side of (9.2) as follows:

$$x * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k).$$

Next, we perform a change of variable. Let $v = n-k$ which implies that $k = n-v$. Using this change of variable, we can rewrite the previous equation as

$$\begin{aligned}
x * h(n) &= \sum_{v=n+\infty}^{n-\infty} x(n-v)h(v) \\
&= \sum_{v=\infty}^{-\infty} x(n-v)h(v) \\
&= \sum_{v=-\infty}^{\infty} x(n-v)h(v) \\
&= \sum_{v=-\infty}^{\infty} h(v)x(n-v) \\
&= h * x(n).
\end{aligned}$$

Thus, we have proven that convolution is commutative. ■

Theorem 9.2 (Associativity of convolution)**.** *Convolution is associative. That is, for any three sequences x, h_1, and h_2,*

$$(x * h_1) * h_2 = x * (h_1 * h_2). \tag{9.3}$$

In other words, the final result of multiple convolutions does not depend on how the convolution operations are grouped.

Proof. To begin, we use the definition of the convolution operation to expand the left-hand side of (9.3) as follows:

$$\begin{aligned}
([x * h_1] * h_2)(n) &= \sum_{\ell=-\infty}^{\infty} [x * h_1(\ell)]h_2(n-\ell) \\
&= \sum_{\ell=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x(k)h_1(\ell-k) \right) h_2(n-\ell).
\end{aligned}$$

Now, we change the order of summation to obtain

$$([x * h_1] * h_2)(n) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} x(k)h_1(\ell-k)h_2(n-\ell).$$

Pulling the factor of $x(k)$ out of the inner summation yields

$$([x * h_1] * h_2)(n) = \sum_{k=-\infty}^{\infty} x(k) \sum_{\ell=-\infty}^{\infty} h_1(\ell-k)h_2(n-\ell).$$

Next, we perform a change of variable. Let $\lambda = \ell - k$ which implies that $\ell = \lambda + k$. Using this change of variable, we can write

$$\begin{aligned}([x*h_1]*h_2)(n) &= \sum_{k=-\infty}^{\infty} x(k) \sum_{\lambda=-\infty-k}^{\infty-k} h_1(\lambda)h_2(n-\lambda-k) \\ &= \sum_{k=-\infty}^{\infty} x(k) \sum_{\lambda=-\infty}^{\infty} h_1(\lambda)h_2(n-\lambda-k) \\ &= \sum_{k=-\infty}^{\infty} x(k) \left(\sum_{\lambda=-\infty}^{\infty} h_1(\lambda)h_2([n-k]-\lambda) \right) \\ &= \sum_{k=-\infty}^{\infty} x(k) \left[h_1 * h_2(n-k) \right] \\ &= (x*[h_1*h_2])(n).\end{aligned}$$

Thus, we have proven that convolution is associative. ■

Theorem 9.3 (Distributivity of convolution)**.** *Convolution is distributive. That is, for any three sequences x, h_1, and h_2,*

$$x*(h_1+h_2) = x*h_1 + x*h_2. \tag{9.4}$$

In other words, convolution can be distributed across addition.

Proof. The proof of this property is relatively simple. Expanding the left-hand side of (9.4), we have:

$$\begin{aligned}(x*[h_1+h_2])(n) &= \sum_{k=-\infty}^{\infty} x(k)[h_1(n-k)+h_2(n-k)] \\ &= \sum_{k=-\infty}^{\infty} x(k)h_1(n-k) + \sum_{k=-\infty}^{\infty} x(k)h_2(n-k) \\ &= x*h_1(n) + x*h_2(n).\end{aligned}$$

Thus, we have shown that convolution is distributive. ■

The identity for an operation defined on elements of a set is often extremely helpful to know. Consider the operations of addition and multiplication as defined for real numbers. For any real number a, $a+0=a$. Since adding zero to a has no effect (i.e., the result is a), we call 0 the **additive identity**. For any real number a, $1\cdot a = a$. Since multiplying a by 1 has no effect (i.e., the result is a), we call 1 the **multiplicative identity**. Imagine for a moment how difficult arithmetic would be if we did not know that $a+0=a$ or $1\cdot a=a$. For this reason, identity values are clearly of fundamental importance.

Earlier, we were introduced to a new operation known as convolution. So, in light of the above, it is natural to wonder if there is a convolutional identity. In fact, there is, as given by the theorem below.

Theorem 9.4 (Convolutional identity)**.** *For any sequence x,*

$$x*\delta = x. \tag{9.5}$$

In other words, δ is the convolutional identity (i.e., convolving any sequence x with δ simply yields x).

Proof. Suppose that we have an arbitrary sequence x. From the definition of convolution, we can write

$$x*\delta(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k).$$

Now, let us employ a change of variable. Let $\lambda = -k$ so that $k = -\lambda$. Applying the change of variable, we obtain

$$\begin{aligned} x * \delta(n) &= \sum_{\lambda=-(-\infty)}^{-(\infty)} x(-\lambda)\delta(n+\lambda) \\ &= \sum_{\lambda=\infty}^{-\infty} x(-\lambda)\delta(n+\lambda) \\ &= \sum_{\lambda=-\infty}^{\infty} x(-\lambda)\delta(\lambda+n). \end{aligned} \tag{9.6}$$

From the equivalence property of δ, we can rewrite the preceding equation as

$$\begin{aligned} x * \delta(n) &= \sum_{\lambda=-\infty}^{\infty} x(-[-n])\delta(\lambda+n) \\ &= \sum_{\lambda=-\infty}^{\infty} x(n)\delta(\lambda+n). \end{aligned}$$

Factoring $x(n)$ out of the summation, we obtain

$$x * \delta(n) = x(n) \sum_{\lambda=-\infty}^{\infty} \delta(\lambda+n).$$

Since $\sum_{\lambda=-\infty}^{\infty} \delta(\lambda) = 1$ implies that $\sum_{\lambda=-\infty}^{\infty} \delta(\lambda+n) = 1$, we have

$$x * \delta(n) = x(n).$$

Thus, δ is the convolutional identity (i.e., $x * \delta = x$). (Alternatively, we could have directly applied the sifting property to (9.6) to show the desired result.) ■

9.4 Periodic Convolution

The convolution of two periodic sequences is usually not well defined. This motivates an alternative notion of convolution for periodic sequences known as periodic convolution. The **periodic convolution** of the N-periodic sequences x and h, denoted $x \circledast h$, is defined as

$$x \circledast h(n) = \sum_{k=\langle N \rangle} x(k)h(n-k),$$

where $\sum_{k=\langle N \rangle}$ denotes summation over an interval of length N. Equivalently, the periodic convolution can be written as

$$x \circledast h(n) = \sum_{k=0}^{N-1} x(k)h(\operatorname{mod}(n-k,N)),$$

where $\operatorname{mod}(a,b)$ is the remainder after division when a is divided by b. The periodic convolution and (linear) convolution of the N-periodic sequences x and h are related as

$$x \circledast h(n) = x_0 * h(n) \quad \text{where} \quad x(n) = \sum_{k=-\infty}^{\infty} x_0(n-Nk)$$

(i.e., $x_0(n)$ equals $x(n)$ over a single period of x and is zero elsewhere).

9.5 Characterizing LTI Systems and Convolution

As a matter of terminology, the **impulse response** h of a system $\mathcal{H}$ is defined as

$$h = \mathcal{H}\delta.$$

In other words, the impulse response of a system is the output that the system produces when presented with δ as an input. As it turns out, a LTI system has a very special relationship between its input, output, and impulse response, as given by the theorem below.

Theorem 9.5 (LTI systems and convolution)**.** *A LTI system $\mathcal{H}$ with impulse response h is such that*

$$\mathcal{H}x = x * h.$$

In other words, a LTI system computes a convolution. In particular, the output of the system is given by the convolution of the input and impulse response.

Proof. To begin, we assume that $\mathcal{H}$ is LTI (i.e., $\mathcal{H}$ is both linear and time invariant). Using the fact that δ is the convolutional identity, we have

$$\mathcal{H}x = \mathcal{H}\{x * \delta\}.$$

From the definition of convolution, we have

$$\mathcal{H}x = \mathcal{H}\left\{\sum_{k=-\infty}^{\infty} x(k)\delta(\cdot - k)\right\}.$$

Since $\mathcal{H}$ is linear, we can pull the summation and $x(k)$ (which is a constant with respect to the operation performed by $\mathcal{H}$) outside $\mathcal{H}$ to obtain

$$\mathcal{H}x = \sum_{k=-\infty}^{\infty} x(k)\mathcal{H}\{\delta(\cdot - k)\}. \tag{9.7}$$

Since $\mathcal{H}$ is time invariant, we can interchange the order of $\mathcal{H}$ and the time shift of δ by k. That is, we have

$$\mathcal{H}\{\delta(\cdot - k)\} = h(\cdot - k).$$

Thus, we can rewrite (9.7) as

$$\begin{aligned}\mathcal{H}x &= \sum_{k=-\infty}^{\infty} x(k)h(\cdot - k) \\ &= x * h.\end{aligned}$$

Thus, we have shown that $\mathcal{H}x = x * h$, where $h = \mathcal{H}\delta$. ■

By Theorem 9.5 above, the behavior of a LTI system is completely characterized by its impulse response. That is, if the impulse response of a system is known, we can determine the response of the system to *any* input. Consequently, the impulse response provides a very powerful tool for the study of LTI systems.

Example 9.6. Consider a LTI system $\mathcal{H}$ with impulse response

$$h(n) = u(n). \tag{9.8}$$

Show that $\mathcal{H}$ is characterized by the equation

$$\mathcal{H}x(n) = \sum_{k=-\infty}^{n} x(k) \tag{9.9}$$

(i.e., $\mathcal{H}$ corresponds to an ideal accumulator).

Solution. Since the system is LTI, we have that

$$\mathcal{H}x(n) = x * h(n).$$

Substituting (9.8) into the preceding equation, and simplifying we obtain

$$\begin{aligned}
\mathcal{H}x(n) &= x * h(n) \\
&= x * u(n) \\
&= \sum_{k=-\infty}^{\infty} x(k)u(n-k) \\
&= \sum_{k=-\infty}^{n} x(k)u(n-k) + \sum_{k=n+1}^{\infty} x(k)u(n-k) \\
&= \sum_{k=-\infty}^{n} x(k).
\end{aligned}$$

Therefore, the system with the impulse response h given by (9.8) is, in fact, the ideal accumulator given by (9.9). ■

Example 9.7. Consider a LTI system with input x, output y, and impulse response h, where

$$h(n) = \begin{cases} 1 & 0 \le n \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find and plot the response of the system to the particular input x given by

$$x(n) = \begin{cases} n+1 & 0 \le n \le 3 \\ 7-n & 4 \le n \le 6 \\ 0 & \text{otherwise.} \end{cases}$$

Solution. Since the system is LTI, we have

$$y = x * h.$$

So, in this example, we are essentially being asked to compute $x * h$. This convolution is likely most easily performed by using a graphical or tabular approach. In what follows, we elect to use a tabular approach. By constructing a table that shows x and the various time-reversed and shifted versions of h, we can easily compute the elements of $x * h$. The result of this process is shown in Table 9.3. From this table, we have

$$x * h(n) = \begin{cases} 1 & n = 0 \\ 3 & n = 1 \\ 6 & n = 2 \\ 9 & n = 3 \\ 10 & n = 4 \\ 9 & n = 5 \\ 6 & n = 6 \\ 3 & n = 7 \\ 1 & n = 8 \\ 0 & \text{otherwise.} \end{cases}$$ ■

Table 9.3: Convolution computation for Example 9.7

$n \backslash k$	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	
	$x(k)$													
				1	2	3	4	3	2	1				
	$h(k)$													
				1	1	1								
	$h(-k)$													
		1	1	1										
	$h(n-k)$													$x * h(n)$
-1	1	1	1											0
0		1	1	1										$0+0+1=1$
1			1	1	1									$0+1+2=3$
2				1	1	1								$1+2+3=6$
3					1	1	1							$2+3+4=9$
4						1	1	1						$3+4+3=10$
5							1	1	1					$4+3+2=9$
6								1	1	1				$3+2+1=6$
7									1	1	1			$2+1+0=3$
8										1	1	1		$1+0+0=1$
9											1	1	1	0

9.6 Unit Step Response of LTI Systems

The **step response** s of a system $\mathcal{H}$ is defined as

$$s = \mathcal{H}u$$

(i.e., the step response of a system is the output it produces for a unit-step sequence input). In the case of a LTI system, it turns out that the step response is closely related to the impulse response, as given by the theorem below.

Theorem 9.6. *The step response s and impulse response h of a LTI system are related as*

$$h(n) = s(n) - s(n-1) \quad \text{and} \quad s(n) = \sum_{k=-\infty}^{n} h(k).$$

That is, the impulse response h is the first difference of the step response s.

Proof. Using the fact that $s = u * h$, we can write

$$\begin{aligned} s(n) &= u * h(n) \\ &= h * u(n) \\ &= \sum_{k=-\infty}^{\infty} h(k)u(n-k) \\ &= \sum_{k=-\infty}^{n} h(k). \end{aligned}$$

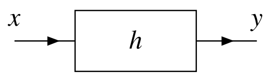


Figure 9.5: Block diagram representation of discrete-time LTI system with input x, output y, and impulse response h.

Thus, s can be obtained by accumulating h. Taking the first difference of s, we obtain

$$\begin{aligned}s(n)-s(n-1) &= \sum_{k=-\infty}^{n} h(k) - \sum_{k=-\infty}^{n-1} h(k) \\ &= h(n) + \sum_{k=-\infty}^{n-1} h(k) - \sum_{k=-\infty}^{n-1} h(k) \\ &= h(n).\end{aligned}$$

Thus, h is the first difference of s. ■

The step response is often of great practical interest, since it can be used to determine the impulse response of a LTI system. In particular, the impulse response can be determined from the step response via differencing.

9.7 Block Diagram Representation of Discrete-Time LTI Systems

Frequently, it is convenient to represent discrete-time LTI systems in block diagram form. Since a LTI system is completely characterized by its impulse response, we often label such a system with its impulse response in a block diagram. That is, we represent a LTI system with input x, output y, and impulse response h, as shown in Figure 9.5.

9.8 Interconnection of Discrete-Time LTI Systems

Suppose that we have a LTI system with input x, output y, and impulse response h. We know that x and y are related as $y = x * h$. In other words, the system can be viewed as performing a convolution operation. From the properties of convolution introduced earlier, we can derive a number of equivalences involving the impulse responses of series- and parallel-interconnected systems.

Consider two LTI systems with impulse responses h_1 and h_2 that are connected in a series configuration, as shown on the left-side of Figure 9.6(a). From the block diagram on the left side of Figure 9.6(a), we have

$$y = (x * h_1) * h_2.$$

Due to the associativity of convolution, however, this is equivalent to

$$y = x * (h_1 * h_2).$$

Thus, the series interconnection of two LTI systems behaves as a single LTI system with impulse response $h_1 * h_2$. In other words, we have the equivalence shown in Figure 9.6(a).

Consider two LTI systems with impulse responses h_1 and h_2 that are connected in a series configuration, as shown on the left-side of Figure 9.6(b). From the block diagram on the left side of Figure 9.6(b), we have

$$y = (x * h_1) * h_2.$$

Due to the associativity and commutativity of convolution, this is equivalent to

$$\begin{aligned}y &= x * (h_1 * h_2) \\ &= x * (h_2 * h_1) \\ &= (x * h_2) * h_1.\end{aligned}$$

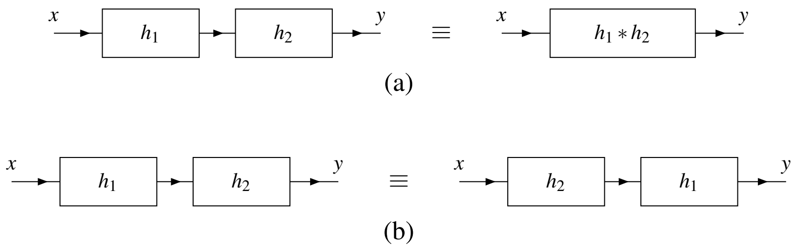


Figure 9.6: Equivalences for the series interconnection of discrete-time LTI systems. The (a) first and (b) second equivalences.

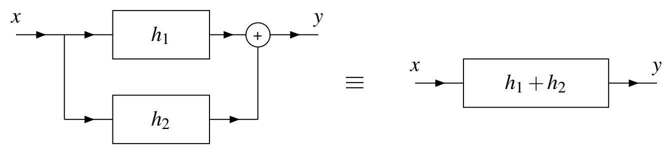


Figure 9.7: Equivalence for the parallel interconnection of discrete-time LTI systems.

Thus, interchanging the two LTI systems does not change the behavior of the overall system with input x and output y. In other words, we have the equivalence shown in Figure 9.6(b).

Consider two LTI systems with impulse responses h_1 and h_2 that are connected in a parallel configuration, as shown on the left-side of Figure 9.7. From the block diagram on the left side of Figure 9.7, we have

$$y = x * h_1 + x * h_2.$$

Due to convolution being distributive, however, this equation can be rewritten as

$$y = x * (h_1 + h_2).$$

Thus, the parallel interconnection of two LTI systems behaves as a single LTI system with impulse response $h_1 + h_2$. In other words, we have the equivalence shown in Figure 9.7.

Example 9.8. Consider the system with input x, output y, and impulse response h as shown in Figure 9.8. Each subsystem in the block diagram is LTI and labelled with its impulse response. Find h.

Solution. From the left half of the block diagram, we can write

$$\begin{aligned} v &= x + x * h_1 + x * h_2 \\ &= x * \delta + x * h_1 + x * h_2 \\ &= x * (\delta + h_1 + h_2). \end{aligned}$$

Similarly, from the right half of the block diagram, we can write

$$y = v * h_3.$$

Substituting the expression for v into the preceding equation we obtain

$$\begin{aligned} y &= v * h_3 \\ &= (x * [\delta + h_1 + h_2]) * h_3 \\ &= x * ([\delta + h_1 + h_2] * h_3) \\ &= x * (h_3 + h_1 * h_3 + h_2 * h_3). \end{aligned}$$

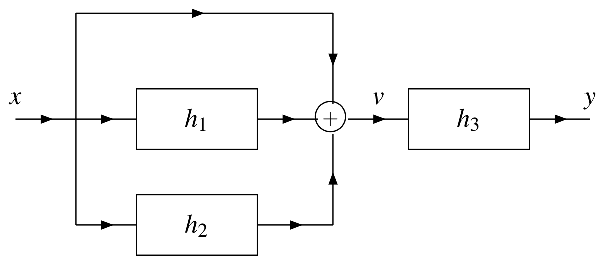


Figure 9.8: System interconnection example.

Thus, the impulse response h of the overall system is

$$h = h_3 + h_1 * h_3 + h_2 * h_3. \qquad \blacksquare$$

9.9 Properties of Discrete-Time LTI Systems

In the previous chapter, we introduced a number of properties that might be possessed by a system (e.g., memory, causality, BIBO stability, and invertibility). Since a LTI system is completely characterized by its impulse response, one might wonder if there is a relationship between some of the properties introduced previously and the impulse response. In what follows, we explore some of these relationships.

9.9.1 Memory

The first system property to be considered is memory.

Theorem 9.7 (Memorylessness of LTI system)**.** *A LTI system with impulse response h is memoryless if and only if*

$$h(n) = 0 \textit{ for all } n \neq 0.$$

Proof. Recall that a system is memoryless if its output y at any arbitrary time depends only on the value of its input x at that same time. Suppose now that we have a LTI system with input x, output y, and impulse response h. The output y at some arbitrary time n_0 is given by

$$\begin{aligned} y(n_0) &= x * h(n_0) \\ &= h * x(n_0) \\ &= \sum_{k=-\infty}^{\infty} h(k)x(n_0 - k). \end{aligned}$$

Consider the summation in the above equation. In order for the system to be memoryless, the result of the summation is allowed to depend on $x(n)$ only for $n = n_0$. This, however, is only possible if

$$h(n) = 0 \quad \text{for all } n \neq 0. \qquad \blacksquare$$

From the preceding theorem, it follows that a memoryless LTI system must have an impulse response h of the form

$$h = K\delta \tag{9.10}$$

where K is a complex constant. As a consequence of this fact, we also have that all memoryless LTI systems must have an input-output relation of the form

$$\begin{aligned} y &= x * (K\delta) \\ &= K(x*\delta) \\ &= Kx. \end{aligned}$$

In other words, a memoryless LTI system must be an ideal amplifier (i.e., a system that simply performs amplitude scaling).

Example 9.9. Consider the LTI system with the impulse response h given by

$$h(n) = e^{-an}u(n),$$

where a is a real constant. Determine whether this system has memory.

Solution. The system has memory since $h(n) \neq 0$ for some $n \neq 0$ (e.g., $h(1) = e^{-a} \neq 0$). ■

Example 9.10. Consider the LTI system with the impulse response h given by

$$h(n) = \delta(n).$$

Determine whether this system has memory.

Solution. Clearly, h is only nonzero at the origin. This follows immediately from the definition of the unit-impulse sequence δ. Therefore, the system is memoryless (i.e., does not have memory). ■

9.9.2 Causality

The next system property to be considered is causality.

Theorem 9.8 (Causality of LTI system)**.** *A LTI system with impulse response h is causal if and only if*

$$h(n) = 0 \textit{ for all } n < 0.$$

(i.e., h is causal).

Proof. Recall that a system is causal if its output y at any arbitrary time n_0 does not depend on its input x at a time later than n_0. Suppose that we have the LTI system with input x, output y, and impulse response h. The value of the output y at n_0 is given by

$$\begin{aligned} y(n_0) &= x*h(n_0) \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n_0-k) \\ &= \sum_{k=-\infty}^{n_0} x(k)h(n_0-k) + \sum_{k=n_0+1}^{\infty} x(k)h(n_0-k). \end{aligned} \tag{9.11}$$

In order for the expression for $y(n_0)$ in (9.11) not to depend on $x(n)$ for $n > n_0$, we must have that

$$h(n) = 0 \quad \text{for } n < 0 \tag{9.12}$$

(i.e., h is causal). In this case, (9.11) simplifies to

$$y(n_0) = \sum_{k=-\infty}^{n_0} x(k)h(n_0-k).$$

Clearly, the result of this integration does not depend on $x(n)$ for $n > n_0$ (since k varies from $-\infty$ to n_0). Therefore, a LTI system is causal if its impulse response h satisfies (9.12). ■

Example 9.11. Consider the LTI system with impulse response h given by

$$h(n) = e^{-an}u(n),$$

where a is a real constant. Determine whether this system is causal.

Solution. Clearly, $h(n) = 0$ for $n < 0$ (due to the $u(n)$ factor in the expression for $h(n)$). Therefore, the system is causal. ■

Example 9.12. Consider the LTI system with impulse response h given by

$$h(n) = \delta(n+n_0),$$

where n_0 is a strictly positive real constant. Determine whether this system is causal.

Solution. From the definition of δ, we can easily deduce that $h(n) = 0$ except at $n = -n_0$. Since $-n_0 < 0$, the system is not causal. ■

9.9.3 Invertibility

The next system property to be considered is invertibility.

Theorem 9.9 (Inverse of LTI system)**.** *Let $\mathcal{H}$ be a LTI system with impulse response h. If the inverse $\mathcal{H}^{-1}$ of $\mathcal{H}$ exists, $\mathcal{H}^{-1}$ is LTI and has an impulse response h_{inv} that satisfies*

$$h * h_{\mathsf{inv}} = \delta.$$

Proof. To begin, we need to show that the inverse of a LTI system, if it exists, must also be LTI. This part of the proof, however, is left as an exercise for the reader in Exercise 9.14. (The general approach to take for this problem is to show that: 1) the inverse of a linear system, if it exists, is linear; and 2) the inverse of a time-invariant system, if it exists, is time invariant.) We assume that this part of the proof has been demonstrated and proceed.

Suppose now that the inverse system $\mathcal{H}^{-1}$ exists. We have that

$$\mathcal{H}x = x * h \quad \text{and} \quad \mathcal{H}^{-1}x = x * h_{\mathsf{inv}}.$$

From the definition of an inverse system, we have that, for every sequence x,

$$\mathcal{H}^{-1}\mathcal{H}x = x.$$

Expanding the left-hand side of the preceding equation, we obtain

$$\begin{aligned} &\mathcal{H}^{-1}[x*h] = x \\ \Leftrightarrow\quad & x * h * h_{\mathsf{inv}} = x. \end{aligned} \tag{9.13}$$

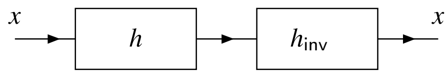


Figure 9.9: System in cascade with its inverse.

This relationship is expressed diagrammatically in Figure 9.9. Since the unit-impulse sequence is the convolutional identity, we can equivalently rewrite (9.13) as

$$x * h * h_{\mathsf{inv}} = x * \delta.$$

This equation, however, must hold for arbitrary x. Thus, by comparing the left- and right-hand sides of this equation, we conclude

$$h * h_{\mathsf{inv}} = \delta. \tag{9.14}$$

Therefore, if $\mathcal{H}^{-1}$ exists, it must have an impulse response h_{inv} that satisfies (9.14). This completes the proof. ■

From the preceding theorem, we have the following result:

Theorem 9.10 (Invertibility of LTI system)**.** *A LTI system $\mathcal{H}$ with impulse response h is invertible if and only if there exists a sequence h_{inv} satisfying*

$$h * h_{\mathsf{inv}} = \delta.$$

Proof. The proof follows immediately from the result of Theorem 9.9 by simply observing that $\mathcal{H}$ being invertible is equivalent to the existence of $\mathcal{H}^{-1}$. ■

Example 9.13. Consider the LTI system $\mathcal{H}$ with impulse response h given by

$$h(n) = A\delta(n - n_0),$$

where A is a nonzero real constant and n_0 is an integer constant. Determine if $\mathcal{H}$ is invertible, and if it is, find the impulse response h_{inv} of the system $\mathcal{H}^{-1}$.

Solution. If the system $\mathcal{H}^{-1}$ exists, its impulse response h_{inv} is given by the solution to the equation

$$h * h_{\mathsf{inv}} = \delta. \tag{9.15}$$

So, let us attempt to solve this equation for h_{inv}. Substituting the given sequence h into (9.15) and using straightforward algebraic manipulation, we can write

$$\begin{aligned}
& h * h_{\mathsf{inv}}(n) = \delta(n) \\
\Rightarrow\quad & \sum_{k=-\infty}^{\infty} h(k) h_{\mathsf{inv}}(n-k) = \delta(n) \\
\Rightarrow\quad & \sum_{k=-\infty}^{\infty} A\delta(k-n_0) h_{\mathsf{inv}}(n-k) = \delta(n) \\
\Rightarrow\quad & \sum_{k=-\infty}^{\infty} \delta(k-n_0) h_{\mathsf{inv}}(n-k) = \tfrac{1}{A}\delta(n).
\end{aligned}$$

Using the sifting property of the unit-impulse sequence, we can simplify the summation on the left-hand side of the preceding equation to obtain

$$h_{\mathsf{inv}}(n - n_0) = \tfrac{1}{A}\delta(n). \tag{9.16}$$

Substituting $n + n_0$ for n in the preceding equation yields

$$\begin{aligned}
& h_{\mathsf{inv}}([n+n_0] - n_0) = \tfrac{1}{A}\delta(n+n_0) \\
\Rightarrow\quad & h_{\mathsf{inv}}(n) = \tfrac{1}{A}\delta(n+n_0).
\end{aligned}$$

Since $A \neq 0$, the sequence h_{inv} is always well defined. Thus, $\mathcal{H}^{-1}$ exists and consequently $\mathcal{H}$ is invertible. ■

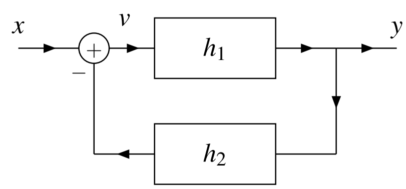


Figure 9.10: Feedback system with input x and output y.

Example 9.14. Consider the system with the input x and output y as shown in Figure 9.10. Each subsystem in the block diagram is LTI and labelled with its impulse response. Use the notion of an inverse system in order to express y in terms of x.

Solution. From Figure 9.10, we can write:

$$v = x - y * h_2 \quad \text{and} \tag{9.17}$$

$$y = v * h_1. \tag{9.18}$$

Substituting (9.17) into (9.18), and simplifying we obtain

$$\begin{aligned} & y = [x - y * h_2] * h_1 \\ \Rightarrow \quad & y = x * h_1 - y * h_2 * h_1 \\ \Rightarrow \quad & y + y * h_2 * h_1 = x * h_1 \\ \Rightarrow \quad & y * \delta + y * h_2 * h_1 = x * h_1 \\ \Rightarrow \quad & y * [\delta + h_2 * h_1] = x * h_1. \end{aligned} \tag{9.19}$$

For convenience, we now define the sequence g as

$$g = \delta + h_2 * h_1. \tag{9.20}$$

So, we can rewrite (9.19) as

$$y * g = x * h_1. \tag{9.21}$$

Thus, we have almost solved for y in terms of x. To complete the solution, we need to eliminate g from the left-hand side of the equation. To do this, we use the notion of an inverse system. Consider the inverse of the system with impulse response g. This inverse system has an impulse response g_{inv} given by

$$g * g_{\mathsf{inv}} = \delta. \tag{9.22}$$

This relationship follows from the definition of an inverse system. Now, we use g_{inv} in order to simplify (9.21) as follows:

$$\begin{aligned} & y * g = x * h_1 \\ \Rightarrow \quad & y * g * g_{\mathsf{inv}} = x * h_1 * g_{\mathsf{inv}} \\ \Rightarrow \quad & y * \delta = x * h_1 * g_{\mathsf{inv}} \\ \Rightarrow \quad & y = x * h_1 * g_{\mathsf{inv}}. \end{aligned}$$

Thus, we can express the output y in terms of the input x as

$$y = x * h_1 * g_{\mathsf{inv}},$$

where g_{inv} is given by (9.22) and g is given by (9.20). ∎

9.9.4 BIBO Stability

The last system property to be considered is BIBO stability.

Theorem 9.11 (BIBO stability of LTI system)**.** *A LTI system with impulse response h is BIBO stable if and only if*

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \tag{9.23}$$

(i.e., h is absolutely summable).

Proof. Recall that a system is BIBO stable if every bounded input to the system produces a bounded output. Suppose that we have a LTI system with input x, output y, and impulse response h.

First, we consider the sufficiency of (9.23) for BIBO stability. Assume that $|x(n)| \leq A < \infty$ for all n (i.e., x is bounded). We can write

$$\begin{aligned} y(n) &= x * h(n) \\ &= h * x(n) \\ &= \sum_{k=-\infty}^{\infty} h(k)x(n-k). \end{aligned}$$

By taking the magnitude of both sides of the preceding equation, we obtain

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right|. \tag{9.24}$$

One can show, for any two sequences f_1 and f_2, that

$$\left| \sum_{n=-\infty}^{\infty} f_1(n) f_2(n) \right| \leq \sum_{n=-\infty}^{\infty} |f_1(n) f_2(n)|.$$

Using this inequality, we can rewrite (9.24) as

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)x(n-k)| = \sum_{k=-\infty}^{\infty} |h(k)| \, |x(n-k)|.$$

We know (by assumption) that $|x(n)| \leq A$ for all n, so we can replace $|x(n)|$ by its bound A in the above inequality to obtain

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)| \, |x(n-k)| \leq \sum_{k=-\infty}^{\infty} A \, |h(k)| = A \sum_{k=-\infty}^{\infty} |h(k)|. \tag{9.25}$$

Thus, we have

$$|y(n)| \leq A \sum_{k=-\infty}^{\infty} |h(k)|. \tag{9.26}$$

Since A is finite, we can deduce from (9.26) that y is bounded if

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \tag{9.27}$$

(i.e., h is absolutely summable). Thus, the absolute summability of the impulse response h is a sufficient condition for BIBO stability.

Now, we consider the necessity of (9.23) for BIBO stability. Suppose that h is not absolutely summable. That is, suppose that

$$\sum_{k=-\infty}^{\infty} |h(k)| = \infty.$$

If such is the case, we can show that the system is not BIBO stable. To begin, consider the particular input x given by

$$x(n) = e^{-j\arg[h(-n)]}.$$

Since $\left|e^{j\theta}\right| = 1$ for all real θ, x is bounded (i.e., $|x(n)| \leq 1$ for all n). The output y is given by

$$\begin{aligned} y(n) &= x * h(n) \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= \sum_{k=-\infty}^{\infty} \left[e^{-j\arg[h(-k)]}\right] h(n-k). \end{aligned} \tag{9.28}$$

Now, consider the output value $y(n)$ at $n = 0$. From (9.28), we have

$$y(0) = \sum_{k=-\infty}^{\infty} \left[e^{-j\arg[h(-k)]}\right] h(-k). \tag{9.29}$$

Since $e^{-j\arg z}z = |z|$ for all complex z, we have that $[e^{-j\arg[h(-k)]}]h(-k) = |h(-k)|$ and we can simplify (9.29) to obtain

$$\begin{aligned} y(0) &= \sum_{k=-\infty}^{\infty} |h(-k)| \\ &= \sum_{k=-\infty}^{\infty} |h(k)| \\ &= \infty. \end{aligned}$$

Thus, we have shown that the bounded input x will result in an unbounded output y (where $y(n)$ is unbounded for $n = 0$). Thus, the absolute summability of h is also necessary for BIBO stability. This completes the proof. ■

Example 9.15. Consider the LTI system with impulse response h given by

$$h(n) = e^{an}u(n),$$

where a is a real constant. Determine for what values of a the system is BIBO stable.

Solution. We need to determine for what values of a the impulse response h is absolutely summable. Suppose that $a \neq 0$. We can write

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |h(n)| &= \sum_{n=-\infty}^{\infty} |e^{an}u(n)| \\ &= \sum_{n=-\infty}^{\infty} e^{an}u(n) \\ &= \sum_{n=-\infty}^{-1} 0 + \sum_{n=0}^{\infty} e^{an} \\ &= \sum_{n=0}^{\infty} e^{an} \\ &= \sum_{n=0}^{\infty} (e^{a})^n. \end{aligned}$$

The right-hand side is an infinite geometric sequence. This converges to a finite value if and only if

$$e^a < 1.$$

Taking the logarithm of both sides, we obtain

$$a < \log 1 = 0.$$

Therefore, h is absolutely summable if $a < 0$ and infinite if $a \geq 0$. Consequently, the system is BIBO stable if and only if $a < 0$. ■

Example 9.16 (Ideal accumulator)**.** Consider the LTI system with input x and output y defined by

$$y(n) = \sum_{k=-\infty}^{n} x(k).$$

Determine whether this system is BIBO stable.

Solution. First, we find the impulse response h of the system. We have

$$\begin{aligned} h(n) &= \sum_{k=-\infty}^{n} \delta(k) \\ &= \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \\ &= u(n). \end{aligned}$$

Using this expression for h, we now check to see if h is absolutely summable. We have

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |h(n)| &= \sum_{n=-\infty}^{\infty} |u(n)| \\ &= \sum_{n=0}^{\infty} 1 \\ &= \infty. \end{aligned}$$

Thus, h is not absolutely summable. Therefore, the system is not BIBO stable. ■

9.10 Eigensequences of Discrete-Time LTI Systems

Earlier, in Section 8.7.7, we were introduced to notion of eigensequences of systems. Given that eigensequences have the potential to simplify the mathematics associated with systems, it is natural to wonder what eigensequences LTI systems might have. In this regard, the following theorem is enlightening.

Theorem 9.12 (Eigensequences of LTI systems)**.** *For an arbitrary LTI system $\mathcal{H}$ with impulse response h and a sequence of the form $x(n) = z^n$, where z is an arbitrary complex constant (i.e., x is an arbitrary complex exponential), the following holds:*

$$\mathcal{H}x(n) = H(z)z^n,$$

where

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}. \tag{9.30}$$

That is, x is an eigensequence of $\mathcal{H}$ with the corresponding eigenvalue $H(z)$.

Proof. To begin, we observe that a system $\mathcal{H}$ is LTI if and only if it computes a convolution (i.e., $\mathcal{H}x = x * h$ for some h). We have

$$\begin{aligned}
\mathcal{H}x(n) &= x * h(n) \\
&= h * x(n) \\
&= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\
&= \sum_{k=-\infty}^{\infty} h(k)z^{n-k} \\
&= z^n \sum_{k=-\infty}^{\infty} h(k)z^{-k} \\
&= z^n H(z).
\end{aligned}$$

■

As a matter of terminology, the function H that appears in the preceding theorem (i.e., Theorem 9.12) is referred to as the **system function** (or **transfer function**) of the system $\mathcal{H}$. The system function completely characterizes the behavior of a LTI system. Consequently, system functions are often useful when working with LTI systems. As it turns out, a summation of the form appearing in (9.30) is of great importance, as it defines what is called the z transform. We will study the z transform in great depth later in Chapter 12.

Note that a LTI system can have eigensequences other than complex exponentials. For example, the system in Example 8.38 is LTI and has every sequence as an eigensequence. Also, a system that has every complex exponential as an eigensequence is not necessarily LTI. This is easily demonstrated by the example below (as well as Exercise 9.17).

Example 9.17. Let S denote the set of all complex exponential sequences (i.e., S is the set of all sequences x of the form $x(n) = ba^n$ where $a, b \in \mathbb{C}$). Consider the system $\mathcal{H}$ given by

$$\mathcal{H}x = \begin{cases} x & x \in S \\ 1 & \text{otherwise.} \end{cases}$$

For any sequence $x \in S$, we have $\mathcal{H}x = x$, implying that x is an eigensequence of $\mathcal{H}$ with eigenvalue 1. Therefore, every complex exponential sequence is an eigensequence of $\mathcal{H}$.

Now, we show that $\mathcal{H}$ is not linear. In what follows, let a denote an arbitrary complex constant. Consider the sequence $x(n) = n$. Clearly, $x \notin S$. Since $x \notin S$, we have $\mathcal{H}x = 1$, which implies that

$$a\mathcal{H}x = a.$$

Next, consider the sequence $ax(n) = an$. Since $ax \notin S$, we have

$$\mathcal{H}(ax) = 1.$$

From the above equations, however, we conclude that $\mathcal{H}(ax) = a\mathcal{H}x$ only in the case that $a = 1$. Therefore, $\mathcal{H}$ is not homogeneous and consequently not linear. So, $\mathcal{H}$ is an example of a system that has every complex exponential as an eigensequence, but is not LTI. ■

Let us now consider an application of eigensequences. Since convolution can often be quite painful to handle at the best of times, let us exploit eigensequences in order to devise a means to avoid having to deal with convolution directly in certain circumstances.

Suppose that we have a LTI system with input x, output y, and impulse response h. Suppose now that we can express some arbitrary input x as a sum of complex exponentials as follows:

$$x(n) = \sum_k a_k z_k^n.$$

From the eigensequence properties of LTI systems, the response to the input $a_k z_k^n$ is $a_k H(z_k) z^n$. By using this knowledge and the superposition property, we can write

$$\begin{aligned} y(n) &= \mathcal{H}x(n) \\ &= \mathcal{H}\left\{\sum_k a_k z_k^n\right\}(n) \\ &= \sum_k a_k \mathcal{H}\{z_k^n\}(n) \\ &= \sum_k a_k H(z_k) z_k^n. \end{aligned}$$

Thus, if an input to a LTI system can be represented as a linear combination of complex exponentials, the output can also be represented as linear combination of the same complex exponentials.

Example 9.18. Consider the LTI system $\mathcal{H}$ with the impulse response h given by

$$h(n) = \delta(n-1). \tag{9.31}$$

(a) Find the system function H of the system $\mathcal{H}$. (b) Use the system function H to determine the response y of the system $\mathcal{H}$ to the particular input x given by

$$x(n) = e^n \cos(\pi n).$$

Solution. (a) Substituting (9.31) into (9.30), we obtain

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \delta(n-1) z^{-n} \\ &= z^{-1}. \end{aligned}$$

(b) To begin, we can rewrite x as

$$\begin{aligned} x(n) &= e^n \left[\tfrac{1}{2}\left(e^{j\pi n} + e^{-j\pi n}\right)\right] \\ &= \tfrac{1}{2}\left(e^{1+j\pi}\right)^n + \tfrac{1}{2}\left(e^{1-j\pi}\right)^n. \end{aligned}$$

So, the input x is now expressed in the form

$$x(n) = \sum_{k=0}^{1} a_k z_k^n,$$

where

$$a_0 = a_1 = \tfrac{1}{2}, \quad \text{and} \quad z_k = \begin{cases} e^{1+j\pi} & k=0 \\ e^{1-j\pi} & k=1. \end{cases}$$

In part (a), we found the system function H to be $H(z) = z^{-1}$. So we can calculate y by using the system function as follows:

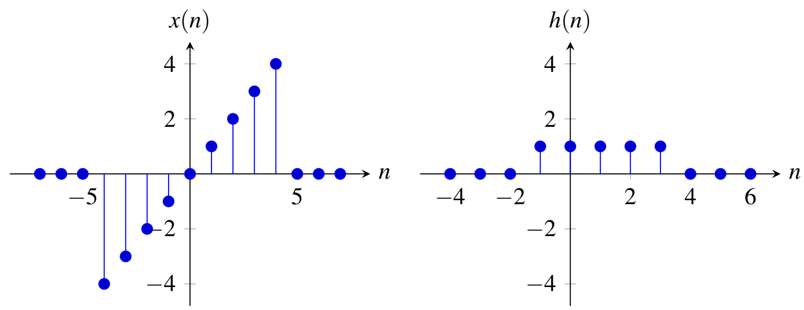
$$\begin{aligned}
y(n) &= \sum_k a_k H(z_k) z_k^n \\
&= a_0 H(z_0) z_0^n + a_1 H(z_1) z_1^n \\
&= \tfrac{1}{2} H(e^{1+j\pi}) e^{(1+j\pi)n} + \tfrac{1}{2} H(e^{1-j\pi}) e^{(1-j\pi)n} \\
&= \tfrac{1}{2} e^{-(1+j\pi)} e^{(1+j\pi)n} + \tfrac{1}{2} e^{-(1-j\pi)} e^{(1-j\pi)n} \\
&= \tfrac{1}{2} e^{n-1+j\pi n-j\pi} + \tfrac{1}{2} e^{n-1-j\pi n+j\pi} \\
&= \tfrac{1}{2} e^{n-1} e^{j\pi(n-1)} + \tfrac{1}{2} e^{n-1} e^{-j\pi(n-1)} \\
&= e^{n-1} \left[\tfrac{1}{2} \left(e^{j\pi(n-1)} + e^{-j\pi(n-1)} \right) \right] \\
&= e^{n-1} \cos[\pi(n-1)].
\end{aligned}$$

Observe that the output y is just the input x time shifted by 1. This is not a coincidence because, as it turns out, a LTI system with the system function $H(z) = z^{-1}$ is an ideal unit delay (i.e., a system that performs a time shift of 1). ■

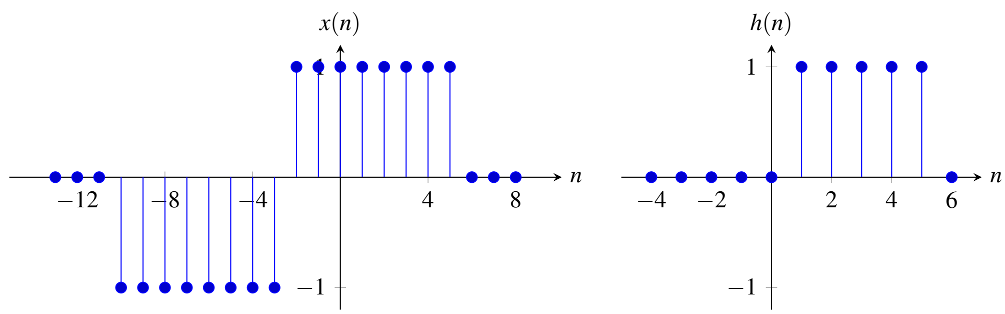
9.11 Exercises

9.11.1 Exercises Without Answer Key

9.1 For each pair of sequences x and h given in the figures below, compute $x * h$.



(a)



(b)

9.2 Compute $x * h$ for each pair of sequences x and h given below.
(a) $x(n) = n[u(n+2) - u(n-5)]$ and $h(n) = u(n)$;
(b) $x(n) = 2^n u(-n)$ and $h(n) = u(n)$;
(c) $x(n) = \begin{cases} n & 0 \leq n \leq 3 \\ 6-n & 4 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases}$ and $h(n) = u(n) - u(n-6)$; and
(d) $x(n) = u(n)$ and $h(n) = u(n-3)$.

9.3 Let x, y, h, and v be sequences such that $y = x * h$ and

$$v(n) = \sum_{k=-\infty}^{\infty} x(-k-b)h(k+an),$$

where a and b are integer constants. Express v in terms of y.

9.4 Consider the convolution $y = x * h$. Assuming that the convolution y exists, prove that each of the following assertions is true:
(a) If x is periodic, then y is periodic.
(b) If x is even and h is odd, then y is odd.

9.5 Let $\mathcal{D}x(n) = x(n) - x(n-1)$ (i.e., $\mathcal{D}$ is the first-difference operator). From the definition of convolution, show that if $y = x * h$, then $\mathcal{D}y = x * \mathcal{D}h$.

9.6 Let x and h be sequences satisfying

$$x(n) = 0 \quad \text{for } n < A_1 \text{ or } n > A_2, \quad \text{and}$$
$$h(n) = 0 \quad \text{for } n < B_1 \text{ or } n > B_2$$

(i.e., x and h are finite duration). Determine for which values of n the quantity $x * h(n)$ must be zero.

9.7 Find the impulse response of the LTI system $\mathcal{H}$ characterized by each of the equations below.

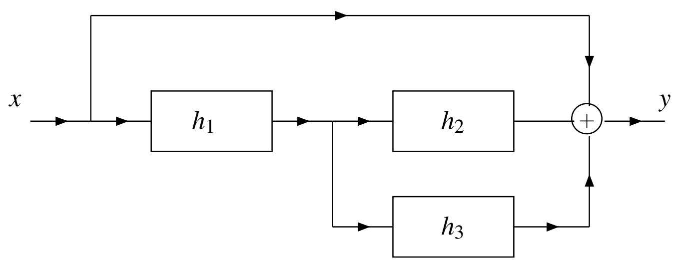
(a) $\mathcal{H}x(n) = \sum_{k=-\infty}^{n+1} x(k)$;

(b) $\mathcal{H}x(n) = \sum_{k=-\infty}^{\infty} x(k+5)e^{k-n+1}u(n-k-2)$;

(c) $\mathcal{H}x(n) = \sum_{k=-\infty}^{n} x(k)v(n-k)$; and

(d) $\mathcal{H}x(n) = \sum_{k=n-1}^{n} x(k)$.

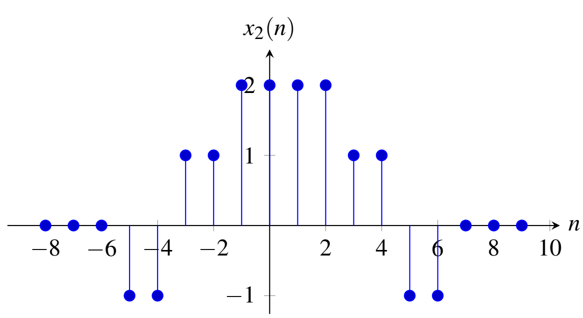
9.8 Consider the system with input x and output y as shown in the figure below. Each system in the block diagram is LTI and labelled with its impulse response.



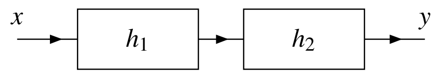
(a) Find the impulse response h of the overall system in terms of h_1, h_2, and h_3.
(b) Determine the impulse response h in the specific case that

$$h_1(n) = \delta(n+1), \quad h_2(n) = \delta(n), \quad \text{and} \quad h_3(n) = \delta(n).$$

9.9 Consider a LTI system whose response to the sequence $x_1(n) = u(n) - u(n-2)$ is the sequence y_1. Determine the response y_2 of the system to the input x_2 shown in the figure below in terms of y_1.



9.10 Consider the system shown in the figure below with input x and output y. This system is formed by the series interconnection of two LTI systems with the impulse responses h_1 and h_2.



For each pair of h_1 and h_2 given below, find the output y if the input $x(n) = u(n)$.
(a) $h_1(n) = \delta(n)$ and $h_2(n) = \delta(n)$;
(b) $h_1(n) = \delta(n+1)$ and $h_2(n) = \delta(n+1)$; and
(c) $h_1(n) = 2^{-n}u(n)$ and $h_2(n) = \delta(n)$.

9.11 Determine whether the LTI system with each impulse response h given below is causal and/or memoryless.
(a) $h(n) = (n+1)u(n-1)$;
(b) $h(n) = 2\delta(n+1)$;
(c) $h(n) = 2^{-n}u(n-1)$;
(d) $h(n) = 2^{n}u(-n-1)$;
(e) $h(n) = 2^{-3|n|}$; and
(f) $h(n) = 3\delta(n)$.

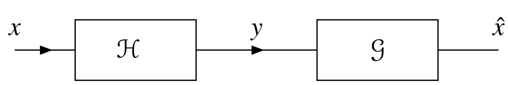
9.12 Determine whether the LTI system with each impulse response h given below is BIBO stable.
(a) $h(n) = 2^{an}u(-n)$, where a is a strictly positive real constant;
(b) $h(n) = n^{-1}u(n-1)$;
(c) $h(n) = 2^{n}u(n)$;
(d) $h(n) = \delta(n-10)$;
(e) $h(n) = u(n) - u(n-9)$; and
(f) $h(n) = 2^{-|n|}$.

9.13 Suppose that we have two LTI systems with impulse responses

$$h_1(n) = \tfrac{1}{2}\delta(n-1) \quad \text{and} \quad h_2(n) = 2\delta(n+1).$$

Determine whether these systems are inverses of one another.

9.14 Consider the system shown in the figure below, where $\mathcal{H}$ is a LTI system and $\mathcal{G}$ is known to be the inverse system of $\mathcal{H}$. Let $y_1 = \mathcal{H}x_1$ and $y_2 = \mathcal{H}x_2$.



(a) Determine the response of the system $\mathcal{G}$ to the input $y'(n) = a_1y_1(n) + a_2y_2(n)$, where a_1 and a_2 are complex constants.
(b) Determine the response of the system $\mathcal{G}$ to the input $y_1'(n) = y_1(n-n_0)$, where n_0 is an integer constant.
(c) Using the results of the previous parts of this question, determine whether the system $\mathcal{G}$ is linear and/or time invariant.

9.15 Suppose that we have the systems $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$, whose responses to a complex exponential input $x(n) = e^{j2n}$ are given by

$$\mathcal{H}_1x(n) = 2e^{j2n}, \quad \mathcal{H}_2x(n) = ne^{j2n}, \quad \mathcal{H}_3x(n) = e^{j2n+\pi/3}, \quad \text{and} \quad \mathcal{H}_4x(n) = \cos(2n).$$

Indicate which of these systems cannot be LTI.

9.16 Show that, for any sequence x, $x * v(n) = x(n-n_0)$, where $v(n) = \delta(n-n_0)$ and n_0 is an arbitrary integer constant.

9.17 A system that has every complex exponential sequence as an eigensequence is not necessarily LTI. In this exercise, we prove this fact by way of counterexample. Consider the system $\mathcal{H}$ given by

$$\mathcal{H}x(n) = \begin{cases} \frac{x^2(n)}{x(n-1)} & \text{if } x(n-1) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that every complex exponential sequence is an eigensequence of $\mathcal{H}$.
(b) Show that $\mathcal{H}$ is not linear (and therefore not LTI).

9.11.2 Exercises With Answer Key

Currently, there are no exercises available with an answer key.

9.12 MATLAB Exercises

Currently, there are no MATLAB exercises.

Chapter 10

Discrete-Time Fourier Series

10.1 Introduction

One very important tool in the study of signals and systems is the (DT) Fourier series. A very large class of sequences can be represented using Fourier series, namely most practically useful periodic sequences. The Fourier series represents a periodic sequence as a linear combination of complex sinusoids. This is often desirable since complex sinusoids are easy sequences with which to work. This is mainly due to the fact that complex sinusoids have important properties in relation to LTI systems. In particular, complex sinusoids are eigensequences of LTI systems. Therefore, the response of a LTI system to a complex sinusoid is the same complex sinusoid multiplied by a complex constant.

10.2 Definition of Discrete-Time Fourier Series

Consider a set of **harmonically-related** complex sinusoids of the form

$$\phi_k(n) = e^{j(2\pi/N)kn} \quad \text{for all } k \in \mathbb{Z},$$

where N is a (strictly) positive integer constant. Since $e^{j\theta}$ is 2π-periodic in the variable θ, $\phi_k = \phi_{k+mN}$ for all $m \in \mathbb{Z}$. Consequently, the above set of sequences contains only N distinct elements, which can be obtained by choosing k as any set of N consecutive integers (e.g., $k \in [0\,..\,N-1]$). Furthermore, since $\left(\frac{2\pi}{N}k\right)/(2\pi) = \frac{k}{N}$ is a rational number, each ϕ_k is periodic. In particular, ϕ_k is periodic with the fundamental period $\frac{N}{\gcd(k,N)}$. Since it follows from the definition of the GCD that N must by an integer multiple of $\frac{N}{\gcd(k,N)}$, each ϕ_k must be N periodic. Since the sum of periodic sequences with the same period must be periodic with that period, any linear combination of ϕ_k must be N periodic. So, for example, a sum of the following form must be N-periodic:

$$\sum_{k=K_0}^{K_0+N-1} a_k \phi_k(n) = \sum_{k=K_0}^{K_0+N-1} a_k e^{j(2\pi/N)kn},$$

where the a_k are complex constants and K_0 is an integer constant.

Suppose now that we can represent a complex N-periodic sequence x as a linear combination of harmonically-related complex sinusoids as

$$x(n) = \sum_{k=\langle N\rangle} c_k e^{jk(2\pi/N)n}, \tag{10.1}$$

where c is a complex N-periodic sequence and $\sum_{k=\langle N\rangle}$ denotes summation over any set of N consecutive integers. Such a representation is known as a (DT) **Fourier series**. More specifically, this is the **complex exponential form** of the Fourier series. As a matter of terminology, we refer to (10.1) as the **Fourier-series synthesis equation**.

Since we often work with Fourier series, it is sometimes convenient to have an abbreviated notation to indicate that a sequence is associated with particular Fourier-series coefficients. If a sequence x has the Fourier-series coefficient sequence c, we sometimes indicate this using the notation

$$x(n) \overset{\text{DTFS}}{\longleftrightarrow} c_k.$$

10.3 Determining the Fourier-Series Representation of a Sequence

Given an arbitrary periodic sequence x, we need some means for finding its corresponding Fourier-series representation. In other words, we need a method for calculating the Fourier-series coefficient sequence corresponding to x. Such a method is given by the theorem below.

Theorem 10.1 (Fourier-series analysis equation)**.** *The Fourier-series coefficient sequence c of an N-periodic sequence x is given by*

$$c_k = \tfrac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn} \quad \textit{for all } k \in \mathbb{Z}. \tag{10.2}$$

Proof. Recalling the Fourier-series synthesis equation, we have

$$x(n) = \sum_{\ell=\langle N \rangle} c_\ell e^{j(2\pi/N)\ell n}.$$

Multiplying both sides of this equation by $e^{-j(2\pi/N)kn}$ yields

$$\begin{aligned} x(n) e^{-j(2\pi/N)kn} &= \sum_{\ell=\langle N \rangle} c_\ell e^{j(2\pi/N)\ell n} e^{-j(2\pi/N)kn} \\ &= \sum_{\ell=\langle N \rangle} c_\ell e^{j(2\pi/N)(\ell-k)n}. \end{aligned}$$

Summing both sides of this equation over one period N of x, we obtain

$$\sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn} = \sum_{n=\langle N \rangle} \sum_{\ell=\langle N \rangle} c_\ell e^{j(2\pi/N)(\ell-k)n}.$$

Interchanging the order of the summations on the right-hand side yields

$$\sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn} = \sum_{\ell=\langle N \rangle} c_\ell \left(\sum_{n=\langle N \rangle} e^{j(2\pi/N)(\ell-k)n} \right).$$

Rewriting k as $k = Nk_1 + k_0$, where k_1 and k_0 are integers and $k_0 \in [0\,..\,N-1]$ (i.e., $k_1 = N\lfloor k/N \rfloor$ and $k_0 = \operatorname{mod}(k, N)$), and performing the outer summation on the right-hand side over $\ell \in [0\,..\,N-1]$, we obtain

$$\begin{aligned}\sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn} &= \sum_{\ell=0}^{N-1} c_\ell \left(\sum_{n=\langle N\rangle} e^{j(2\pi/N)(\ell-[Nk_1+k_0])n}\right) \\ &= \sum_{\ell=0}^{N-1} c_\ell \left(\sum_{n=\langle N\rangle} e^{j(2\pi/N)(\ell-Nk_1-k_0)n}\right) \\ &= \sum_{\ell=0}^{N-1} c_\ell \left(\sum_{n=\langle N\rangle} e^{j(2\pi/N)(\ell-k_0)n-j2\pi k_1 n}\right) \\ &= \sum_{\ell=0}^{N-1} c_\ell \left(\sum_{n=\langle N\rangle} e^{-j2\pi k_1 n} e^{j(2\pi/N)(\ell-k_0)n}\right) \\ &= \sum_{\ell=0}^{N-1} c_\ell \left(\sum_{n=\langle N\rangle} 1^{k_1 n} e^{j(2\pi/N)(\ell-k_0)n}\right) \\ &= \sum_{\ell=0}^{N-1} c_\ell \left(\sum_{n=\langle N\rangle} e^{j(2\pi/N)(\ell-k_0)n}\right). \end{aligned} \tag{10.3}$$

Consider now the inner summation on the right-hand side of this equation. We observe that

$$\sum_{n=\langle N\rangle} e^{j(2\pi/N)(\ell-k_0)n} = \begin{cases} N & (\ell-k_0)/N \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases} \tag{10.4}$$

(The proof of (10.4) is left as an exercise for the reader in Exercise A.11.) Moreover, since $\ell - k_0 \in [-(N-1)\,..\,N-1]$, the only time that $(\ell-k_0)/N \in \mathbb{Z}$ is when $\ell - k_0 = 0$ (i.e., $\ell = k_0$). Thus, the right-hand side of (10.4) can be simplified to $N\delta(\ell-k_0)$. Substituting (10.4) into (10.3), we have

$$\begin{aligned}\sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn} &= \sum_{\ell=0}^{N-1} c_\ell [N\delta(\ell-k_0)] \\ &= c_{k_0} N.\end{aligned}$$

Rearranging, we obtain

$$c_{k_0} = \tfrac{1}{N} \sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn}.$$

Since $k = k_0$ for $k \in [0\,..\,N-1]$, we have

$$c_k = \tfrac{1}{N} \sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn} \quad \text{for } k \in [0\,..\,N-1].$$

Since c is N-periodic and the formula for c_k given by the preceding equation is N-periodic in k, this formula must be valid for all $k \in \mathbb{Z}$. Thus, we have

$$c_k = \tfrac{1}{N} \sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn} \quad \text{for all } k \in \mathbb{Z}.$$ ■

As a matter of terminology, we refer to (10.2) as the **Fourier-series analysis equation**.

Suppose that we have a complex-valued N-periodic sequence x with Fourier-series coefficient sequence c. One can easily show that the coefficient c_0 is the average value of x over a single period N. The proof is trivial. Consider the Fourier-series analysis equation given by (10.2). Substituting $k=0$ into this equation, we obtain

$$\begin{aligned} c_0 &= \left.\left[\tfrac{1}{N}\sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn}\right]\right|_{k=0} \\ &= \tfrac{1}{N}\sum_{n=\langle N\rangle} x(n)e^{0} \\ &= \tfrac{1}{N}\sum_{n=\langle N\rangle} x(n). \end{aligned}$$

Thus, c_0 is simply the average value of x over a single period.

Example 10.1. Find the Fourier-series representation of the sequence

$$x(n) = \sin\left(\tfrac{2\pi}{7}n\right).$$

Solution. To begin, we can first confirm that x has a Fourier-series representation. Since $(2\pi)/\left(\frac{2\pi}{7}\right) = (2\pi)\left(\frac{7}{2\pi}\right) = 7$ is rational, the sequence x is periodic. In particular, the period $N=7$. So, x does have a Fourier-series representation. Using Euler's relation, we can express x as

$$\begin{aligned} \sin\left(\tfrac{2\pi}{7}n\right) &= \tfrac{1}{2j}\left[e^{j(2\pi/7)n} - e^{-j(2\pi/7)n}\right] \\ &= \tfrac{j}{2}e^{-j(2\pi/7)n} - \tfrac{j}{2}e^{j(2\pi/7)n} \\ &= \tfrac{j}{2}e^{j(2\pi/7)(-1)n} - \tfrac{j}{2}e^{j(2\pi/7)(1)n}. \end{aligned}$$

Thus, x has the Fourier-series representation

$$x(n) = \sum_{k=-3}^{3} c_k e^{j(2\pi/7)kn}$$

where

$$c_k = \begin{cases} \frac{j}{2} & k=-1 \\ -\frac{j}{2} & k=1 \\ 0 & k\in\{-3,-2,0,2,3\} \end{cases} \quad \text{and} \quad c_k = c_{k+7}.$$

Since c is 7-periodic, we could also change the summation to be taken over any set of 7 consecutive integers and specify c_k for $k \in [0\,..\,6]$ to yield

$$x(n) = \sum_{k=\langle 7\rangle} c_k e^{j(2\pi/7)kn}$$

where

$$c_k = \begin{cases} -\frac{j}{2} & k=1 \\ \frac{j}{2} & k=6 \\ 0 & k\in\{0,2,3,4,5\} \end{cases} \quad \text{and} \quad c_k = c_{k+7}.$$

■

Example 10.2 (Periodic impulse train)**.** Find the Fourier-series representation of the periodic impulse train

$$x(n) = \sum_{\ell=-\infty}^{\infty} \delta(n-N\ell),$$

where N is a strictly positive integer constant.

Solution. Due to the form of the formula for x, x is clearly N-periodic. Recalling the Fourier-series analysis equation, we have

$$c_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}.$$

Choosing to take the summation over $[0\,..\,N-1]$ and substituting the given formula for x, we have

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} \delta(n) e^{-j(2\pi/N)kn}.$$

From the sifting property of the δ sequence, we have

$$\begin{aligned} c_k &= \tfrac{1}{N} e^0 \\ &= \tfrac{1}{N}. \end{aligned}$$

Thus, x has the Fourier-series representation

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$$

where

$$c_k = \frac{1}{N} \quad \text{for all } k \in \mathbb{Z}.$$

■

Example 10.3. Find the Fourier-series representation of the 5-periodic sequence x given by

$$x(n) = \begin{cases} -\frac{1}{2} & n = -1 \\ 1 & n = 0 \\ \frac{1}{2} & n = 1 \\ 0 & n \in \{-2, 0, 2\}. \end{cases}$$

Solution. Recalling the Fourier-series analysis equation, we have

$$c_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}.$$

Choosing to take the summation over $[-2\,..\,2]$ and substituting the given formula for x, we have

$$\begin{aligned} c_k &= \tfrac{1}{5} \sum_{n=-2}^{2} x(n) e^{-j(2\pi/5)kn} \\ &= \tfrac{1}{5} \left(-\tfrac{1}{2} e^{-j(2\pi/5)(-1)k} + e^{-j(2\pi/5)(0)k} + \tfrac{1}{2} e^{-j(2\pi/5)(1)k} \right). \end{aligned}$$

Simplifying, we obtain

$$\begin{aligned} c_k &= \tfrac{1}{5} \left(-\tfrac{1}{2} e^{j(2\pi/5)k} + 1 + \tfrac{1}{2} e^{-j(2\pi/5)k} \right) \\ &= \tfrac{1}{5} \left(1 - \tfrac{1}{2} (e^{j(2\pi/5)k} - e^{-j(2\pi/5)k}) \right) \\ &= \tfrac{1}{5} \left(1 - \tfrac{1}{2} \left[2j \sin\left(\tfrac{2\pi}{5}k\right) \right] \right) \\ &= \tfrac{1}{5} \left[1 - j \sin\left(\tfrac{2\pi}{5}k\right) \right]. \end{aligned}$$

Thus, x has the Fourier-series representation

$$x(n) = \sum_{k=\langle 5 \rangle} c_k e^{j(2\pi/5)kn},$$

where

$$c_k = \tfrac{1}{5}\left[1 - j\sin\left(\tfrac{2\pi}{5}k\right)\right] \quad \text{for all } k \in \mathbb{Z}.$$

■

Example 10.4. Find the Fourier-series representation of the 8-periodic sequence x given by

$$x(n) = \begin{cases} 1 & n \in [0\,..\,3] \\ 0 & n \in [4\,..\,7]. \end{cases}$$

Solution. Recalling the Fourier-series analysis equation, we have

$$c_k = \tfrac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}.$$

Taking the summation over $[0\,..\,7]$, we have

$$\begin{aligned} c_k &= \tfrac{1}{8} \sum_{n=0}^{7} x(n) e^{-j(2\pi/8)kn} \\ &= \tfrac{1}{8} \sum_{n=0}^{3} e^{-j(\pi/4)kn} = \tfrac{1}{8} \sum_{n=0}^{3} \left(e^{-j(\pi/4)n}\right)^k. \end{aligned}$$

Using the formula for the sum of a geometric sequence, we have

$$c_k = \begin{cases} \tfrac{1}{8}\left(\dfrac{(e^{-j(\pi/4)k})^4 - 1}{e^{-j(\pi/4)k} - 1}\right) & k \neq 0 \\ \tfrac{1}{8}\displaystyle\sum_{k=0}^{3} 1 & k = 0. \end{cases}$$

Now, we must simplify this expression for c_k for the cases of $k \neq 0$ and $k = 0$. First, we consider the case of $k \neq 0$. We have

$$\begin{aligned} c_k &= \tfrac{1}{8}\left(\frac{(e^{-j(\pi/4)k})^4 - 1}{e^{-j(\pi/4)k} - 1}\right) = \tfrac{1}{8}\left(\frac{e^{-j\pi k} - 1}{e^{-j(\pi/4)k} - 1}\right) \\ &= \tfrac{1}{8}\left(\frac{e^{-j(\pi/2)k}\left[e^{-j(\pi/2)k} - e^{j(\pi/2)k}\right]}{e^{-j(\pi/8)k}\left[e^{-j(\pi/8)k} - e^{j(\pi/8)k}\right]}\right) = \tfrac{1}{8}\left(\frac{e^{-j(\pi/2)k}[2j\sin\left(-\frac{\pi}{2}k\right)]}{e^{-j(\pi/8)k}[2j\sin\left(-\frac{\pi}{8}k\right)]}\right) \\ &= \frac{\sin\left(\frac{\pi}{2}k\right)}{8e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)}. \end{aligned} \tag{10.5}$$

Now, we consider the case of $k = 0$. We have

$$\begin{aligned} c_0 &= \tfrac{1}{8}\sum_{k=0}^{3} 1 = \tfrac{4}{8} \\ &= \tfrac{1}{2}. \end{aligned}$$

Furthermore, we observe that using (10.5) to compute c_k for $k=0$ yields

$$\begin{aligned}
\left.\frac{\sin\left(\frac{\pi}{2}k\right)}{8e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)}\right|_{k=0} &= \left.\frac{\frac{\pi}{2}\cos\left(\frac{\pi}{2}k\right)}{8\left[\frac{3j\pi}{8}e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)+\frac{\pi}{8}\cos\left(\frac{\pi}{8}k\right)e^{j(3\pi/8)k}\right]}\right|_{k=0} \\
&= \frac{\frac{\pi}{2}}{8[0+\frac{\pi}{8}]} = \frac{\frac{\pi}{2}}{\pi} \\
&= \tfrac{1}{2}.
\end{aligned}$$

So, as it turns out, (10.5) happens to yield the correct result for $k=0$. Consequently, c_k is given by (10.5) for all $k \in \mathbb{Z}$. Thus, x has the Fourier-series representation

$$x(n) = \sum_{k=\langle 8\rangle} c_k e^{j(\pi/4)kn},$$

where

$$c_k = \frac{\sin\left(\frac{\pi}{2}k\right)}{8e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)} \quad \text{for all } k \in \mathbb{Z}.$$

■

Example 10.5. Find the Fourier-series representation of the 8-periodic sequence x given by

$$x(n) = \begin{cases} n & n \in [-2\,..\,2] \\ 0 & n=-3 \text{ or } n \in [3\,..\,4]. \end{cases}$$

Solution. Recalling the Fourier-series analysis equation, we have

$$c_k = \tfrac{1}{N}\sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn}.$$

Performing the summation over $[-3\,..\,4]$, we have

$$\begin{aligned}
c_k &= \tfrac{1}{8}\sum_{n=-3}^{4} x(n)e^{-j(2\pi/8)kn} \\
&= \tfrac{1}{8}\sum_{n=-2}^{2} ne^{-j(\pi/4)kn} \\
&= \tfrac{1}{8}\left[-2e^{-j(\pi/4)(-2)k} - e^{-j(\pi/4)(-1)k} + e^{-j(\pi/4)(1)k} + 2e^{-j(\pi/4)(2)k}\right] \\
&= \tfrac{1}{8}\left[-2e^{j(\pi/2)k} - e^{j(\pi/4)k} + e^{-j(\pi/4)k} + 2e^{-j(\pi/2)k}\right] \\
&= \tfrac{1}{8}\left[2e^{-j(\pi/2)k} - 2e^{j(\pi/2)k} + e^{-j(\pi/4)k} - e^{j(\pi/4)k}\right] \\
&= \tfrac{1}{8}\left[2\left(e^{-j(\pi/2)k} - e^{j(\pi/2)k}\right) + \left(e^{-j(\pi/4)k} - e^{j(\pi/4)k}\right)\right] \\
&= \tfrac{1}{8}\left[4j\sin\left(-\tfrac{\pi}{2}k\right) + 2j\sin\left(-\tfrac{\pi}{4}k\right)\right] \\
&= \tfrac{1}{8}\left[-4j\sin\left(\tfrac{\pi}{2}k\right) - 2j\sin\left(\tfrac{\pi}{4}k\right)\right] \\
&= -\tfrac{j}{2}\sin\left(\tfrac{\pi}{2}k\right) - \tfrac{j}{4}\sin\left(\tfrac{\pi}{4}k\right) \\
&= -j\left[\tfrac{1}{2}\sin\left(\tfrac{\pi}{2}k\right) + \tfrac{1}{4}\sin\left(\tfrac{\pi}{4}k\right)\right].
\end{aligned}$$

Thus, x has the Fourier-series representation

$$x(n) = \sum_{k=\langle 8\rangle} c_k e^{j(\pi/4)kn},$$

where

$$c_k = -j\left[\tfrac{1}{2}\sin\left(\tfrac{\pi}{2}k\right) + \tfrac{1}{4}\sin\left(\tfrac{\pi}{4}k\right)\right] \quad \text{for all } k \in \mathbb{Z}. \qquad \blacksquare$$

Example 10.6 (Fourier series of an even real sequence)**.** Let x be an even real N-periodic sequence with the Fourier-series coefficient sequence c. Show that

- c is real (i.e., $\operatorname{Im}\{c_k\} = 0$ for all $k \in \mathbb{Z}$);
- c is even (i.e., $c_k = c_{-k}$ for all $k \in \mathbb{Z}$); and
- $c_0 = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)$.

Solution. From the Fourier-series analysis equation (10.2) and using Euler's relation, we can write

$$\begin{aligned}
c_k &= \tfrac{1}{N}\sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn} \\
&= \tfrac{1}{N}\sum_{n=\langle N\rangle} x(n)\left[\cos\left(-\tfrac{2\pi}{N}kn\right) + j\sin\left(-\tfrac{2\pi}{N}kn\right)\right] \\
&= \tfrac{1}{N}\sum_{n=\langle N\rangle} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right].
\end{aligned}$$

At this point, we split the further manipulation of the preceding equation into two cases: 1) N is even and 2) N is odd.

First, we consider the case of even N. We have

$$c_k = \tfrac{1}{N}\sum_{n=-(N/2-1)}^{N/2} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right].$$

Splitting the summation into four parts (i.e., for n in $\left[-(\frac{N}{2}-1)\,..\,-1\right]$, $\{0\}$, $\left[1\,..\,\frac{N}{2}-1\right]$, and $\left\{\frac{N}{2}\right\}$), we obtain

$$\begin{aligned}
c_k &= \tfrac{1}{N}\Bigg[x(0)[\cos(0) - j\sin(0)] + \sum_{n=-(N/2-1)}^{-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right] \\
&\qquad + x\left(\tfrac{N}{2}\right)[\cos(\pi k) - j\sin(\pi k)] + \sum_{n=1}^{N/2-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\Bigg] \\
&= \tfrac{1}{N}\left[x(0) + (-1)^k x\left(\tfrac{N}{2}\right) + \sum_{n=-(N/2-1)}^{-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{N/2-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right].
\end{aligned}$$

Performing a change of variable in the first summation and using the fact that x and cos are even and sin is odd, we have

$$c_k = \tfrac{1}{N}\left[x(0) + (-1)^k x\left(\tfrac{N}{2}\right) + \sum_{n=1}^{N/2-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) + j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{N/2-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right].$$

Combining the two summations and simplifying, we obtain

$$c_k = \tfrac{1}{N}\left[x(0) + (-1)^k x\left(\tfrac{N}{2}\right) + 2\sum_{n=1}^{N/2-1} x(n)\cos\left(\tfrac{2\pi}{N}kn\right)\right].$$

Next, we consider the case of odd N. We have

$$c_k = \tfrac{1}{N}\sum_{n=-(N-1)/2}^{(N-1)/2} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right].$$

Splitting the summation into three parts (i.e., for n in $\left[-\frac{N-1}{2} \mathinner{..} -1\right]$, $\{0\}$, and $\left[1 \mathinner{..} \frac{N-1}{2}\right]$), we obtain

$$c_k = \frac{1}{N}\left[x(0) + \sum_{n=-(N-1)/2}^{-1} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{(N-1)/2} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right].$$

Performing a change of variable in the first summation and using the fact that x and cos are even and sin is odd, we have

$$c_k = \frac{1}{N}\left[x(0) + \sum_{n=1}^{(N-1)/2} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) + j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{(N-1)/2} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right].$$

Combining the two summations and simplifying, we obtain

$$c_k = \frac{1}{N}\left[x(0) + 2\sum_{n=1}^{(N-1)/2} x(n)\cos\left(\tfrac{2\pi}{N}kn\right)\right].$$

Combining the results for the cases of N even and N odd, we have

$$c_k = \begin{cases} \frac{1}{N}\left[x(0) + (-1)^k x\left(\frac{N}{2}\right) + 2\sum_{n=1}^{N/2-1} x(n)\cos\left(\tfrac{2\pi}{N}kn\right)\right] & N \text{ even} \\ \frac{1}{N}\left[x(0) + 2\sum_{n=1}^{(N-1)/2} x(n)\cos\left(\tfrac{2\pi}{N}kn\right)\right] & N \text{ odd.} \end{cases}$$

Since x and cos are real, the quantity c_k must also be real. Thus, we have that $\operatorname{Im}(c_k) = 0$. Also, since $(-1)^k$ and $\cos\left(\frac{2\pi}{N}kn\right)$ are even, c is even.

Consider now the quantity c_0. Substituting $k = 0$ into the Fourier-series analysis equation, we have

$$\begin{aligned} c_0 &= \left.\frac{1}{N}\sum_{n=\langle N\rangle} x(n)e^{j(2\pi/N)kn}\right|_{k=0} \\ &= \frac{1}{N}\sum_{n=\langle N\rangle} x(n). \end{aligned}$$

Thus, c_0 has the stated value. ∎

Example 10.7 (Fourier series of an odd real sequence)**.** Let x be an odd real N-periodic sequence with the Fourier-series coefficient sequence c. Show that

- c is purely imaginary (i.e., $\operatorname{Re}\{c_k\} = 0$ for all $k \in \mathbb{Z}$);
- c is odd (i.e., $c_k = -c_{-k}$ for all $k \in \mathbb{Z}$); and
- $c_0 = 0$.

Solution. From the Fourier-series analysis equation (10.2) and using Euler's relation, we can write

$$\begin{aligned} c_k &= \frac{1}{N}\sum_{n=\langle N\rangle} x(n)e^{-j(2\pi/N)kn} \\ &= \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\left[\cos\left(-\tfrac{2\pi}{N}kn\right) + j\sin\left(-\tfrac{2\pi}{N}kn\right)\right] \\ &= \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]. \end{aligned}$$

At this point, we split the further manipulation of the preceding equation into two cases: 1) N is even and 2) N is odd.

First, we consider the case of even N. We have

$$c_k = \frac{1}{N} \sum_{n=-(N/2-1)}^{N/2} x(n) \left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right].$$

Splitting the summation into four parts (i.e., for n in $\left[-(\frac{N}{2}-1)\,..\,-1\right]$, $\{0\}$, $\left[1\,..\,\frac{N}{2}-1\right]$, and $\left\{\frac{N}{2}\right\}$) and using the fact that $x(0) = 0$ and $x\left(\frac{N}{2}\right) = 0$, we obtain

$$\begin{aligned}
c_k &= \frac{1}{N}\left[x(0)[\cos 0 - j\sin 0] + \sum_{n=-(N/2-1)}^{-1} x(n) \left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right. \\
&\qquad \left. x\left(\tfrac{N}{2}\right)[\cos(\pi k) - j\sin(\pi k)] + \sum_{n=1}^{N/2-1} x(n) \left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right] \\
&= \frac{1}{N}\left[\sum_{n=-(N/2-1)}^{-1} x(n) \left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{N/2-1} x(n) \left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right].
\end{aligned}$$

(Note that $x(0) = x\left(\frac{N}{2}\right) = 0$ due to Theorem 8.3.) Performing a change of variable in the first summation and using the fact that x and sin are odd and cos is even, we have

$$c_k = \frac{1}{N}\left[-\sum_{n=1}^{N/2-1} x(n) \left[\cos\left(\tfrac{2\pi}{N}kn\right) + j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{N/2-1} x(n) \left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right].$$

Combining the two summations and simplifying, we obtain

$$c_k = \frac{1}{N}\left[-2j\sum_{n=1}^{N/2-1} x(n)\sin\left(\tfrac{2\pi}{N}kn\right)\right].$$

Next, we consider the case of odd N. We have

$$c_k = \frac{1}{N} \sum_{n=-(N-1)/2}^{(N-1)/2} x(n) \left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right].$$

Splitting the summation into three parts (i.e., for n in $\left[-\frac{(N-1)}{2}\,..\,-1\right]$, $\{0\}$, and $\left[1\,..\,\frac{N-1}{2}\right]$) and using the fact that $x(0) = 0$, we obtain

$$c_k = \frac{1}{N}\left[\sum_{n=-(N-1)/2}^{-1} x(n) \left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{(N-1)/2} x(n) \left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right].$$

Performing a change of variable in the first summation and using the fact that x and sin are odd and cos is even, we have

$$c_k = \frac{1}{N}\left[-\sum_{n=1}^{(N-1)/2} x(n) \left[\cos\left(\tfrac{2\pi}{N}kn\right) + j\sin\left(\tfrac{2\pi}{N}kn\right)\right] + \sum_{n=1}^{(N-1)/2} x(n) \left[\cos\left(\tfrac{2\pi}{N}kn\right) - j\sin\left(\tfrac{2\pi}{N}kn\right)\right]\right].$$

Combining the two summations and simplifying, we obtain

$$c_k = \frac{1}{N}\left[-2j\sum_{n=1}^{(N-1)/2} x(n)\sin\left(\tfrac{2\pi}{N}kn\right)\right].$$

Combining the above results, we have

$$c_k = \begin{cases} -\frac{2j}{N} \displaystyle\sum_{n=1}^{N/2-1} x(n) \sin\left(\tfrac{2\pi}{N}kn\right) & N \text{ even} \\ -\frac{2j}{N} \displaystyle\sum_{n=1}^{(N-1)/2} x(n) \sin\left(\tfrac{2\pi}{N}kn\right) & N \text{ odd.} \end{cases}$$

Since x and sin are real, the quantity c_k must be imaginary. Thus, we have that $\operatorname{Re}(c_k) = 0$. Since sin is odd, c is odd.

Consider now the quantity c_0. Substituting $k = 0$ into the above formula for c_k, we have

$$c_0 = 0.$$

Thus, c_0 has the stated value. ■

10.4 Comments on Convergence of Discrete-Time Fourier Series

Since the analysis and synthesis equations for (DT) Fourier series involve only finite sums (as opposed to infinite series), convergence is not a significant concern. If an N-periodic sequence is bounded (i.e., is finite in value), its Fourier-series coefficient sequence will exist and be bounded and the Fourier-series analysis and synthesis equations must converge.

10.5 Properties of Discrete-Time Fourier Series

Fourier-series representations possess a number of important properties. In the sections that follow, we introduce several of these properties. For convenience, these properties are also summarized later in Table 10.1 (on page 404).

10.5.1 Linearity

Arguably, the most important property of Fourier series is linearity, as introduced below.

Theorem 10.2 (Linearity)**.** *Let x and y denote two N-periodic sequences. If*

$$x(n) \xleftrightarrow{\text{DTFS}} a_k \quad \textit{and} \quad y(n) \xleftrightarrow{\text{DTFS}} b_k,$$

then

$$Ax(n) + By(n) \xleftrightarrow{\text{DTFS}} Aa_k + Bb_k,$$

where A and B are complex constants. In other words, a linear combination of sequences produces the same linear combination of their Fourier-series coefficients.

Proof. To prove the above property, we proceed as follows. First, we express x and y in terms of their corresponding Fourier series as

$$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \quad \text{and} \quad y(n) = \sum_{k=\langle N\rangle} b_k e^{jk(2\pi/N)n}.$$

Now, we determine the Fourier series of $Ax + By$. Using the Fourier-series representations of x and y, we have

$$\begin{aligned} Ax(n) + By(n) &= A \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} + B \sum_{k=\langle N\rangle} b_k e^{jk(2\pi/N)n} \\ &= \sum_{k=\langle N\rangle} Aa_k e^{jk(2\pi/N)n} + \sum_{k=\langle N\rangle} Bb_k e^{jk(2\pi/N)n} \\ &= \sum_{k=\langle N\rangle} (Aa_k + Bb_k) e^{jk(2\pi/N)n}. \end{aligned}$$

Now, we observe that the right-hand side of the preceding equation is a Fourier series with coefficient sequence $c'_k = Aa_k + Bb_k$. Therefore, we have that $Ax(n) + By(n) \stackrel{\text{DTFS}}{\longleftrightarrow} Aa_k + Bb_k$. ■

10.5.2 Translation (Time Shifting)

The next property of Fourier series to be introduced is the translation (i.e., time-shifting) property, as given below.

Theorem 10.3 (Translation (i.e., time shifting))**.** *Let x denote an N-periodic sequence. If*

$$x(n) \stackrel{\text{DTFS}}{\longleftrightarrow} a_k,$$

then

$$x(n-n_0) \stackrel{\text{DTFS}}{\longleftrightarrow} e^{-jk(2\pi/N)n_0} a_k,$$

where n_0 is an integer constant.

Proof. To prove the translation property, we proceed as follows. The Fourier series of x is given by

$$x(n) = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}.$$

Substituting $n - n_0$ for n in this equation, we have

$$\begin{aligned} x(n-n_0) &= \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)(n-n_0)} \\ &= \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} e^{-jk(2\pi/N)n_0} \\ &= \sum_{k=\langle N \rangle} \left(a_k e^{-jk(2\pi/N)n_0}\right) e^{jk(2\pi/N)n}. \end{aligned}$$

Now, we observe that the right-hand side of the preceding equation is a Fourier series with coefficient sequence $a'_k = a_k e^{-jk(2\pi/N)n_0}$. Therefore, we have that $x(n-n_0) \stackrel{\text{DTFS}}{\longleftrightarrow} e^{-jk(2\pi/N)n_0} a_k$. ■

From the above theorem, we can see that time shifting a periodic sequence does not change the magnitude of its Fourier-series coefficients (since $\left|e^{j\theta}\right| = 1$ for all real θ).

10.5.3 Modulation (Frequency Shifting)

The next property of Fourier series to be introduced is the modulation (i.e., frequency-shifting) property, as given below.

Theorem 10.4 (Modulation (i.e., frequency shifting))**.** *Let x denote an N-periodic sequence. If*

$$x(n) \stackrel{\text{DTFS}}{\longleftrightarrow} a_k,$$

then

$$e^{jM(2\pi/N)n} x(n) \stackrel{\text{DTFS}}{\longleftrightarrow} a_{k-M},$$

where M is an integer constant.

Proof. To prove the modulation property, we proceed as follows. From the definition of Fourier series, we have

$$x(n) = \sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)kn}.$$

Multiplying both sides of this equation by $e^{j(2\pi/N)Mn}$, we can write

$$\begin{aligned}
e^{j(2\pi/N)Mn}x(n) &= e^{j(2\pi/N)Mn} \sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)kn} \\
&= \sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)Mn} e^{j(2\pi/N)kn} \\
&= \sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)(k+M)n}.
\end{aligned}$$

Now, we employ a change of variable. Let $\ell = k+M$ so that $k = \ell - M$. With this change of variable, a summation over $k \in [\eta \mathinner{..} \eta+N-1]$ becomes a summation over $\ell \in [-(\eta+N-1) \mathinner{..} -\eta]$, which is still a summation over a single period of length N. Applying the change of variable, we obtain

$$e^{j(2\pi/N)Mn}x(n) = \sum_{\ell=\langle N \rangle} a_{\ell-M} e^{j(2\pi/N)\ell n}.$$

Now, we observe that the right-hand side of this equation is a Fourier series with the coefficient sequence $a'_\ell = a_{\ell-M}$. So, the Fourier-series coefficient sequence for $e^{j(2\pi/N)Mn}x(n)$ (i.e., the left-hand side of the equation) is $a'_k = a_{k-M}$. Therefore, we have that $e^{jM(2\pi/N)n}x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_{k-M}$. ■

10.5.4 Reflection (Time Reversal)

The next property of Fourier series to be introduced is the reflection (i.e., time-reversal) property, as given below.

Theorem 10.5 (Reflection (i.e., time reversal))**.** *Let x denote an N-periodic sequence. If*

$$x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k,$$

then

$$x(-n) \overset{\text{DTFS}}{\longleftrightarrow} a_{-k}.$$

Proof. To prove the time-reversal property, we proceed in the following manner. The Fourier series of x is given by

$$x(n) = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}.$$

Substituting $-n$ for n yields

$$x(-n) = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)(-n)}.$$

Now, we employ a change of variable. Let $\ell = -k$ so that $k = -\ell$. With this change of variable, a summation over $k \in [\eta \mathinner{..} \eta+N-1]$ becomes a summation over $\ell \in [-(\eta+N-1) \mathinner{..} -\eta]$, which is still a summation over a single period of length N. Applying this change of variable, we have

$$\begin{aligned}
x(-n) &= \sum_{\ell=\langle N \rangle} a_{-\ell} e^{j(-\ell)(2\pi/N)(-n)} \\
&= \sum_{\ell=\langle N \rangle} a_{-\ell} e^{j\ell(2\pi/N)n}.
\end{aligned}$$

Now, we observe that the right-hand side of this equation is a Fourier series with the coefficient sequence $a'_\ell = a_{-\ell}$. Therefore, we have that $x(-n) \overset{\text{DTFS}}{\longleftrightarrow} a_{-k}$. ■

In other words, the above theorem states that time reversing a sequence time reverses the corresponding sequence of Fourier-series coefficients.

10.5.5 Conjugation

The next property of Fourier series to be introduced is the conjugation property, as given below.

Theorem 10.6 (Conjugation)**.** *For an N-periodic sequence x with Fourier-series coefficient sequence c,*

$$x^*(n) \overset{\text{DTFS}}{\longleftrightarrow} c^*_{-k}.$$

Proof. From the definition of Fourier series, we have

$$x(n) = \sum_{k=\langle N\rangle} c_k e^{jk(2\pi/N)n}.$$

Taking the complex conjugate of both sides of the preceding equation, we obtain

$$\begin{aligned} x^*(n) &= \left(\sum_{k=\langle N\rangle} c_k e^{jk(2\pi/N)n}\right)^* \\ &= \sum_{k=\langle N\rangle} \left(c_k e^{jk(2\pi/N)n}\right)^* \\ &= \sum_{k=\langle N\rangle} c_k^* e^{-jk(2\pi/N)n}. \end{aligned}$$

Now, we employ a change of variable. Let $\ell = -k$ so that $k = -\ell$. With this change of variable, a summation over $k \in [\eta \mathinner{..} \eta+N-1]$ becomes a summation over $\ell \in [-(\eta+N-1) \mathinner{..} -\eta]$, which is still a summation over a single period of length N. Applying the change of variable, we obtain

$$x^*(n) = \sum_{\ell=\langle N\rangle} c^*_{-\ell} e^{j\ell(2\pi/N)n}.$$

Now, we observe that the right-hand side of the preceding equation is a Fourier series with the coefficient sequence $c'_\ell = c^*_{-\ell}$. So, the Fourier-series coefficient sequence c' of x^* (i.e., the left-hand side of the equation) is $c'_k = c^*_{-k}$. Therefore, we have shown that $x^*(n) \overset{\text{DTFS}}{\longleftrightarrow} c^*_{-k}$. ■

In other words, the above theorem states that conjugating a sequence has the effect of time reversing and conjugating the corresponding Fourier-series coefficient sequence.

10.5.6 Duality

The next property of Fourier series to be introduced is the duality property, as given below.

Theorem 10.7 (Duality)**.** *Let x be an N-periodic sequence with the corresponding Fourier-series coefficient sequence a. Then,*

$$a_n \overset{\text{DTFS}}{\longleftrightarrow} \tfrac{1}{N} x(-k).$$

Proof. From the definition of Fourier series, we have

$$x(k) = \sum_{\ell=\langle N\rangle} a_\ell e^{j(2\pi/N)\ell k}.$$

Substituting $-k$ for k, we have

$$\begin{aligned} x(-k) &= \sum_{\ell=\langle N\rangle} a_\ell e^{j(2\pi/N)\ell(-k)} \\ &= \sum_{\ell=\langle N\rangle} a_\ell e^{-j(2\pi/N)\ell k}. \end{aligned}$$

Multiplying both sides of the equation by $\frac{1}{N}$, we obtain

$$\tfrac{1}{N}x(-k) = \tfrac{1}{N}\sum_{\ell=\langle N\rangle} a_\ell e^{-j(2\pi/N)\ell k}.$$

Now, we observe that the right-hand side is the (Fourier-series analysis) formula for computing the kth Fourier-series coefficient of a. Thus, the duality property holds. ■

The duality property stated in the preceding theorem follows from the high degree of similarity in the equations for the Fourier-series analysis and synthesis equations, given by (10.2) and (10.1), respectively. To make this similarity more obvious, we can rewrite the Fourier-series analysis and synthesis equations, respectively, as

$$X(m) = \tfrac{1}{N}\sum_{\ell=\langle N\rangle}^{\infty} x(\ell)e^{-j(2\pi/N)\ell m} \quad \text{and} \quad x(m) = \sum_{\ell=\langle N\rangle} X(\ell)e^{j(2\pi/N)\ell m}.$$

Observe that these two equations are identical except for: 1) a factor of N; and 2) a different sign in the parameter for the exponential function. Consequently, if we were to accidentally use one equation in place of the other, we would obtain an almost correct result. In fact, this almost correct result could be made to be correct by compensating for the above two differences (i.e., the factor of N and the sign difference in the exponential function). This is, in effect, what the duality property states.

Although the relationship $x(n) \overset{\text{DTFS}}{\longleftrightarrow} X(k)$ only directly provides us with the Fourier-series coefficient sequence X of the sequence x, the duality property allows us to indirectly infer the Fourier-series coefficient sequence of X. Consequently, the duality property can be used to effectively double the number of Fourier-series relationships that we know.

10.5.7 Periodic Convolution

The next property of Fourier series to be introduced is the periodic-convolution property, as given below.

Theorem 10.8 (Periodic convolution)**.** *Let x and y be N-periodic sequences with their respective Fourier-series representations given by*

$$x(n) = \sum_{k=\langle N\rangle} a_k e^{j(2\pi/N)kn} \quad \textit{and} \quad y(n) = \sum_{k=\langle N\rangle} b_k e^{j(2\pi/N)kn}.$$

Let $z(n) = x \circledast y(n)$, where z has the Fourier-series representation

$$z(n) = \sum_{k=\langle N\rangle} c_k e^{j(2\pi/N)kn}.$$

The sequences a, b, and c are related as

$$c_k = Na_k b_k.$$

Proof. From the definition of periodic convolution, we have

$$x \circledast y(n) = \sum_{m=\langle N\rangle} x(m)y(n-m).$$

Expanding x and y in terms of their Fourier-series representations, we have

$$x \circledast y(n) = \sum_{m=\langle N\rangle}\left(\sum_{\ell=\langle N\rangle} a_\ell e^{j(2\pi/N)\ell m}\right)\left(\sum_{k=\langle N\rangle} b_k e^{j(2\pi/N)k(n-m)}\right).$$

Changing the order of summations and rearranging, we have

$$\begin{aligned}
x \circledast y(n) &= \sum_{m=\langle N\rangle} \sum_{k=\langle N\rangle} \sum_{\ell=\langle N\rangle} a_\ell b_k e^{j(2\pi/N)\ell m} e^{j(2\pi/N)k(n-m)} \\
&= \sum_{m=\langle N\rangle} \sum_{k=\langle N\rangle} \sum_{\ell=\langle N\rangle} a_\ell b_k e^{j(2\pi/N)kn} e^{j(2\pi/N)(\ell-k)m} \\
&= \sum_{k=\langle N\rangle} \sum_{\ell=\langle N\rangle} a_\ell b_k e^{j(2\pi/N)kn} \sum_{m=\langle N\rangle} e^{j(2\pi/N)(\ell-k)m}.
\end{aligned}$$

Taking the two outermost summations over $[0\,..\,N-1]$, we have

$$x \circledast y(n) = \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} a_\ell b_k e^{j(2\pi/N)kn} \sum_{m=\langle N\rangle} e^{j(2\pi/N)(\ell-k)m}.$$

Now, we observe that

$$\sum_{m=\langle N\rangle} e^{j(2\pi/N)(\ell-k)m} = \begin{cases} N & (\ell-k)/N \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

(The proof of this fact is left as an exercise for the reader in Exercise A.11.) Moreover, since $\ell-k \in [-(N-1)\,..\,N-1]$, $(\ell-k)/N \in \mathbb{Z}$ implies that $\ell-k=0$ (i.e., $\ell=k$). Using these facts, we can simplify the above expression for $x \circledast y$ to obtain

$$\begin{aligned}
x \circledast y(n) &= \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} a_\ell b_k e^{j(2\pi/N)kn} N\delta(\ell-k) \\
&= \sum_{k=0}^{N-1} a_k b_k e^{j(2\pi/N)kn} N \\
&= \sum_{k=0}^{N-1} N a_k b_k e^{j(2\pi/N)kn} \\
&= \sum_{k=\langle N\rangle} N a_k b_k e^{j(2\pi/N)kn}.
\end{aligned}$$

Now, we simply observe that the right-hand side of the preceding equation is a Fourier series with the coefficient sequence $c_k = Na_kb_k$. Therefore, the Fourier-series coefficient sequence c of $x \circledast y$ (i.e., the left-hand side of the equation) is given by $c_k = Na_kb_k$. ■

10.5.8 Multiplication

The next property of Fourier series to be considered is the multiplication property, as given below.

Theorem 10.9 (Multiplication)**.** *Let x and y be N-periodic sequences given by*

$$x(n) = \sum_{k=\langle N\rangle} a_k e^{j(2\pi/N)kn} \quad \text{and} \quad y(n) = \sum_{k=\langle N\rangle} b_k e^{j(2\pi/N)kn}.$$

Let $z(n) = x(n)y(n)$, where z has the Fourier-series representation

$$z(n) = \sum_{k=\langle N\rangle} c_k e^{j(2\pi/N)kn}.$$

The sequences a, b, and c are related as

$$c_k = \sum_{n=\langle N\rangle} a_n b_{k-n} \quad \text{(i.e., } c = a \circledast b\text{)}.$$

Proof. From the Fourier-series analysis equation, we can write

$$c_k = \frac{1}{N} \sum_{\ell=\langle N \rangle} x(\ell)y(\ell)e^{-j(2\pi/N)k\ell}.$$

Replacing x by its Fourier-series representation, we obtain

$$\begin{aligned} c_k &= \frac{1}{N} \sum_{\ell=\langle N \rangle} \left(\sum_{n=\langle N \rangle} a_n e^{j(2\pi/N)n\ell} \right) y(\ell)e^{-j(2\pi/N)k\ell} \\ &= \frac{1}{N} \sum_{\ell=\langle N \rangle} \sum_{n=\langle N \rangle} a_n e^{j(2\pi/N)n\ell} y(\ell)e^{-j(2\pi/N)k\ell}. \end{aligned}$$

Reversing the order of the two summations and rearranging, we have

$$\begin{aligned} c_k &= \frac{1}{N} \sum_{n=\langle N \rangle} \sum_{\ell=\langle N \rangle} a_n e^{j(2\pi/N)n\ell} y(\ell)e^{-j(2\pi/N)k\ell} \\ &= \sum_{n=\langle N \rangle} a_n \left(\frac{1}{N} \sum_{\ell=\langle N \rangle} y(\ell)e^{-j(2\pi/N)(k-n)\ell} \right). \end{aligned}$$

Observing that the expression on the preceding line in the large pair of parenthesis is simply the formula for computing the $(k-n)$th Fourier-series coefficient of y, we conclude

$$c_k = \sum_{n=\langle N \rangle} a_n b_{k-n}. \qquad \blacksquare$$

10.5.9 Parseval's Relation

Another important property of Fourier series relates to the energy of sequences, as given by the theorem below.

Theorem 10.10 (Parseval's relation)**.** *An N-periodic sequence x and its corresponding Fourier-series coefficient sequence c satisfy the relationship*

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 = \sum_{k=\langle N \rangle} |c_k|^2$$

(i.e., the energy in x and the energy in c are equal up to a scale factor).

Proof. Let x, y, and z denote N-periodic sequences with the Fourier series given by

$$\begin{aligned} x(n) &= \sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)kn}, \\ y(n) &= \sum_{k=\langle N \rangle} b_k e^{j(2\pi/N)kn}, \quad \text{and} \\ z(n) = x(n)y(n) &= \sum_{k=\langle N \rangle} c_k e^{j(2\pi/N)kn}. \end{aligned}$$

From the multiplication property of Fourier series (i.e., Theorem 10.9), we have

$$c_k = \sum_{n=\langle N \rangle} a_n b_{k-n}. \tag{10.6}$$

Now, let $y(n) = x^*(n)$ so that $z(n) = x(n)x^*(n) = |x(n)|^2$. From the conjugation property of Fourier series (i.e., Theorem 10.6), since $y(n) = x^*(n)$, we know

$$b_k = a^*_{-k}.$$

So, we can rewrite (10.6) as

$$\begin{aligned} c_k &= \sum_{n=\langle N \rangle} a_n a^*_{-(k-n)} \\ &= \sum_{n=\langle N \rangle} a_n a^*_{n-k}. \end{aligned} \tag{10.7}$$

From the Fourier-series analysis equation, we have

$$c_k = \frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 e^{-j(2\pi/N)kn}. \tag{10.8}$$

Equating (10.7) and (10.8), we obtain

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 e^{-j(2\pi/N)kn} = \sum_{n=\langle N \rangle} a_n a^*_{n-k}.$$

Letting $k=0$ in the preceding equation yields

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 = \sum_{n=\langle N \rangle} a_n a^*_n = \sum_{n=\langle N \rangle} |a_n|^2. \qquad \blacksquare$$

The above theorem is simply stating that the amount of energy in x and the amount of energy in the Fourier-series coefficient sequence c are equal up to a scale factor. In other words, the transformation between a sequence and its Fourier series coefficient sequence preserves energy (up to a scale factor).

10.5.10 Even/Odd Symmetry

Next, we consider the relationship between Fourier series and even/odd symmetry. As it turns out, Fourier series preserve signal symmetry. In other words, we have the result below.

Theorem 10.11 (Even/odd symmetry)**.** *For an N-periodic sequence x with Fourier-series coefficient sequence c, the following properties hold:*

x is even if and only if c is even; and
x is odd if and only if c is odd.

Proof. The proof is left as an exercise for the reader in Exercise 10.3. $\blacksquare$

In other words, the above theorem states that the even/odd symmetry properties of x and c always match (i.e., Fourier series preserve symmetry).

10.5.11 Real Sequences

Consider the Fourier-series representation of the periodic sequence x given by (10.1). In the most general case, x is a complex-valued sequence, but let us now suppose that x is real-valued. In the case of real-valued sequences, an important relationship exists between the Fourier-series coefficients c_k and c_{-k} as given by the theorem below.

Theorem 10.12 (Fourier series of a real-valued sequence)**.** *Let x be a periodic sequence with Fourier-series coefficient sequence c. The sequence x is real-valued if and only if*

$$c_k = c^*_{-k} \text{ for all } k \in \mathbb{Z} \tag{10.9}$$

(i.e., c is conjugate symmetric).

Proof. First, we show that x being real-valued implies that c is conjugate symmetric. Assume that x is real-valued. From the Fourier series analysis equation, we have

$$c_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}.$$

Substituting $-k$ for k in the preceding equation and taking the conjugate of both sides, we obtain

$$\begin{aligned} c_{-k}^* &= \left(\frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)(-k)n} \right)^* \\ &= \left(\frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{j(2\pi/N)kn} \right)^* \\ &= \frac{1}{N} \sum_{n=\langle N \rangle} x^*(n) e^{-j(2\pi/N)kn}. \end{aligned}$$

Since x is real-valued (i.e., $x^* = x$), we have

$$\begin{aligned} c_{-k}^* &= \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn} \\ &= c_k. \end{aligned}$$

Thus, c is conjugate symmetric. Therefore, x being real-valued implies that c is conjugate symmetric.

Now, we show that c being conjugate symmetric implies that x is real-valued. Assume that c is conjugate symmetric. From the Fourier series synthesis equation, we have

$$x(n) = \sum_{k=\langle N \rangle} c_k e^{j(2\pi/N)kn}.$$

Taking the complex conjugate of both sides of the preceding equation, we obtain

$$\begin{aligned} x^*(n) &= \left(\sum_{k=\langle N \rangle} c_k e^{j(2\pi/N)kn} \right)^* \\ &= \sum_{k=\langle N \rangle} \left(c_k e^{j(2\pi/N)kn} \right)^* \\ &= \sum_{k=\langle N \rangle} c_k^* e^{-j(2\pi/N)kn} \\ &= \sum_{k=N_0}^{N_0+N-1} c_k^* e^{-j(2\pi/N)kn}. \end{aligned}$$

Now, we employ a change of variable. Let $k' = -k$ so that $k = -k'$. Applying the change of variable and dropping the primes, we obtain

$$\begin{aligned} x^*(n) &= \sum_{k=-N_0-N+1}^{-N_0} c_{-k}^* e^{-j(2\pi/N)(-k)n} \\ &= \sum_{k=-N_0-N+1}^{-N_0} c_{-k}^* e^{j(2\pi/N)kn}. \end{aligned}$$

Since each term in the summation on the right-hand side of the preceding equation is N-periodic in k, the summation can be taken over any N consecutive integers. Thus, we have

$$x^*(n) = \sum_{k=\langle N \rangle} c_{-k}^* e^{j(2\pi/N)kn}.$$

Since $c_k = c^*_{-k}$ for all $k \in \mathbb{Z}$, we have

$$\begin{aligned} x^*(n) &= \sum_{k=\langle N \rangle} c_k e^{j(2\pi/N)kn} \\ &= x(n). \end{aligned}$$

Thus, x is real-valued. Therefore, c being conjugate symmetric implies that x is real-valued. This completes the proof. ■

Using the relationship in (10.9), we can derive an alternative form of the Fourier series for the case of real-valued sequences. In particular, the Fourier series of a sequence x can be expressed as

$$x(n) = \begin{cases} \alpha_0 + \sum_{k=1}^{N/2-1} \left[\alpha_k \cos\left(\frac{2\pi}{N}kn\right) + \beta_k \sin\left(\frac{2\pi}{N}kn\right)\right] + \alpha_{N/2}\cos(\pi n) & N \text{ even} \\ \alpha_0 + \sum_{k=1}^{(N-1)/2} \left[\alpha_k \cos\left(\frac{2\pi}{N}kn\right) + \beta_k \sin\left(\frac{2\pi}{N}kn\right)\right] & N \text{ odd,} \end{cases}$$

where

$$\alpha_0 = a_0, \quad \alpha_{N/2} = a_{N/2}, \quad \alpha_k = 2\operatorname{Re} a_k, \quad \text{and} \quad \beta_k = -2\operatorname{Im} a_k.$$

This is known as the **trigonometric form** of a Fourier series. Note that the trigonometric form of a Fourier series only involves real quantities, whereas the exponential form involves some complex quantities. For this reason, the trigonometric form may sometimes be preferred when dealing with Fourier series of real-valued sequences.

As noted earlier in Theorem 10.12, the Fourier series of a real-valued sequence has a special structure. In particular, a sequence x is real-valued if and only if its Fourier-series coefficient sequence c is conjugate symmetric (i.e., $c_k = c^*_{-k}$ for all k). From properties of complex numbers, one can show that

$$c_k = c^*_{-k} \text{ for all } k$$

is equivalent to

$$|c_k| = |c_{-k}| \text{ for all } k \quad \text{and} \quad \arg c_k = -\arg c_{-k} \text{ for all } k$$

(i.e., $|c_k|$ is even and $\arg c_k$ is odd). Note that x being real-valued does *not* necessarily imply that c is real-valued.

As it turns out, the Fourier-series coefficient sequence corresponding to a real-valued sequence has a number of other special properties, as given by the theorem below.

Theorem 10.13 (Fourier series of a real-valued sequence)**.** *Let x denote a real-valued N-periodic sequence with the corresponding Fourier-series coefficient sequence c. Then, the following assertions can be made about c_k for the single period of c corresponding to $k \in [0\,..\,N-1]$:*

1. *$c_k = c^*_{N-k}$ for $k \in [1\,..\,N-1]$;*
2. *of the N coefficients c_k for $k \in [0\,..\,N-1]$, only $\left\lfloor \frac{N}{2} \right\rfloor + 1$ coefficients are independent; for example, c_k for $k \in \left[0\,..\left\lfloor \frac{N}{2} \right\rfloor\right]$ completely determines c_k for all $k \in [0\,..\,N-1]$;*
3. *c_0 is real; and*
4. *if N is even, $c_{N/2}$ is real.*

Proof. From Theorem 10.12, we have

$$c_k = c^*_{-k} \quad \text{for all } k \in \mathbb{Z}. \tag{10.10}$$

Since c is N-periodic, $c_{-k} = c_{N-k}$ and we can rewrite the above equation as

$$c_k = c^*_{N-k} \quad \text{for all } k \in \mathbb{Z}. \tag{10.11}$$

This proves assertion 1 of the theorem.

Evaluating (10.10) at $k = 0$, we obtain

$$c_0 = c_0^*.$$

Thus, c_0 must be real. This proves assertion 3 of the theorem.

Evaluating (10.11) at $k = \frac{N}{2}$ where $\frac{N}{2} \in \mathbb{Z}$, we obtain

$$\begin{aligned} c_{N/2} &= c^*_{N-N/2}. \\ &= c^*_{N/2}. \end{aligned}$$

Thus, $c_{N/2}$ must be real. This proves assertion 4 of the theorem.

Evaluating (10.11) for $k \in \left[\left\lfloor \frac{N}{2} \right\rfloor + 1 \,..\, N-1\right]$, we have

$$\begin{aligned} c_{\lfloor N/2 \rfloor + 1} &= c^*_{N-(\lfloor N/2 \rfloor + 1)} = c^*_{N-\lfloor N/2 \rfloor - 1} = c^*_{N+\lceil -N/2 \rceil - 1} = c^*_{\lceil N-1-N/2 \rceil} \\ &= c^*_{\lceil N/2-1 \rceil} = c^*_{\lceil N/2 \rceil - 1} = c^*_{\lfloor (N-1)/2 \rfloor + 1 - 1} \\ &= c^*_{\lfloor (N-1)/2 \rfloor}, \\ &\;\;\vdots \\ c_{N-3} &= c^*_{N-(N-3)} \\ &= c^*_3, \\ c_{N-2} &= c^*_{N-(N-2)} \\ &= c^*_2, \quad \text{and} \\ c_{N-1} &= c^*_{N-(N-1)} \\ &= c^*_1. \end{aligned}$$

(Note that, in the simplification of the expression for $c_{\lfloor N/2 \rfloor + 1}$ above, we used some properties of the floor and ceiling functions introduced in Section 3.5.10.) Thus, c_k for $k \in \left[\left\lfloor \frac{N}{2} \right\rfloor + 1 \,..\, N-1\right]$ is completely determined from c_k for $k \in \left[0 \,..\, \left\lfloor \frac{N-1}{2} \right\rfloor\right]$. Therefore, only $\left\lfloor \frac{N}{2} \right\rfloor + 1$ of the N coefficients c_k for $k \in [0 \,..\, N-1]$ are independent. This proves assertion 2 of the theorem. ■

The above theorem (i.e., Theorem 10.13) shows that the Fourier-series coefficient sequence c corresponding to a real-valued sequence has a high degree of redundancy. In particular, approximately half of the coefficients of c over a single period of c are redundant.

10.6 Discrete Fourier Transform (DFT)

Closely related to DT Fourier series is the discrete Fourier transform (DFT) (not to be confused with the discrete-time Fourier transform, which is introduced later in Chapter 11). The DFT is essentially a slightly modified definition of (DT) Fourier series that is typically useful in the context of the DT Fourier transform (to be discussed later). In what follows, we briefly introduce the DFT.

Consider the Fourier-series synthesis and analysis equations given by

$$x(n) = \sum_{k=\langle N \rangle} c_k e^{jk(2\pi/N)n} \quad \text{and} \quad c_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}.$$

Table 10.1: Properties of DT Fourier series

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(n)+\beta y(n)$	$\alpha a_k+\beta b_k$
Translation	$x(n-n_0)$	$e^{-jk(2\pi/N)n_0}a_k$
Modulation	$e^{j(2\pi/N)k_0 n}x(n)$	a_{k-k_0}
Reflection	$x(-n)$	a_{-k}
Conjugation	$x^*(n)$	a^*_{-k}
Duality	a_n	$\frac{1}{N}x(-k)$
Periodic Convolution	$x \circledast y(n)$	Na_kb_k
Multiplication	$x(n)y(n)$	$a \circledast b_k$

Property	
Parseval's Relation	$\frac{1}{N}\sum\limits_{n=\langle N\rangle}\|x(n)\|^2=\sum\limits_{k=\langle N\rangle}\|a_k\|^2$
Even Symmetry	x is even $\Leftrightarrow$ a is even
Odd Symmetry	x is odd $\Leftrightarrow$ a is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ a is conjugate symmetric

Now let us define a sequence a' as $a'_k = Na_k$. In other words, a' is just a renormalized (i.e., scaled) version of the original Fourier-series coefficient sequence a. Rewriting the Fourier-series synthesis and analysis equations from above in terms of a' (instead of a) with both of the summations taken over $[0\,..\,N-1]$, we obtain

$$x(n) = \tfrac{1}{N}\sum_{k=0}^{N-1} a'_k e^{j(2\pi/N)kn} \quad \text{and} \quad a'_k = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn}.$$

Since x and a' are both N-periodic, each of these sequences is completely characterized by its N samples over a single period. If we only consider the behavior of x and a' over a single period, this leads to the equations

$$x(n) = \tfrac{1}{N}\sum_{k=0}^{N-1} a'_k e^{j(2\pi/N)kn} \quad \text{for } n \in [0\,..\,N-1] \quad \text{and}$$
$$a'_k = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn} \quad \text{for } k \in [0\,..\,N-1].$$

As it turns out, the above two equations define what is known as the discrete Fourier transform (DFT).

The **discrete Fourier transform (DFT)** X of the sequence x is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn} \quad \text{for } k \in [0\,..\,N-1].$$

The preceding equation is known as the **DFT analysis equation**. The **inverse DFT** x of the sequence X is given by

$$x(n) = \tfrac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j(2\pi/N)kn} \quad \text{for } n \in [0\,..\,N-1].$$

The preceding equation is known as the **DFT synthesis equation**. The DFT maps a finite-length sequence of N elements to another finite-length sequence of N elements. The DFT will be considered in more detail later in Chapter 11.

Since the DFT is essentially a renormalized (i.e., rescaled) version of DT Fourier series, the properties of the DFT and Fourier series are very similar. The only differences (aside from the sequence indices being taken modulo N in the DFT case) are some differing scale factors appearing in the formulas associated with a few properties. The properties of the DFT are summarized in Table 10.2. As can be seen from this table, most of the properties of the DFT are similar to their counterparts listed earlier for Fourier series in Table 10.1. The only notable differences are the duality, periodic convolution, and multiplication properties and Parseval's relation. Some of the constants appearing the formulas associated with these properties differ by a factor of N relative to their Fourier-series counterparts.

10.7 Fourier Series and Frequency Spectra

The Fourier series represents a (periodic) sequence in terms of harmonically-related complex sinusoids. In this sense, the Fourier series captures information about the frequency content of a sequence. Each complex sinusoid is associated with a particular frequency (which is some integer multiple of the fundamental frequency). So, these coefficients indicate at which frequencies the information/energy in a sequence is concentrated. For example, if only the Fourier-series coefficients for the low-order harmonics have large magnitudes, then the sequence is mostly associated with low frequencies. On the other hand, if a function has many large magnitude coefficients for high-order harmonics, then the sequence has a considerable amount of information/energy associated with high frequencies. In this way, the Fourier-series representation provides a means for measuring the frequency content of a sequence. The distribution of the energy/information in a sequence over different frequencies is referred to as the **frequency spectrum** of the sequence.

To gain further insight into the role played by the Fourier-series coefficients c_k in the context of the frequency

Table 10.2: Properties of the Discrete Fourier Transform

Property	Time Domain	Fourier Domain
Linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(k)+a_2X_2(k)$
Translation	$x(n-n_0)$	$e^{-jk(2\pi/N)n_0}X(k)$
Modulation	$e^{j(2\pi/N)k_0n}x(n)$	$X(k-k_0)$
Reflection	$x(-n)$	$X(-k)$
Conjugation	$x^*(n)$	$X^*(-k)$
Duality	$X(n)$	$Nx(-k)$
Periodic Convolution	$x_1 \circledast x_2(n)$	$X_1(k)X_2(k)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{N}X_1 \circledast X_2(k)$

Property	
Parseval's Relation	$\sum_{n=0}^{N-1}\|x(n)\|^2 = \frac{1}{N}\sum_{k=0}^{N-1}\|X(k)\|^2$
Even Symmetry	x is even $\Leftrightarrow$ X is even
Odd Symmetry	x is odd $\Leftrightarrow$ X is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ X is conjugate symmetric

spectrum of the sequence x, it is helpful to write the Fourier series with the c_k expressed in polar form as follows:

$$\begin{aligned} x(n) &= \sum_{k=\langle N\rangle} c_k e^{j(2\pi/N)kn} \\ &= \sum_{k=\langle N\rangle} |c_k| e^{j\arg c_k} e^{j(2\pi/N)kn} \\ &= \sum_{k=\langle N\rangle} |c_k| e^{j((2\pi/N)kn+\arg c_k)}. \end{aligned}$$

Clearly (from the last line of the above equation), the kth term in the summation corresponds to a complex sinusoid with the frequency $\frac{2\pi}{N}k$ that has had its amplitude scaled by a factor of $|c_k|$ and has been time-shifted by an amount that depends on $\arg c_k$. For a given k, the larger $|c_k|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j(2\pi/N)kn}$, and therefore the larger the contribution the kth term (which is associated with frequency $\frac{2\pi}{N}k$) will make to the overall summation. Consequently, $|c_k|$ can be used to quantify how much information a sequence x has at the frequency $\frac{2\pi}{N}k$. The quantity $\arg c_k$ is also important because its affects how much different complex sinusoids in the Fourier series are time-shifted relative to one another before being added together.

To formalize the notion of frequency spectrum, the **frequency spectrum** of a sequence x is essentially its corresponding Fourier-series coefficients c_k. Due to the above interpretation of a Fourier series in terms of the polar form, we are often interested in $|c_k|$ and $\arg c_k$. As a matter of terminology, we refer to $|c_k|$ as the **magnitude spectrum** of x and $\arg c_k$ as the **phase spectrum** of x.

Since the graphical presentation of information is often helpful for visualization purposes, we often want to plot frequency spectra of sequences. Since three-dimensional plots are usually more difficult to generate (especially by hand) than two-dimensional ones and can often be more difficult to interpret accurately, we usually present frequency spectra in graphical form using only two-dimensional plots. In the case that the frequency spectrum is either purely real or purely imaginary, we typically plot the frequency spectrum directly on a single pair of axes. Most often, however, the frequency spectrum will be complex (but neither purely real nor purely imaginary), in which case we plot the frequency spectrum in polar form by using two plots, one showing the magnitude spectrum and one showing the phase spectrum. When plotting frequency spectra (including magnitude and phase spectra), the horizontal axis is labelled with the frequency corresponding to Fourier-series coefficient indices, rather than the indices themselves. Frequency is used for the quantity corresponding to the horizontal axis since frequency has a direct physical meaning (i.e., the rate at which something is oscillating), whereas the Fourier-series coefficient index is just an integer that has no direct physical meaning.

Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, we only have values to plot for these particular frequencies. In other words, the frequency spectrum is discrete in the independent variable (i.e., frequency). For this reason, we use a stem graph to plot such functions. Due to the general appearance of the graph (i.e., a number of vertical lines at various frequencies) we refer to such spectra as **line spectra**.

Recall that, for a real sequence x, the Fourier-series coefficient sequence c is conjugate symmetric (i.e., $c_k = c_{-k}^*$ for all k). This, however, implies that $|c_k| = |c_{-k}|$ and $\arg c_k = -\arg c_{-k}$. Since $|c_k| = |c_{-k}|$, the magnitude spectrum of a real sequence is always even. Similarly, since $\arg c_k = -\arg c_{-k}$, the phase spectrum of a real sequence is always odd.

Example 10.8. The 8-periodic sequence x in Example 10.4 has the Fourier-series coefficient sequence c given by

$$c_k = \frac{\sin\left(\frac{\pi}{2}k\right)}{8e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)}.$$

Find and plot the magnitude and phase spectra of x. Determine at what frequency (or frequencies) in $(-\pi,\pi]$ the sequence x has the most information.

Solution. First, we compute the magnitude spectrum of x, which is given by $|c_k|$. Taking the magnitude of c_k, we have

$$\begin{aligned}
|c_k| &= \left| \frac{\sin\left(\frac{\pi}{2}k\right)}{8e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)} \right| \\
&= \frac{\left|\sin\left(\frac{\pi}{2}k\right)\right|}{\left|8e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)\right|} \\
&= \frac{\left|\sin\left(\frac{\pi}{2}k\right)\right|}{8\left|e^{j(3\pi/8)k}\right|\left|\sin\left(\frac{\pi}{8}k\right)\right|} \\
&= \frac{\left|\sin\left(\frac{\pi}{2}k\right)\right|}{8\left|\sin\left(\frac{\pi}{8}k\right)\right|}.
\end{aligned}$$

Next, we compute the phase spectrum of x, which is given by $\arg c_k$. Taking the argument of c_k, We have

$$\begin{aligned}
\arg c_k &= \arg\left(\frac{\sin\left(\frac{\pi}{2}k\right)}{8e^{j(3\pi/8)k}\sin\left(\frac{\pi}{8}k\right)} \right) \\
&= \arg\left[\sin\left(\tfrac{\pi}{2}k\right)\right] - \arg\left[8e^{j(3\pi/8)k}\sin\left(\tfrac{\pi}{8}k\right)\right] \\
&= \arg\left[\sin\left(\tfrac{\pi}{2}k\right)\right] - \left(\arg 8e^{j(3\pi/8)k} + \arg\left[\sin\left(\tfrac{\pi}{8}k\right)\right]\right) \\
&= \arg\left[\sin\left(\tfrac{\pi}{2}k\right)\right] - \left(\tfrac{3\pi}{8}k + \arg\left[\sin\left(\tfrac{\pi}{8}k\right)\right]\right) \\
&= \arg\left[\sin\left(\tfrac{\pi}{2}k\right)\right] - \arg\left[\sin\left(\tfrac{\pi}{8}k\right)\right] - \tfrac{3\pi}{8}k.
\end{aligned}$$

The magnitude and phase spectra of x are plotted in Figures 10.1(a) and (b), respectively. Note that the magnitude spectrum has even symmetry, while the phase spectrum has odd symmetry. This is what we should expect, since x is real. Considering only frequencies in $(-\pi, \pi]$, the sequence x has the most information at a frequency of 0 as this corresponds to the Fourier-series coefficient with the largest magnitude (i.e., $\frac{1}{2}$). ■

Example 10.9. Earlier, in Example 10.2, we saw that the N-periodic sequence

$$x(n) = \sum_{\ell=-\infty}^{\infty} \delta(n - N\ell)$$

has the Fourier-series coefficient sequence

$$c_k = \frac{1}{N}.$$

For the case that $N = 10$, find the magnitude and phase spectra of x, and plot the frequency spectrum of x.

Solution. The magnitude and phase spectra of x are given by the magnitude and argument of c_k, respectively. Taking the magnitude of c_k, we have

$$|c_k| = \left|\tfrac{1}{10}\right| = \tfrac{1}{10}.$$

Similarly, taking the argument of c_k, we have

$$\arg c_k = \arg \tfrac{1}{10} = 0.$$

Since all of the c_k are real, we can plot the frequency spectrum of x directly using a single two-dimensional plot (instead of using two two-dimensional plots with one for each of the magnitude and phase spectra). The frequency spectrum of x is plotted in Figure 10.2. ■

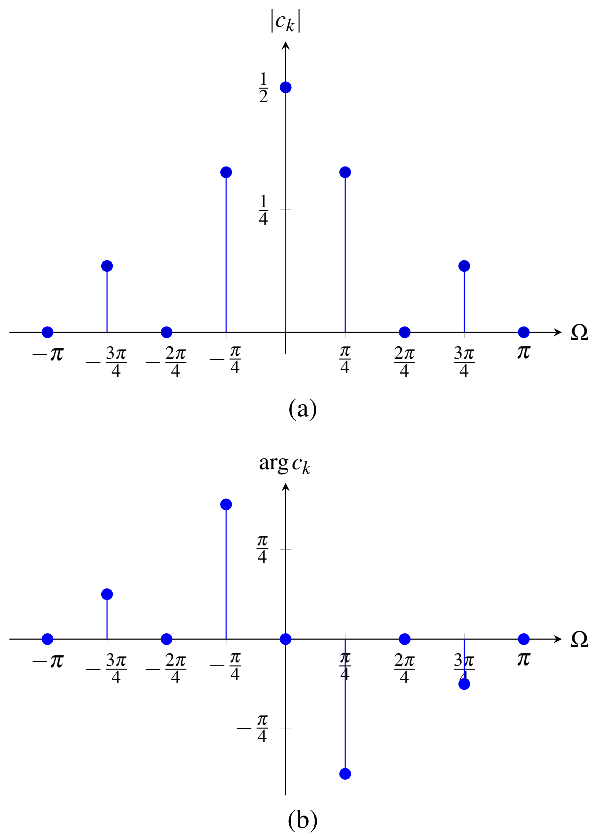


Figure 10.1: Frequency spectrum of x. (a) Magnitude spectrum and (b) phase spectrum.

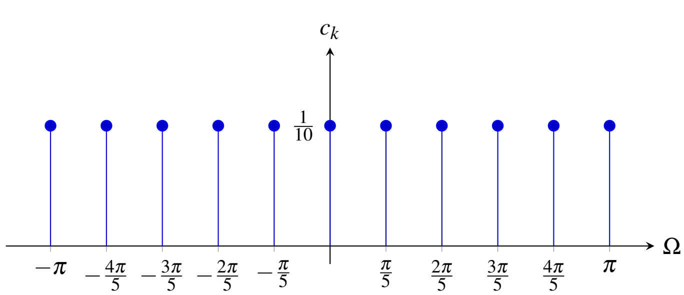


Figure 10.2: Frequency spectrum of x.

10.8 Fourier Series and LTI Systems

From earlier, in Theorem 9.12, we know that complex exponentials are eigensequences of LTI systems. Since complex sinusoids are a special case of complex exponentials, it follows that complex sinusoids are also eigensequences of LTI systems. In particular, we have the result below.

Corollary 10.1. *For an arbitrary LTI system $\mathcal{H}$ with impulse response h and a sequence of the form $x(n) = e^{j\Omega n}$, where Ω is an arbitrary real constant (i.e., x is an arbitrary complex sinusoid), the following holds:*

$$\mathcal{H}x(n) = H(\Omega)e^{j\Omega n},$$

where

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega n}. \tag{10.12}$$

That is, x is an eigensequence of $\mathcal{H}$ with the corresponding eigenvalue $H(\Omega)$.

The preceding result (i.e., Corollary 10.1) is simply a special case of Theorem 9.12 for $z = e^{j\Omega}$. Note that, in order to obtain more convenient notation, the function H in Corollary 10.1 is defined differently from the function H in Theorem 9.12. In particular, letting H_{F} and H_{Z} denote the function H that appears in each of Corollary 10.1 and Theorem 9.12, respectively, we have the relationship $H_{\mathsf{F}}(\Omega) = H_{\mathsf{Z}}(e^{j\Omega})$.

As a matter of terminology, the function H in (10.12) is referred to as the **frequency response** of the system $\mathcal{H}$. The frequency response completely characterizes the behavior of a LTI system. Consequently, the frequency response is often useful when working with LTI systems. As it turns out, (10.12) is of fundamental importance, as it defines what is called the (DT) Fourier transform. We will study the (DT) Fourier transform in great depth later in Chapter 11.

Let us now consider an application of eigensequences. Since convolution can often be quite painful to handle at the best of times, let us exploit eigensequences in order to devise a means to avoid having to deal with convolution directly in certain circumstances. Suppose that we have an N-periodic sequence x represented in terms of a Fourier series as

$$x(n) = \sum_{k=\langle N\rangle} c_k e^{j(2\pi/N)kn}.$$

Using (10.12) and the superposition property, we can determine the system response y to the input x as follows:

$$\begin{aligned}
y(n) &= \mathcal{H}x(n) \\
&= \mathcal{H}\left\{\sum_{k=\langle N\rangle} c_k e^{j(2\pi/N)kn}\right\}(n) \\
&= \sum_{k=\langle N\rangle} \mathcal{H}\left\{c_k e^{j(2\pi/N)kn}\right\}(n) \\
&= \sum_{k=\langle N\rangle} c_k \mathcal{H}\left\{e^{j(2\pi/N)kn}\right\}(n) \\
&= \sum_{k=\langle N\rangle} c_k H\left(\tfrac{2\pi}{N}k\right) e^{j(2\pi/N)kn}.
\end{aligned}$$

Therefore, we can view a LTI system as an entity that operates on the individual coefficients of a Fourier series. In particular, the system forms its output by multiplying each Fourier series coefficient by the value of the frequency response function at the frequency to which the Fourier-series coefficient corresponds. In other words, if

$$x(n) \xleftrightarrow{\text{DTFS}} c_k$$

then

$$y(n) \xleftrightarrow{\text{DTFS}} H\left(\tfrac{2\pi}{N}k\right) c_k.$$

Example 10.10. A LTI system has the impulse response

$$h(n) = \left(\tfrac{1}{2}\right)^{n+1} u(n).$$

Find the response y of this system to the input x, where

$$x(n) = 1 + \tfrac{1}{2}\cos\left(\tfrac{\pi}{2}n\right) + \tfrac{1}{8}\cos(\pi n).$$

Solution. To begin, we find the frequency response H of the system. Recalling (10.12), we have

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega n}.$$

Substituting the given expression for h, we obtain

$$\begin{aligned} H(\Omega) &= \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^{n+1} u(n)e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{n+1} e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} \tfrac{1}{2}\left(\tfrac{1}{2}e^{-j\Omega}\right)^{n}. \end{aligned}$$

Using the formula for the sum of an infinite geometric sequence, we have

$$\begin{aligned} H(\Omega) &= \tfrac{1}{2}\left(\frac{1}{1-\tfrac{1}{2}e^{-j\Omega}}\right) \\ &= \frac{1}{2-e^{-j\Omega}}. \end{aligned}$$

Now, we find the Fourier-series representation of x. The fundamental period N of x is $\operatorname{lcm}(4,2) = 4$. We can rewrite x as

$$\begin{aligned} x(n) &= 1 + \tfrac{1}{2}\left[\tfrac{1}{2}\left(e^{j(\pi/2)n} + e^{-j(\pi/2)n}\right)\right] + \tfrac{1}{8}\left[\tfrac{1}{2}\left(e^{j\pi n} + e^{-j\pi n}\right)\right] \\ &= 1 + \tfrac{1}{4}e^{j(\pi/2)n} + \tfrac{1}{4}e^{-j(\pi/2)n} + \tfrac{1}{8}\left(\tfrac{1}{2}\right)(2)e^{j\pi n} \\ &= \tfrac{1}{4}e^{-j(\pi/2)n} + 1 + \tfrac{1}{4}e^{j(\pi/2)n} + \tfrac{1}{8}e^{j\pi n} \\ &= \tfrac{1}{4}e^{j(2\pi/4)(-1)n} + 1 + \tfrac{1}{4}e^{j(2\pi/4)(1)n} + \tfrac{1}{8}e^{j(2\pi/4)(2)n}. \end{aligned}$$

Thus, the Fourier series for x is given by

$$x(n) = \sum_{k=-1}^{2} a_k e^{j(2\pi/4)kn},$$

where

$$a_k = \begin{cases} \tfrac{1}{4} & k=-1 \\ 1 & k=0 \\ \tfrac{1}{4} & k=1 \\ \tfrac{1}{8} & k=2. \end{cases}$$

The magnitude spectrum of x is plotted in Figure 10.3 along with the magnitude response $|H(\cdot)|$.

From the eigensequence properties of LTI systems, we have

$$y(n) = \sum_{k=-1}^{2} b_k e^{j(2\pi/4)kn} \quad \text{where} \quad b_k = a_k H\left(\tfrac{2\pi}{4}k\right).$$

Computing the b_k, we have

$$\begin{aligned}
b_{-1} &= a_{-1} H\left(\tfrac{2\pi}{4}[-1]\right) = a_{-1} H\left(-\tfrac{\pi}{2}\right) \\
&= \left(\tfrac{1}{4}\right)\left(\frac{1}{2-e^{-j(-\pi/2)}}\right) = \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2-j}\right) = \tfrac{1}{8-4j} = \left(4\sqrt{5}e^{-j\pi/4}\right)^{-1} \\
&= \tfrac{1}{4\sqrt{5}} e^{j\pi/4}, \\
b_0 &= a_0 H\left(\tfrac{2\pi}{4}[0]\right) = a_0 H(0) \\
&= (1)\left(\frac{1}{2-e^{-j0}}\right) \\
&= 1, \\
b_1 &= a_1 H\left(\tfrac{2\pi}{4}[1]\right) = a_1 H\left(\tfrac{\pi}{2}\right) \\
&= \left(\tfrac{1}{4}\right)\left(\frac{1}{2-e^{-j\pi/2}}\right) = \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2+j}\right) = \tfrac{1}{8+4j} = \left(4\sqrt{5}e^{j\pi/4}\right)^{-1} \\
&= \tfrac{1}{4\sqrt{5}} e^{-j\pi/4}, \quad \text{and} \\
b_2 &= a_2 H\left(\tfrac{2\pi}{4}[2]\right) = a_2 H(\pi) \\
&= \left(\tfrac{1}{8}\right)\left(\frac{1}{2-e^{-j\pi}}\right) = \left(\tfrac{1}{8}\right)\left(\tfrac{1}{2+1}\right) \\
&= \tfrac{1}{24}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
y(n) &= \sum_{k=-1}^{2} b_k e^{j(2\pi/4)kn} \\
&= b_{-1} e^{j(2\pi/4)(-1)n} + b_0 e^{j(2\pi/4)(0)n} + b_1 e^{j(2\pi/4)(1)n} + b_2 e^{j(2\pi/4)(2)n} \\
&= \tfrac{1}{4\sqrt{5}} e^{j\pi/4} e^{-j(\pi/2)n} + 1 + \tfrac{1}{4\sqrt{5}} e^{-j\pi/4} e^{j(\pi/2)n} + \tfrac{1}{24} e^{j\pi n} \\
&= 1 + \tfrac{1}{4\sqrt{5}} \left[e^{-j\pi/4} e^{j(\pi/2)n} + e^{j\pi/4} e^{j(-\pi/2)n}\right] + \tfrac{1}{24}\cos(\pi n) \\
&= 1 + \tfrac{1}{4\sqrt{5}} \left[e^{j[(\pi/2)n-\pi/4]} + e^{-j[(\pi/2)n-\pi/4]}\right] + \tfrac{1}{24}\cos(\pi n) \\
&= 1 + \tfrac{1}{4\sqrt{5}} \cos\left(\tfrac{\pi}{2}n - \tfrac{\pi}{4}\right) + \tfrac{1}{24}\cos(\pi n).
\end{aligned}$$

The magnitude spectrum of the output y is shown in Figure 10.4.

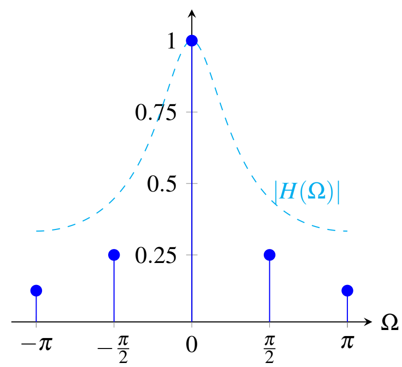


Figure 10.3: Magnitude spectrum of input sequence x.

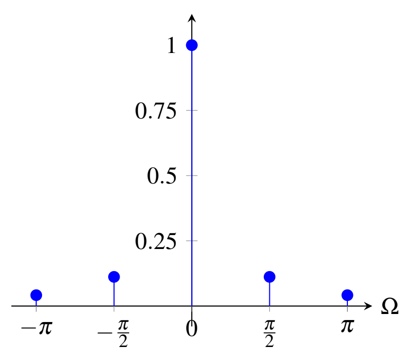


Figure 10.4: Magnitude spectrum of output sequence y.

In passing, we note that H can be written in Cartesian form as

$$\begin{aligned}
H(\Omega) &= \tfrac{1}{2}\left(\frac{1}{1-\frac{1}{2}[\cos(-\Omega)+j\sin(-\Omega)]}\right) \\
&= \tfrac{1}{2}\left(\frac{1}{1-\frac{1}{2}\cos\Omega+\frac{j}{2}\sin\Omega}\right) \\
&= \tfrac{1}{2}\left(\frac{1-\frac{1}{2}\cos\Omega-\frac{j}{2}\sin\Omega}{(1-\frac{1}{2}\cos\Omega)^2+(-\frac{1}{2}\sin\Omega)^2}\right) \\
&= \tfrac{1}{2}\left(\frac{1-\frac{1}{2}\cos\Omega-\frac{j}{2}\sin\Omega}{1-\cos\Omega+\frac{1}{4}\cos^2\Omega+\frac{1}{4}\sin^2\Omega}\right) \\
&= \tfrac{1}{2}\left(\frac{1-\frac{1}{2}\cos\Omega-\frac{j}{2}\sin\Omega}{\frac{5}{4}-\cos\Omega}\right) \\
&= \frac{2-\cos\Omega-j\sin\Omega}{5-4\cos\Omega} \\
&= \frac{\cos\Omega-2}{4\cos\Omega-5}+j\frac{\sin\Omega}{4\cos\Omega-5}.
\end{aligned}$$

This expression was used to assist in the generation the plot of $|H(\cdot)|$ in Figure 10.3. ■

Example 10.11. Consider a LTI system with the frequency response

$$H(\Omega) = e^{-j\Omega}.$$

Find the response y of the system to the input x, where

$$x(n) = \tfrac{1}{2}\cos\left(\tfrac{2\pi}{5}n\right).$$

Solution. To begin, we observe that x is 5-periodic. We rewrite x as

$$x(n) = \tfrac{1}{4}\left(e^{j(2\pi/5)n}+e^{-j(2\pi/5)n}\right).$$

Thus, the Fourier series for x is given by

$$x(n) = \sum_{k=-2}^{2} c_k e^{j(2\pi/5)kn},$$

where

$$c_k = \begin{cases} \frac{1}{4} & k\in\{-1,1\} \\ 0 & k\in\{-2,0,2\}. \end{cases}$$

Thus, we can write

$$
\begin{aligned}
y(n) &= \sum_{k=-2}^{2} c_k H(\tfrac{2\pi}{5}k) e^{j(2\pi/5)kn} \\
&= c_{-1} H(-\tfrac{2\pi}{5}) e^{-j(2\pi/5)n} + c_1 H(\tfrac{2\pi}{5}) e^{j(2\pi/5)n} \\
&= \tfrac{1}{4} H(-\tfrac{2\pi}{5}) e^{-j(2\pi/5)n} + \tfrac{1}{4} H(\tfrac{2\pi}{5}) e^{j(2\pi/5)n} \\
&= \tfrac{1}{4} e^{j(2\pi/5)} e^{-j(2\pi/5)n} + \tfrac{1}{4} e^{-j(2\pi/5)} e^{j(2\pi/5)n} \\
&= \tfrac{1}{4} [e^{-j(2\pi/5)(n-1)} + e^{j(2\pi/5)(n-1)}] \\
&= \tfrac{1}{4} \left(2\cos\left[\tfrac{2\pi}{5}(n-1)\right]\right) \\
&= \tfrac{1}{2} \cos\left[\tfrac{2\pi}{5}(n-1)\right].
\end{aligned}
$$

Observe that $y(n) = x(n-1)$. This is not a coincidence because, as it turns out, a LTI system with the frequency response $H(\Omega) = e^{-j\Omega}$ is an ideal unit delay (i.e., a system that performs a time shift of 1). ■

10.9 Filtering

In some applications, we want to change the magnitude or phase of the frequency components of a sequence or possibly eliminate some frequency components altogether. This process of modifying the frequency components of a sequence is referred to as **filtering** and the system that performs such processing is called a **filter**.

For the sake of simplicity, we consider only LTI filters here. If a filter is LTI, then it is completely characterized by its frequency response. Since the frequency spectra of sequences are 2π-periodic, we only need to consider frequencies over an interval of length 2π. Normally, we choose this interval to be centered about the origin, namely, $(-\pi, \pi]$. When we consider this interval, low frequencies are those closer to the origin, while high frequencies are those closer to $\pm\pi$.

Various types of filters exist. Frequency-selective filters pass some frequency components with little or no distortion, while significantly attenuating others. Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

An **ideal lowpass filter** eliminates all frequency components with a frequency greater in magnitude than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$
H(\Omega) = \begin{cases} 1 & |\Omega| \leq \Omega_c \\ 0 & \Omega_c < |\Omega| \leq \pi, \end{cases}
$$

where Ω_c is the cutoff frequency. A plot of this frequency response is shown in Figure 10.5(a).

The **ideal highpass filter** eliminates all frequency components with a frequency lesser in magnitude than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$
H(\Omega) = \begin{cases} 1 & \Omega_c < |\Omega| \leq \pi \\ 0 & |\Omega| \leq \Omega_c, \end{cases}
$$

where Ω_c is the cutoff frequency. A plot of this frequency response is shown in Figure 10.5(b).

An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$
H(\Omega) = \begin{cases} 1 & \Omega_{c1} \leq |\Omega| \leq \Omega_{c2} \\ 0 & |\Omega| < \Omega_{c1} \text{ or } \Omega_{c2} < |\Omega| < \pi, \end{cases}
$$

where the limits of the passband are Ω_{c1} and Ω_{c2}. A plot of this frequency response is shown in Figure 10.5(c).

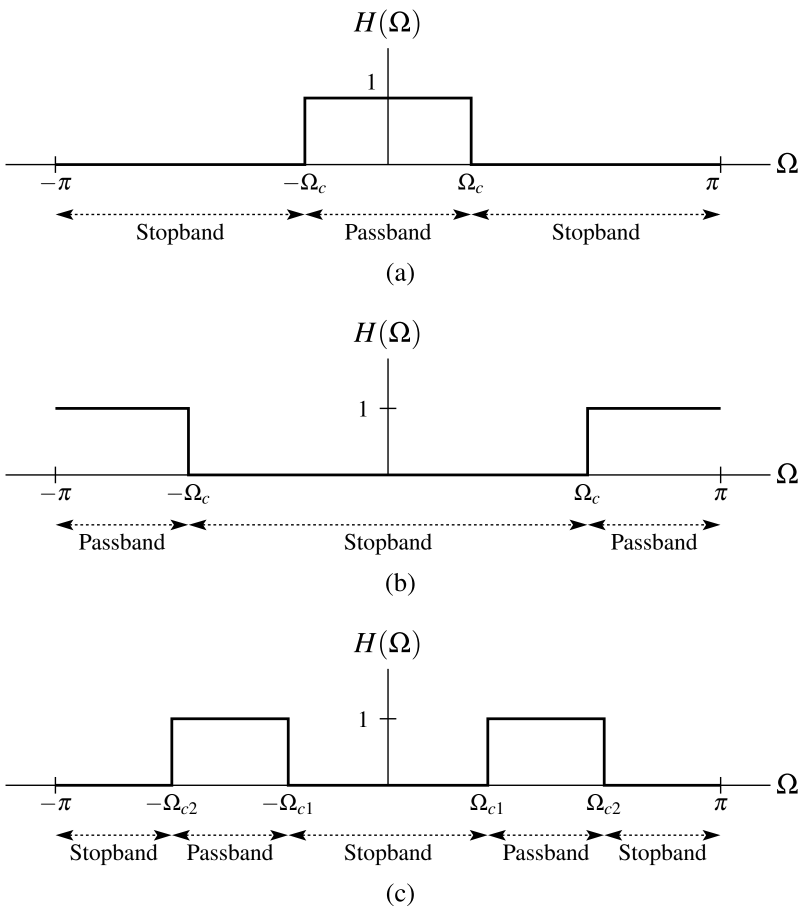


Figure 10.5: Frequency responses of (a) ideal lowpass, (b) ideal highpass, and (c) ideal bandpass filters.

Example 10.12 (Lowpass filtering)**.** Consider a LTI system with input x, output y, and frequency response H, where

$$H(\Omega) = \begin{cases} 1 & |\Omega| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\Omega| \leq \pi. \end{cases}$$

Suppose that the input x is the periodic sequence

$$x(n) = 1 + \cos\left(\tfrac{\pi}{4}n\right) + \tfrac{1}{2}\cos\left(\tfrac{3\pi}{4}n\right) + \tfrac{1}{5}\cos(\pi n).$$

(a) Find the Fourier-series representation of x. (b) Use this representation in order to find the response y of the system to the input x. (c) Plot the frequency spectra of x and y.

Solution. (a) We begin by finding the Fourier-series representation of x. The fundamental period N of x is given by

$$N = \operatorname{lcm}\{8,8,2\} = \operatorname{lcm}\{2^3,2^3,2^1\} = 2^3 = 8.$$

Using Euler's formula, we can re-express x as

$$\begin{aligned} x(n) &= 1 + \cos\left(\tfrac{\pi}{4}n\right) + \tfrac{1}{2}\cos\left(\tfrac{3\pi}{4}n\right) + \tfrac{1}{5}\cos(\pi n) \\ &= 1 + \left[\tfrac{1}{2}\left(e^{j(\pi/4)n} + e^{-j(\pi/4)n}\right)\right] + \tfrac{1}{2}\left[\tfrac{1}{2}\left(e^{j(3\pi/4)n} + e^{-j(3\pi/4)n}\right)\right] + \tfrac{1}{5}e^{j\pi n} \\ &= 1 + \tfrac{1}{2}\left(e^{j(\pi/4)n} + e^{-j(\pi/4)n}\right) + \tfrac{1}{4}\left(e^{j(3\pi/4)n} + e^{-j(3\pi/4)n}\right) + \tfrac{1}{5}e^{j\pi n} \\ &= \tfrac{1}{4}e^{-j(3\pi/4)n} + \tfrac{1}{2}e^{-j(\pi/4)n} + 1 + \tfrac{1}{2}e^{j(\pi/4)n} + \tfrac{1}{4}e^{j(3\pi/4)n} + \tfrac{1}{5}e^{j\pi n} \\ &= \tfrac{1}{4}e^{j(2\pi/8)(-3)n} + \tfrac{1}{2}e^{j(2\pi/8)(-1)n} + 1 + \tfrac{1}{2}e^{j(2\pi/8)(1)n} + \tfrac{1}{4}e^{j(2\pi/8)(3)n} + \tfrac{1}{5}e^{j(2\pi/8)(4)n}. \end{aligned}$$

Thus, we have that the Fourier series of x is given by

$$x(n) = \sum_{k=-3}^{4} a_k e^{j(2\pi/8)kn},$$

where

$$a_k = \begin{cases} \frac{1}{4} & k \in \{-3,3\} \\ \frac{1}{2} & k \in \{-1,1\} \\ 1 & k = 0 \\ \frac{1}{5} & k = 4 \\ 0 & k \in \{-2,2\}. \end{cases}$$

(b) Since the system is LTI, we know that the output y has the form

$$y(n) = \sum_{k=-\infty}^{\infty} b_k e^{j(2\pi/8)kn}$$

where

$$b_k = a_k H\left(\tfrac{2\pi}{8}k\right).$$

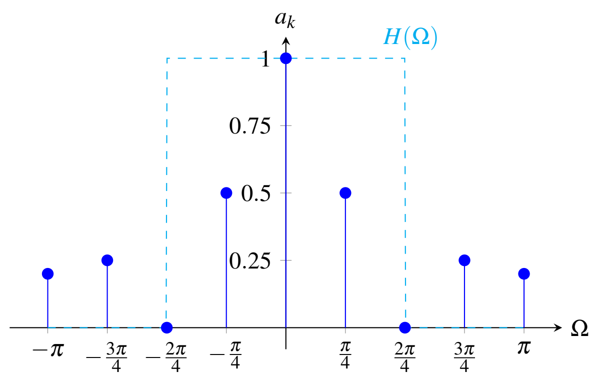
For each nonzero a_k, we compute the corresponding b_k to obtain:

$$\begin{aligned}
b_0 &= a_0 H\left[\left(\tfrac{\pi}{4}\right)(0)\right] = 1(1) \\
&= 1, \\
b_1 &= a_1 H\left[\left(\tfrac{\pi}{4}\right)(1)\right] = \left(\tfrac{1}{2}\right)(1) \\
&= \tfrac{1}{2}, \\
b_{-1} &= a_{-1} H\left[\left(\tfrac{\pi}{4}\right)(-1)\right] = \left(\tfrac{1}{2}\right)(1) \\
&= \tfrac{1}{2}, \\
b_3 &= a_3 H\left[\left(\tfrac{\pi}{4}\right)(3)\right] = \left(\tfrac{1}{4}\right)(0) \\
&= 0, \\
b_{-3} &= a_{-3} H\left[\left(\tfrac{\pi}{4}\right)(-3)\right] = \left(\tfrac{1}{4}\right)(0) \\
&= 0, \\
b_4 &= a_4 H\left[\left(\tfrac{\pi}{4}\right)(4)\right] = \left(\tfrac{1}{5}\right)(0) \\
&= 0.
\end{aligned}$$

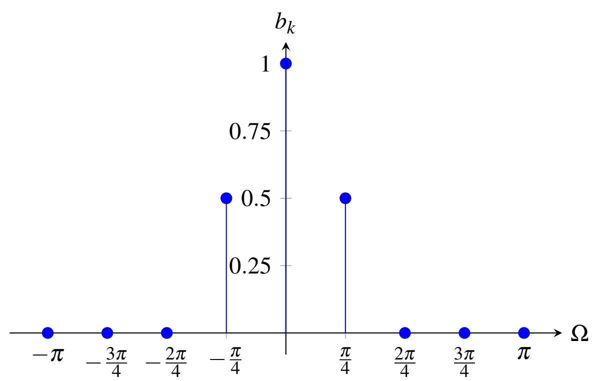
Thus, we have

$$b_k = \begin{cases} 1 & k = 0 \\ \frac{1}{2} & k \in \{-1, 1\} \\ 0 & k \in \{-3, -2, 2, 3, 4\}. \end{cases}$$

(c) Lastly, we plot the frequency spectra of x and y in Figures 10.6(a) and (b), respectively. The frequency response H is superimposed on the plot of the frequency spectrum of x for illustrative purposes. ■



(a)



(b)

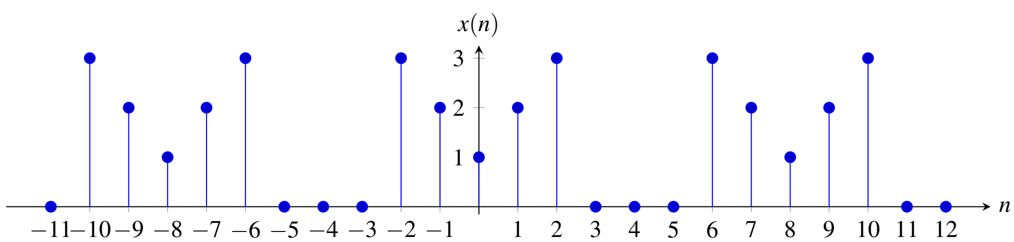
Figure 10.6: Frequency spectra of the (a) input sequence x and (b) output sequence y.

10.10 Exercises

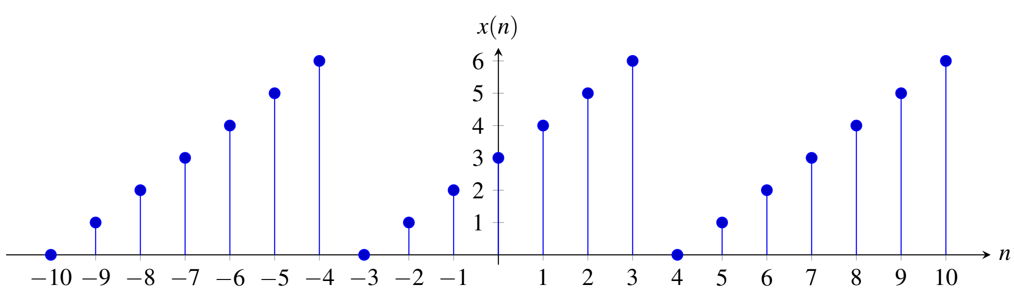
10.10.1 Exercises Without Answer Key

10.1 Find the Fourier series representation of each sequence x given below.
(a) $x(n) = \left(\frac{1}{2}\right)^n$ for $n \in [0\,..\,7]$ and $x(n) = x(n+8)$;
(b) $x(n) = n$ for $n \in [-3\,..\,3]$ and $x(n) = x(n+7)$;
(c) $x(n) = 1 + \cos\left(\frac{2\pi}{7}n\right) + \sin\left(\frac{4\pi}{7}n - \frac{\pi}{6}\right)$; and
(d) $x(n) = \begin{cases} 1 & n \in [-4\,..\,4] \\ 0 & n \in [-15\,..\,-5] \text{ or } [5\,..\,16] \end{cases}$ and $x(n) = x(n) + 32$.

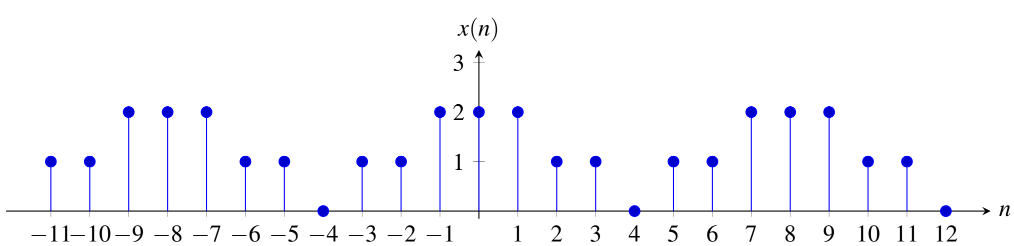
10.2 For each periodic sequence x shown in the figures below, find the corresponding Fourier series coefficient sequence a. The number of samples appearing in each plot is an integer multiple of the period. Properties of Fourier series may be used if helpful. [Hint: For part (c) use the result of Exercise 10.11.]



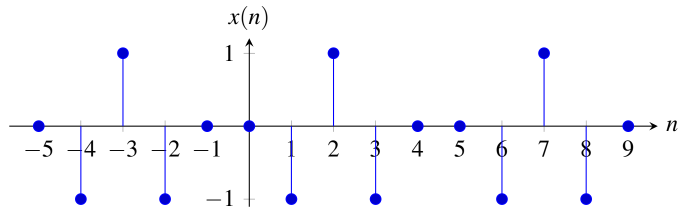
(a)



(b)



(c)



(d)

10.3 Show that, for a complex periodic sequence x with the Fourier series coefficient sequence a:
(a) x is even if and only if a is even; and
(b) x is odd if and only if a is odd.

10.4 Let x and y denote N-periodic sequences with the Fourier-series coefficient sequences a and b, respectively. For each sequence y given below, find an expression for b in terms of a.
(a) $y(n) = x(n) - x(n-1)$; and
(b) $y(n) = x(n+1) - x(n-1)$.

10.5 Let x be a 31-periodic sequence with the Fourier-series coefficient sequence a given by

$$a_k = k^2\left(2^k + 2^{-k}\right)e^{j(2\pi/31)k} \quad \text{for } k \in [-15\,..\,15].$$

Using Fourier series properties as appropriate, determine if each of the following assertions is true:
(a) x is real;
(b) x is even; and
(c) the sequence $y(n) = x(n-1)$ is even.

10.6 An N-periodic sequence x, where N is even, has the Fourier series coefficient sequence c.
(a) Show that if, $c_k = 0$ for all even $k \in \mathbb{Z}$, then $x(n) = -x(n - \frac{N}{2})$ for all $n \in \mathbb{Z}$.
(b) Show that if $x(n) = -x(n - \frac{N}{2})$ for all $n \in \mathbb{Z}$, then $c_k = 0$ for all even $k \in \mathbb{Z}$.

10.7 Let x and y denote N-periodic sequences with the Fourier series coefficient sequences a and b, respectively. For each sequence y given below, find b in terms of a.
(a) $y(n) = \operatorname{Even}\{x\}(n)$; and
(b) $y(n) = \operatorname{Re}\{x\}(n)$.

10.8 Find and plot the magnitude and phase spectrum of each sequence x given below for frequencies in $[-\pi, \pi]$.
(a) $x(n) = 1 + \sin\left(\frac{\pi}{5}n + \frac{\pi}{3}\right)$;
(b) $x(n) = \frac{3}{4} + \frac{1}{2}\cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right) + \frac{1}{4}\sin\left(\frac{\pi}{2}n\right) + \frac{1}{8}\cos(\pi n)$; and
(c) $x(n) = \begin{cases} -1 & n \in [-3\,..\,0] \\ 1 & n \in [1\,..\,4] \end{cases} \quad \text{and} \quad x(n) = x(n+8)$.

10.9 Consider a LTI system with frequency response

$$H(\Omega) = \begin{cases} 1 & |\Omega| \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ 0 & |\Omega| \in \left(\frac{\pi}{2}, \pi\right]. \end{cases}$$

Find the response y of the system to the input x, where

$$x(n) = 1 + \tfrac{1}{3}\cos\left(\tfrac{4\pi}{9}n\right) + \tfrac{1}{6}\cos\left(\tfrac{6\pi}{9}n\right).$$

10.10 A LTI system has the impulse response

$$h(n) = \left(\tfrac{1}{2}\right)^{n+1} u(n).$$

Find the response y of this system to the input x, where

$$x(n) = 1 + \cos\left(\tfrac{\pi}{4}n\right) + \sin\left(\tfrac{\pi}{2}n\right).$$

10.10.2 Exercises With Answer Key

10.11 Find the Fourier-series coefficient sequence a of an N-periodic square-wave sequence x of the form

$$x(n) = \begin{cases} 1 & n \in [N_1 \,..\, N_2] \\ 0 & n \in [N_0 \,..\, N_1 - 1] \cup [N_2 + 1 \,..\, N_0 + N - 1], \end{cases}$$

where N_0 is an integer, $N_1, N_2 \in [N_0 \,..\, N_0 + N - 1]$ and $N_1 \leq N_2$. (Note that this exercise is closely related to Exercise A.12.)

Short Answer.
$a_k = \begin{cases} \frac{1}{N} e^{-j\pi(N_1+N_2)k/N} \left[\frac{\sin[\pi(N_2-N_1+1)k/N]}{\sin(\pi k/N)}\right] & \frac{k}{N} \notin \mathbb{Z} \\ \frac{N_2-N_1+1}{N} & \frac{k}{N} \in \mathbb{Z}. \end{cases}$

10.11 MATLAB Exercises

Currently, there are no MATLAB exercises.

Chapter 11

Discrete-Time Fourier Transform

11.1 Introduction

The (DT) Fourier series provides an extremely useful representation for periodic sequences. Often, however, we need to deal with sequences that are not periodic. A more general tool than the Fourier series is needed in this case. In this chapter, we will introduce a tool for representing arbitrary (i.e., possibly aperiodic) sequences, known as the (DT) Fourier transform.

11.2 Development of the Discrete-Time Fourier Transform for Aperiodic Sequences

As demonstrated earlier, the Fourier series is an extremely useful representation for sequences. Unfortunately, this representation can only be used for sequences that are periodic, since a Fourier series is inherently periodic. Many sequences, however, are not periodic. Therefore, one might wonder if we can somehow use the Fourier series to develop a representation for aperiodic sequences. As it turns out, this is possible. In order to understand why, we must make the following key observation. An aperiodic sequence can be viewed as a periodic sequence with a period of infinity. By viewing an aperiodic sequence as this limiting case of a periodic sequence where the period is infinite, we can use the Fourier series to develop a more general representation for sequences that can be used in the aperiodic case. (In what follows, our development of the Fourier transform is not completely rigorous, as we assume that various integrals, summations, and limits converge. Such assumptions are not valid in all cases. Our development is mathematically sound, however, provided that the Fourier transform of the sequence being considered exists.)

Suppose that we have an aperiodic sequence x. From x, let us define the sequence x_M as

$$x_M(n) = \begin{cases} x(n) & n \in [-M \mathinner{..} M-1] \\ 0 & \text{otherwise.} \end{cases} \tag{11.1}$$

Essentially, x_M is identical in value to x for the $2M$ elements (approximately) centered about the origin. Let us now repeat the portion of $x_M(n)$ for $n \in [-M \mathinner{..} M-1]$ to form a $2M$-periodic sequence $\tilde{x}$. That is, we define $\tilde{x}$ as

$$\tilde{x}(n) = x_M(n) \;\; \text{for } n \in [-M \mathinner{..} M-1] \quad \text{and} \quad \tilde{x}(n) = \tilde{x}(n+2M).$$

Before proceeding further, we make two important observations that we will use later. First, from the definition of x_M, we have

$$\lim_{M\to\infty} x_M(n) = x(n). \tag{11.2}$$

Second, from the definition of x_M and $\tilde{x}$, we have

$$\lim_{M\to\infty} \tilde{x}(n) = x(n). \tag{11.3}$$

Now, let us consider the sequence $\tilde{x}$. Since $\tilde{x}$ is $2M$-periodic, we can represent it using a Fourier series as

$$\begin{aligned}\tilde{x}(n) &= \sum_{k=\langle 2M\rangle} a_k e^{j[2\pi/(2M)]kn} \\ &= \sum_{k=\langle 2M\rangle} a_k e^{j(\pi/M)kn} \\ &= \sum_{k=-M}^{M-1} a_k e^{j(\pi/M)kn}. \end{aligned} \tag{11.4}$$

The coefficient sequence a is then given by

$$\begin{aligned}a_k &= \tfrac{1}{2M}\sum_{\ell=\langle 2M\rangle} \tilde{x}(\ell)e^{-j(2\pi/(2M))k\ell} \\ &= \tfrac{1}{2M}\sum_{\ell=\langle 2M\rangle} \tilde{x}(\ell)e^{-j(\pi/M)k\ell} \\ &= \tfrac{1}{2M}\sum_{\ell=-M}^{M-1} \tilde{x}(\ell)e^{-j(\pi/M)k\ell}.\end{aligned}$$

Moreover, since $x_M(\ell) = \tilde{x}(\ell)$ for $\ell \in [-M\,..\,M-1]$, we can rewrite the preceding equation for a_k as

$$a_k = \tfrac{1}{2M}\sum_{\ell=-M}^{M-1} x_M(\ell)e^{-j(\pi/M)k\ell}.$$

Substituting this expression for a_k into (11.4) and rearranging, we obtain the following Fourier series representation for $\tilde{x}$:

$$\tilde{x}(n) = \sum_{k=-M}^{M-1}\left(\tfrac{1}{2M}\sum_{\ell=-M}^{M-1} x_M(\ell)e^{-j(\pi/M)k\ell}\right)e^{j(\pi/M)kn}.$$

Now, we define $\Delta\Omega = \frac{2\pi}{2M} = \frac{\pi}{M}$. Rewriting the previous equation for $\tilde{x}$ in terms of $\Delta\Omega$, we obtain

$$\begin{aligned}\tilde{x}(n) &= \sum_{k=-M}^{M-1}\left(\tfrac{1}{2\pi}\Delta\Omega\sum_{\ell=-M}^{M-1} x_M(\ell)e^{-j\Delta\Omega k\ell}\right)e^{j\Delta\Omega kn} \\ &= \tfrac{1}{2\pi}\sum_{k=-M}^{M-1}\left(\Delta\Omega\sum_{\ell=-M}^{M-1} x_M(\ell)e^{-j\Delta\Omega k\ell}\right)e^{j\Delta\Omega kn}.\end{aligned}$$

Substituting the above expression for $\tilde{x}$ into (11.3), we obtain

$$x(n) = \lim_{M\to\infty}\tfrac{1}{2\pi}\sum_{k=-M}^{M-1}\left(\Delta\Omega\sum_{\ell=-M}^{M-1} x_M(\ell)e^{-j\Delta\Omega k\ell}\right)e^{j\Delta\Omega kn} \tag{11.5}$$

Now, we must evaluate the above limit. As $M\to\infty$, we have that $\Delta\Omega\to 0$. Thus, in the limit above, $k\Delta\Omega$ becomes a continuous variable which we denote as Ω, $\Delta\Omega$ becomes the infinitesimal $d\Omega$, and the summation becomes an integral whose lower and upper limits are respectively given by

$$\lim_{M\to\infty}\left(k\Delta\Omega|_{k=-M}\right) = \lim_{M\to\infty} -M\Delta\Omega = \lim_{M\to\infty} -M\left(\tfrac{\pi}{M}\right) = -\pi \quad \text{and}$$
$$\lim_{M\to\infty}\left(k\Delta\Omega|_{k=M-1}\right) = \lim_{M\to\infty}(M-1)\Delta\Omega = \lim_{M\to\infty}(M-1)\left(\tfrac{\pi}{M}\right) = \pi.$$

From (11.2), as $M\to\infty$, we have that $x_M\to x$. Combining these results, we can rewrite (11.5) to obtain

$$x(n) = \tfrac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty} x(\ell)e^{-j\Omega\ell}\right)e^{j\Omega n}d\Omega,$$

Since the integrand is 2π-periodic, we can rewrite this equation as

$$x(n) = \frac{1}{2\pi}\int_{2\pi}\left(\sum_{\ell=-\infty}^{\infty} x(\ell)e^{-j\Omega\ell}\right)e^{j\Omega n}d\Omega,$$

Thus, we have that

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega,$$

where

$$X(\Omega) = \sum_{\ell=-\infty}^{\infty} x(\ell)e^{-j\Omega\ell}.$$

Thus, we have found a representation of the aperiodic sequence x in terms of complex sinusoids at all frequencies. We call this the (DT) Fourier transform representation of the sequence x.

11.3 Generalized Fourier Transform

In the previous section, we used a limiting process involving the analysis and synthesis equations for Fourier series in order to develop a new mathematical tool known as the (DT) Fourier transform. As it turns out, many sequences of practical interest do not have a Fourier transform in the sense of the definition developed previously. That is, for a given sequence x, the Fourier transform summation

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

may fail to converge, in which case the Fourier transform X of x does not exist. For example, the preceding summation does not converge if x is any of the following (as well as many other possibilities):

- a nonzero constant sequence;
- a periodic sequence (e.g., a real or complex sinusoid); or
- the unit-step sequence (i.e., u).

Sequences such these are of great practical interest, however. Therefore, it is highly desirable to have a mathematical tool that can handle such sequences. This motivates the development of what is called the **generalized Fourier transform**. The generalized Fourier transform exists for periodic sequences, nonzero constant sequences, and many other types of sequences as well. The underlying math associated with the generalized Fourier transform is quite complicated, however. So, we will not attempt to formally develop the generalized Fourier transform here. Although not entirely correct, one can think of the generalized Fourier transform as being defined by the same formulas as the classical Fourier transform. So, for this and other reasons, we can mostly ignore the distinction between the generalized Fourier transform and classical Fourier transform, and think of them as being one and the same. In what follows, we will avoid making a distinction between the classical Fourier transform and generalized Fourier transform, except in a very small number of places where it is beneficial to do so. The main disadvantage of not formally introducing the generalized Fourier transform is that some results presented later (which actually rely on the use of the generalized Fourier transform) must be accepted on faith since their proof would require formal knowledge of the generalized Fourier transform, which is not introduced herein. As long as the generalized Fourier transform is used, both periodic and aperiodic sequences can be handled, and in this sense we have a more general tool than Fourier series (which require periodic sequences). Later, when we discuss the Fourier transform of periodic sequences, we will implicitly be using the generalized Fourier transform in that context. In fact, in much of what follows, when we speak of the Fourier transform, we are often referring to the generalized Fourier transform.

11.4 Definition of the Discrete-Time Fourier Transform

Earlier, we developed the Fourier transform representation of a sequence. This representation expresses a sequence in terms of complex sinusoids at all frequencies. More formally, the **Fourier transform** of the sequence x, denoted as $\mathcal{F}x$ or X, is defined as

$$\mathcal{F}x(\Omega) = X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}. \tag{11.6}$$

Similarly, the inverse Fourier transform of X, denoted as $\mathcal{F}^{-1}X$ or x, is given by

$$\mathcal{F}^{-1}X(n) = x(n) = \tfrac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega. \tag{11.7}$$

We refer to (11.6) as the **Fourier transform analysis equation** and (11.7) as the **Fourier transform synthesis equation**.

To denote that a sequence x has the Fourier transform X, we can write

$$x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega).$$

As a matter of terminology, x and X are said to constitute a **Fourier transform pair**.

Example 11.1 (Fourier transform of a shifted and scaled delta sequence)**.** Find the Fourier transform X of the sequence

$$x(n) = A\delta(n-n_0),$$

where A is a real constant and n_0 is an integer constant. Then, from this result, write the Fourier transform representation of x.

Solution. From the definition of the Fourier transform, we can write

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} A\delta(n-n_0)e^{-j\Omega n} \\ &= A\sum_{n=-\infty}^{\infty} \delta(n-n_0)e^{-j\Omega n}. \end{aligned}$$

Using the sifting property of the delta sequence, we can simplify the above result to obtain

$$X(\Omega) = Ae^{-j\Omega n_0}.$$

Thus, we have shown that

$$A\delta(n-n_0) \overset{\text{DTFT}}{\longleftrightarrow} Ae^{-j\Omega n_0}.$$

From the Fourier transform analysis and synthesis equations, we have that the Fourier transform representation of x is given by

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega, \quad \text{where} \quad X(\Omega) = Ae^{-j\Omega n_0}. \qquad \blacksquare$$

Example 11.2 (Fourier transform of a rectangular pulse)**.** Find the Fourier transform X of the sequence

$$x(n) = u(n-a) - u(n-b),$$

where a and b are integer constants such that $a < b$.

Solution. To begin, we observe that

$$x(n) = \begin{cases} 1 & n \in [a\,..\,b) \\ 0 & \text{otherwise.} \end{cases}$$

From the definition of the Fourier transform, we can write

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \\ &= \sum_{n=a}^{b-1} e^{-j\Omega n} \\ &= \sum_{n=a}^{b-1} \left(e^{-j\Omega}\right)^n \\ &= e^{-ja\Omega} \sum_{n=0}^{b-a-1} \left(e^{-j\Omega}\right)^n. \end{aligned}$$

The summation on the right-hand side corresponds to the sum of a geometric sequence. Using the formula for the sum of a geometric sequence (F.2), we can write

$$\begin{aligned} X(\Omega) &= e^{-j\Omega a} \frac{\left(e^{-j\Omega}\right)^{b-a} - 1}{e^{-j\Omega} - 1} \\ &= \frac{e^{-jb\Omega} - e^{-ja\Omega}}{e^{-j\Omega} - 1} \\ &= \frac{e^{-ja\Omega} - e^{-jb\Omega}}{1 - e^{-j\Omega}}. \end{aligned}$$

From the relationship between exponentials and sinusoids, we can rewrite this as

$$\begin{aligned} X(\Omega) &= \frac{e^{-j(a+b)\Omega/2}\left(e^{-j(a-b)\Omega/2} - e^{j(a-b)\Omega/2}\right)}{e^{-j\Omega/2}\left(e^{j\Omega/2} - e^{-j\Omega/2}\right)} \\ &= \frac{e^{-j(a+b)\Omega/2}\left(2j\sin\left[\frac{(b-a)\Omega}{2}\right]\right)}{e^{-j\Omega/2}\left(2j\sin\left[\frac{\Omega}{2}\right]\right)} \\ &= \frac{e^{-j(a+b)\Omega/2}\sin\left[\frac{(b-a)\Omega}{2}\right]}{e^{-j\Omega/2}\sin\left[\frac{\Omega}{2}\right]} \\ &= e^{-j(a+b-1)\Omega/2}\left(\frac{\sin\left[\frac{(b-a)\Omega}{2}\right]}{\sin\left[\frac{\Omega}{2}\right]}\right). \end{aligned}$$

Thus, we have shown that

$$u(n-a) - u(n-b) \overset{\text{DTFT}}{\longleftrightarrow} e^{-j(a+b-1)\Omega/2}\left(\frac{\sin\left[\frac{1}{2}(b-a)\Omega\right]}{\sin\left[\frac{1}{2}\Omega\right]}\right).$$ ■

Example 11.3. Find the Fourier transform X of the sequence

$$x(n) = a^n u(n),$$

where a is a real constant satisfying $|a| < 1$.

Solution. From the definition of the Fourier transform, we can write

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} \left(ae^{-j\Omega}\right)^n. \end{aligned}$$

The summation on the right-hand side of the preceding equation is a geometric series. Using the formula for the sum of a geometric series (F.3), we can simplify the preceding equation (for $|a| < 1$) to obtain

$$\begin{aligned} X(\Omega) &= \frac{1}{1-ae^{-j\Omega}} \\ &= \frac{e^{j\Omega}}{e^{j\Omega}-a}. \end{aligned}$$

Thus, we have shown that

$$a^n u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega}-a} \quad \text{for } |a| < 1.$$ ■

Example 11.4. Find the Fourier transform X of the sequence

$$x(n) = a^{|n|},$$

where a is a real constant satisfying $|a| < 1$.

Solution. From the definition of the Fourier transform, we can write

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\Omega n} + \sum_{n=0}^{\infty} a^n e^{-j\Omega n} \\ &= \sum_{n=1}^{\infty} a^n e^{j\Omega n} + \sum_{n=0}^{\infty} a^n e^{-j\Omega n} \\ &= \sum_{n=1}^{\infty} \left(ae^{j\Omega}\right)^n + \sum_{n=0}^{\infty} \left(ae^{-j\Omega}\right)^n. \end{aligned}$$

Each of the summations on the right-hand side of this equation is a geometric series. Using the formula for the sum of a geometric series (F.3), we can simplify the preceding equation (for $|a| < 1$) to obtain

$$\begin{aligned} X(\Omega) &= \frac{ae^{j\Omega}}{1-ae^{j\Omega}} + \frac{1}{1-ae^{-j\Omega}} \\ &= \frac{1-ae^{j\Omega}+ae^{j\Omega}(1-ae^{-j\Omega})}{(1-ae^{j\Omega})(1-ae^{-j\Omega})} \\ &= \frac{1-ae^{j\Omega}+ae^{j\Omega}-a^2}{1-ae^{-j\Omega}-ae^{j\Omega}+a^2} \\ &= \frac{1-a^2}{1-a(e^{j\Omega}+e^{-j\Omega})+a^2} \\ &= \frac{1-a^2}{1-2a\cos\Omega+a^2}. \end{aligned}$$

Thus, we have shown that

$$a^{|n|} \overset{\text{DTFT}}{\longleftrightarrow} \frac{1-a^2}{1-2a\cos\Omega + a^2} \quad \text{for } |a| < 1. \qquad \blacksquare$$

Example 11.5. Find the inverse Fourier transform x of the (2π-periodic) sequence

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k).$$

(Note that, in the preceding formula, δ denotes the delta function, not the delta sequence.)

Solution. From the definition of the inverse Fourier transform, we can write

$$\begin{aligned} x(n) &= \tfrac{1}{2\pi} \int_{2\pi} \left[2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \right] e^{j\Omega n} d\Omega \\ &= \int_{2\pi} \delta(\Omega) e^{j\Omega n} d\Omega \\ &= \int_{-\pi}^{\pi} \delta(\Omega) e^{j\Omega n} d\Omega. \end{aligned}$$

Using the sifting property of the delta function, we can simplify the preceding equation to obtain

$$x(n) = e^0 = 1.$$

Thus, we have that

$$1 \overset{\text{DTFT}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k). \qquad \blacksquare$$

11.5 Remarks on Notation Involving the Fourier Transform

The Fourier transform operator $\mathcal{F}$ maps a sequence to a (2π-periodic) function, and the inverse Fourier transform operator $\mathcal{F}^{-1}$ maps a (2π-periodic) function to a sequence. Consequently, the operand for each of these operators must be a function/sequence (not a number). Consider the unnamed sequence that maps n to $e^{-|n/10|}$ as shown in Figure 11.1. Suppose that we would like to write an expression that denotes the Fourier transform of this sequence. At first, we might be inclined to write "$\mathcal{F}\{e^{-|n/10|}\}$". Strictly speaking, however, this notation is not correct, since the Fourier transform operator requires a sequence as an operand and "$e^{-|n/10|}$" (strictly speaking) denotes a number (i.e., the value of the sequence in the figure evaluated at n). Essentially, the cause of our problems here is that the sequence in question does not have a name (such as "x") by which it can be referred. To resolve this problem, we could define a sequence x using the equation $x(n) = e^{-|n/10|}$ and then write the Fourier transform as "$\mathcal{F}x$". Unfortunately, introducing a new sequence name just for the sake of strictly correct notation is often undesirable as it frequently leads to overly verbose writing.

One way to avoid overly verbose writing when referring to functions or sequences without names is offered by dot notation, introduced earlier in Section 2.1. Again, consider the sequence from Figure 11.1 that maps n to $e^{-|n/10|}$. Using strictly correct notation, we could write the Fourier transform of this sequence as "$\mathcal{F}\left\{e^{-|\cdot/10|}\right\}$". In other words, we can indicate that an expression refers to a sequence (as opposed to the value of sequence) by using the interpunct symbol (as discussed in Section 2.1). Some examples of the use of dot notation can be found below in Example 11.6. Dot notation is often extremely beneficial when one wants to employ precise (i.e., strictly correct) notation without being overly verbose.

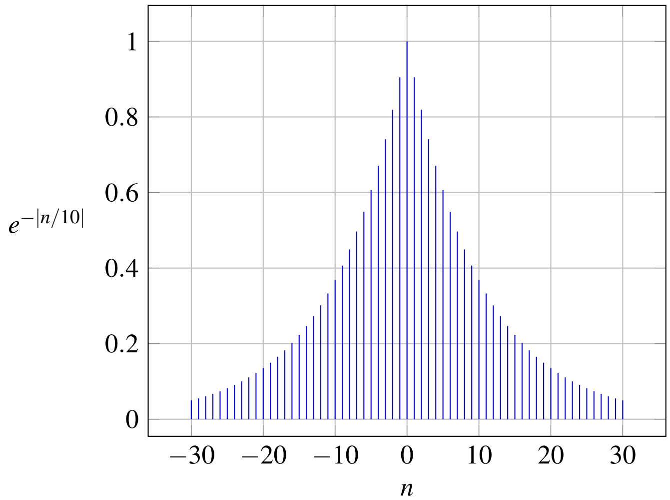


Figure 11.1: A plot of $e^{-|n/10|}$ versus n.

Example 11.6 (Dot notation)**.** Several examples of the use of dot notation are as follows:

1. To denote the Fourier transform of the sequence x defined by the equation $x(n) = e^{2|n|+3}$ (without the need to introduce the named sequence x), we can write: $\mathcal{F}\left\{e^{2|\cdot|+3}\right\}$.

2. To denote the Fourier transform of the sequence x defined by the equation $x(n) = e^{2|n|+3}$ evaluated at $2\Omega - 3$ (without the need to introduce the named sequence x), we can write: $\mathcal{F}\left\{e^{2|\cdot|+3}\right\}(2\Omega - 3)$.

3. To denote the inverse Fourier transform of the function X defined by the equation $X(\Omega) = \frac{1-a^2}{1-2a\cos\Omega+a^2}$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1-a^2}{1-2a\cos(\cdot)+a^2}\right\}$.

4. To denote the inverse Fourier transform of the function X defined by the equation $X(\Omega) = \frac{1-a^2}{1-2a\cos\Omega+a^2}$ evaluated at $n-3$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1-a^2}{1-2a\cos(\cdot)+a^2}\right\}(n-3)$. ■

If the reader is comfortable with dot notation, the author would encourage the reader to use it when appropriate. Since some readers may find the dot notation to be confusing, however, this book (for the most part) attempts to minimize the use of dot notation. Instead, as a compromise solution, this book adopts the following notational conventions in order to achieve conciseness and a reasonable level of clarity without the need to use dot notation pervasively:

- unless indicated otherwise, in an expression for the operand of the Fourier transform operator $\mathcal{F}$, the variable "n" is assumed to be the independent variable for the sequence to which the Fourier transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "n" were a "$\cdot$");

- unless indicated otherwise, in an expression for the operand of the inverse Fourier transform operator $\mathcal{F}^{-1}$, the variable "Ω" is assumed to be the independent variable for the function to which the inverse Fourier transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "Ω" were a "$\cdot$")

Some examples of using these book-sanctioned notational conventions can be found below in Example 11.7. Admittedly, these book-sanctioned conventions are not ideal, as they abuse mathematical notation somewhat, but they seem to be the best compromise in order to accommodate those who may prefer not to use dot notation.

Example 11.7 (Book-sanctioned notation)**.** Several examples of using the notational conventions that are employed throughout most of this book (as described above) are as follows:

1. To denote the Fourier transform of the sequence x defined by the equation $x(n) = e^{2|n|+3}$ (without the need to introduce the named sequence x), we can write: $\mathcal{F}\left\{e^{2|n|+3}\right\}$.

2. To denote the Fourier transform of the sequence x defined by the equation $x(n) = e^{2|n|+3}$ evaluated at $2\Omega - 3$ (without the need to introduce the named sequence x), we can write: $\mathcal{F}\left\{e^{2|n|+3}\right\}(2\Omega - 3)$.

3. To denote the inverse Fourier transform of the function X defined by the equation $X(\Omega) = \frac{1-a^2}{1-2a\cos\Omega+a^2}$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1-a^2}{1-2a\cos\Omega+a^2}\right\}$.

4. To denote the inverse Fourier transform of the function X defined by the equation $X(\Omega) = \frac{1-a^2}{1-2a\cos\Omega+a^2}$ evaluated at $n-3$ (without the need to introduce the named function X), we can write: $\mathcal{F}^{-1}\left\{\frac{1-a^2}{1-2a\cos\Omega+a^2}\right\}(n-3)$. ■

Since applying the Fourier transform operator or inverse Fourier transform operator to a sequence/function yields another function/sequence, we can evaluate this other function/sequence at some value. Again, consider the sequence from Figure 11.1 that maps n to $e^{-|n/10|}$. To denote the value of the Fourier transform of this sequence evaluated at $\Omega - 1$, we would write "$\mathcal{F}\{e^{-|\cdot/10|}\}(\Omega-1)$" using dot notation or "$\mathcal{F}\{e^{-|n/10|}\}(\Omega-1)$" using the book-sanctioned notational conventions described above.

11.6 Convergence Issues Associated with the Discrete-Time Fourier Transform

When deriving the Fourier transform representation earlier, we implicitly made some assumptions about the convergence of the summations/integrals and other expressions involved. These assumptions are not always valid. For this reason, a more careful examination of the convergence properties of the Fourier transform is in order.

Suppose that we have an arbitrary sequence x. This sequence has the Fourier transform representation $\hat{x}$ given by

$$\hat{x}(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega, \quad \text{where} \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}.$$

Now, we need to concern ourselves with the convergence properties of this representation. In other words, we want to know when $\hat{x}$ is a valid representation of x.

The first important result concerning convergence is given by the theorem below.

Theorem 11.1. *If a sequence x is absolutely summable (i.e., $\sum_{n=-\infty}^{\infty}|x(n)| < \infty$), then the Fourier transform X of x converges uniformly.*

Proof. A rigorous proof of this result is beyond the scope of this book and is therefore omitted here. ■

Since, in practice, we often encounter sequences that are not absolutely summable, the above result is sometimes not helpful. This motivates us to consider additional results concerning convergence.

The next important result concerning convergence relates to finite-energy sequences as stated by the theorem below.

Theorem 11.2 (Convergence of Fourier transform (finite-energy case))**.** *If a sequence x is of finite energy (i.e., $\sum_{n=-\infty}^{\infty}|x(n)|^2 < \infty$), then its Fourier transform representation $\hat{x}$ converges in the MSE sense.*

Proof. A rigorous proof of this result is beyond the scope of this book and is therefore omitted here. ■

11.7 Properties of the Discrete-Time Fourier Transform

The Fourier transform has a number of important properties. In the sections that follow, we introduce several of these properties. For convenience, these properties are also later summarized in Table 11.1 (on page 451).

11.7.1 Periodicity

A particularly important property of the (DT) Fourier transform is that is it always periodic, as elucidated by the theorem below.

Theorem 11.3 (Periodicity)**.** *If* $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$*, then*

$$X(\Omega) = X(\Omega + 2\pi)$$

(i.e., X is 2π-periodic).

Proof. To prove this property, we proceed as follows. From the definition of the Fourier transform, we have

$$\begin{aligned} X(\Omega + 2\pi) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j(\Omega+2\pi)n} \\ &= \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi n} e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x(n) \left(e^{-j2\pi}\right)^n e^{-j\Omega n}. \end{aligned}$$

Since $e^{-j2\pi} = 1$, we have

$$\begin{aligned} X(\Omega + 2\pi) &= \sum_{n=-\infty}^{\infty} x(n)(1)^n e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \\ &= X(\Omega). \end{aligned}$$

Thus, $X(\Omega + 2\pi) = X(\Omega)$ (i.e., X is 2π-periodic). ■

11.7.2 Linearity

Arguably, one of the most important properties of the Fourier transform is linearity, as introduced below.

Theorem 11.4 (Linearity)**.** *If* $x_1(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)$ *and* $x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_2(\Omega)$*, then*

$$a_1 x_1(n) + a_2 x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} a_1 X_1(\Omega) + a_2 X_2(\Omega),$$

where a_1 and a_2 are arbitrary complex constants. This is known as the ***linearity property*** *of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = a_1 x_1(n) + a_2 x_2(n)$ and let $Y = \mathcal{F}y$. We have

$$\begin{aligned} Y(\Omega) &= \sum_{n=-\infty}^{\infty} \left[a_1 x_1(n) + a_2 x_2(n)\right] e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} a_1 x_1(n) e^{-j\Omega n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) e^{-j\Omega n} \\ &= a_1 \sum_{n=-\infty}^{\infty} x_1(n) e^{-j\Omega n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) e^{-j\Omega n} \\ &= a_1 X_1(\Omega) + a_2 X_2(\Omega). \end{aligned}$$

Thus, we have shown that the linearity property holds. ■

Example 11.8. Using the Fourier transform pairs

$$\delta(n) \overset{\text{DTFT}}{\longleftrightarrow} 1 \quad \text{and} \quad u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k),$$

find the Fourier transform X of the sequence

$$x(n) = 2\delta(n) - u(n).$$

Solution. Taking the Fourier transform of x, we trivially have

$$X(\Omega) = \mathcal{F}\{2\delta(n) - u(n)\}(\Omega).$$

Using the linearity property of the Fourier transform, we can write

$$X(\Omega) = 2\mathcal{F}\delta(\Omega) - \mathcal{F}u(\Omega).$$

Using the given Fourier transform pairs, we obtain

$$\begin{aligned} X(\Omega) &= 2(1) - \left[\frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)\right] \\ &= \frac{2e^{j\Omega} - 2 - e^{j\Omega}}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \\ &= \frac{e^{j\Omega} - 2}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k). \end{aligned}$$

■

11.7.3 Translation (Time Shifting)

The next property of the Fourier transform to be introduced is the translation (or time-domain shifting) property, as given below.

Theorem 11.5 (Translation (i.e., time shifting))**.** *If* $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, *then*

$$x(n - n_0) \overset{\text{DTFT}}{\longleftrightarrow} e^{-j\Omega n_0} X(\Omega),$$

where n_0 is an arbitrary integer. This is known as the ***translation property*** *(or* ***time-domain shifting property****) of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = x(n - n_0)$, and let Y denote the Fourier transform of y. From the definition of the Fourier transform, we can write

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x(n - n_0) e^{-j\Omega n}.$$

Now, we employ a change of variable. Let $\lambda = n - n_0$ so that $n = \lambda + n_0$. Applying the change of variable, we obtain

$$\begin{aligned} Y(\Omega) &= \sum_{\lambda=-\infty}^{\infty} x(\lambda) e^{-j\Omega(\lambda + n_0)} \\ &= e^{-j\Omega n_0} \sum_{\lambda=-\infty}^{\infty} x(\lambda) e^{-j\Omega\lambda} \\ &= e^{-j\Omega n_0} X(\Omega). \end{aligned}$$

Thus, we have proven that the translation property holds. ■

Example 11.9. Using the Fourier transform pair

$$a^n u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega} - a} \quad \text{for } |a| < 1,$$

find the Fourier transform X of the sequence

$$x(n) = a^n u(n-3),$$

where a is a complex constant satisfying $|a| < 1$.

Solution. To begin, we observe that

$$x(n) = a^3 a^{n-3} u(n-3)$$

Define the sequence

$$v_1(n) = a^n u(n)$$

Using the definition of v, we can rewrite x as

$$x(n) = a^3 v_1(n-3).$$

Taking the Fourier transforms of v_1 and x, we obtain

$$V_1(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - a} \quad \text{and}$$
$$X(\Omega) = a^3 e^{-j3\Omega} V_1(\Omega).$$

Substituting the formula for V_1 into the formula for X, we obtain

$$\begin{aligned} X(\Omega) &= a^3 e^{-j3\Omega} V_1(\Omega) \\ &= a^3 e^{-j3\Omega} \left(\frac{e^{j\Omega}}{e^{j\Omega} - a} \right) \\ &= \frac{a^3 e^{-j2\Omega}}{e^{j\Omega} - a}. \end{aligned}$$

■

11.7.4 Modulation (Frequency-Domain Shifting)

The next property of the Fourier transform to be introduced is the modulation (i.e., frequency-domain shifting) property, as given below.

Theorem 11.6 (Modulation (i.e., frequency-domain shifting))**.** *If* $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$*, then*

$$e^{j\Omega_0 n} x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega - \Omega_0),$$

where Ω_0 *is an arbitrary real constant. This is known as the* ***modulation property*** *(or* ***frequency-domain shifting property****) of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = e^{j\Omega_0 n} x(n)$, and let Y denote the Fourier transform of y. From the definition of the Fourier transform, we have

$$\begin{aligned} Y(\Omega) &= \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n} x(n) e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x(n) e^{-j(\Omega - \Omega_0)n} \\ &= X(\Omega - \Omega_0). \end{aligned}$$

Thus, the modulation property holds. ■

Example 11.10. Using the Fourier transform pair,

$$1 \xleftrightarrow{\text{DTFT}} 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k),$$

find the Fourier transform X of the sequence

$$x(n) = \cos(\Omega_0 n),$$

where Ω_0 is a nonzero real constant.

Solution. To begin, we rewrite x as

$$x(n) = \tfrac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right).$$

Taking the Fourier transform of x, we trivially have

$$X(\Omega) = \mathcal{F}\left\{\tfrac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right)\right\}(\Omega).$$

From the linearity property of the Fourier transform, we have

$$X(\Omega) = \tfrac{1}{2}\left[\mathcal{F}\{1e^{j\Omega_0 n}\}(\Omega) + \mathcal{F}\{1e^{-j\Omega_0 n}\}(\Omega)\right].$$

Using the modulation property of the Fourier transform, we can write

$$X(\Omega) = \tfrac{1}{2}\left[\mathcal{F}\{1\}(\Omega - \Omega_0) + \mathcal{F}\{1\}(\Omega + \Omega_0)\right].$$

Substituting the result from the given Fourier transform pair, we obtain

$$\begin{aligned} X(\Omega) &= \tfrac{1}{2}\left(2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) + 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2\pi k)\right) \\ &= \pi \sum_{k=-\infty}^{\infty} \left[\delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k)\right]. \end{aligned}$$

■

11.7.5 Conjugation

The next property of the Fourier transform to be introduced is the conjugation property, as given below.

Theorem 11.7 (Conjugation)**.** *If* $x(n) \xleftrightarrow{\text{DTFT}} X(\Omega)$*, then*

$$x^*(n) \xleftrightarrow{\text{DTFT}} X^*(-\Omega).$$

This is known as the ***conjugation property*** *of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = x^*(n)$, and let Y denote the Fourier transform of y. From the definition of the Fourier transform, we have

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x^*(n) e^{-j\Omega n}.$$

From the properties of conjugation, we can rewrite this equation as

$$\begin{aligned} Y(\Omega) &= \left(\sum_{n=-\infty}^{\infty} x^*(n)e^{-j\Omega n}\right)^{**} \\ &= \left(\sum_{n=-\infty}^{\infty} x(n)e^{j\Omega n}\right)^{*} \\ &= \left(\sum_{n=-\infty}^{\infty} x(n)e^{-j(-\Omega)n}\right)^{*} \\ &= X^*(-\Omega). \end{aligned}$$

Thus, we have shown that the conjugation property holds. ■

Example 11.11 (Fourier transform of a real sequence)**.** Show that the Fourier transform X of a real sequence x is conjugate symmetric (i.e., $X(\Omega) = X^*(-\Omega)$ for all Ω).

Solution. From the conjugation property of the Fourier transform, we have

$$\mathcal{F}\{x^*(n)\}(\Omega) = X^*(-\Omega).$$

Since x is real, we can replace x^* with x to yield

$$\mathcal{F}x(\Omega) = X^*(-\Omega).$$

So, we have

$$X(\Omega) = X^*(-\Omega).$$

Thus, the Fourier transform of a real sequence is conjugate symmetric. ■

11.7.6 Time Reversal

The next property of the Fourier transform to be introduced is the time-reversal (i.e., reflection) property, as given below.

Theorem 11.8 (Time reversal (i.e., reflection))**.** *If* $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$*, then*

$$x(-n) \overset{\text{DTFT}}{\longleftrightarrow} X(-\Omega).$$

This is known as the ***time-reversal property*** *of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = x(-n)$, and let Y denote the Fourier transform of y. From the definition of the Fourier transform, we have

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x(-n)e^{-j\Omega n}.$$

Now, we employ a change of variable. Let $n' = -n$ so that $n = -n'$. Applying the change of variable and dropping the primes, we have

$$\begin{aligned} Y(\Omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(-\Omega)n} \\ &= X(-\Omega). \end{aligned}$$

Thus, we have shown that the time-reversal property holds. ■

Example 11.12. Using properties of the Fourier transform and the fact that

$$a^n u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega}-a},$$

find the Fourier transform X of the sequence

$$x(n) = a^{|n|}.$$

Solution. We begin by rewriting x as

$$x(n) = a^n u(n) + a^{-n} u(-n) - \delta(n).$$

Taking the Fourier transform of both sides of this equation, we obtain

$$X(\Omega) = \mathcal{F}\{a^n u(n) + a^{-n} u(-n) - \delta(n)\}(\Omega).$$

Using the linearity property of the Fourier transform, we can rewrite the preceding equation as

$$X(\Omega) = \mathcal{F}\{a^n u(n)\}(\Omega) + \mathcal{F}\{a^{-n} u(-n)\}(\Omega) - \mathcal{F}\delta(\Omega).$$

Using the time-reversal property of the Fourier transform, we have

$$X(\Omega) = \mathcal{F}\{a^n u(n)\}(\Omega) + \mathcal{F}\{a^n u(n)\}(-\Omega) - \mathcal{F}\delta(\Omega).$$

Using the given Fourier transform pair and the fact that $\mathcal{F}\delta(\Omega) = 1$, we have

$$\begin{aligned}
X(\Omega) &= \frac{e^{j\Omega}}{e^{j\Omega}-a} + \frac{e^{-j\Omega}}{e^{-j\Omega}-a} - 1 \\
&= \frac{e^{j\Omega}(e^{-j\Omega}-a) + e^{-j\Omega}(e^{j\Omega}-a) + (e^{j\Omega}-a)(e^{-j\Omega}-a)}{(e^{j\Omega}-a)(e^{-j\Omega}-a)} \\
&= \frac{1 - ae^{j\Omega} + 1 - ae^{-j\Omega} - (1 - ae^{j\Omega} - ae^{-j\Omega} + a^2)}{1 - ae^{j\Omega} - ae^{-j\Omega} + a^2} \\
&= \frac{1-a^2}{1 - a(e^{j\Omega} + e^{-j\Omega}) + a^2} \\
&= \frac{1-a^2}{1 - 2a\cos\Omega + a^2}.
\end{aligned}$$

■

11.7.7 Upsampling

The next property of the Fourier transform to be introduced is the upsampling property, as given below.

Theorem 11.9 (Upsampling). *If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then*

$$(\uparrow M)x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(M\Omega),$$

*where M is a (strictly) positive integer. This is known as the **upsampling property** of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = (\uparrow M)x(n)$, and let Y denote the Fourier transform of y. From the definition of the Fourier transform, we have

$$\begin{aligned}
Y(\Omega) &= \sum_{n=-\infty}^{\infty} y(n) e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} \left[(\uparrow M)x\right](n) e^{-j\Omega n}.
\end{aligned}$$

From the definition of upsampling, we have

$$Y(\Omega) = \sum_{\substack{n \in \mathbb{Z}: \\ M \text{ divides } n}} x(n/M)e^{-j\Omega n}.$$

Rewriting the summation so that only the terms where M divides n are considered, we obtain

$$\begin{aligned} Y(\Omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-jM\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(M\Omega)n} \\ &= X(M\Omega). \end{aligned}$$

Thus, we have shown that the upsampling property holds. ■

Example 11.13 (Upsampling property)**.** Using the Fourier transform pair

$$u(n) \overset{\text{DTFT}}{\longleftrightarrow} \tfrac{e^{j\Omega}}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k),$$

find the Fourier transform X of the sequence

$$x(n) = \begin{cases} 1 & n \geq 0 \text{ and } 3 \text{ divides } n \text{ (i.e., } n/3 \text{ is an integer)} \\ 0 & \text{otherwise.} \end{cases}$$

Solution. To begin, we observe that

$$x(n) = (\uparrow 3)u(n).$$

So, we have

$$X(\Omega) = \mathcal{F}\{(\uparrow 3)u(n)\}(\Omega).$$

From the upsampling property of the Fourier transform, we can write

$$X(\Omega) = \mathcal{F}u(3\Omega).$$

Using the given Fourier transform pair, we have

$$\begin{aligned} X(\Omega) &= \left. \left[\frac{e^{j\lambda}}{e^{j\lambda}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\lambda - 2\pi k) \right] \right|_{\lambda = 3\Omega} \\ &= \frac{e^{j3\Omega}}{e^{j3\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(3\Omega - 2\pi k). \end{aligned}$$ ■

11.7.8 Downsampling

The next property of the Fourier transform to be introduced is the downsampling property, as given below.

Theorem 11.10 (Downsampling)**.** *If* $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$*, then*

$$x(Mn) \overset{\text{DTFT}}{\longleftrightarrow} \tfrac{1}{M} \sum_{k=0}^{M-1} X\left[\tfrac{1}{M}(\Omega - 2\pi k)\right],$$

where M is a (strictly) positive integer. This is known as the ***downsampling property*** *of the Fourier transform.*

Proof. The proof immediately follows from Theorem 12.7 in the special case that $z = e^{j\Omega}$. ■

Example 11.14. Find the Fourier transform Y of the sequence

$$y(n) = (\downarrow 2)x(n),$$

where

$$x(n) = \tfrac{1}{4}\operatorname{sinc}\left(\tfrac{\pi}{4}n\right).$$

Solution. To begin, we compute the Fourier transform X of x. From Table 11.2, we have

$$\begin{aligned}X(\Omega) &= \mathcal{F}\left\{\tfrac{1}{4}\operatorname{sinc}\left(\tfrac{\pi}{4}n\right)\right\}(\Omega)\\ &= \sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{2}{\pi}(\Omega-2\pi k)\right].\end{aligned}$$

Now, we consider Y. From the downsampling property of the Fourier transform, we have

$$\begin{aligned}Y(\Omega) &= \tfrac{1}{2}\sum_{\ell=0}^{1} X\left[\tfrac{1}{2}(\Omega-2\pi\ell)\right]\\ &= \tfrac{1}{2}X\left(\tfrac{1}{2}\Omega\right)+\tfrac{1}{2}X\left[\tfrac{1}{2}(\Omega-2\pi)\right].\end{aligned}$$

Substituting the expression for X computed above into the preceding equation for Y, we have

$$\begin{aligned}Y(\Omega) &= \tfrac{1}{2}\sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{2}{\pi}(\tfrac{1}{2}\Omega-2\pi k)\right]+\tfrac{1}{2}\sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{2}{\pi}\left(\left[\tfrac{1}{2}(\Omega-2\pi)\right]-2\pi k\right)\right]\\ &= \tfrac{1}{2}\sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{2}{\pi}\left(\tfrac{1}{2}\right)(\Omega-4\pi k)\right]+\tfrac{1}{2}\sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{2}{\pi}\left(\tfrac{1}{2}\right)([(\Omega-2\pi)]-4\pi k)\right]\\ &= \tfrac{1}{2}\sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{1}{\pi}(\Omega-4\pi k)\right]+\tfrac{1}{2}\sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{1}{\pi}([(\Omega-2\pi)]-4\pi k)\right]\\ &= \tfrac{1}{2}\sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{1}{\pi}(\Omega-4\pi k)\right]+\tfrac{1}{2}\sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{1}{\pi}(\Omega-2\pi(2k+1))\right].\end{aligned}$$

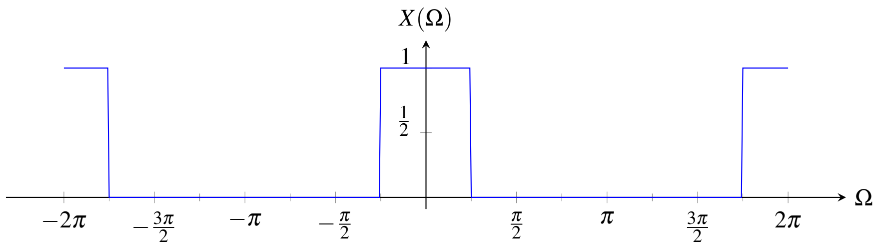
Note that each of the two summations on the right-hand side of the preceding equation are 4π-periodic functions. Consider $\Omega \in (-\pi,\pi]$. Over this interval, only the $k=0$ term from the first summation is nonzero and none of the terms in the second summation are nonzero. Therefore, we can simplify the preceding expression for Y to obtain

$$Y(\Omega) = \tfrac{1}{2}\operatorname{rect}\left(\tfrac{1}{\pi}\Omega\right) \quad \text{for } \Omega \in (-\pi,\pi].$$

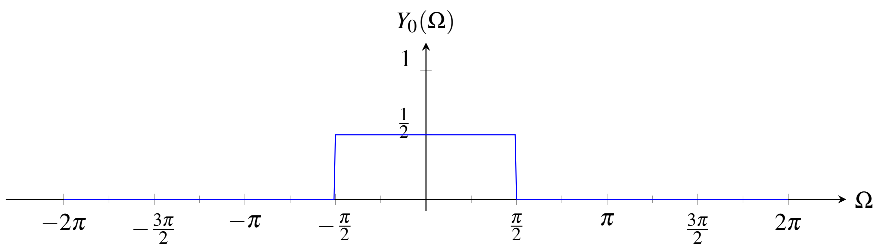
To visualize how the two infinite summations collapse into a single term, some graphs are helpful. The spectrum X is shown in Figure 11.2(a). The two 4π-periodic functions from above are denoted as Y_0 and Y_1 so that $Y = Y_0 + Y_1$. One full period of each of Y_0 and Y_1 is shown in Figures 11.2(b) and (c). Observe that of all of the rectangular pulses in the 4π-periodic functions Y_0 and Y_1, only one pulse (namely, from Y_0) falls in the interval $(-\pi,\pi]$. The spectrum $Y = Y_0 + Y_1$ is shown in Figure 11.2(d). ■

11.7.9 Convolution

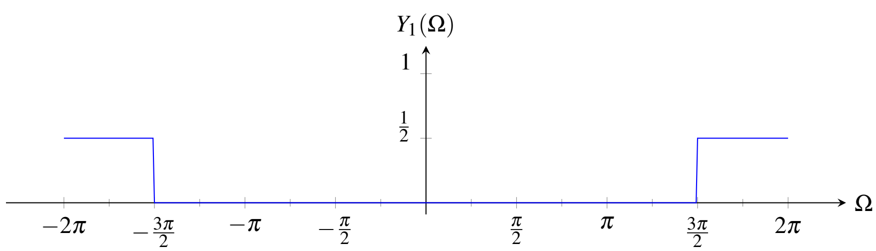
The next property of the Fourier transform to be introduced is the convolution property, as given below.



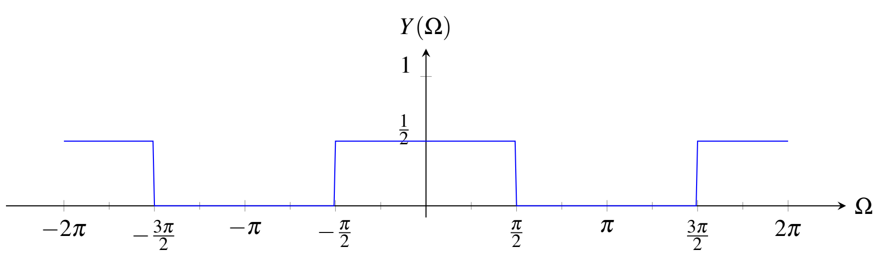
(a)



(b)



(c)



(d)

Figure 11.2: Spectra for downsampling example. (a) Spectrum X of x. (b) First summation Y_0 in expression for Y. (c) Second summation Y_1 in expression for Y. (d) Spectrum Y of y.

Theorem 11.11 (Convolution). *If* $x_1(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)$ *and* $x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_2(\Omega)$*, then*

$$x_1 * x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)X_2(\Omega).$$

This is known as the ***convolution property*** *of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = x_1 * x_2(n)$, and let Y denote the Fourier transform of y. From the definition of the Fourier transform, we have

$$\begin{aligned} Y(\Omega) &= \sum_{n=-\infty}^{\infty} [x_1 * x_2(n)]\, e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)e^{-j\Omega n}. \end{aligned}$$

Now, we employ a change of variable. Let $\lambda = n-k$ so that $n = \lambda + k$. Applying the change of variable, we obtain

$$\begin{aligned} Y(\Omega) &= \sum_{\lambda=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)x_2(\lambda)e^{-j\Omega(\lambda+k)} \\ &= \sum_{\lambda=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)x_2(\lambda)e^{-j\Omega\lambda}e^{-j\Omega k} \\ &= \sum_{k=-\infty}^{\infty} \sum_{\lambda=-\infty}^{\infty} x_1(k)x_2(\lambda)e^{-j\Omega\lambda}e^{-j\Omega k} \\ &= \left[\sum_{k=-\infty}^{\infty} x_1(k)e^{-j\Omega k}\right]\left[\sum_{\lambda=-\infty}^{\infty} x_2(\lambda)e^{-j\Omega\lambda}\right] \\ &= X_1(\Omega)X_2(\Omega). \end{aligned}$$

Thus, we have shown that the time-domain convolution property holds. ■

The convolution property of the Fourier transform has important practical implications. Since the Fourier transform effectively converts a convolution into a multiplication, the Fourier transform can be used as a means to avoid directly dealing with convolution operations. This is often extremely helpful when solving problems involving LTI systems, for example, since such problems almost inevitably involve convolution (due to the fact that a LTI system computes a convolution).

Example 11.15 (Convolution property). Find the Fourier transform X of the sequence

$$x(n) = x_1 * x_2(n)$$

where

$$x_1(n) = a^n u(n), \quad x_2(n) = a^{|n|},$$

and a is a complex constant satisfying $|a| < 1$.

Solution. We have

$$\begin{aligned} X(\Omega) &= \mathcal{F}\{x_1 * x_2\}(\Omega) \\ &= \mathcal{F}x_1(\Omega)\mathcal{F}x_2(\Omega). \end{aligned}$$

Using the Fourier transform pairs in Table 11.2, we have

$$\begin{aligned} X(\Omega) &= \left(\frac{e^{j\Omega}}{e^{j\Omega}-a}\right)\left(\frac{1-a^2}{1-2a\cos\Omega+a^2}\right) \\ &= \frac{(e^{j\Omega})(1-a^2)}{(e^{j\Omega}-a)(1-2a\cos\Omega+a^2)}. \end{aligned}$$ ■

11.7.10 Multiplication

The next property of the Fourier transform to be introduced is the multiplication property, as given below.

Theorem 11.12 (Multiplication)**.** *If $x_1(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)$ and $x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_2(\Omega)$, then*

$$x_1(n)x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{2\pi}\int_{2\pi} X_1(\theta)X_2(\Omega-\theta)d\theta.$$

This is known as the ***multiplication property*** *of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = x_1(n)x_2(n)$, and let Y denote the Fourier transform of y. We have

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n)e^{-j\Omega n}.$$

Rewriting $x_1(n)$ in terms of the formula for the inverse Fourier transform of X_1, we obtain

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} x_2(n)\left[\frac{1}{2\pi}\int_{2\pi} X_1(\lambda)e^{j\lambda n}d\lambda\right]e^{-j\Omega n}.$$

Interchanging the order of the integration and summation, we have

$$\begin{aligned}
Y(\Omega) &= \frac{1}{2\pi}\int_{2\pi} X_1(\lambda)\sum_{n=-\infty}^{\infty} x_2(n)e^{j\lambda n}d\lambda e^{-j\Omega n} \\
&= \frac{1}{2\pi}\int_{2\pi} X_1(\lambda)\left[\sum_{n=-\infty}^{\infty} x_2(n)e^{j\lambda n}e^{-j\Omega n}\right]d\lambda \\
&= \frac{1}{2\pi}\int_{2\pi} X_1(\lambda)\left[\sum_{n=-\infty}^{\infty} x_2(n)e^{-j(\Omega-\lambda)n}\right]d\lambda \\
&= \frac{1}{2\pi}\int_{2\pi} X_1(\lambda)X_2(\Omega-\lambda)d\lambda.
\end{aligned}$$

Thus, we have shown that the multiplication property holds. ∎

From the multiplication property in the preceding theorem, we can see that the Fourier transform effectively converts a multiplication operation into a convolution operation. Since convolution is significantly more complicated than multiplication, we normally prefer to avoid using this property in a manner that would result in the introduction of additional convolution operations into our work.

Example 11.16. Using the Fourier transform pair

$$\tfrac{B}{\pi}\operatorname{sinc}(Bn) \overset{\text{DTFT}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} \operatorname{rect}\left(\tfrac{1}{2B}[\Omega-2\pi k]\right) \quad \text{for } 0 < B < \pi,$$

find the Fourier transform X of the sequence

$$x(n) = \tfrac{B}{2\pi}\operatorname{sinc}^2\left(\tfrac{B}{2}n\right),$$

where B is a real constant satisfying $0 < B < \frac{\pi}{2}$.

Solution. We were given the Fourier transform pair

$$\tfrac{B}{\pi}\operatorname{sinc}(Bn) \overset{\text{DTFT}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} \operatorname{rect}\left(\tfrac{1}{2B}[\Omega-2\pi k]\right).$$

Substituting $B/2$ for B in this pair yields

$$\tfrac{B}{2\pi}\operatorname{sinc}\left(\tfrac{B}{2}n\right) \overset{\text{DTFT}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} \operatorname{rect}\left(\tfrac{1}{B}[\Omega-2\pi k]\right).$$

So, from the linearity property of the Fourier transform, we can infer

$$\operatorname{sinc}\left(\tfrac{B}{2}n\right) \overset{\text{DTFT}}{\longleftrightarrow} \tfrac{2\pi}{B}\sum_{k=-\infty}^{\infty} \operatorname{rect}\left(\tfrac{1}{B}[\Omega-2\pi k]\right).$$

Taking the Fourier transform of x, we have

$$\begin{aligned} X(\Omega) &= \mathcal{F}\left\{\tfrac{B}{2\pi}\operatorname{sinc}^2\left(\tfrac{B}{2}n\right)\right\}(\Omega) \\ &= \tfrac{B}{2\pi}\mathcal{F}\left\{\operatorname{sinc}^2\left(\tfrac{B}{2}n\right)\right\}(\Omega). \end{aligned}$$

Using the multiplication property of the Fourier transform and choosing the interval $[-\pi,\pi]$ for the resulting integration, we obtain

$$\begin{aligned} X(\Omega) &= \tfrac{B}{2\pi}\left(\tfrac{1}{2\pi}\left[\mathcal{F}\left\{\operatorname{sinc}\left(\tfrac{B}{2}n\right)\right\} \circledast \mathcal{F}\left\{\operatorname{sinc}\left(\tfrac{B}{2}n\right)\right\}\right](\Omega)\right) \\ &= \tfrac{B}{4\pi^2}\int_{-\pi}^{\pi}\left(\tfrac{2\pi}{B}\sum_{k=-\infty}^{\infty}\operatorname{rect}\left(\tfrac{1}{B}[\theta-2\pi k]\right)\right)\left(\tfrac{2\pi}{B}\sum_{\ell=-\infty}^{\infty}\operatorname{rect}\left(\tfrac{1}{B}[\Omega-\theta-2\pi\ell]\right)\right)d\theta \\ &= \tfrac{1}{B}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty}\operatorname{rect}\left(\tfrac{1}{B}[\theta-2\pi k]\right)\right)\left(\sum_{\ell=-\infty}^{\infty}\operatorname{rect}\left(\tfrac{1}{B}[\Omega-\theta-2\pi\ell]\right)\right)d\theta. \end{aligned}$$

Since, for $\theta \in [-\pi,\pi]$, the left summation is zero for $k \neq 0$, we can rewrite the preceding equation as

$$X(\Omega) = \tfrac{1}{B}\int_{-\pi}^{\pi}\operatorname{rect}\left(\tfrac{1}{B}\theta\right)\left(\sum_{\ell=-\infty}^{\infty}\operatorname{rect}\left(\tfrac{1}{B}[\Omega-\theta-2\pi\ell]\right)\right)d\theta.$$

Since, for $\theta \in [-\pi,\pi]$, $\operatorname{rect}\left(\tfrac{1}{B}\theta\right)$ and the right summation are never both simultaneously nonzero for $\ell \neq 0$, we can rewrite the preceding equation as

$$X(\Omega) = \tfrac{1}{B}\int_{-\pi}^{\pi}\operatorname{rect}\left(\tfrac{1}{B}\theta\right)\operatorname{rect}\left(\tfrac{1}{B}[\Omega-\theta]\right)d\theta.$$

Using the fact that the $\operatorname{rect}\left(\tfrac{1}{B}\theta\right)$ factor in the integrand is zero everywhere outside of the integration interval, we can rewrite the preceding equation as

$$\begin{aligned} X(\Omega) &= \tfrac{1}{B}\int_{-\infty}^{\infty}\operatorname{rect}\left(\tfrac{1}{B}\theta\right)\operatorname{rect}\left(\tfrac{1}{B}[\Omega-\theta]\right)d\theta \\ &= \tfrac{1}{B}\left\{\operatorname{rect}\left[\tfrac{1}{B}(\cdot)\right] * \operatorname{rect}\left[\tfrac{1}{B}(\cdot)\right]\right\}(\Omega). \end{aligned}$$

One can show that $\operatorname{rect}\left[\tfrac{1}{B}(\cdot)\right] * \operatorname{rect}\left[\tfrac{1}{B}(\cdot)\right] = B\operatorname{tri}\left[\tfrac{1}{2B}(\cdot)\right]$. (The proof of this is left as an exercise for the reader in Exercise 4.19(b).) Using this fact, we can write

$$\begin{aligned} X(\Omega) &= \tfrac{1}{B}\left[B\operatorname{tri}\left(\tfrac{1}{2B}\Omega\right)\right] \\ &= \operatorname{tri}\left(\tfrac{1}{2B}\Omega\right). \end{aligned}$$

Thus, for all Ω, we have that

$$X(\Omega) = \sum_{k=-\infty}^{\infty}\operatorname{tri}\left(\tfrac{1}{2B}[\Omega-2\pi k]\right).$$

Therefore, we have that

$$\tfrac{B}{2\pi}\operatorname{sinc}^2\left(\tfrac{B}{2}n\right) \overset{\text{DTFT}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty}\operatorname{tri}\left(\tfrac{1}{2B}[\Omega-2\pi k]\right) \quad \text{for } 0 < B < \tfrac{\pi}{2}.$$ ■

11.7.11 Frequency-Domain Differentiation

The next property of the Fourier transform to be introduced is the frequency-domain differentiation property, as given below.

Theorem 11.13 (Frequency-domain differentiation)**.** *If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then*

$$nx(n) \overset{\text{DTFT}}{\longleftrightarrow} j\tfrac{d}{d\Omega}X(\Omega).$$

*This is known as the **frequency-domain differentiation property** of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = nx(n)$, and let Y denote the Fourier transform of y. From the definition of the Fourier transform, we have

$$\begin{aligned}
Y(\Omega) &= \sum_{n=-\infty}^{\infty} nx(n)e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} x(n)\left(ne^{-j\Omega n}\right).
\end{aligned}$$

Using that fact that $ne^{-j\Omega n} = \frac{d}{d\Omega}je^{-j\Omega n}$, we can rewrite the preceding equation as

$$\begin{aligned}
Y(\Omega) &= \sum_{n=-\infty}^{\infty} x(n)\tfrac{d}{d\Omega}je^{-j\Omega n} \\
&= j\tfrac{d}{d\Omega}\sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} \\
&= j\tfrac{d}{d\Omega}X(\Omega).
\end{aligned}$$

Thus, we have shown that the frequency-domain differentiation property holds. (Alternatively, we could have simply differentiated both sides of (11.6) with respect to Ω to show the desired result.) ■

Example 11.17 (Frequency-domain differentiation property)**.** Using the Fourier transform pair

$$a^n u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega}-a} \quad \text{for } |a| < 1$$

and properties of the Fourier transform, find the Fourier transform X of the sequence

$$x(n) = na^n u(n).$$

Solution. Let $v_1(n) = a^n u(n)$ so that

$$x(n) = nv_1(n).$$

From the given Fourier transform pair, we have

$$V_1(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega}-a}.$$

For convenience, let the prime symbol denote derivative in what follows. Taking the Fourier transform of x with the help of the frequency-domain differentiation property, we obtain

$$\begin{aligned}
X(\Omega) &= jV_1'(\Omega) \\
&= j\tfrac{d}{d\Omega}\left[\frac{e^{j\Omega}}{e^{j\Omega}-a}\right] \\
&= j\tfrac{d}{d\Omega}\left[(e^{j\Omega})(e^{j\Omega}-a)^{-1}\right] \\
&= j\left[(je^{j\Omega})(e^{j\Omega}-a)^{-1}+[(-1)(e^{j\Omega}-a)^{-2}(je^{j\Omega})](e^{j\Omega})\right] \\
&= -\frac{e^{j\Omega}}{e^{j\Omega}-a}+\frac{e^{j2\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{-e^{j\Omega}(e^{j\Omega}-a)+e^{j2\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{-e^{j2\Omega}+ae^{j\Omega}+e^{j2\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2}.
\end{aligned}$$

Thus, we conclude that

$$na^nu(n) \stackrel{\text{DTFT}}{\longleftrightarrow} \frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2} \text{ for } |a|<1.$$ ■

11.7.12 Differencing

The next property of the Fourier transform to be introduced is the differencing property, as given below.

Theorem 11.14 (Differencing)**.** *If* $x(n) \stackrel{\text{DTFT}}{\longleftrightarrow} X(\Omega)$*, then*

$$x(n)-x(n-1) \stackrel{\text{DTFT}}{\longleftrightarrow} \left(1-e^{-j\Omega}\right)X(\Omega)$$

This is known as the ***differencing property*** *of the Fourier transform.*

Proof. The result of this theorem follows immediately from the linearity and translation properties of the Fourier transform. ■

Example 11.18 (Differencing property)**.** Given the Fourier transform pair

$$a^{|n|}, |a|<1 \stackrel{\text{DTFT}}{\longleftrightarrow} \frac{1-a^2}{1-2a\cos\Omega+a^2},$$

find the Fourier transform X of the sequence

$$x(n)=a^{|n|}-a^{|n-1|},$$

where a is a complex constant satisfying $|a|<1$.

Solution. From the differencing property, we have

$$X(\Omega)=\frac{(1-e^{-j\Omega})(1-a^2)}{1-2a\cos\Omega+a^2}.$$ ■

11.7.13 Accumulation

The next property of the Fourier transform to be introduced is the accumulation property, as given below.

Theorem 11.15 (Accumulation)**.** *If* $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, *then*

$$\sum_{k=-\infty}^{n} x(k) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega}-1}X(\Omega) + \pi X(0) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k).$$

This is known as the ***accumulation property*** *of the Fourier transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = \sum_{k=-\infty}^{n} x(k)$, and let Y denote the Fourier transform of y. To begin, we observe that

$$y(n) = x * u(n).$$

That is, we have

$$\begin{aligned} y(n) &= x * u(n) \\ &= \sum_{k=-\infty}^{\infty} x(k)u(n-k) \\ &= \sum_{k=-\infty}^{n} x(k)u(n-k) + \sum_{k=n+1}^{\infty} x(k)u(n-k) \\ &= \sum_{k=-\infty}^{n} x(k). \end{aligned}$$

Thus, from the convolution property of the Fourier transform, we have

$$Y(\Omega) = X(\Omega)U(\Omega).$$

From Table 11.2, we have the Fourier transform pair

$$u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega}-1} + \sum_{k=-\infty}^{\infty} \pi\delta(\Omega - 2\pi k).$$

Using this pair, we can simplify the above expression for Y as follows:

$$\begin{aligned} Y(\Omega) &= X(\Omega)\left[\frac{e^{j\Omega}}{e^{j\Omega}-1} + \sum_{k=-\infty}^{\infty} \pi\delta(\Omega - 2\pi k)\right] \\ &= \frac{e^{j\Omega}}{e^{j\Omega}-1}X(\Omega) + \sum_{k=-\infty}^{\infty} \pi X(\Omega)\delta(\Omega - 2\pi k) \\ &= \frac{e^{j\Omega}}{e^{j\Omega}-1}X(\Omega) + \sum_{k=-\infty}^{\infty} \pi X(2\pi k)\delta(\Omega - 2\pi k) \\ &= \frac{e^{j\Omega}}{e^{j\Omega}-1}X(\Omega) + \pi X(0)\sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k). \end{aligned}$$

Thus, we have shown that the accumulation property holds. ■

Example 11.19 (Accumulation property)**.** Find the Fourier transform X of the sequence

$$x(n) = u(n).$$

Solution. To begin, we observe that

$$x(n) = \sum_{k=-\infty}^{n} \delta(k).$$

So, using the accumulation property of the Fourier transform and the fact that $\mathcal{F}\delta(\Omega) = 1$ (from Table 11.2), we can write

$$\begin{aligned}
X(\Omega) &= \frac{e^{j\Omega}}{e^{j\Omega}-1}\mathcal{F}\delta(\Omega) + \pi\mathcal{F}\delta(0) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \\
&= \frac{e^{j\Omega}}{e^{j\Omega}-1}(1) + \pi(1) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \\
&= \frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k).
\end{aligned}$$

■

11.7.14 Parseval's Relation

The next property of the Fourier transform to be introduced, given below, relates to signal energy and is known as Parseval's relation.

Theorem 11.16 (Parseval's relation)**.** *If* $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$*, then*

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \tfrac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 \, d\Omega. \tag{11.8}$$

This is known as ***Parseval's relation***.

Proof. To prove the above property, we proceed as follows. We start from the expression on the left-hand side of (11.8). From properties of complex numbers, we have

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x^*(n)x(n).$$

Rewriting $x(n)$ in terms of the inverse Fourier transform of X, we have

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x^*(n) \tfrac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega.$$

Interchanging the order of integration and summation, we obtain

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |x(n)|^2 &= \tfrac{1}{2\pi} \int_{2\pi} X(\Omega) \sum_{n=-\infty}^{\infty} x^*(n) e^{j\Omega n} d\Omega \\
&= \tfrac{1}{2\pi} \int_{2\pi} X(\Omega) \left(\sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \right)^* d\Omega \\
&= \tfrac{1}{2\pi} \int_{2\pi} X(\Omega) X^*(\Omega) d\Omega \\
&= \tfrac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 \, d\Omega.
\end{aligned}$$

Thus, we have shown that Parseval's relation holds. ■

Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform preserves energy (up to a scale factor). For example, if we are solving a problem in the Fourier domain, we do not have to return to the time domain to compute energy, since we can do this directly in the Fourier domain by using Parseval's relation.

Example 11.20. Find the energy E of the sequence

$$x(n) = \operatorname{sinc}\left(\tfrac{1}{2}n\right).$$

Solution. We could try to find E directly from its definition in terms of x using

$$\begin{aligned} E &= \sum_{n=-\infty}^{\infty} |x(n)|^2 \\ &= \sum_{n=-\infty}^{\infty} \left|\operatorname{sinc}\left(\tfrac{1}{2}n\right)\right|^2 \\ &= \sum_{n=-\infty}^{\infty} \left|\frac{2\sin\left(\tfrac{1}{2}n\right)}{n}\right|^2. \end{aligned}$$

This sum, however, would be rather tedious to compute. So, we instead use Parseval's relation to compute E in terms of X. From Table 11.2, we know that

$$2\pi\left[\frac{1}{2\pi}\operatorname{sinc}\left(\frac{n}{2}\right)\right] \overset{\text{DTFT}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} \operatorname{rect}(\Omega - 2\pi k).$$

Thus, we have

$$X(\Omega) = 2\pi \operatorname{rect}\Omega \quad \text{for } \Omega \in [-\pi, \pi).$$

Using this fact, we can find E as follows:

$$\begin{aligned} E &= \tfrac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2 \, d\Omega \\ &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} |2\pi \operatorname{rect}\Omega|^2 \, d\Omega \\ &= \tfrac{1}{2\pi}\int_{-1/2}^{1/2} (2\pi)^2 d\Omega \\ &= \left(\tfrac{1}{2\pi}\right)(4\pi^2) \\ &= 2\pi. \end{aligned}$$

■

11.7.15 Even/Odd Symmetry

The Fourier transform preserves symmetry. In other words, we have the result below.

Theorem 11.17 (Even/odd symmetry)**.** *For a sequence x with Fourier transform X, the following assertions hold:*

- *x is even if and only if X is even; and*
- *x is odd if and only if X is odd.*

Proof. First, we show that, if a sequence x is even/odd, then its Fourier transform X is even/odd. From the definition of the Fourier transform, we have

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}.$$

Since x is even/odd, we have that $x(n) = \pm x(-n)$, where the plus case and minus case in the "$\pm$" correspond to x being even and odd, respectively. Using this, we can rewrite the above expression for $X(\Omega)$ as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \pm x(-n)e^{-j\Omega n}.$$

Now, we employ a change of variable. Let $n' = -n$ so that $n = -n'$. Applying this change of variable and dropping the primes, we obtain

$$\begin{aligned}
X(\Omega) &= \sum_{n=-\infty}^{\infty} \pm x(n)e^{-j\Omega(-n)} \\
&= \pm \sum_{n=-\infty}^{\infty} x(n)e^{j\Omega n} \\
&= \pm \sum_{n=-\infty}^{\infty} x(n)e^{-j(-\Omega)n} \\
&= \pm X(-\Omega).
\end{aligned}$$

Therefore, X is even/odd.

Next, we show that if X is even/odd, then x is even/odd. From the definition of the inverse Fourier transform, we have

$$x(n) = \tfrac{1}{2\pi} \int_{2\pi} X(\Omega)e^{j\Omega n} d\Omega.$$

Since X is even/odd, we have that $X(\Omega) = \pm X(-\Omega)$, where the plus case and minus case in the "$\pm$" correspond to X being even and odd, respectively. Using this, we can rewrite the above expression for $x(n)$ as

$$\begin{aligned}
x(n) &= \tfrac{1}{2\pi} \int_{2\pi} \pm X(-\Omega)e^{j\Omega n} d\Omega \\
&= \tfrac{1}{2\pi} \int_{a}^{a+2\pi} \pm X(-\Omega)e^{j\Omega n} d\Omega.
\end{aligned}$$

Now, we employ a change of variable. Let $\lambda = -\Omega$ so that $\Omega = -\lambda$ and $d\Omega = -d\lambda$. Applying this change of variable, we obtain

$$\begin{aligned}
x(n) &= \tfrac{1}{2\pi} \int_{-a}^{-a-2\pi} \pm X(\lambda)e^{-j\lambda n}(-1)d\lambda \\
&= \pm \tfrac{1}{2\pi} \int_{-a-2\pi}^{-a} X(\lambda)e^{-j\lambda n} d\lambda \\
&= \pm \tfrac{1}{2\pi} \int_{2\pi} X(\lambda)e^{j\lambda(-n)} d\lambda \\
&= \pm x(-n).
\end{aligned}$$

Therefore, x is even/odd. This completes the proof. ∎

In other words, the preceding theorem simply states that the forward and inverse Fourier transforms preserve even/odd symmetry.

11.7.16 Real Sequences

As it turns out, the Fourier transform of a real-valued sequence has a special structure, as given by the theorem below.

Theorem 11.18 (Real-valued sequences)**.** *A sequence x is real-valued if and only if its Fourier transform X satisfies*

$$X(\Omega) = X^*(-\Omega) \text{ for all } \Omega$$

(i.e., X is conjugate symmetric).

Proof. From the definition of the Fourier transform, we have

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}. \tag{11.9}$$

Substituting $-\Omega$ for Ω in the preceding equation, we have

$$X(-\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{j\Omega n}.$$

Conjugating both sides of this equation, we obtain

$$X^*(-\Omega) = \sum_{n=-\infty}^{\infty} x^*(n)e^{-j\Omega n}. \tag{11.10}$$

First, we show that x being real-valued implies that X is conjugate symmetric. Suppose that x is real-valued. Since x is real-valued, we can replace x^* with x in (11.10) to yield

$$X^*(-\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}.$$

Observing that the right-hand side is simply $X(\Omega)$, we have

$$X^*(-\Omega) = X(\Omega).$$

Thus, x being real-valued implies that X is conjugate symmetric.

Next, we show that X being conjugate symmetric implies that x is real-valued. Suppose that X is conjugate symmetric. Since X is conjugate symmetric, the expressions for $X(\Omega)$ in (11.9) and $X^*(-\Omega)$ in (11.10) must be equal. Thus, we can write

$$\begin{aligned}
& X(\Omega) - X^*(-\Omega) = 0 \\
\Rightarrow \quad & \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} - \sum_{n=-\infty}^{\infty} x^*(n)e^{-j\Omega n} = 0 \\
\Rightarrow \quad & \sum_{n=-\infty}^{\infty} [x(n) - x^*(n)]e^{-j\Omega n} = 0.
\end{aligned}$$

This implies that $x^* = x$. Therefore, x is real-valued. Thus, X being conjugate symmetric implies that x is real-valued. This completes the proof. ∎

Suppose that X is the Fourier transform of a real-valued sequence x so that X is conjugate symmetric. From properties of complex numbers, we can show that that X being conjugate symmetric is equivalent to

$$|X(\Omega)| = |X(-\Omega)| \quad \text{for all } \Omega \in \mathbb{R} \quad \text{and} \tag{11.11a}$$

$$\arg X(\Omega) = -\arg X(-\Omega) \quad \text{for all } \Omega \in \mathbb{R} \tag{11.11b}$$

(i.e., the magnitude and argument of X are even and odd, respectively).

Since the Fourier transform X of a real-valued sequence x is conjugate symmetric, the graph of X for negative values is completely redundant and can be determined from the graph of X for nonnegative values. Lastly, note that x being real-valued does not necessarily imply that X is real-valued, since a conjugate-symmetric function need not be real-valued.

Table 11.1: Properties of the DT Fourier transform

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(\Omega)+a_2X_2(\Omega)$
Translation	$x(n-n_0)$	$e^{-j\Omega n_0}X(\Omega)$
Modulation	$e^{j\Omega_0 n}x(n)$	$X(\Omega-\Omega_0)$
Conjugation	$x^*(n)$	$X^*(-\Omega)$
Time Reversal	$x(-n)$	$X(-\Omega)$
Upsampling	$(\uparrow M)x(n)$	$X(M\Omega)$
Downsampling	$x(Mn)$	$\frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\Omega-2\pi k}{M}\right)$
Convolution	$x_1 * x_2(n)$	$X_1(\Omega)X_2(\Omega)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{2\pi} X_1(\theta)X_2(\Omega-\theta)d\theta$
Frequency-Domain Differentiation	$nx(n)$	$j\frac{d}{d\Omega}X(\Omega)$
Differencing	$x(n)-x(n-1)$	$\left(1-e^{-j\Omega}\right)X(\Omega)$
Accumulation	$\sum_{k=-\infty}^{n} x(k)$	$\frac{e^{j\Omega}}{e^{j\Omega}-1}X(\Omega)+\pi X(0)\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$

Property	
Periodicity	$X(\Omega)=X(\Omega+2\pi)$
Parseval's Relation	$\sum_{n=-\infty}^{\infty} \lvert x(n)\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(\Omega)\rvert^2 d\Omega$
Even Symmetry	x is even $\Leftrightarrow$ X is even
Odd Symmetry	x is odd $\Leftrightarrow$ X is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ X is conjugate symmetric

11.8 Discrete-Time Fourier Transform of Periodic Sequences

By making use of the generalized Fourier transform briefly discussed in Section 11.3, the Fourier transform can also be applied to periodic sequences. In particular, the Fourier transform of a periodic sequence can be computed using the result below.

Theorem 11.19 (Fourier transform of a periodic sequence)**.** *Let x be an N-periodic sequence with the corresponding Fourier-series coefficient sequence a. Let x_N denote the sequence*

$$x_N(n) = \begin{cases} x(n) & n \in [0\,..\,N-1] \\ 0 & \text{otherwise} \end{cases}$$

(i.e., x_N is a truncated/windowed version of the sequence x). (Note that x_N is a sequence equal to x over a single period and zero elsewhere.) Let X_N denote the Fourier transform of x_N. The Fourier transform X of x is given by

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \tfrac{2\pi}{N}k\right), \tag{11.12a}$$

or equivalently,

$$X(\Omega) = \tfrac{2\pi}{N} \sum_{k=-\infty}^{\infty} X_N\left(\tfrac{2\pi}{N}k\right) \delta\left(\Omega - \tfrac{2\pi}{N}k\right). \tag{11.12b}$$

Furthermore, a and X_N are related by

$$a_k = \tfrac{1}{N} X_N\left(\tfrac{2\pi}{N}k\right). \tag{11.13}$$

Proof. Since x is N-periodic, we can express it using a Fourier series as

$$x(n) = \sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)kn} \tag{11.14a}$$

where

$$a_k = \tfrac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}. \tag{11.14b}$$

Consider the expression for a_k in (11.14b). Since $x_N(n) = x(n)$ for a single period of x and is zero otherwise, we can rewrite (11.14b) as

$$\begin{aligned} a_k &= \tfrac{1}{N} \sum_{n=-\infty}^{\infty} x_N(n) e^{-j[(2\pi/N)k]n} \\ &= \tfrac{1}{N} X_N\left(\tfrac{2\pi}{N}k\right). \end{aligned} \tag{11.15}$$

Thus, we have shown (11.13) to be correct.

Now, let us consider the Fourier transform X of x. By taking the Fourier transform of both sides of (11.14a), we obtain

$$\begin{aligned} X(\Omega) &= \left(\mathcal{F}\left\{\sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)kn}\right\}\right)(\Omega) \\ &= \sum_{k=\langle N \rangle} a_k \mathcal{F}\left\{e^{j(2\pi/N)kn}\right\}(\Omega) \end{aligned}$$

Now, consider the Fourier transform on the right-hand side of the preceding equation. From Table 11.2 and the modulation property of the Fourier transform, we can deduce

$$\begin{aligned}\mathcal{F}\{e^{j(2\pi/N)kn}\}(\Omega) &= \mathcal{F}\{e^{j(2\pi/N)kn}\cdot 1\}(\Omega)\\ &= 2\pi\sum_{\ell=-\infty}^{\infty}\delta\left(\left[\Omega-\tfrac{2\pi}{N}k\right]-2\pi\ell\right)\\ &= 2\pi\sum_{\ell=-\infty}^{\infty}\delta\left(\Omega-\tfrac{2\pi}{N}k-2\pi\ell\right).\end{aligned}$$

Substituting this Fourier transform into the above expression for $X(\Omega)$, we obtain

$$\begin{aligned}X(\Omega) &= \sum_{k=\langle N\rangle} a_k\mathcal{F}\left\{e^{j(2\pi/N)kn}\right\}(\Omega)\\ &= \sum_{k=\langle N\rangle} a_k\left[2\pi\sum_{\ell=-\infty}^{\infty}\delta\left(\Omega-\tfrac{2\pi}{N}k-2\pi\ell\right)\right].\end{aligned}$$

Reversing the order of the two summations and then choosing the take the summation over k using the range $[0\,..\,N-1]$, we have

$$\begin{aligned}X(\Omega) &= 2\pi\sum_{\ell=-\infty}^{\infty}\sum_{k=0}^{N-1} a_k\delta\left(\Omega-\tfrac{2\pi}{N}k-2\pi\ell\right)\\ &= 2\pi\sum_{\ell=-\infty}^{\infty}\sum_{k=0}^{N-1} a_k\delta\left(\Omega-\tfrac{2\pi}{N}[N\ell+k]\right).\end{aligned}$$

Now, we observe that the combined effect of the two summations is that $N\ell+k$ takes on every integer value exactly once. Consequently, we can combine the two summations into a single summation to yield

$$X(\Omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k\delta\left(\Omega-\tfrac{2\pi}{N}k\right).$$

Thus, (11.12a) holds. Substituting (11.15) into the preceding equation for $X(\Omega)$, we have

$$\begin{aligned}X(\Omega) &= 2\pi\sum_{k=-\infty}^{\infty} a_k\delta\left(\Omega-\tfrac{2\pi}{N}k\right)\\ &= 2\pi\sum_{k=-\infty}^{\infty}\tfrac{1}{N}X_N\left(\tfrac{2\pi}{N}k\right)\delta\left(\Omega-\tfrac{2\pi}{N}k\right)\\ &= \tfrac{2\pi}{N}\sum_{k=-\infty}^{\infty}X_N\left(\tfrac{2\pi}{N}k\right)\delta\left(\Omega-\tfrac{2\pi}{N}k\right).\end{aligned}$$

Thus, (11.12b) holds. This completes the proof. ■

Theorem 11.19 above provides two formulas for computing the Fourier transform X of a periodic sequence x. One formula is in written terms of the Fourier series coefficient sequence a of x, while the other formula is in written in terms of the Fourier transform X_N of a sequence consisting of a single period of x. The choice of which formula to use would be driven by what information is available or most easily determined. For example, if the Fourier series coefficients of x were known, the use of (11.12b) would likely be preferred.

From Theorem 11.19, we can also make a few important observations. First, the Fourier transform of a periodic sequence is a series of impulse functions located at integer multiples of the fundamental frequency $\frac{2\pi}{N}$. The weight of each impulse is 2π times the corresponding Fourier series coefficient. Second, the Fourier series coefficient sequence a of the periodic sequence x is produced by sampling the Fourier transform of x_N at integer multiples of the fundamental frequency $\frac{2\pi}{N}$ and scaling the resulting sequence by $\frac{1}{N}$.

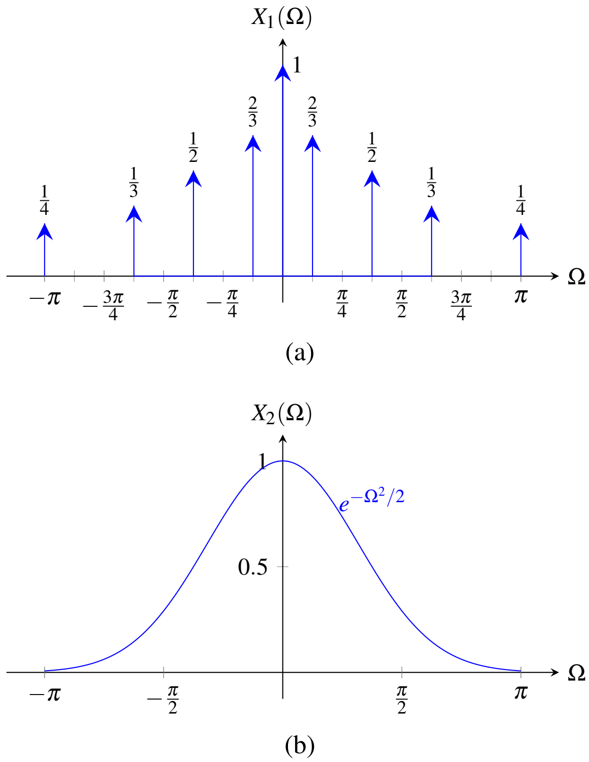


Figure 11.3: Frequency spectra. The frequency spectra (a) X_1 and (b) X_2.

Example 11.21. Let X_1 and X_2 denote the Fourier transforms of x_1 and x_2, respectively. Suppose that X_1 and X_2 are as shown in Figures 11.3(a) and (b). Determine whether x_1 and x_2 are periodic.

Solution. We know that the Fourier transform X of an N-periodic sequence x must be of the form

$$X(\Omega) = \sum_{k=-\infty}^{\infty} \alpha_k \delta\left(\Omega - \tfrac{2\pi}{N}k\right),$$

where the $\{\alpha_k\}$ are complex constants. The spectrum X_1 does have this form, with $N = 16$. That is, impulses reside at integer multiples of the frequency $\frac{2\pi}{16} = \frac{\pi}{8}$. Therefore, x_1 must be 16-periodic. The spectrum X_2 does not have the required form. Therefore, x_2 must not be periodic. ■

Example 11.22. Consider the N-periodic sequence x shown in Figure 11.4, where $N = 16$. Using the Fourier transform, find the Fourier series representation of x.

Solution. Define the sequence y as the truncated/windowed version of x given by

$$\begin{aligned} y(n) &= \begin{cases} x(n) & n \in [0\,..\,15] \\ 0 & \text{otherwise} \end{cases} \\ &= u(n) - u(n-8). \end{aligned}$$

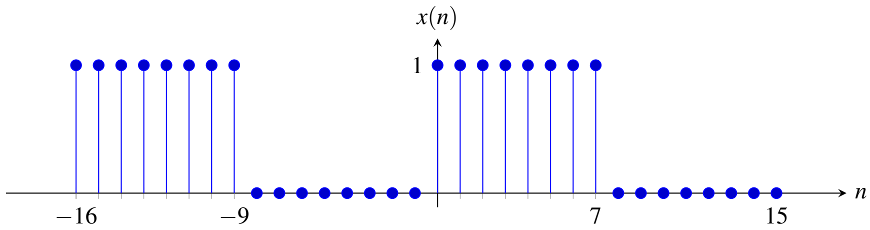


Figure 11.4: The 16-periodic sequence x.

Thus, we have that

$$\begin{aligned} x(n) &= \sum_{k=-\infty}^{\infty} y(n-Nk) \\ &= \sum_{k=-\infty}^{\infty} y(n-16k). \end{aligned}$$

From Table 11.2, the Fourier transform Y of y is given by

$$\begin{aligned} Y(\Omega) &= (\mathcal{F}\{u(n)-u(n-8)\})(\Omega) \\ &= e^{-j\Omega(8-1)/2}\left[\frac{\sin\left(\frac{8}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}\right] \\ &= e^{-j(7/2)\Omega}\left[\frac{\sin(4\Omega)}{\sin\left(\frac{1}{2}\Omega\right)}\right]. \end{aligned}$$

Now, we seek to find the Fourier series representation of x, which has the form

$$\begin{aligned} x(n) &= \sum_{k=\langle N\rangle} c_k e^{j(2\pi/N)kn} \\ &= \sum_{k=\langle 16\rangle} c_k e^{j(\pi/8)kn}. \end{aligned}$$

Using the Fourier transform, we have

$$\begin{aligned} c_k &= \tfrac{1}{N} Y\left(\tfrac{2\pi}{N}k\right) \\ &= \tfrac{1}{16} Y\left(\tfrac{\pi}{8}k\right) \\ &= \tfrac{1}{16} e^{-j(7/2)(\pi/8)k}\left(\frac{\sin\left[4\left(\frac{\pi}{8}k\right)\right]}{\sin\left[\frac{1}{2}\left(\frac{\pi}{8}k\right)\right]}\right) \\ &= \tfrac{1}{16} e^{-j(7\pi/16)k}\left[\frac{\sin\left(\frac{\pi}{2}k\right)}{\sin\left(\frac{\pi}{16}k\right)}\right]. \end{aligned}$$ ■

11.9 More Fourier Transforms

Throughout this chapter, we have derived a number of Fourier transform pairs. Some of these and other important transform pairs are listed in Table 11.2. Using the various Fourier transform properties listed in Table 11.1 and the Fourier transform pairs listed in Table 11.2, we can determine (more easily) the Fourier transforms of more complicated sequences.

Table 11.2: Transform pairs for the DT Fourier transform

Pair	$x(n)$	$X(\Omega)$
1	$\delta(n)$	1
2	1	$2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$
3	$u(n)$	$\frac{e^{j\Omega}}{e^{j\Omega}-1} + \sum_{k=-\infty}^{\infty} \pi\delta(\Omega - 2\pi k)$
4	$a^n u(n), \|a\| < 1$	$\frac{e^{j\Omega}}{e^{j\Omega}-a}$
5	$-a^n u(-n-1), \|a\| > 1$	$\frac{e^{j\Omega}}{e^{j\Omega}-a}$
6	$a^{\|n\|}, \|a\| < 1$	$\frac{1-a^2}{1-2a\cos\Omega + a^2}$
7	$\cos(\Omega_0 n)$	$\pi \sum_{k=-\infty}^{\infty} [\delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k)]$
8	$\sin(\Omega_0 n)$	$j\pi \sum_{k=-\infty}^{\infty} [\delta(\Omega + \Omega_0 - 2\pi k) - \delta(\Omega - \Omega_0 - 2\pi k)]$
9	$\cos(\Omega_0 n)u(n)$	$\frac{e^{j2\Omega} - e^{j\Omega}\cos\Omega_0}{e^{j2\Omega} - 2e^{j\Omega}\cos\Omega_0 + 1} + \frac{\pi}{2} \sum_{k=-\infty}^{\infty} [\delta(\Omega - 2\pi k - \Omega_0) + \delta(\Omega - 2\pi k + \Omega_0)]$
10	$\sin(\Omega_0 n)u(n)$	$\frac{e^{j\Omega}\sin\Omega_0}{e^{j2\Omega} - 2e^{j\Omega}\cos\Omega_0 + 1} + \frac{\pi}{2j} \sum_{k=-\infty}^{\infty} [\delta(\Omega - 2\pi k - \Omega_0) - \delta(\Omega - 2\pi k + \Omega_0)]$
11	$\frac{B}{\pi}\operatorname{sinc}(Bn), 0 < B < \pi$	$\sum_{k=-\infty}^{\infty} \operatorname{rect}\left(\frac{\Omega - 2\pi k}{2B}\right)$
12	$u(n) - u(n-M)$	$e^{-j\Omega(M-1)/2} \left(\frac{\sin(M\Omega/2)}{\sin(\Omega/2)}\right)$
13	$na^n u(n), \|a\| < 1$	$\frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2}$

Example 11.23. Using a table of Fourier transform pairs and properties of the Fourier transform, find the Fourier transform X of the sequence

$$x(n) = n\left(\tfrac{1}{2}\right)^{|n+3|}.$$

Solution. We reexpress x as

$$x(n) = nv_2(n),$$

where

$$v_2(n) = v_1(n+3) \quad \text{and}$$
$$v_1(n) = \left(\tfrac{1}{2}\right)^{|n|}.$$

Let V_1 and V_2 denote the Fourier transforms of v_1 and v_2, respectively. Taking the Fourier transform of v_1 using Table 11.2, we have

$$\begin{aligned}
V_1(\Omega) &= \frac{1-\left(\frac{1}{2}\right)^2}{1-2\left(\frac{1}{2}\right)\cos\Omega+\left(\frac{1}{2}\right)^2} \\
&= \frac{\frac{3}{4}}{1-\cos\Omega+\frac{1}{4}} \\
&= \frac{3}{4-4\cos\Omega+1} \\
&= \frac{3}{5-4\cos\Omega} \\
&= 3(5-4\cos\Omega)^{-1}.
\end{aligned}$$

Taking the Fourier transform of v_2, we have

$$V_2(\Omega) = e^{j3\Omega}V_1(\Omega).$$

Taking the Fourier transform of x, we have

$$X(\Omega) = jV_2'(\Omega).$$

Taking the derivative of V_1 yields

$$\begin{aligned}
V_1'(\Omega) &= \tfrac{d}{d\Omega}\left[3(5-4\cos\Omega)^{-1}\right] \\
&= -3(5-4\cos\Omega)^{-2}(4\sin\Omega) \\
&= \frac{-12\sin\Omega}{(5-4\cos\Omega)^2}.
\end{aligned}$$

Taking the derivative of V_2 yields

$$V_2'(\Omega) = 3je^{j3\Omega}V_1(\Omega) + e^{j3\Omega}V_1'(\Omega).$$

Substituting the preceding formula for V_2' into the above expression for $X(\Omega)$, we have

$$\begin{aligned}
X(\Omega) &= jV_2'(\Omega) \\
&= j\left[3je^{j3\Omega}V_1(\Omega) + e^{j3\Omega}V_1'(\Omega)\right] \\
&= -3e^{j3\Omega}V_1(\Omega) + je^{j3\Omega}V_1'(\Omega) \\
&= -3e^{j3\Omega}\frac{3}{5-4\cos\Omega} + je^{j3\Omega}\frac{-12\sin\Omega}{(5-4\cos\Omega)^2} \\
&= e^{j3\Omega}\left[\frac{-9}{5-4\cos\Omega} + \frac{-12j\sin\Omega}{(5-4\cos\Omega)^2}\right] \\
&= e^{j3\Omega}\left[\frac{-9(5-4\cos\Omega) - 12j\sin\Omega}{(5-4\cos\Omega)^2}\right] \\
&= e^{j3\Omega}\left[\frac{36\cos\Omega - 12j\sin\Omega - 45}{(5-4\cos\Omega)^2}\right] \\
&= 3e^{j3\Omega}\left[\frac{12\cos\Omega - 4j\sin\Omega - 15}{(5-4\cos\Omega)^2}\right].
\end{aligned}$$

■

Example 11.24. Using a table of Fourier transform pairs and properties of the Fourier transform, find the Fourier transform X of the sequence

$$x(n) = (n+1)a^n u(n),$$

where a is complex constant satisfying $|a| < 1$.

Solution. Let $v(n) = a^n u(n)$. We rewrite x as

$$\begin{aligned}
x(n) &= (n+1)a^n u(n) \\
&= na^n u(n) + a^n u(n) \\
&= nv(n) + v(n).
\end{aligned}$$

Let V denote the Fourier transform of v. Taking the Fourier transform of v, we have

$$\begin{aligned}
V(\Omega) &= \frac{e^{j\Omega}}{e^{j\Omega} - a} \\
&= e^{j\Omega}\left(e^{j\Omega} - a\right)^{-1}.
\end{aligned}$$

Taking the Fourier transform of x, we have

$$X(\Omega) = jV'(\Omega) + V(\Omega).$$

Taking the derivative of V, we have

$$\begin{aligned}
V'(\Omega) &= je^{j\Omega}\left(e^{j\Omega}-a\right)^{-1} + (-1)\left(e^{j\Omega}-a\right)^{-2}\left(je^{j\Omega}\right)e^{j\Omega} \\
&= \frac{je^{j\Omega}}{e^{j\Omega}-a} - \frac{je^{j2\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{je^{j\Omega}(e^{j\Omega}-a) - je^{j2\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{je^{j2\Omega} - aje^{j\Omega} - je^{j2\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{-jae^{j\Omega}}{(e^{j\Omega}-a)^2}.
\end{aligned}$$

Substituting the above formulas for V' and V into the expression for $X(\Omega)$ from above, we obtain

$$\begin{aligned}
X(\Omega) &= j\left[\frac{-jae^{j\Omega}}{(e^{j\Omega}-a)^2}\right] + \frac{e^{j\Omega}}{e^{j\Omega}-a} \\
&= \frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2} + \frac{e^{j\Omega}}{e^{j\Omega}-a} \\
&= \frac{ae^{j\Omega} + e^{j\Omega}\left(e^{j\Omega}-a\right)}{(e^{j\Omega}-a)^2} \\
&= \frac{ae^{j\Omega} + e^{j2\Omega} - ae^{j\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{e^{j2\Omega}}{(e^{j\Omega}-a)^2} \\
&= \frac{e^{j2\Omega}}{[e^{j\Omega}(1-ae^{-j\Omega})]^2} \\
&= \frac{e^{j2\Omega}}{e^{j2\Omega}(1-ae^{-j\Omega})^2} \\
&= \frac{1}{(1-ae^{-j\Omega})^2}.
\end{aligned}$$

■

Example 11.25. Consider the N-periodic sequence x shown in Figure 11.5, where $N=7$. Find the Fourier transform X of x.

Solution. In what follows, we will use the prime symbol to denote the derivative. To begin, we observe that x is given by

$$x(n) = \sum_{k=-\infty}^{\infty} y(n-7k),$$

where

$$y(n) = n[u(n+3)-u(n-4)].$$

Now, we re-express y as

$$y(n) = nv_2(n),$$

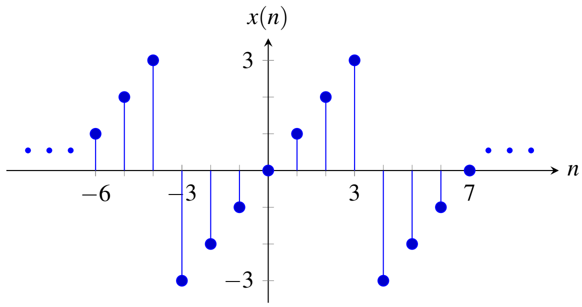


Figure 11.5: The 7-periodic sequence x.

where

$$v_2(n) = v_1(n+3) \quad \text{and}$$
$$v_1(n) = u(n) - u(n-7).$$

Let Y, V_1, and V_2 denote the Fourier transforms of y, v_1, and v_2, respectively. Taking the Fourier transform of the equation for v_1 with the assistance of Table 11.2, we have

$$\begin{aligned} V_1(\Omega) &= e^{-j\Omega(7-1)/2}\left(\frac{\sin\left(\frac{7}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}\right) \\ &= e^{-j3\Omega}\left(\frac{\sin\left(\frac{7}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}\right). \end{aligned}$$

Taking the Fourier transform of the equation for v_2 using the translation property of the Fourier transform, we have

$$\begin{aligned} V_2(\Omega) &= e^{j3\Omega}V_1(\Omega) \\ &= \frac{\sin\left(\frac{7}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}. \end{aligned}$$

Now, we compute the derivative of V_2. We have

$$\begin{aligned} V_2'(\Omega) &= \frac{\left[\sin\left(\frac{1}{2}\Omega\right)\right]\left[\frac{7}{2}\cos\left(\frac{7}{2}\Omega\right)\right] - \left[\sin\left(\frac{7}{2}\Omega\right)\right]\left[\frac{1}{2}\cos\left(\frac{1}{2}\Omega\right)\right]}{\sin^2\left(\frac{1}{2}\Omega\right)} \\ &= \frac{\frac{7}{2}\cos\left(\frac{7}{2}\Omega\right)\sin\left(\frac{1}{2}\Omega\right) - \frac{1}{2}\cos\left(\frac{1}{2}\Omega\right)\sin\left(\frac{7}{2}\Omega\right)}{\sin^2\left(\frac{1}{2}\Omega\right)}. \end{aligned}$$

Taking the Fourier transform of the equation for y using the frequency-domain differentiation property of the Fourier transform, we have

$$\begin{aligned} Y(\Omega) &= jV_2'(\Omega) \\ &= \frac{\frac{7j}{2}\cos\left(\frac{7}{2}\Omega\right)\sin\left(\frac{1}{2}\Omega\right) - \frac{j}{2}\cos\left(\frac{1}{2}\Omega\right)\sin\left(\frac{7}{2}\Omega\right)}{\sin^2\left(\frac{1}{2}\Omega\right)}. \end{aligned}$$

From Theorem 11.19, we know that

$$X(\Omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k\delta\left(\Omega - \tfrac{2\pi}{7}k\right),$$

where

$$a_k = \tfrac{1}{7} Y\left(\tfrac{2\pi}{7} k\right).$$

Evaluating a_k, we have

$$\begin{aligned}
a_k &= \tfrac{1}{7} Y\left(\tfrac{2\pi}{7} k\right) \\
&= \tfrac{1}{7}\left[\frac{\frac{7j}{2}\cos\left[\frac{7}{2}\left(\frac{2\pi}{7}\right)k\right]\sin\left[\frac{1}{2}\left(\frac{2\pi}{7}\right)k\right] - \frac{j}{2}\cos\left[\frac{1}{2}\left(\frac{2\pi}{7}\right)k\right]\sin\left[\frac{7}{2}\left(\frac{2\pi}{7}\right)k\right]}{\sin^2\left[\frac{1}{2}\left(\frac{2\pi}{7}\right)k\right]}\right] \\
&= \tfrac{1}{7}\left[\frac{\frac{7j}{2}\cos(\pi k)\sin\left(\frac{\pi}{7}k\right) - \frac{j}{2}\cos\left(\frac{\pi}{7}k\right)\sin(\pi k)}{\sin^2\left(\frac{\pi}{7}k\right)}\right] \\
&= \frac{\frac{j}{2}\cos(\pi k)\sin\left(\frac{\pi}{7}k\right) - \frac{j}{14}\cos\left(\frac{\pi}{7}k\right)\sin(\pi k)}{\sin^2\left(\frac{\pi}{7}k\right)}.
\end{aligned}$$

If $\frac{k}{7} \notin \mathbb{Z}$, we have

$$\begin{aligned}
a_k &= \frac{\frac{j}{2}\cos(\pi k)\sin\left(\frac{\pi}{7}k\right)}{\sin^2\left(\frac{\pi}{7}k\right)} \\
&= \frac{j(-1)^k \sin\left(\frac{\pi}{7}k\right)}{2\sin^2\left(\frac{\pi}{7}k\right)} \\
&= \frac{j(-1)^k}{2\sin\left(\frac{\pi}{7}k\right)}.
\end{aligned}$$

(In the above simplification, we used the fact $\sin(\pi k) = 0$ and $\cos(\pi k) = (-1)^k$ for all $k \in \mathbb{Z}$.) Otherwise (i.e., if $\frac{k}{7} \in \mathbb{Z}$), we have

$$\begin{aligned}
a_k &= a_0 \\
&= \tfrac{1}{7} Y(0) \\
&= \tfrac{1}{7} \sum_{n=-\infty}^{\infty} y(n) \\
&= 0.
\end{aligned}$$

Thus, we conclude

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \tfrac{2\pi}{7} k\right),$$

where

$$a_k = \begin{cases} \dfrac{j(-1)^k}{2\sin\left(\frac{\pi}{7}k\right)} & \frac{k}{7} \notin \mathbb{Z} \\ 0 & \frac{k}{7} \in \mathbb{Z}. \end{cases}$$

■

Example 11.26. Let x and y denote two sequences related by

$$y(n) = \sum_{k=-\infty}^{n} e^{-j3k} x(-k).$$

Find the Fourier transform Y of y in terms of the Fourier transform X of x.

Solution. We begin with the given equation for y, namely,

$$y(n) = \sum_{k=-\infty}^{n} e^{-j3k} x(-k).$$

Letting $v_1(k) = x(-k)$, we can rewrite the preceding equation for y as

$$y(n) = \sum_{k=-\infty}^{n} e^{-j3k} v_1(k).$$

Letting $v_2(k) = e^{-j3k} v_1(k)$, we can rewrite the preceding equation as

$$y(n) = \sum_{k=-\infty}^{n} v_2(k).$$

Taking the Fourier transform of the equation for each of v_1, v_2, and y, we have

$$\begin{aligned} v_1(n) = x(-n) \quad &\Leftrightarrow \quad V_1(\Omega) = X(-\Omega), \\ v_2(n) = e^{-j3n} v_1(n) \quad &\Leftrightarrow \quad V_2(\Omega) = V_1(\Omega + 3), \quad \text{and} \\ y(n) = \sum_{k=-\infty}^{n} v_2(k) \quad &\Leftrightarrow \quad Y(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - 1} V_2(\Omega) + \pi V_2(0) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k). \end{aligned}$$

Substituting the formulas for V_2 and V_1 into the formula for $Y(\Omega)$ on the preceding line, we obtain

$$\begin{aligned} Y(\Omega) &= \frac{e^{j\Omega}}{e^{j\Omega} - 1} V_2(\Omega) + \pi V_2(0) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \\ &= \frac{e^{j\Omega}}{e^{j\Omega} - 1} V_1(\Omega + 3) + \pi V_1(3) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \\ &= \frac{e^{j\Omega}}{e^{j\Omega} - 1} X(-\Omega - 3) + \pi X(-3) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k). \end{aligned}$$

■

Example 11.27. Let x and y be two sequences related by

$$y(n) = x(n) \cos(\Omega_0 n),$$

where Ω_0 is a nonzero real constant. Let X and Y denote the Fourier transforms of x and y, respectively. Find an expression for Y in terms of X.

Solution. Essentially, we need to take the Fourier transform of both sides of the given equation. There are two different ways in which to do this. One is to use the multiplication property of the Fourier transform, and another is to use the modulation property. We will solve this problem using each method in turn in order to show that the two approaches do not involve an equal amount of effort. The moral of this example is that, when multiple solution techniques are possible for a problem, one should always select the simpler technique, as this saves time and reduces the likelihood of errors.

FIRST SOLUTION (USING AN UNENLIGHTENED APPROACH). We use an approach based on the multiplication property of the Fourier transform. Taking the Fourier transform of both sides of the given equation, we trivially have

$$Y(\Omega) = \mathcal{F}\{x(n) \cos(\Omega_0 n)\}(\Omega).$$

From the multiplication property of the Fourier transform, we have

$$Y(\Omega) = \tfrac{1}{2\pi} \left(X * \mathcal{F}\{\cos(\Omega_0 n)\} \right)(\Omega).$$

Using Table 11.2 to determine $\mathcal{F}\{\cos(\Omega_0 n)\}$, we have

$$\begin{aligned}
Y(\Omega) &= \tfrac{1}{2\pi}\left\{X * \left(\pi \sum_{k=-\infty}^{\infty} [\delta(\cdot - \Omega_0 - 2\pi k) + \delta(\cdot + \Omega_0 - 2\pi k)]\right)\right\}(\Omega) \\
&= \tfrac{1}{2\pi}\int_{2\pi} X(\theta)\left[\pi \sum_{k=-\infty}^{\infty} [\delta(\Omega - \theta - \Omega_0 - 2\pi k) + \delta(\Omega - \theta + \Omega_0 - 2\pi k)]\right] d\theta \\
&= \tfrac{1}{2\pi}\int_{-\pi}^{\pi^-} X(\theta)\left[\pi \sum_{k=-\infty}^{\infty} [\delta(\Omega - \theta - \Omega_0 - 2\pi k) + \delta(\Omega - \theta + \Omega_0 - 2\pi k)]\right] d\theta.
\end{aligned}$$

Since the delta function is even, we can rewrite the preceding equation as

$$\begin{aligned}
Y(\Omega) &= \tfrac{1}{2}\int_{-\pi}^{\pi^-} X(\theta) \sum_{k=-\infty}^{\infty} [\delta(-\Omega + \theta + \Omega_0 + 2\pi k) + \delta(-\Omega + \theta - \Omega_0 + 2\pi k)]\, d\theta \\
&= \tfrac{1}{2}\int_{-\pi}^{\pi^-} X(\theta) \sum_{k=-\infty}^{\infty} [\delta(\theta - [\Omega - \Omega_0 - 2\pi k]) + \delta(\theta - [\Omega + \Omega_0 - 2\pi k])]\, d\theta \\
&= \tfrac{1}{2}\left[\int_{-\pi}^{\pi^-} X(\theta) \sum_{k=-\infty}^{\infty} \delta[\theta - (\Omega - \Omega_0 - 2\pi k)] d\theta + \int_{-\pi}^{\pi^-} X(\theta) \sum_{\ell=-\infty}^{\infty} \delta[\theta - (\Omega + \Omega_0 - 2\pi \ell)] d\theta\right].
\end{aligned}$$

Consider the two integrals on the right-hand side of the preceding equation. The summation in the leftmost integral only contains one term that is nonzero in the integration interval $[-\pi, \pi)$. Let this term correspond to $k = k'$. Similarly, the summation in the rightmost integral only contains one term that is nonzero in the integration interval $[-\pi, \pi)$. Let this term correspond to $\ell = \ell'$. Now, we can rewrite the above equation for Y as

$$Y(\Omega) = \tfrac{1}{2}\left[\int_{-\pi}^{\pi^-} X(\theta)\delta[\theta - (\Omega - \Omega_0 - 2\pi k')] d\theta + \int_{-\pi}^{\pi^-} X(\theta)\delta[\theta - (\Omega + \Omega_0 - 2\pi \ell')] d\theta\right].$$

Using the sifting property of the delta function, we have

$$Y(\Omega) = \tfrac{1}{2}\left[X(\Omega - \Omega_0 - 2\pi k') + X(\Omega + \Omega_0 - 2\pi \ell')\right].$$

Since X is 2π-periodic, we can rewrite the preceding equation as

$$\begin{aligned}
Y(\Omega) &= \tfrac{1}{2}[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)] \\
&= \tfrac{1}{2}X(\Omega - \Omega_0) + \tfrac{1}{2}X(\Omega + \Omega_0).
\end{aligned}$$

Although we have managed to solve the problem at hand, the above solution is quite tedious. Fortunately, there is a much better way to approach the problem, which we consider next.

SECOND SOLUTION (USING AN ENLIGHTENED APPROACH). We use an approach based on the modulation property of the Fourier transform. Taking the Fourier transform of both sides of the given equation, we trivially have

$$Y(\Omega) = \mathcal{F}\{x(n)\cos(\Omega_0 n)\}(\Omega).$$

Expressing $\cos(\Omega_0 n)$ in terms of complex sinusoids, we have

$$\begin{aligned}
Y(\Omega) &= \mathcal{F}\left\{\tfrac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right)x(n)\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{1}{2}e^{j\Omega_0 n}x(n) + \tfrac{1}{2}e^{-j\Omega_0 n}x(n)\right\}(\Omega).
\end{aligned}$$

Using the linearity property of the Fourier transform, we have

$$Y(\Omega) = \tfrac{1}{2}\mathcal{F}\left\{e^{j\Omega_0 n}x(n)\right\}(\Omega) + \tfrac{1}{2}\mathcal{F}\left\{e^{-j\Omega_0 n}x(n)\right\}(\Omega).$$

Using the modulation property of the Fourier transform, we obtain

$$Y(\Omega) = \tfrac{1}{2}X(\Omega - \Omega_0) + \tfrac{1}{2}X(\Omega + \Omega_0).$$

COMMENTARY. Clearly, of the above two solution methods, the second approach is simpler and much less error prone. Generally, the multiplication property of the Fourier transform is usually best avoided whenever possible, since its use introduces convolution into the solution. ■

11.10 Frequency Spectra of Sequences

The Fourier transform representation expresses a sequence in terms of complex sinusoids at all frequencies. In this sense, the Fourier transform representation captures information about the frequency content of a sequence. For example, suppose that we have a sequence x with Fourier transform X. If X is nonzero at some frequency Ω_0, then the sequence x contains some information at the frequency Ω_0. On the other hand, if X is zero at the frequency Ω_0, then the sequence x has no information at that frequency. In this way, the Fourier transform representation provides a means for measuring the frequency content of a sequence. This distribution of information in a sequence over different frequencies is referred to as the **frequency spectrum** of the sequence. That is, X is the frequency spectrum of x.

To gain further insight into the role played by the Fourier transform X in the context of the frequency spectrum of x, it is helpful to write the Fourier transform representation of x with X expressed in polar form as follows:

$$\begin{aligned} x(n) &= \tfrac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega \\ &= \tfrac{1}{2\pi}\int_{2\pi} |X(\Omega)|\, e^{j\arg X(\Omega)}e^{j\Omega n}d\Omega \\ &= \tfrac{1}{2\pi}\int_{2\pi} |X(\Omega)|\, e^{j[\Omega n+\arg X(\Omega)]}d\Omega. \end{aligned}$$

In effect, the quantity $|X(\Omega)|$ is a weight that determines how much the complex sinusoid at frequency Ω contributes to the integration result $x(n)$. Perhaps, this can be more easily seen if we express the above integral as the limit of a sum, derived from an approximation of the integral using the area of rectangles (i.e., $\int_{-\infty}^{\infty} f(x)dx = \lim_{\Delta x\to 0}\sum_{k=-\infty}^{\infty}\Delta x f(k\Delta x)$). Expressing x in this way, we obtain

$$\begin{aligned} x(n) &= \lim_{\Delta\Omega\to 0} \tfrac{1}{2\pi}\sum_{k=-\infty}^{\infty} \Delta\Omega\, |X(k\Delta\Omega)|\, e^{j[k\Delta\Omega n+\arg X(k\Delta\Omega)]} \\ &= \lim_{\Delta\Omega\to 0} \tfrac{1}{2\pi}\sum_{k=-\infty}^{\infty} \Delta\Omega\, \left|X(\Omega')\right|\, e^{j[\Omega' n+\arg X(\Omega')]}, \end{aligned}$$

where $\Omega' = k\Delta\Omega$. From the last line of the above equation, the kth term in the summation (associated with the frequency $\Omega' = k\Delta\Omega$) corresponds to a complex sinusoid with frequency Ω' that has had its amplitude scaled by a factor of $|X(\Omega')|$ and has been time-shifted by an amount that depends on $\arg X(\Omega')$. For a given $\Omega' = k\Delta\Omega$ (which is associated with the kth term in the summation), the larger $|X(\Omega')|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\Omega' n}$ will be, and therefore the larger the contribution the kth term will make to the overall summation. In this way, we can use $|X(\Omega')|$ as a measure of how much information a sequence x has at the frequency Ω'.

To formalize the notion of frequency spectrum, the **frequency spectrum** of a sequence x is given by its Fourier transform X. Due to the above interpretation of the Fourier transform in terms of the polar form, we are often interested in $|X(\cdot)|$ and $\arg X(\cdot)$. As a matter of terminology, we refer to $|X(\cdot)|$ as the **magnitude spectrum** of x and $\arg X(\cdot)$ as the **phase spectrum** of x.

Since the graphical presentation of information is often helpful for visualization purposes, we often want to plot frequency spectra of sequences. Since three-dimensional plots are usually more difficult to generate (especially by hand) than two-dimensional ones and can often be more difficult to interpret accurately, we usually present frequency spectra in graphical form using only two-dimensional plots. In the case that the frequency spectrum is either purely real or purely imaginary, we typically plot the frequency spectrum directly on a single pair of axes. Most often, however, the frequency spectrum will be complex (but neither purely real nor purely imaginary), in which case we plot the frequency spectrum in polar form by using two plots, one showing the magnitude spectrum and one showing the phase spectrum.

Note that, since the Fourier transform X is a function of a real variable, a sequence x can, in the most general case, have information at *any arbitrary* real frequency. This is different from the case of frequency spectra in the Fourier series context (which deals only with periodic sequences), where a sequence can only have information at certain specific frequencies (namely, at integer multiples of the fundamental frequency). There is no inconsistency here, however. As we saw in Section 11.8, in the case of periodic sequences the Fourier transform will also be zero, except possibly at integer multiples of the fundamental frequency.

Recall (from Theorem 11.18) that, for a real-valued sequence x, the Fourier transform X is conjugate symmetric (i.e., $X(\Omega) = X^*(-\Omega)$ for all $\Omega \in \mathbb{R}$). This, however, implies that

$$\begin{aligned} |X(\Omega)| &= |X(-\Omega)| \quad \text{for all } \Omega \in \mathbb{R} \quad \text{and} \\ \arg X(\Omega) &= -\arg X(-\Omega) \quad \text{for all } \Omega \in \mathbb{R} \end{aligned}$$

(i.e., the magnitude and argument of X are even and odd, respectively). (See (11.11a) and (11.11b).) Due to the symmetry in the frequency spectra of real-valued sequences, we typically ignore negative frequencies when dealing with such sequences. In the case of sequences that are complex-valued but not real-valued, frequency spectra do not possess the above symmetry, and negative frequencies become important.

Example 11.28. Find and plot the frequency spectrum of the sequence

$$x(n) = u(n) - u(n-16).$$

Solution. From Table 11.2, the Fourier transform X of x is given by

$$X(\Omega) = e^{-j(15/2)\Omega}\left[\frac{\sin(8\Omega)}{\sin\left(\frac{1}{2}\Omega\right)}\right] \quad \text{for } \Omega \in (-\pi, \pi].$$

In this case, X is neither purely real nor purely imaginary, so we will plot the frequency spectrum X in polar form using two graphs, one for the magnitude spectrum and one for the phase spectrum. Taking the magnitude of X, we have

$$|X(\Omega)| = \left|\frac{\sin(8\Omega)}{\sin\left(\frac{1}{2}\Omega\right)}\right|.$$

Taking the argument of X, we have

$$\arg[X(\Omega)] = -\tfrac{15}{2}\Omega + \arg\left[\frac{\sin(8\Omega)}{\sin\left(\frac{1}{2}\Omega\right)}\right].$$

The magnitude spectrum and phase spectrum are shown plotted in Figures 11.6(a) and (b), respectively. ■

Example 11.29. Find and plot the frequency spectrum of the sequence

$$x(n) = \left(\tfrac{1}{2}\right)^{|n|}.$$

Solution. From Table 11.2, the Fourier transform X of x is given by

$$X(\Omega) = \frac{3}{5 - 4\cos\Omega} \quad \text{for } \Omega \in (-\pi, \pi].$$

Since, in this case, X is real, we can plot the frequency spectrum X on a single graph, as shown in Figure 11.7. ■

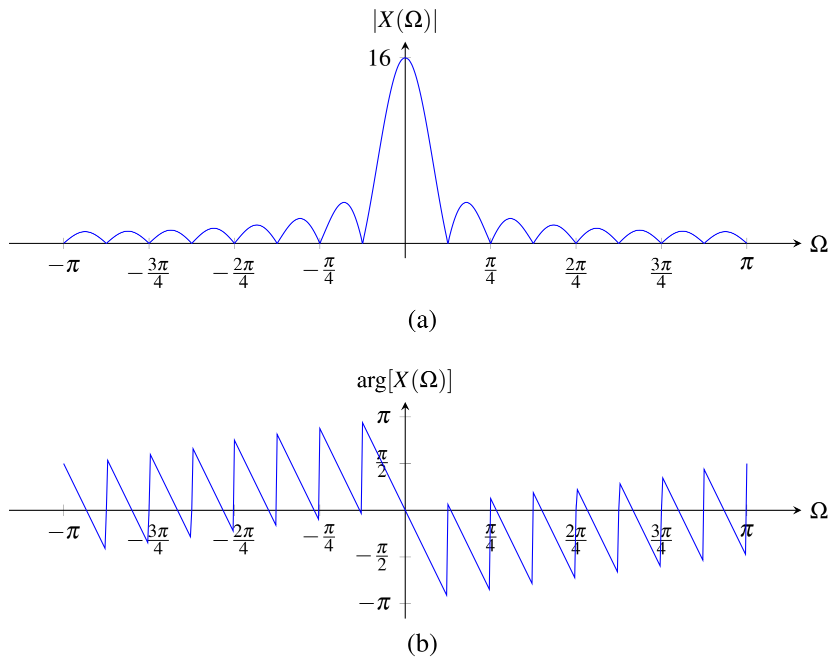


Figure 11.6: Frequency spectrum X of the sequence x. (a) Magnitude spectrum and (b) phase spectrum.

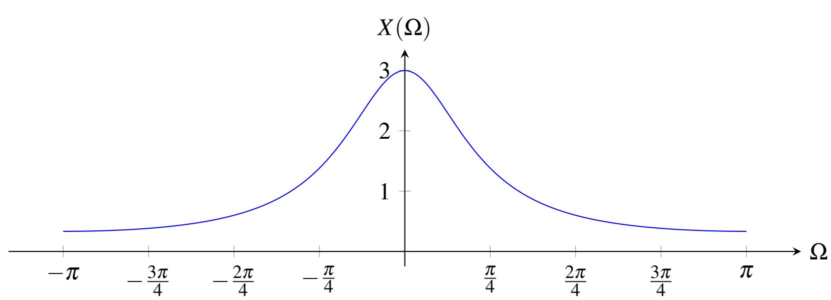


Figure 11.7: Frequency spectrum X of the sequence x.

Example 11.30. The sequence

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n)$$

has the Fourier transform

$$X(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - \frac{1}{2}}.$$

(a) Find and plot the magnitude and phase spectra of x. (b) Determine at what frequency (or frequencies) in the range $(-\pi, \pi]$ the sequence x has the most information.

Solution. (a) First, we find the magnitude spectrum $|X(\Omega)|$. From the expression for $X(\Omega)$, we can write

$$\begin{aligned}
|X(\Omega)| &= \left| \frac{e^{j\Omega}}{e^{j\Omega} - \frac{1}{2}} \right| \\
&= \frac{1}{\left|e^{j\Omega} - \frac{1}{2}\right|} \\
&= \frac{1}{\left|\cos(\Omega) + j\sin(\Omega) - \frac{1}{2}\right|} \\
&= \frac{1}{\sqrt{\left(\cos(\Omega) - \frac{1}{2}\right)^2 + j(\sin\Omega)^2}} \\
&= \frac{1}{\sqrt{\cos^2(\Omega) - \cos\Omega + \frac{1}{4} + \sin^2\Omega}} \\
&= \frac{1}{\sqrt{\frac{5}{4} - \cos\Omega}}.
\end{aligned}$$

Next, we find the phase spectrum $\arg[X(\Omega)]$. From the expression for $X(\Omega)$, we can write

$$\begin{aligned}
\arg[X(\Omega)] &= \arg\left[\frac{e^{j\Omega}}{e^{j\Omega} - \frac{1}{2}}\right] \\
&= \arg\left[\frac{1}{1 - \frac{1}{2}e^{-j\Omega}}\right] \\
&= \arg 1 - \arg\left(1 - \tfrac{1}{2}e^{-j\Omega}\right) \\
&= -\arg\left(1 - \tfrac{1}{2}e^{-j\Omega}\right) \\
&= -\arg\left[1 - \tfrac{1}{2}\left[\cos(-\Omega) + j\sin(-\Omega)\right]\right] \\
&= -\arg\left(\left[1 - \tfrac{1}{2}\cos\Omega\right] + j\left[\tfrac{1}{2}\sin\Omega\right]\right) \\
&= -\arg\left([2 - \cos(\Omega)] + j[\sin(\Omega)]\right) \\
&= -\arctan\left[\frac{\sin(\Omega)}{2 - \cos(\Omega)}\right].
\end{aligned}$$

Finally, using numerical calculation, we can plot the graphs of $|X(\Omega)|$ and $\arg X(\Omega)$ to obtain the results shown in Figures 11.8(a) and (b).

(b) For $\Omega \in (-\pi, \pi]$, $|X(\Omega)|$ is greatest at $\Omega = 0$. Therefore, x has the most information at the frequency 0. ■

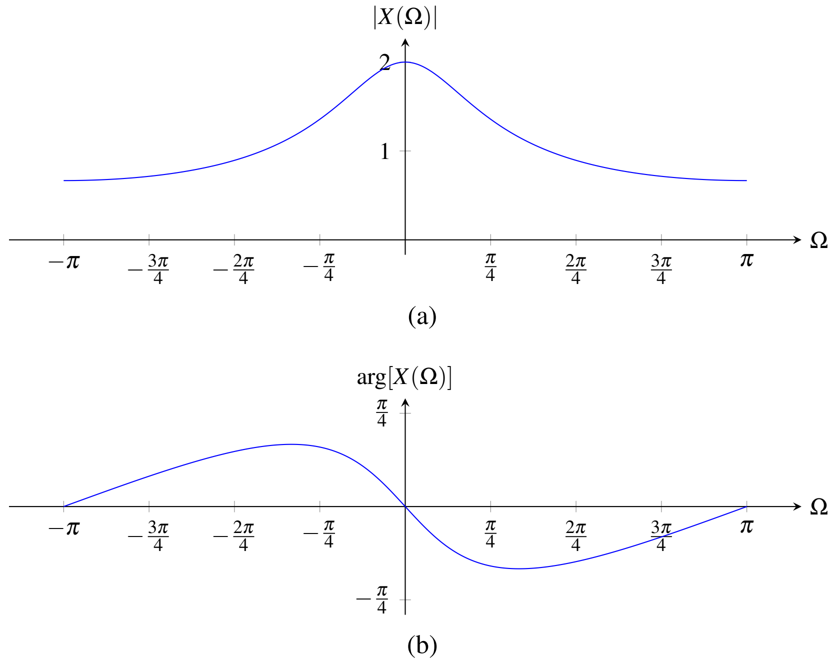


Figure 11.8: Frequency spectrum X of the sequence x. (a) Magnitude spectrum and (b) phase spectrum.

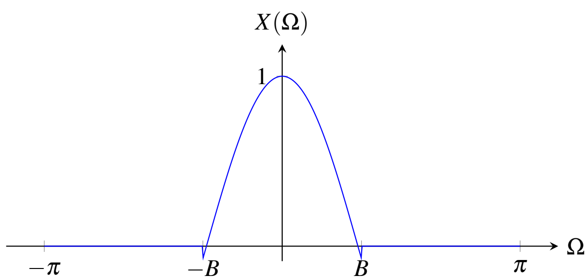


Figure 11.9: Example of the Fourier transform X of a sequence x that is bandlimited to frequencies in $[-B, B]$.

11.11 Bandwidth of Sequences

The range of frequencies over which the spectrum of a sequence is nonzero is often of interest. This is analogous to the notion of bandwidth that was introduced in the context of continuous-time signals (i.e., functions) in Section 6.11. Since, the spectrum of a discrete-time signal (i.e., sequence) is always 2π-periodic, when we speak of the bandwidth of a sequence, we only consider the spectrum over a single interval of length 2π. In particular, we normally consider an interval (of length 2π) centered at the origin (e.g., $(-\pi, \pi]$). When a sequence x has a Fourier transform X satisfying $X(\Omega) = 0$ for all Ω in $(-\pi, \pi]$ except for some interval I, we say that x is **bandlimited** to frequencies in I. Moreover, we define the **bandwidth** of a sequence x with Fourier transform X as the length of the interval in $(-\pi, \pi]$ over which X is nonzero. For example, the sequence x whose Fourier transform X is shown in Figure 11.9 is bandlimited to frequencies in $[-B, B]$ and has bandwidth $B - (-B) = 2B$. Sometimes, when dealing with real sequences, negative frequencies are ignored. Since x is real in this example (as X is conjugate symmetric), we might choose to ignore negative frequencies, in which case x would be deemed to be bandlimited to frequencies in $[0, B]$ and have bandwidth $B - 0 = B$.

11.12 Energy-Density Spectra

Suppose that we have a sequence x with finite energy E and Fourier transform X. By definition, the energy contained in x is given by

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 .$$

We can use Parseval's relation (6.12) to express E in terms of X as

$$E = \tfrac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 \, d\Omega.$$

Thus, the energy E is given by

$$E = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} E_x(\Omega) d\Omega,$$

where

$$E_x(\Omega) = |X(\Omega)|^2 .$$

We refer to E_x as the **energy-density spectrum** of the sequence x. The function E_x indicates how the energy in x is distributed with respect to frequency. For example, the energy contributed by frequencies in the range $[\Omega_1, \Omega_2]$ is given by

$$\tfrac{1}{2\pi} \int_{\Omega_1}^{\Omega_2} E_x(\Omega) d\Omega.$$

Example 11.31. Consider the sequence

$$x(n) = \operatorname{sinc}\left(\tfrac{\pi}{4} n\right).$$

Compute the energy-density spectrum E_x of x. Determine the amount of energy contained in x in the frequency range $\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$. Also, determine the total amount of energy in x.

Solution. First, we compute the Fourier transform X of x. We obtain

$$X(\Omega) = 4 \operatorname{rect}\left(\tfrac{2}{\pi}\Omega\right).$$

Computing the energy-density spectrum E_x, we have

$$\begin{aligned} E_x(\Omega) &= |X(\Omega)|^2 \\ &= \left|4 \operatorname{rect}\left(\tfrac{2}{\pi}\Omega\right)\right|^2 \\ &= 16 \operatorname{rect}^2\left(\tfrac{2}{\pi}\Omega\right) \\ &= 16 \operatorname{rect}\left(\tfrac{2}{\pi}\Omega\right). \end{aligned}$$

Let E_1 denote the energy contained in x for frequencies $\Omega \in \left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$. Then, we have

$$\begin{aligned}
E_1 &= \tfrac{1}{2\pi}\int_{-\pi/8}^{\pi/8} E_x(\Omega)d\Omega \\
&= \tfrac{1}{2\pi}\int_{-\pi/8}^{\pi/8} 16\,\mathrm{rect}\left(\tfrac{2}{\pi}\Omega\right)d\Omega \\
&= \tfrac{8}{\pi}\int_{-\pi/8}^{\pi/8} \mathrm{rect}\left(\tfrac{2}{\pi}\Omega\right)d\Omega \\
&= \tfrac{8}{\pi}\int_{-\pi/8}^{\pi/8} 1d\Omega \\
&= \tfrac{8}{\pi}\left(\tfrac{\pi}{8}+\tfrac{\pi}{8}\right) \\
&= \tfrac{8}{\pi}\left(\tfrac{\pi}{4}\right) \\
&= 2.
\end{aligned}$$

Let E denote the total amount of energy in x. Then, we have

$$\begin{aligned}
E &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} E_x(\Omega)d\Omega \\
&= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} 16\,\mathrm{rect}\left(\tfrac{2}{\pi}\Omega\right)d\Omega \\
&= \tfrac{8}{\pi}\int_{-\pi}^{\pi} \mathrm{rect}\left(\tfrac{2}{\pi}\Omega\right)d\Omega \\
&= \tfrac{8}{\pi}\int_{-\pi/4}^{\pi/4} 1d\Omega \\
&= \tfrac{8}{\pi}\left(\tfrac{\pi}{2}\right) \\
&= 4.
\end{aligned}$$

■

11.13 Characterizing LTI Systems Using the Fourier Transform

Consider a LTI system with input x, output y, and impulse response h. Such a system is depicted in Figure 11.10. The behavior of such a system is governed by the equation

$$y(n) = x * h(n). \tag{11.16}$$

Let X, Y, and H denote the Fourier transforms of x, y, and h, respectively. Taking the Fourier transform of both sides of (11.16) and using the convolution property of the Fourier transform, we obtain

$$Y(\Omega) = X(\Omega)H(\Omega). \tag{11.17}$$

This result provides an alternative way of viewing the behavior of a LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output sequences. In other words, we have a system resembling that in Figure 11.11. In this case, however, the convolution operation from the time domain is replaced by multiplication in the frequency domain. The frequency spectrum (i.e., Fourier transform) of the output is the product of the frequency spectrum (i.e., Fourier transform) of the input and the frequency spectrum (i.e., Fourier transform) of the impulse response. As a matter of terminology, we refer to H as the **frequency response** of the system. The system behavior is completely characterized by the frequency response H. If we know the input, we can compute its Fourier transform X, and then determine the Fourier transform Y of the output. Using the inverse Fourier transform, we can then determine the output y.

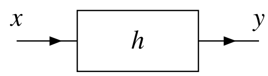


Figure 11.10: Time-domain view of a LTI system with input x, output y, and impulse response h.

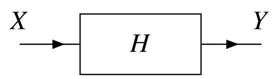


Figure 11.11: Frequency-domain view of a LTI system with input spectrum X, output spectrum Y, and frequency response H.

In the most general case, the frequency response H is a complex-valued function. Thus, we can represent H in terms of its magnitude and argument. We refer to the magnitude of H as the **magnitude response** of the system. Similarly, we refer to the argument of H as the **phase response** of the system.

From (11.17), we can write

$$\begin{aligned} |Y(\Omega)| &= |X(\Omega)H(\Omega)| \\ &= |X(\Omega)|\,|H(\Omega)| \quad \text{and} \end{aligned} \tag{11.18a}$$

$$\begin{aligned} \arg Y(\Omega) &= \arg[X(\Omega)H(\Omega)] \\ &= \arg X(\Omega) + \arg H(\Omega). \end{aligned} \tag{11.18b}$$

From (11.18a), we can see that the magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude response of the system (i.e., the magnitude spectrum of the impulse response). From (11.18b), we can see that the phase spectrum of the output equals the phase spectrum of the input plus the phase response of the system (i.e., the phase spectrum of the impulse response).

Since the frequency response H is simply the frequency spectrum of the impulse response h, for the reasons explained in Section 11.10, if h is real, then

$$\begin{aligned} |H(\Omega)| = |H(-\Omega)| \text{ for all } \Omega \quad \text{and} \\ \arg H(\Omega) = -\arg H(-\Omega) \text{ for all } \Omega \end{aligned}$$

(i.e., the magnitude and phase responses are even and odd, respectively).

Example 11.32. A LTI system has the impulse response

$$h(n) = u(n+5) - u(n-6).$$

Find the frequency response H of the system.

Solution. The frequency response is simply the Fourier transform of h. Using Table 11.2 (or the result of Example 11.2), we can easily determine H to be

$$H(\Omega) = \frac{\sin\left(\frac{11}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}.$$

Observe that H is real. So, we can plot the frequency response H on a single graph, as shown in Figure 11.12. ■

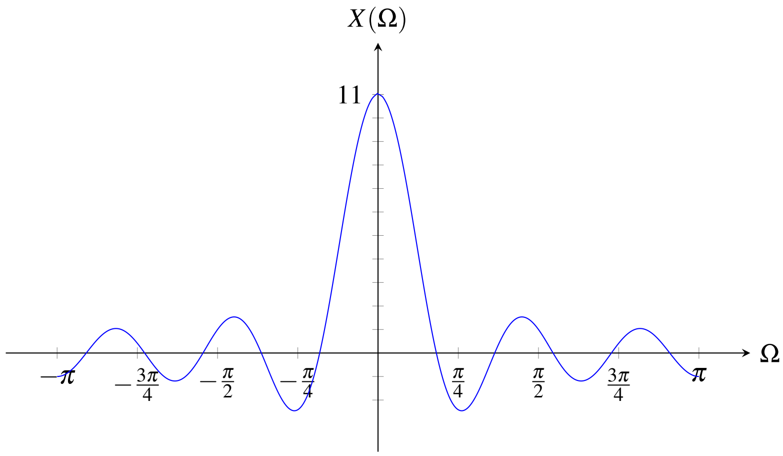


Figure 11.12: Frequency response of example system.

11.13.1 Unwrapped Phase

Since the argument of a complex number is not uniquely determined, the argument of a complex-valued function is also not uniquely determined. Consequently, we have some freedom in how we define a function that corresponds to the phase (i.e., argument) of a complex-valued function. Often, for convenience, we restrict the argument to lie in an interval of length 2π, such as the interval $(-\pi, \pi]$ which corresponds to the principal argument. Defining the phase of a complex-valued function in this way, however, can often result in a phase function with unnecessary discontinuities. This motivates the notion of unwrapped phase. The **unwrapped phase** is simply the phase defined in such a way so as not to restrict the phase to an interval of length 2π and to keep the phase function continuous to the greatest extent possible. An example illustrating the notion of unwrapped phase is given below.

Example 11.33 (Unwrapped phase)**.** Consider the phase response of a LTI system with the frequency response

$$H(\Omega) = e^{j3\Omega}.$$

We can choose to define the phase (i.e., argument) of H by simply using the principal argument (i.e., $\operatorname{Arg} H(\Omega)$). This yields the phase function shown in Figure 11.13(a). Using the principal argument in this way, however, unnecessarily introduces discontinuities into the phase function. For this reason, we sometimes prefer to define the phase function in such a way as to eliminate such unnecessary discontinuities. This motivates the use of the unwrapped phase. The function H has the unwrapped phase Θ given by

$$\Theta(\Omega) = 3\Omega.$$

A plot of Θ is shown in Figure 11.13(b). Unlike the function in Figure 11.13(a) (which has numerous discontinuities), the function in Figure 11.13(b) is continuous. Although the functions in these two figures are distinct, these functions are equivalent in the sense that they correspond to the same physical angular displacement (i.e., $e^{j \operatorname{Arg} H(\Omega)} = e^{j\Theta(\Omega)}$ for all $\Omega \in \mathbb{R}$). ■

11.13.2 Magnitude and Phase Distortion

Recall, from Corollary 10.1, that a LTI system $\mathcal{H}$ with frequency response H is such that

$$\mathcal{H}\left\{e^{j\Omega n}\right\}(n) = H(\Omega)e^{j\Omega n}$$

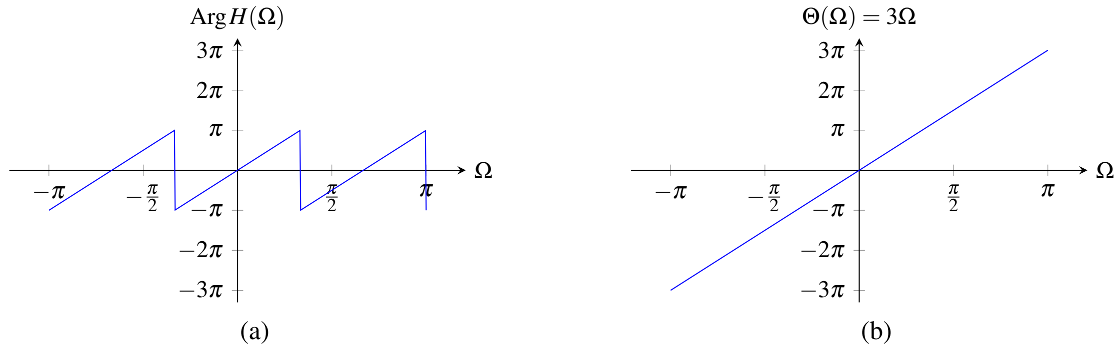


Figure 11.13: Unwrapped phase example. (a) The phase function restricted such that its range is in $(-\pi,\pi]$ and (b) the corresponding unwrapped phase.

(i.e., $e^{j\Omega n}$ is an eigensequence of $\mathcal{H}$ with eigenvalue $H(\Omega)$). Expressing $H(\Omega)$ in polar form, we have

$$\begin{aligned}\mathcal{H}\{e^{j\Omega n}\}(n) &= |H(\Omega)|\, e^{j\arg H(\Omega)} e^{j\Omega n} \\ &= |H(\Omega)|\, e^{j[\Omega n+\arg H(\Omega)]} \\ &= |H(\Omega)|\, e^{j\Omega(n+\arg[H(\Omega)]/\Omega)}.\end{aligned}$$

This equation can be rewritten as

$$\mathcal{H}\{e^{j\Omega n}\}(n) = |H(\Omega)|\, e^{j\Omega[n-\tau_{\mathsf{p}}(\Omega)]}, \tag{11.19a}$$

where

$$\tau_{\mathsf{p}}(\Omega) = -\frac{\arg H(\Omega)}{\Omega}. \tag{11.19b}$$

Thus, the response of the system to the sequence $e^{j\Omega n}$ is produced by applying two transformations to this sequence:

- (amplitude) scaling by $|H(\Omega)|$; and
- if $\tau_{\mathsf{p}}(\Omega) \in \mathbb{Z}$, translating by $\tau_{\mathsf{p}}(\Omega)$; otherwise (i.e., $\tau_{\mathsf{p}}(\Omega) \notin \mathbb{Z}$), the corresponding continuous-time complex sinusoid is translated by $\tau_{\mathsf{p}}(\Omega)$ and then sampled (i.e., bandlimited interpolation is performed).

Therefore, the magnitude response determines how different complex sinusoids are (amplitude) scaled by the system. Similarly, the phase response determines how different complex sinusoids are translated (i.e., delayed/advanced) by the system (possibly using interpolation).

A system for which $|H(\Omega)| = 1$ for all Ω is said to be **allpass**[1]. In the case of an allpass system, the magnitude spectra of the system's input and output are identical. If a system is not allpass, it modifies the magnitude spectrum in some way. In situations where the magnitude spectrum is changed in an undesirable manner, **magnitude distortion** (i.e., distortion of the magnitude spectrum) is said to occur. If $|H(\Omega)| = a$ for all Ω, where a is a constant, every complex sinusoid is scaled by the same amount a when passing through the system. In practice, this type of change to the magnitude spectrum may sometimes be undesirable if $a \neq 1$. If $|H(\Omega)|$ is not a constant, different complex sinusoids are scaled by different amounts. In practice, this type of change to the magnitude spectrum is usually undesirable and deemed to constitute magnitude distortion.

The function τ_{p} appearing in (11.19b) is known as the **phase delay** of the system. A system for which $\tau_{\mathsf{p}}(\Omega) = 0$ for all Ω is said to have **zero phase**. In the case of a system having zero phase, the phase spectra of the system's input and output are identical. In the case that the system does not have zero phase, the phase spectra of the system's input

[1]Some authors (e.g., [5, 7]) define an allpass system as one for which $|H(\Omega)| = c$ for all Ω, where c is a constant (and c is not necessarily 1).

and output differ. In situations where the phase spectrum is changed in an undesirable manner, **phase distortion** (i.e., distortion of the phase spectrum) is said to occur. If $\tau_\mathsf{p}(\Omega) = n_\mathsf{d}$ for all Ω, where n_d is a constant, the system shifts all complex sinusoids by the same amount n_d. Note that $\tau_\mathsf{p}(\Omega) = n_\mathsf{d}$ is equivalent to the (unwrapped) phase response being of the form

$$\arg H(\Omega) = -n_\mathsf{d}\Omega,$$

which is a linear function with a zero constant term. For this reason, a system with a constant phase delay is said to have **linear phase**. If $\tau_\mathsf{p}(\Omega)$ is not a constant, different complex sinusoids are shifted by different amounts. In many practical applications, shifting different complex sinusoids by different amounts is undesirable. Therefore, systems that are not linear phase are typically deemed to introduce phase distortion. For this reason, in contexts where phase spectra are important, systems with either zero phase or linear phase are typically used.

Example 11.34 (Distortionless transmission)**.** Consider a LTI system with input x and output y given by

$$y(n) = x(n-n_0),$$

where n_0 is an integer constant. That is, the output of the system is simply the input delayed by n_0. This type of system behavior is referred to as distortionless transmission, since the system allows the input to pass through to the output unmodified, except for a delay being introduced. This type of behavior is the ideal for which we strive in real-world communication systems (i.e., the received signal y equals a delayed version of the transmitted signal x). Taking the Fourier transform of the above equation, we have

$$Y(\Omega) = e^{-j\Omega n_0}X(\Omega).$$

Thus, the system has the frequency response H given by

$$H(\Omega) = e^{-j\Omega n_0}.$$

Since $|H(\Omega)| = 1$ for all Ω, the system is allpass and does not introduce any magnitude distortion. The phase delay τ_p of the system is given by

$$\begin{aligned}\tau_\mathsf{p}(\Omega) &= -\frac{\arg H(\Omega)}{\Omega} \\ &= -\left(\frac{-\Omega n_0}{\Omega}\right) \\ &= n_0.\end{aligned}$$

Since the phase delay is a constant, the system has linear phase and does not introduce any phase distortion (except for a trivial time shift of n_0). ■

11.14 Interconnection of LTI Systems

From the properties of the Fourier transform and the definition of the frequency response, we can derive a number of equivalences involving the frequency response and series- and parallel-interconnected systems.

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with frequency responses H_1 and H_2, respectively, that are connected in a series configuration as shown in the left-hand side of Figure 11.14(a). Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(n) = h_1 * h_2(n).$$

Taking the Fourier transform of both sides of this equation yields

$$\begin{aligned}H(\Omega) &= \mathcal{F}\{h_1 * h_2\}(\Omega) \\ &= \mathcal{F}h_1(\Omega)\mathcal{F}h_2(\Omega) \\ &= H_1(\Omega)H_2(\Omega).\end{aligned}$$

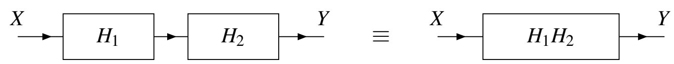


Figure 11.14: Equivalence involving frequency responses and the series interconnection of LTI systems.

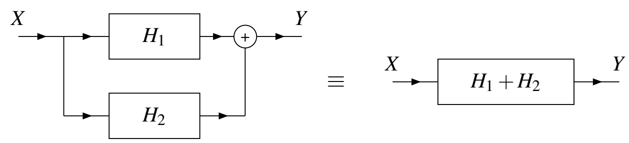


Figure 11.15: Equivalence involving frequency responses and the parallel interconnection of LTI systems.

Thus, we have the equivalence shown in Figure 11.14.

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with frequency responses H_1 and H_2 that are connected in a parallel configuration as shown on the left-hand side of Figure 11.15. Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(n) = h_1(n) + h_2(n).$$

Taking the Fourier transform of both sides of this equation yields

$$\begin{aligned} H(\Omega) &= \mathcal{F}\{h_1 + h_2\}(\Omega) \\ &= \mathcal{F}h_1(\Omega) + \mathcal{F}h_2(\Omega) \\ &= H_1(\Omega) + H_2(\Omega). \end{aligned}$$

Thus, we have the equivalence shown in Figure 11.15.

11.15 LTI Systems and Difference Equations

Many LTI systems of practical interest can be represented using an Nth-order linear difference equation with constant coefficients. Suppose that we have such a system with input x and output y. Then, the input-output behavior of the system is given by an equation of the form

$$\sum_{k=0}^{N} b_k y(n-k) = \sum_{k=0}^{M} a_k x(n-k)$$

(where $M \leq N$). Let X and Y denote the Fourier transforms of x and y, respectively. Taking the Fourier transform of both sides of the above equation yields

$$\mathcal{F}\left\{\sum_{k=0}^{N} b_k y(n-k)\right\}(\Omega) = \mathcal{F}\left\{\sum_{k=0}^{M} a_k x(n-k)\right\}(\Omega).$$

Using the linearity property of the Fourier transform, we can rewrite this as

$$\sum_{k=0}^{N} b_k \mathcal{F}\{y(n-k)\}(\Omega) = \sum_{k=0}^{M} a_k \mathcal{F}\{x(n-k)\}(\Omega).$$

Using the differencing property of the Fourier transform, we can re-express this as

$$\sum_{k=0}^{N} b_k e^{-j\Omega k} Y(\Omega) = \sum_{k=0}^{M} a_k e^{-j\Omega k} X(\Omega).$$

Then, factoring we have

$$Y(\Omega)\sum_{k=0}^{N} b_k e^{-j\Omega k} = X(\Omega)\sum_{k=0}^{M} a_k e^{-j\Omega k}.$$

Rearranging this equation, we obtain

$$\frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{k=0}^{M} a_k e^{-j\Omega k}}{\sum_{k=0}^{N} b_k e^{-j\Omega k}}.$$

Since $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$, the frequency response H is given by

$$H(\Omega) = \frac{\sum_{k=0}^{M} a_k e^{-j\Omega k}}{\sum_{k=0}^{N} b_k e^{-j\Omega k}} = \frac{\sum_{k=0}^{M} a_k (e^{-j\Omega})^k}{\sum_{k=0}^{N} b_k (e^{-j\Omega})^k}.$$

Observe that each of the numerator and denominator of H is a polynomial in the variable $e^{-j\Omega}$. Thus, H is a (proper) rational function in the variable $e^{-j\Omega}$. As it turns out, this is one reason why rational functions are of particular interest in the study of signals and systems.

Example 11.35 (Difference equation to frequency response)**.** A LTI system with input x and output y is characterized by the difference equation

$$5y(n) + 2y(n-1) + 3y(n-2) = x(n) - 2x(n-1).$$

Find the frequency response H of this system.

Solution. Let $X = \mathcal{F}x$ and $Y = \mathcal{F}y$. Taking the Fourier transform of the given difference equation, we obtain

$$5Y(\Omega) + 2e^{-j\Omega}Y(\Omega) + 3e^{-j2\Omega}Y(\Omega) = X(\Omega) - 2e^{-j\Omega}X(\Omega).$$

Rearranging the terms and factoring, we have

$$(5 + 2e^{-j\Omega} + 3e^{-j2\Omega})Y(\Omega) = (1 - 2e^{-j\Omega})X(\Omega).$$

Dividing both sides of the equation by $5 + 2e^{-j\Omega} + 3e^{-j2\Omega}$ and $X(\Omega)$, we have

$$\frac{Y(\Omega)}{X(\Omega)} = \frac{1 - 2e^{-j\Omega}}{5 + 2e^{-j\Omega} + 3e^{-j2\Omega}}.$$

Since $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$, we have

$$H(\Omega) = \frac{1 - 2e^{-j\Omega}}{5 + 2e^{-j\Omega} + 3e^{-j2\Omega}}.$$

■

Example 11.36 (Frequency response to difference equation)**.** A LTI system with input x and output y has the frequency response

$$H(\Omega) = \frac{e^{j2\Omega} - e^{j\Omega}}{e^{j2\Omega} - e^{j\Omega} + \frac{1}{4}}.$$

Find the differential equation that characterizes this system.

Solution. Let $X = \mathcal{F}x$ and $Y = \mathcal{F}y$. Since $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$, we have (from the given frequency response H)

$$\frac{Y(\Omega)}{X(\Omega)} = \frac{e^{j2\Omega} - e^{j\Omega}}{e^{j2\Omega} - e^{j\Omega} + \frac{1}{4}}.$$

Multiplying both sides of the equation by $e^{j2\Omega} - e^{j\Omega} + \frac{1}{4}$ and $X(\Omega)$, we have

$$e^{j2\Omega}Y(\Omega) - e^{j\Omega}Y(\Omega) + \tfrac{1}{4}Y(\Omega) = e^{j2\Omega}X(\Omega) - e^{j\Omega}X(\Omega).$$

Multiplying both sides by $e^{-j2\Omega}$ (so that the largest power of $e^{j\Omega}$ is zero), we obtain

$$Y(\Omega) - e^{-j\Omega}Y(\Omega) + \tfrac{1}{4}e^{-j2\Omega}Y(\Omega) = X(\Omega) - e^{-j\Omega}X(\Omega).$$

Taking the inverse Fourier transform of this equation, we obtain

$$y(n) - y(n-1) + \tfrac{1}{4}y(n-2) = x(n) - x(n-1).$$ ■

11.16 Filtering

In some applications, we want to change the magnitude or phase of the frequency components of a sequence or possibly eliminate some frequency components altogether. This process of modifying the frequency components of a sequence is referred to as **filtering** and the system that performs such processing is called a **filter**.

For the sake of simplicity, we consider only LTI filters here. If a filter is LTI, then it is completely characterized by its frequency response. Since the frequency spectra of sequences are 2π-periodic, we only need to consider frequencies over an interval of length 2π. Normally, we choose this interval to be centered about the origin, namely, $(-\pi, \pi]$. When we consider this interval, low frequencies are those closer to the origin, while high frequencies are those closer to $\pm\pi$.

Many types of filters exist. One important class of filters are those that are frequency selective. Frequency selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies. Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

An **ideal lowpass filter** eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\Omega) = \begin{cases} 1 & |\Omega| \in [0, \Omega_c] \\ 0 & |\Omega| \in (\Omega_c, \pi], \end{cases}$$

where Ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 11.16(a).

An **ideal highpass filter** eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\Omega) = \begin{cases} 1 & |\Omega| \in [\Omega_c, \pi] \\ 0 & |\Omega| \in [0, \Omega_c), \end{cases}$$

where Ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 11.16(b).

An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie between two cutoff frequencies, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\Omega) = \begin{cases} 1 & |\Omega| \in [\Omega_{c1}, \Omega_{c2}] \\ 0 & |\Omega| \in [0, \Omega_{c1}) \cup (\Omega_{c2}, \pi], \end{cases}$$

where Ω_{c1} and Ω_{c2} are the cutoff frequencies. A plot of this frequency response is given in Figure 11.16(c).

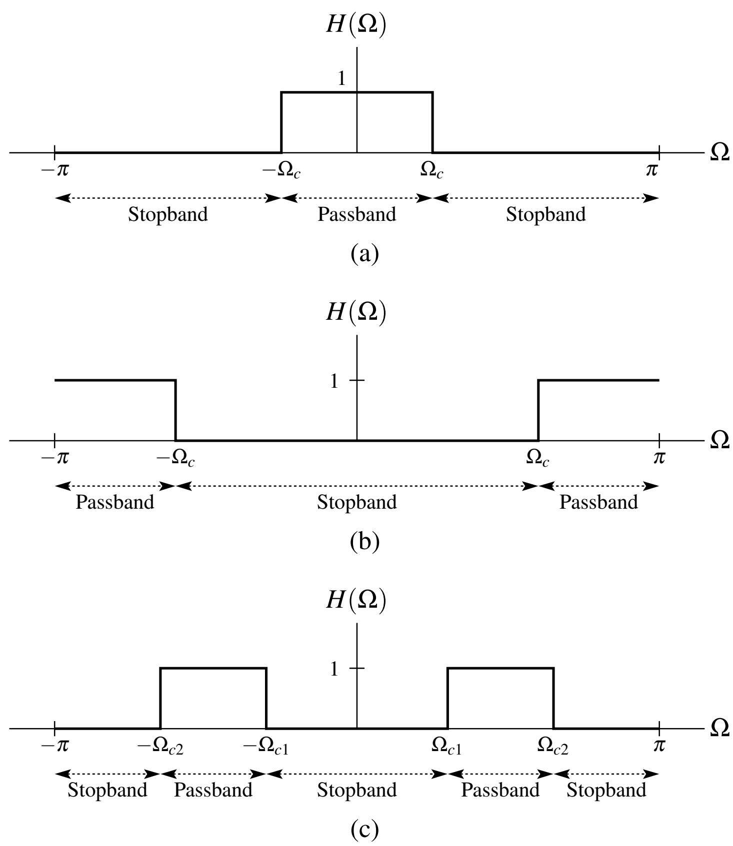


Figure 11.16: Frequency responses of (a) ideal lowpass, (b) ideal highpass, and (c) ideal bandpass filters.

Example 11.37 (Ideal filters)**.** For each LTI system whose impulse response h is given below, find and plot the frequency response H of the system, and identify the type of frequency-selective filter to which the system corresponds.
(a) $h(n) = \frac{\Omega_c}{\pi}\operatorname{sinc}(\Omega_c n)$, where Ω_c is a real constant in the interval $(0,\pi)$;
(b) $h(n) = \delta(n) - \frac{\Omega_c}{\pi}\operatorname{sinc}(\Omega_c n)$, where Ω_c is a real constant in the interval $(0,\pi)$; and
(c) $h(n) = \frac{2\Omega_b}{\pi}\operatorname{sinc}(\Omega_b n)\cos(\Omega_a n)$, where Ω_a and Ω_b are real constants in the interval $(0,\pi)$.

Solution. In what follows, let us denote the input and output of the system as x and y, respectively. Also, let X and Y denote the Fourier transforms of x and y, respectively.

(a) The frequency response H of the system is simply the Fourier transform of the impulse response h. Thus, we have

$$\begin{aligned}
H(\Omega) &= \mathcal{F}\left\{\tfrac{\Omega_c}{\pi}\operatorname{sinc}(\Omega_c n)\right\}(\Omega) \\
&= \sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{1}{2\Omega_c}(\Omega-2\pi k)\right] \\
&= \operatorname{rect}\left(\tfrac{1}{2\Omega_c}\Omega\right) \quad \text{for } |\Omega| \in (-\pi,\pi] \\
&= \begin{cases} 1 & |\Omega| \in [0,\Omega_c] \\ 0 & |\Omega| \in (\Omega_c,\pi]. \end{cases}
\end{aligned}$$

The frequency response H is plotted in Figure 11.17(a). Since $Y(\Omega) = H(\Omega)X(\Omega)$ and $H(\Omega) = 0$ for $|\Omega| \in (\Omega_c,\pi]$, Y will contain only those frequency components in X that lie in the frequency range $|\Omega| \in [0,\Omega_c]$. In other words, only the lower frequency components from X are kept. Thus, the system corresponds to a lowpass filter.

(b) The frequency response H of the system is simply the Fourier transform of the impulse response h. Thus, we have

$$\begin{aligned}
H(\Omega) &= \mathcal{F}\left\{\delta(n) - \tfrac{\Omega_c}{\pi}\operatorname{sinc}(\Omega_c n)\right\}(\Omega) \\
&= \mathcal{F}\delta(\Omega) - \mathcal{F}\left\{\tfrac{\Omega_c}{\pi}\operatorname{sinc}(\Omega_c n)\right\}(\Omega) \\
&= 1 - \sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{1}{2\Omega_c}(\Omega-2\pi k)\right] \\
&= 1 - \operatorname{rect}\left(\tfrac{1}{2\Omega_c}\Omega\right) \quad \text{for } \Omega \in (-\pi,\pi] \\
&= \begin{cases} 1 & |\Omega| \in [\Omega_c,\pi] \\ 0 & |\Omega| \in [0,\Omega_c). \end{cases}
\end{aligned}$$

The frequency response H is plotted in Figure 11.17(b). Since $Y(\Omega) = H(\Omega)X(\Omega)$ and $H(\Omega) = 0$ for $|\Omega| \in [0,\Omega_c)$, Y will contain only those frequency components in X that lie in the frequency range $|\Omega| \in [\Omega_c,\pi]$. In other words, only the higher frequency components from X are kept. Thus, the system corresponds to a highpass filter.

(c) The frequency response H of the system is simply the Fourier transform of the impulse response h. Thus, we

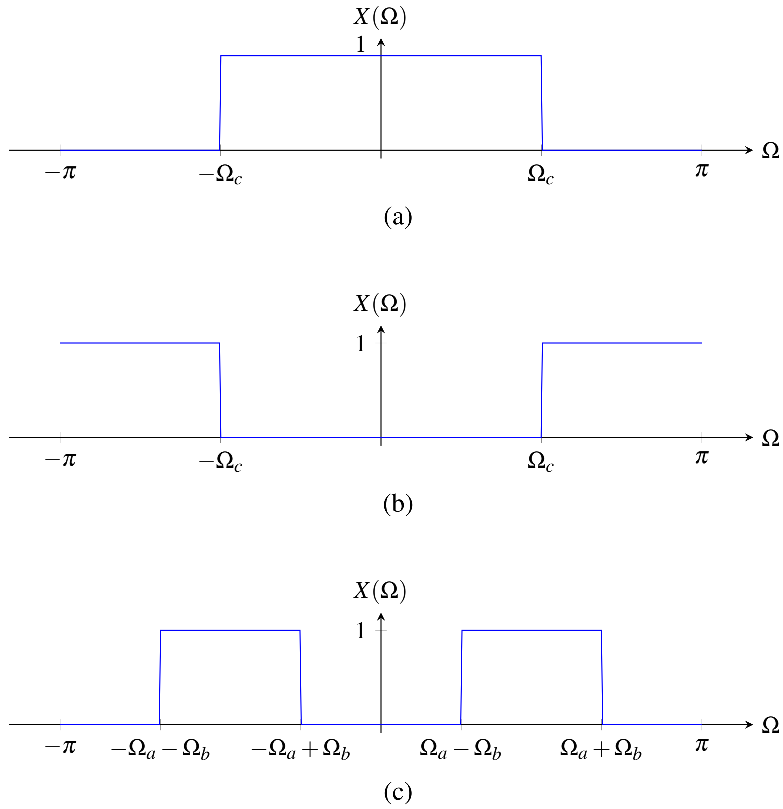


Figure 11.17: Frequency responses of each of the (a) first, (b) second, and (c) third systems from the example.

have

$$
\begin{aligned}
H(\Omega) &= \mathcal{F}\left\{\tfrac{2\Omega_b}{\pi}\operatorname{sinc}(\Omega_b n)\cos(\Omega_a n)\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{\Omega_b}{\pi}\operatorname{sinc}(\Omega_b n)[2\cos(\Omega_a n)]\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{\Omega_b}{\pi}\operatorname{sinc}(\Omega_b n)\left[e^{j\Omega_a n}+e^{-j\Omega_a n}\right]\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{\Omega_b}{\pi}e^{j\Omega_a n}\operatorname{sinc}(\Omega_b n)\right\}(\Omega)+\mathcal{F}\left\{\tfrac{\Omega_b}{\pi}e^{-j\Omega_a n}\operatorname{sinc}(\Omega_b n)\right\}(\Omega) \\
&= \sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{1}{2\Omega_b}([\Omega-\Omega_a]-2\pi k)\right]+\sum_{k=-\infty}^{\infty}\operatorname{rect}\left[\tfrac{1}{2\Omega_b}([\Omega+\Omega_a]-2\pi k)\right] \\
&= \operatorname{rect}\left[\tfrac{1}{2\Omega_b}(\Omega-\Omega_a)\right]+\operatorname{rect}\left[\tfrac{1}{2\Omega_b}(\Omega+\Omega_a)\right] \quad \text{for } \Omega\in(-\pi,\pi] \\
&= \begin{cases} 1 & |\Omega|\in[\Omega_a-\Omega_b,\Omega_a+\Omega_b] \\ 0 & |\Omega|\in[0,\Omega_a-\Omega_b)\cup(\Omega_a+\Omega_b,\pi]. \end{cases}
\end{aligned}
$$

The frequency response H is plotted in Figure 11.17(c). Since $Y(\Omega)=H(\Omega)X(\Omega)$ and $H(\Omega)=0$ for $|\Omega|\in[0,\Omega_a-\Omega_b)\cup(\Omega_a+\Omega_b,\pi]$, Y will contain only those frequency components in X that lie in the frequency range $|\Omega|\in[\Omega_a-\Omega_b,\Omega_a+\Omega_b]$. In other words, only the middle frequency components of X are kept. Thus, the system corresponds to a bandpass filter. ■

Example 11.38 (Lowpass filtering)**.** Consider a LTI system with impulse response

$$h(n) = \tfrac{1}{3}\operatorname{sinc}\left(\tfrac{\pi}{3}n\right).$$

Using frequency-domain methods, find the response y of the system to the input

$$x(n) = \tfrac{1}{2} + \tfrac{2}{3}\cos\left(\tfrac{\pi}{4}n\right) + \tfrac{1}{2}\cos\left(\tfrac{3\pi}{4}n\right) + \tfrac{1}{6}\cos\left(\pi n\right).$$

Solution. To begin, we find the Fourier transform X of x. Computing $X(\Omega)$ for $\Omega \in (-\pi,\pi]$, we have

$$\begin{aligned}
X(\Omega) &= \mathcal{F}\left\{\tfrac{1}{2} + \tfrac{2}{3}\cos\left(\tfrac{\pi}{4}n\right) + \tfrac{1}{2}\cos\left(\tfrac{3\pi}{4}n\right) + \tfrac{1}{6}\cos(\pi n)\right\}(\Omega) \\
&= \tfrac{1}{2}\mathcal{F}\{1\}(\Omega) + \tfrac{2}{3}\mathcal{F}\left\{\cos(\tfrac{\pi}{4}n)\right\}(\Omega) + \tfrac{1}{2}\mathcal{F}\left\{\cos(\tfrac{3\pi}{4}n)\right\}(\Omega) + \tfrac{1}{6}\mathcal{F}\left\{\cos(\pi n)\right\}(\Omega) \\
&= \tfrac{1}{2}[2\pi\delta(\Omega)] + \tfrac{2}{3}\left(\pi\left[\delta\left(\Omega+\tfrac{\pi}{4}\right) + \delta\left(\Omega-\tfrac{\pi}{4}\right)\right]\right) + \tfrac{1}{2}\left(\pi\left[\delta\left(\Omega+\tfrac{3\pi}{4}\right) + \delta\left(\Omega-\tfrac{3\pi}{4}\right)\right]\right) + \tfrac{1}{6}[2\pi\delta(\Omega-\pi)] \\
&= \pi\delta(\Omega) + \tfrac{2\pi}{3}\delta\left(\Omega+\tfrac{\pi}{4}\right) + \tfrac{2\pi}{3}\delta\left(\Omega-\tfrac{\pi}{4}\right) + \tfrac{\pi}{2}\delta\left(\Omega+\tfrac{3\pi}{4}\right) + \tfrac{\pi}{2}\delta\left(\Omega-\tfrac{3\pi}{4}\right) + \tfrac{\pi}{3}\delta\left(\Omega-\pi\right) \\
&= \tfrac{\pi}{2}\delta\left(\Omega+\tfrac{3\pi}{4}\right) + \tfrac{2\pi}{3}\delta\left(\Omega+\tfrac{\pi}{4}\right) + \pi\delta\left(\Omega\right) + \tfrac{2\pi}{3}\delta\left(\Omega-\tfrac{\pi}{4}\right) + \tfrac{\pi}{2}\delta\left(\Omega-\tfrac{3\pi}{4}\right) + \tfrac{\pi}{3}\delta\left(\Omega-\pi\right).
\end{aligned}$$

A plot of the frequency spectrum X is shown in Figure 11.18(a). Now, we compute the Fourier transform H of h. For $\Omega \in (-\pi,\pi]$, we have

$$\begin{aligned}
H(\Omega) &= \mathcal{F}\left\{\tfrac{1}{3}\operatorname{sinc}\left(\tfrac{\pi}{3}n\right)\right\} \\
&= \operatorname{rect}\left(\tfrac{3}{2\pi}\Omega\right) \\
&= \begin{cases} 1 & |\Omega| \in [0, \tfrac{\pi}{3}] \\ 0 & |\Omega| \in (\tfrac{\pi}{3}, \pi]. \end{cases}
\end{aligned}$$

The frequency response H is shown in Figure 11.18(b). The frequency spectrum Y of the output can be computed as

$$\begin{aligned}
Y(\Omega) &= H(\Omega)X(\Omega) \\
&= \tfrac{2\pi}{3}\delta\left(\Omega+\tfrac{\pi}{4}\right) + \pi\delta\left(\Omega\right) + \tfrac{2\pi}{3}\delta\left(\Omega-\tfrac{\pi}{4}\right).
\end{aligned}$$

The frequency spectrum Y is shown in Figure 11.18(c). Taking the inverse Fourier transform of Y yields

$$\begin{aligned}
y(n) &= \mathcal{F}^{-1}\left\{\pi\delta\left(\Omega\right) + \tfrac{2\pi}{3}\left[\delta\left(\Omega+\tfrac{\pi}{4}\right) + \delta\left(\Omega-\tfrac{\pi}{4}\right)\right]\right\}(n) \\
&= \tfrac{1}{2}\mathcal{F}^{-1}\left\{2\pi\delta\left(\Omega\right)\right\}(n) + \tfrac{2}{3}\mathcal{F}^{-1}\left\{\pi\left[\delta\left(\Omega+\tfrac{\pi}{4}\right) + \delta\left(\Omega-\tfrac{\pi}{4}\right)\right]\right\}(n) \\
&= \tfrac{1}{2} + \tfrac{2}{3}\cos\left(\tfrac{\pi}{4}n\right).
\end{aligned}$$

■

Example 11.39 (Bandpass filtering)**.** Consider a LTI system with the impulse response

$$h(n) = \tfrac{8}{5\pi}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\cos(\tfrac{6}{5}n).$$

Using frequency-domain methods, find the response y of the system to the input

$$x(n) = \tfrac{1}{4} + \cos\left(\tfrac{4}{5}n\right) + \tfrac{2}{3}\cos\left(\tfrac{8}{5}n\right) + \tfrac{1}{4}\cos\left(\tfrac{12}{5}n\right) + \tfrac{1}{6}\cos(\pi n).$$

In passing, we note that x is not periodic.

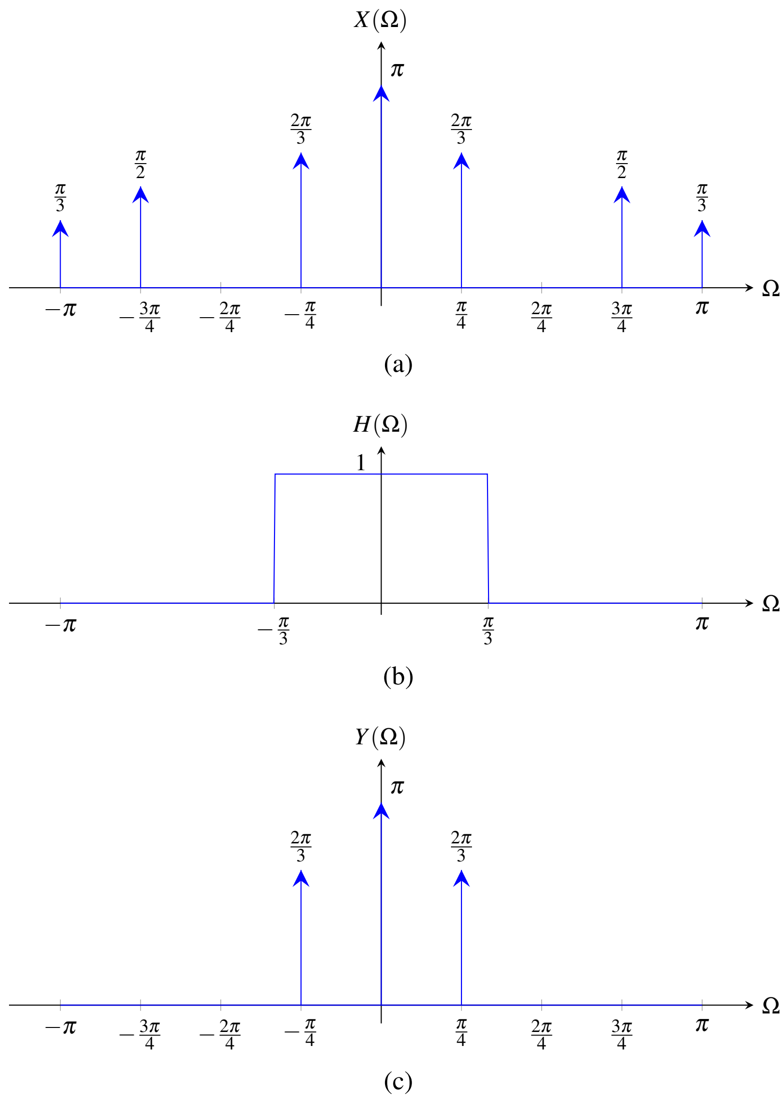


Figure 11.18: Frequency spectra for the lowpass filtering example. (a) Frequency spectrum X of the input x. (b) Frequency response H of the system. (c) Frequency spectrum Y of the output y.

Solution. Taking the Fourier transform of x, we have

$$\begin{aligned}
X(\Omega) &= \mathcal{F}\left\{\tfrac{1}{4}+\cos\left(\tfrac{4}{5}n\right)+\tfrac{2}{3}\cos\left(\tfrac{8}{5}n\right)+\tfrac{1}{4}\cos\left(\tfrac{12}{5}n\right)+\tfrac{1}{6}\cos(\pi n)\right\}(\Omega) \\
&= \tfrac{1}{4}\mathcal{F}\{1\}(\Omega)+\mathcal{F}\left\{\cos\left(\tfrac{4}{5}n\right)\right\}(\Omega)+\tfrac{2}{3}\mathcal{F}\left\{\cos\left(\tfrac{8}{5}n\right)\right\}(\Omega)+\tfrac{1}{4}\mathcal{F}\left\{\cos\left(\tfrac{12}{5}n\right)\right\}(\Omega) \\
&\quad+\tfrac{1}{6}\mathcal{F}\{\cos(\pi n)\}(\Omega) \\
&= \tfrac{1}{4}[2\pi\delta(\Omega)]+\left(\pi\left[\delta\left(\Omega-\tfrac{4}{5}\right)+\delta\left(\Omega+\tfrac{4}{5}\right)\right]\right)+\tfrac{2}{3}\left(\pi\left[\delta\left(\Omega-\tfrac{8}{5}\right)+\delta\left(\Omega+\tfrac{8}{5}\right)\right]\right) \\
&\quad+\tfrac{1}{4}\left(\pi\left[\delta\left(\Omega-\tfrac{12}{5}\right)+\delta\left(\Omega+\tfrac{12}{5}\right)\right]\right)+\tfrac{1}{6}[2\pi\delta(\Omega-\pi)] \\
&= \tfrac{\pi}{2}\delta(\Omega)+\pi\left[\delta\left(\Omega-\tfrac{4}{5}\right)+\delta\left(\Omega+\tfrac{4}{5}\right)\right]+\tfrac{2\pi}{3}\left[\delta\left(\Omega-\tfrac{8}{5}\right)+\delta\left(\Omega+\tfrac{8}{5}\right)\right] \\
&\quad+\tfrac{\pi}{4}\left(\left[\delta\left(\Omega-\tfrac{12}{5}\right)+\delta\left(\Omega+\tfrac{12}{5}\right)\right]\right)+\tfrac{\pi}{3}[\delta(\Omega-\pi)] \\
&= \tfrac{\pi}{2}\delta(\Omega)+\pi\delta\left(\Omega+\tfrac{4}{5}\right)+\pi\delta\left(\Omega-\tfrac{4}{5}\right)+\tfrac{2\pi}{3}\delta\left(\Omega+\tfrac{8}{5}\right)+\tfrac{2\pi}{3}\delta\left(\Omega-\tfrac{8}{5}\right)+\tfrac{\pi}{4}\delta\left(\Omega+\tfrac{12}{5}\right) \\
&\quad+\tfrac{\pi}{4}\delta\left(\Omega-\tfrac{12}{5}\right)+\tfrac{\pi}{3}\delta(\Omega-\pi) \\
&= \tfrac{\pi}{4}\delta\left(\Omega+\tfrac{12}{5}\right)+\tfrac{2\pi}{3}\delta\left(\Omega+\tfrac{8}{5}\right)+\pi\delta\left(\Omega+\tfrac{4}{5}\right)+\tfrac{\pi}{2}\delta(\Omega)+\pi\delta\left(\Omega-\tfrac{4}{5}\right)+\tfrac{2\pi}{3}\delta\left(\Omega-\tfrac{8}{5}\right) \\
&\quad+\tfrac{\pi}{4}\delta\left(\Omega-\tfrac{12}{5}\right)+\tfrac{\pi}{3}\delta(\Omega-\pi).
\end{aligned}$$

The frequency spectrum X is shown in Figure 11.19(a). Now, we compute the frequency response H of the system. We have

$$\begin{aligned}
H(\Omega) &= \mathcal{F}\left\{\tfrac{8}{5\pi}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\cos\left(\tfrac{6}{5}n\right)\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{8}{5\pi}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\left[\tfrac{1}{2}\left(e^{j(6/5)n}+e^{-j(6/5)n}\right)\right]\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{4}{5\pi}e^{j(6/5)n}\operatorname{sinc}\left(\tfrac{4}{5}n\right)+\tfrac{4}{5\pi}e^{-j(6/5)n}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{4}{5\pi}e^{j(6/5)n}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\right\}(\Omega)+\mathcal{F}\left\{\tfrac{4}{5\pi}e^{-j(6/5)n}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\right\}(\Omega) \\
&= \mathcal{F}\left\{\tfrac{4}{5\pi}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\right\}\left(\Omega-\tfrac{6}{5}\right)+\mathcal{F}\left\{\tfrac{4}{5\pi}\operatorname{sinc}\left(\tfrac{4}{5}n\right)\right\}\left(\Omega+\tfrac{6}{5}\right) \\
&= \operatorname{rect}\left[\tfrac{5}{8}\left(\Omega-\tfrac{6}{5}\right)\right]+\operatorname{rect}\left[\tfrac{5}{8}\left(\Omega+\tfrac{6}{5}\right)\right] \\
&= \begin{cases} 1 & |\Omega|\in\left[\tfrac{2}{5},\tfrac{10}{5}\right] \\ 0 & |\Omega|\in\left[0,\tfrac{2}{5}\right)\cup\left(\tfrac{10}{5},\pi\right]. \end{cases}
\end{aligned}$$

The frequency response H is shown in Figure 11.19(b). The frequency spectrum Y of the output is given by

$$\begin{aligned}
Y(\Omega) &= H(\Omega)X(\Omega) \\
&= \tfrac{2\pi}{3}\delta\left(\Omega+\tfrac{8}{5}\right)+\pi\delta\left(\Omega+\tfrac{4}{5}\right)+\pi\delta\left(\Omega-\tfrac{4}{5}\right)+\tfrac{2\pi}{3}\delta\left(\Omega-\tfrac{8}{5}\right).
\end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned}
y(n) &= \mathcal{F}^{-1}\left\{\tfrac{2\pi}{3}\delta\left(\Omega+\tfrac{8}{5}\right)+\pi\delta\left(\Omega+\tfrac{4}{5}\right)+\pi\delta\left(\Omega-\tfrac{4}{5}\right)+\tfrac{2\pi}{3}\delta\left(\Omega-\tfrac{8}{5}\right)\right\}(n) \\
&= \mathcal{F}^{-1}\left\{\tfrac{2\pi}{3}\delta\left(\Omega+\tfrac{8}{5}\right)+\tfrac{2\pi}{3}\delta\left(\Omega-\tfrac{8}{5}\right)\right\}(n)+\mathcal{F}^{-1}\left\{\pi\delta\left(\Omega+\tfrac{4}{5}\right)+\pi\delta\left(\Omega-\tfrac{4}{5}\right)\right\}(n) \\
&= \tfrac{2}{3}\mathcal{F}^{-1}\left\{\pi\left[\delta\left(\Omega+\tfrac{8}{5}\right)+\delta\left(\Omega-\tfrac{8}{5}\right)\right]\right\}(n)+\mathcal{F}^{-1}\left\{\pi\left[\delta\left(\Omega+\tfrac{4}{5}\right)+\delta\left(\Omega-\tfrac{4}{5}\right)\right]\right\}(n) \\
&= \cos\left(\tfrac{4}{5}n\right)+\tfrac{2}{3}\cos\left(\tfrac{8}{5}n\right).
\end{aligned}$$

■

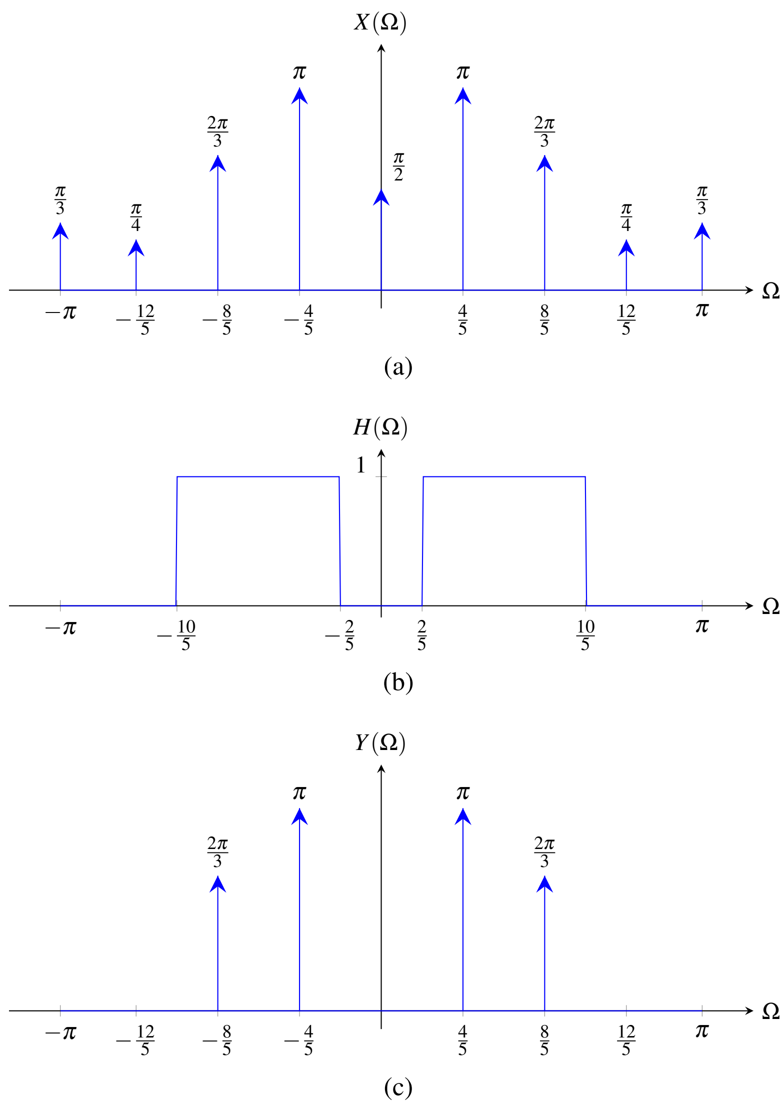


Figure 11.19: Frequency spectra for the bandpass filtering example. (a) Frequency spectrum X of the input x. (b) Frequency response H of the system. (c) Frequency spectrum Y of the output y.

11.17 Relationship Between DT Fourier Transform and CT Fourier Series

A duality relationship exists between the DT Fourier transform and CT Fourier series. Recall that the DT Fourier transform analysis and synthesis equations are, respectively, given by

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x(k)e^{-jk\Omega} \quad \text{and} \tag{11.20a}$$

$$x(n) = \tfrac{1}{2\pi}\int_{2\pi} X(\Omega)e^{jn\Omega}d\Omega. \tag{11.20b}$$

Also, recall that the CT Fourier series synthesis and analysis equations are, respectively, given by

$$y(t) = \sum_{k=-\infty}^{\infty} Y(k)e^{jk(2\pi/T)t} \quad \text{and}$$

$$Y(n) = \tfrac{1}{T}\int_{T} y(t)e^{-jn(2\pi/T)t}dt,$$

which can be rewritten, respectively, as

$$y(t) = \sum_{k=-\infty}^{\infty} Y(-k)e^{-jk(2\pi/T)t} \quad \text{and} \tag{11.21a}$$

$$Y(-n) = \tfrac{1}{T}\int_{T} y(t)e^{jn(2\pi/T)t}dt. \tag{11.21b}$$

Observe that, if $T = 2\pi$, the pair of equations in (11.20) is essentially the same as the pair of equations in (11.21). In particular, (11.21a) with $T = 2\pi$ is identical to (11.20a) with $X = y$, $\Omega = t$, and $x(n) = Y(-n)$. Furthermore, (11.21b) with $T = 2\pi$ is identical to (11.20b) with $X = y$, $\Omega = t$, and $x(n) = Y(-n)$. Consequently, the DT Fourier transform X of the sequence x can be viewed as a CT Fourier-series representation of the 2π-periodic spectrum X. We can formalize this result in the following theorem.

Theorem 11.20. *Let X be a 2π-periodic function and let x be a sequence. Then, the following are equivalent:*

1. $X \overset{\text{CTFS}}{\longleftrightarrow} x$
2. $\mathcal{R}x \overset{\text{DTFT}}{\longleftrightarrow} X$, *where $\mathcal{R}$ denotes time reversal (i.e., $\mathcal{R}x(n) = x(-n)$).*

Proof. The result of this theorem follows immediately from the definition of the analysis and synthesis equations for the DT Fourier transform and CT Fourier series (as explained above). ■

The duality relationship in the preceding theorem can be quite helpful in some situations. We consider one such situation below. In particular, we use this relationship to assist in the computation of a DT Fourier transform.

Example 11.40 (Fourier transform of sinc sequence)**.** Let B denote a real constant in the interval $(0, \pi)$.
(a) Show that the 2π-periodic function

$$y(t) = \sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{1}{2B}(t - 2\pi k)\right],$$

has the Fourier series coefficient sequence

$$Y(k) = \tfrac{B}{\pi}\operatorname{sinc}(Bk).$$

(b) Use the result of part (a) to find the DT Fourier transform X of the sequence

$$x(n) = \tfrac{B}{\pi}\operatorname{sinc}(Bn).$$

Solution. (a) From the Fourier series analysis equation, we have

$$\begin{aligned}
Y(k) &= \tfrac{1}{T}\int_T y(t)e^{-j(2\pi/T)kt}dt \\
&= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} y(t)e^{-jkt}dt \\
&= \tfrac{1}{2\pi}\int_{-B}^{B} e^{-jkt}dt \\
&= \begin{cases} \tfrac{1}{2\pi}\left[\tfrac{1}{-jk}e^{-jkt}\right]\Big|_{-B}^{B} & k \neq 0 \\ \tfrac{1}{2\pi}(2B) & k = 0. \end{cases}
\end{aligned}$$

Now, we simplify the preceding formula for Y for each of the cases $k \neq 0$ and $k = 0$. First, we consider $k \neq 0$. We have

$$\begin{aligned}
Y(k) = {} & \tfrac{1}{2\pi}\left[\frac{1}{-jk}e^{-jkt}\right]\Bigg|_{-B}^{B} \\
&= \tfrac{j}{2\pi k}e^{-jkt}\Big|_{-B}^{B} \\
&= \tfrac{j}{2\pi k}\left[e^{-jBk} - e^{jBk}\right] \\
&= \tfrac{j}{2\pi k}\left[2j\sin(-Bk)\right] \\
&= -\tfrac{1}{\pi k}\sin(-Bk) \\
&= \tfrac{1}{\pi k}\sin(Bk) \\
&= \tfrac{B}{\pi}\left[\frac{\sin(Bk)}{Bk}\right] \\
&= \tfrac{B}{\pi}\operatorname{sinc}(Bk).
\end{aligned}$$

Next, we consider $k = 0$. We have

$$\begin{aligned}
Y(k) &= \tfrac{1}{2\pi}(2B) \\
&= \tfrac{B}{\pi}.
\end{aligned}$$

Since, by chance, the formula for $Y(k)$ for $k \neq 0$ yields the correct result for $k = 0$, we can use this formula for both cases. Thus, we conclude

$$y(t) = \sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{1}{2B}(t - 2\pi k)\right] \quad \stackrel{\text{CTFS}}{\longleftrightarrow} \quad Y(k) = \tfrac{B}{\pi}\operatorname{sinc}(Bk).$$

(b) Since $x = Y$, X is the Fourier transform of Y. So, in effect, we are being asked to find the Fourier transform of Y. Since $y \stackrel{\text{CTFS}}{\longleftrightarrow} Y$, we have by duality (i.e., Theorem 11.20) that $\mathcal{R}Y \stackrel{\text{DTFT}}{\longleftrightarrow} y$, where $\mathcal{R}$ denotes time reversal (i.e., $\mathcal{R}x(n) = x(-n)$). Moreover, since Y is even, $\mathcal{R}Y = Y$, which implies $Y \stackrel{\text{DTFT}}{\longleftrightarrow} y$, or equivalently, $x \stackrel{\text{DTFT}}{\longleftrightarrow} y$. Thus, $X = y$. Therefore, we conclude

$$\begin{aligned}
X(\Omega) &= y(\Omega) \\
&= \sum_{k=-\infty}^{\infty} \operatorname{rect}\left[\tfrac{1}{2B}(\Omega - 2\pi k)\right].
\end{aligned}$$

■

11.18 Relationship Between DT and CT Fourier Transforms

Earlier, in Section 6.20, we were introduced to impulse sampling. For a bandlimited function x, a relationship exists between the Fourier transform of the function obtained from impulse sampling x and the Fourier transform of the sequence obtained from sampling x. This relationship is given by the theorem below.

Theorem 11.21. *Let x be a bandlimited function and let T denote a sampling period for x that satisfies the Nyquist condition. Let $\tilde{y}$ be the function obtained by impulse sampling x with sampling period T. That is,*

$$\tilde{y}(t) = \sum_{n=-\infty}^{\infty} x(Tn)\delta(t - Tn).$$

Let y denote the sequence obtaining by sampling x with sampling period T. That is,

$$y(n) = x(Tn).$$

Let $\tilde{Y}$ denote the (CT) Fourier transform of $\tilde{y}$ and let Y denote the (DT) Fourier transform of y. Then, the following relationship holds:

$$Y(\Omega) = \tilde{Y}\left(\tfrac{\Omega}{T}\right) \quad \text{for all } \Omega \in \mathbb{R}.$$

Proof. We take the (CT) Fourier transform of $\tilde{y}$ to obtain

$$\begin{aligned}
\tilde{Y}(\omega) &= \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} x(Tn)\delta(\cdot - Tn)\right\}(\omega) \\
&= \sum_{n=-\infty}^{\infty} x(Tn)\mathcal{F}\{\delta(\cdot - Tn)\}(\omega) \\
&= \sum_{n=-\infty}^{\infty} y(n)e^{-jTn\omega} \\
&= \sum_{n=-\infty}^{\infty} y(n)e^{-j(T\omega)n} \\
&= Y(T\omega).
\end{aligned}$$

Thus, we have shown that

$$Y(T\omega) = \tilde{Y}(\omega).$$

Letting $\omega = \frac{\Omega}{T}$ in the preceding equation yields

$$Y(\Omega) = \tilde{Y}\left(\tfrac{\Omega}{T}\right). \qquad \blacksquare$$

11.19 Relationship Between DT Fourier Transform and DFT

As it turns out, an important relationship exists between the DT Fourier transform and the DFT, as given by the theorem below.

Theorem 11.22. *Let x be a finite-duration sequence such that $x(n) = 0$ for all $n \notin [0\,..\,M-1]$ and let X denote the (DT) Fourier transform of x. Let $\tilde{X}$ denote the N-point DFT of X. That is,*

$$\tilde{X}(k) = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn} \quad \text{for } k \in [0\,..\,N-1].$$

Suppose now that $N \geq M$. Then, the following relationship holds:

$$X\left(\tfrac{2\pi}{N}k\right) = \tilde{X}(k) \quad \text{for } k \in [0\,..\,N-1].$$

In other words, the elements of the sequence $\tilde{X}$ correspond to uniformly-spaced samples of the function X.

Proof. From the definition of the DFT, we have

$$\tilde{X}(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn}.$$

Since $x(n) = 0$ for all $n \notin [0\,..\,M-1]$ and $N \geq M$, it must be the case that $x(n) = 0$ for all $n \notin [0\,..\,N-1]$. Therefore, we can equivalently rewrite the summation on the right-hand side of the above equation to be taken over all integers, yielding

$$\begin{aligned}\tilde{X}(k) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j(2\pi/N)kn} \\ &= \left.\sum_{k=-\infty}^{\infty} x(n) e^{-j\Omega n}\right|_{\Omega=(2\pi/N)k} \\ &= X\left(\tfrac{2\pi}{N}k\right).\end{aligned}$$

■

The above theorem (i.e., Theorem 11.22) has extremely important practical implications. Most systems that we build these days have a digital computer at their core. Digital computers, being discrete-time systems, are quite adept at processing and manipulating finite-length sequences, which can be represented as a finite-size array of numbers. In contrast, digital computers have a much more difficult time algebraically (i.e., symbolically) manipulating functions. Consequently, it is much easier for a computer to calculate a DFT than a DT Fourier transform. So, in practice, the DT Fourier transform is often evaluated using the DFT. The DFT is used to obtain uniformly-spaced samples of the DT Fourier transform, and then interpolation techniques can be used to assign values to the Fourier transform at points between these samples. When computing an N-point DFT, we are free to choose N however we like. Therefore, we can choose N so that the Fourier transform can be determined on a fine enough grid that interpolation will yield a sufficiently accurate approximation of the Fourier transform for the application at hand.

Example 11.41. Consider the sequence

$$x(n) = u(n) - u(n-4).$$

This sequence can be shown to have the Fourier transform

$$X(\Omega) = e^{-j(3/2)\Omega}\left[\frac{\sin(2\Omega)}{\sin\left(\frac{1}{2}\Omega\right)}\right].$$

Clearly, $x(n) = 0$ for all $n \notin [0\,..\,3]$. Therefore, we can determine samples of X using an N-point DFT, where $N \geq 4$. The N-point DFT of x is shown plotted in Figures 11.20, 11.21, 11.22, and 11.23 for N chosen as 4, 8, 16, and 64, respectively. To show relationship between the DFT and Fourier transform, the Fourier transform is also superimposed on each plot. We have used the 2π-periodicity of X in order to plot X over the interval $(-\pi, \pi]$ (instead of $[0, 2\pi)$). Examining the plots, we can see that the sampled frequency spectrum obtained from the DFT represents the spectrum more faithfully as N increases. Of course, the downside to increasing N is greater computational and memory costs.

■

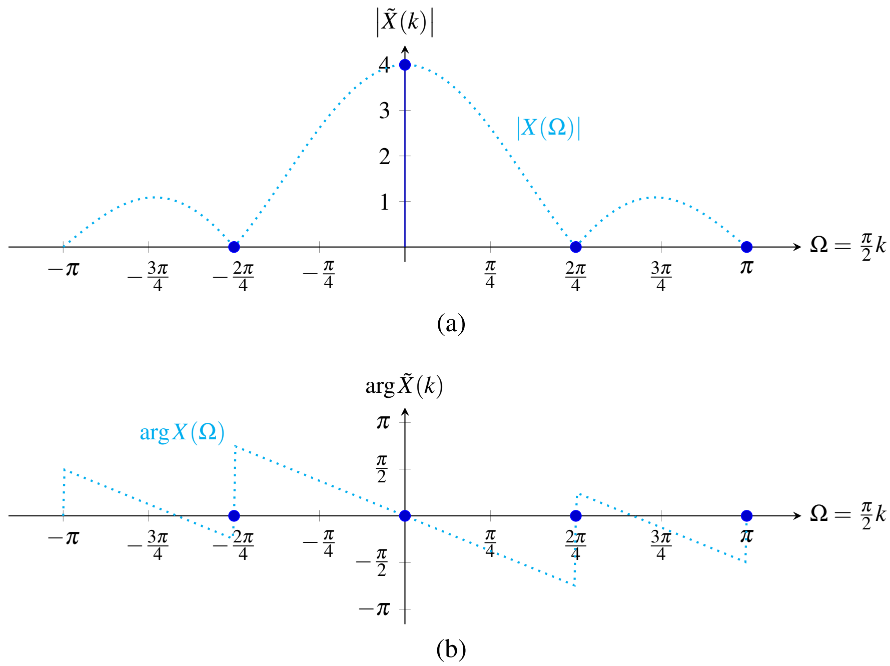


Figure 11.20: The sampled DT Fourier transform obtained from the DFT when $N = 4$. (a) Magnitude spectrum. (b) Phase spectrum.

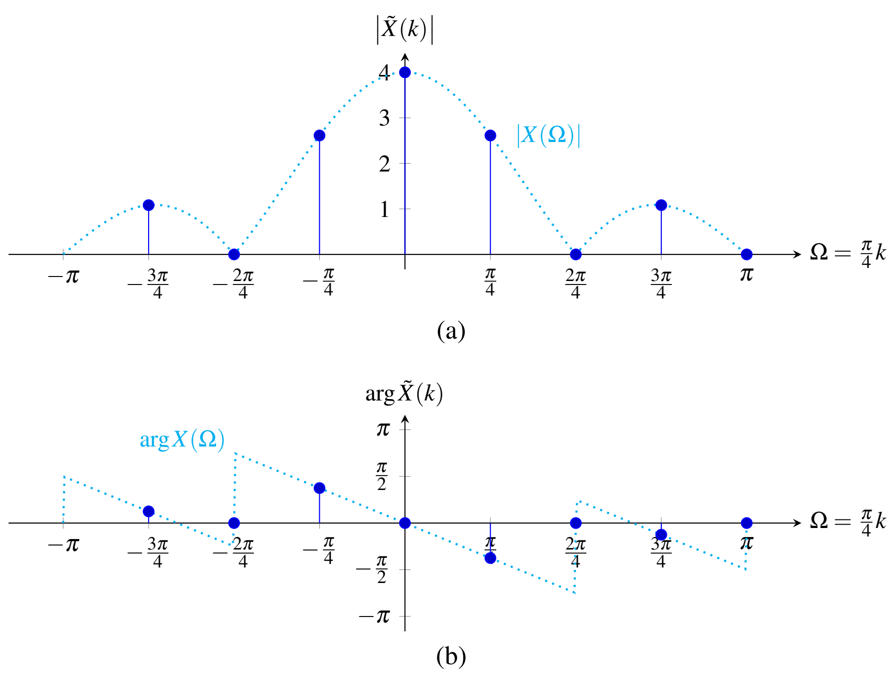
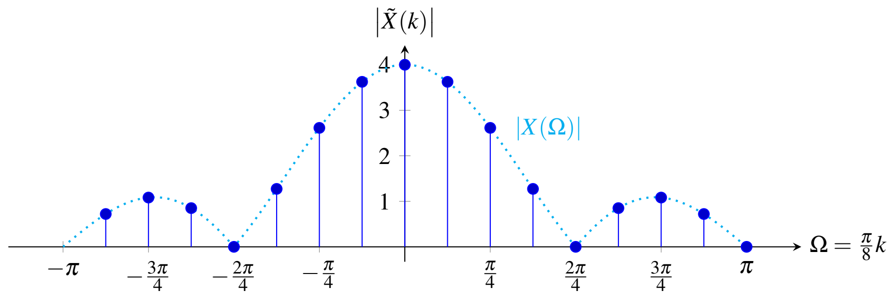
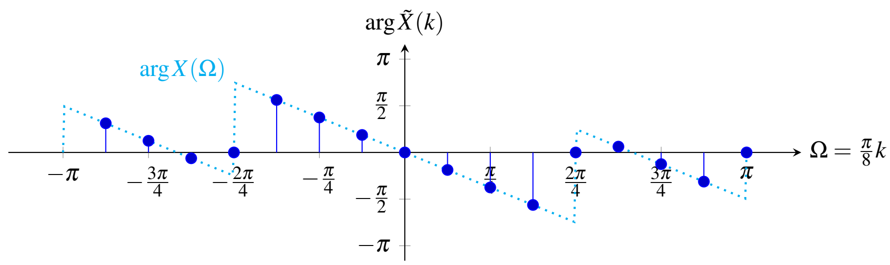


Figure 11.21: The sampled DT Fourier transform obtained from the DFT when $N = 8$. (a) Magnitude spectrum. (b) Phase spectrum.

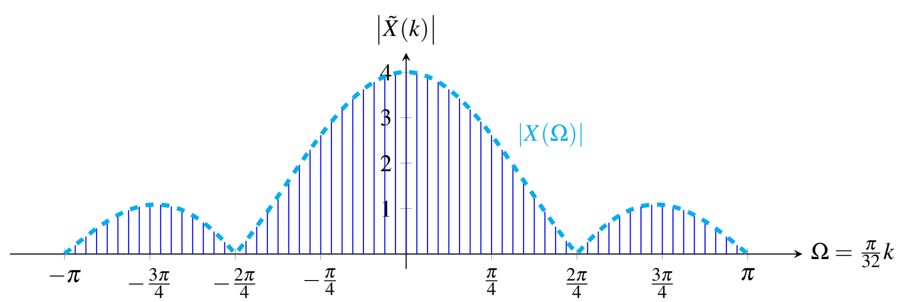


(a)

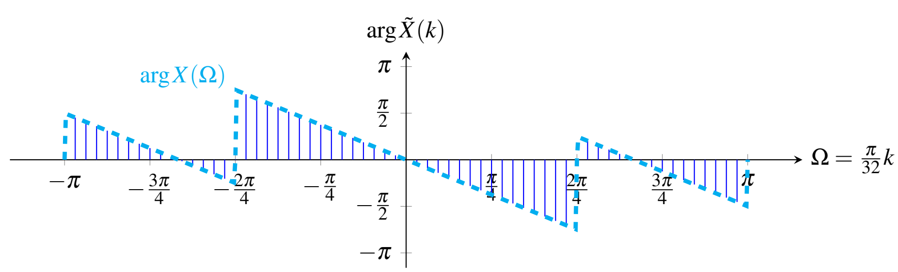


(b)

Figure 11.22: The sampled DT Fourier transform obtained from the DFT when $N = 16$. (a) Magnitude spectrum. (b) Phase spectrum.



(a)



(b)

Figure 11.23: The sampled DT Fourier transform obtained from the DFT when $N = 64$. (a) Magnitude spectrum. (b) Phase spectrum.

11.20 Exercises

11.20.1 Exercises Without Answer Key

11.1 Using the Fourier transform analysis equation, find the Fourier transform X of each sequence x given below.
(a) $x(n) = n[u(n) - u(n-N)]$, where N is an integer constant [Hint: Use (F.8)];
(b) $x(n) = |n|\,[u(n+M) - u(n-M-1)]$, where M is an integer constant [Hint: Use (F.8)];
(c) $x(n) = a^n[u(n) - n(n-M)]$, where a is a complex constant satisfying $a \neq 1$, and M is an integer constant;
(d) $x(n) = a^{|2n|}$, where a is a complex constant satisfying $|a| < 1$;
(e) $x(n) = a^n u(n-1)$, where a is a complex constant and $|a| < 1$;
(f) $x(n) = \delta(n+a) + \delta(n-a)$, where a is an integer; and
(g) $x(n) = \delta(n+a) - \delta(n-a)$, where a is an integer.

11.2 Using a table of Fourier transform pairs and and properties of the Fourier transform, find the Fourier transform X of each sequence x given below.
(a) $x(n) = \cos\left(\frac{2\pi}{7}n - \frac{\pi}{3}\right)$;
(b) $x(n) = u(n+5) - u(n-5)$;
(c) $x(n) = x_1 * x_2(n)$, where $x_1(n) = \delta(n) - \frac{1}{3}\operatorname{sinc}\left(\frac{\pi}{3}n\right)$ and $x_2(n) = \frac{2}{3}\operatorname{sinc}\left(\frac{2\pi}{3}n\right)$;
(d) $x(n) = \cos^2\left(\frac{\pi}{5}n\right)$;
(e) $x(n) = (\uparrow 3)x_1(n)$, where $x_1(n) = \frac{1}{12}\operatorname{sinc}\left(\frac{\pi}{12}n\right)$;
(f) $x(n) = \left(\frac{1}{3}\right)^n \cos\left(\frac{\pi}{7}n\right) u(n)$;
(g) $x(n) = \operatorname{sinc}\left(\frac{\pi}{6}n\right)\cos\left(\frac{\pi}{2}n\right)$;
(h) $x(n) = \left(\frac{3}{4}\right)^n u(n-2)$;
(i) $x(n) = (n-2)\left(\frac{1}{3}\right)^{n-2} u(n-2)$;
(j) $x(n) = n\left(\frac{2}{3}\right)^{n+1} u(n+1)$;
(k) $x(n) = (n+1)\left(\frac{3}{4}\right)^n u(n)$;
(l) $x(n) = \left(\frac{1}{3}\right)^{n-2} u(n-2)$;
(m) $x(n) = \left(\frac{1}{3}\right)^{n+1} u(n)$; and
(n) $x(n) = (-1)^n$.

11.3 Using a table of Fourier transform pairs and and properties of the Fourier transform, find the Fourier transform X of each sequence x given below.
(a) $x(n) = (n-m)a^{n-m}u(n-m)$, where $m \in \mathbb{Z}$ and $a \in \mathbb{C}$ such that $|a| < 1$;
(b) $x(n) = na^{n-m}u(n-m)$, where $m \in \mathbb{Z}$ and $a \in \mathbb{C}$ such that $|a| < 1$;
(c) $x(n) = (n-m)a^n u(n)$, where $m \in \mathbb{Z}$ and $a \in \mathbb{C}$ such that $|a| < 1$;
(d) $x(n) = a^{n-m}u(n-m)$, where $m \in \mathbb{Z}$ and $a \in \mathbb{C}$ such that $|a| < 1$;
(e) $x(n) = a^{n-m}u(n)$, where $m \in \mathbb{Z}$ and $a \in \mathbb{C}$ such that $|a| < 1$; and
(f) $x(n) = a^n u(n-m)$, where $m \in \mathbb{Z}$ and $a \in \mathbb{C}$ such that $|a| < 1$.

11.4 For each sequence y given below, find the Fourier transform Y of y in terms of the Fourier transform X of the sequence x.
(a) $y(n) = nx(n-3)$;
(b) $y(n) = n^2 x(n+1)$;
(c) $y(n) = x(1-n) + x(n-1)$;
(d) $y(n) = \frac{1}{2}[x^*(n) + x(n)]$;
(e) $y(n) = (n+1)x(n+1)$;

(f) $y(n) = x^*(-n)$;
(g) $y(n) = x(n+m) + x(n-m)$, where $m \in \mathbb{Z}$; and
(h) $y(n) = x(n+m) - x(n-m)$, where $m \in \mathbb{Z}$.

11.5 Using the Fourier transform synthesis equation, find the inverse Fourier transform x of each function X given below.

(a) $X(\Omega) = \begin{cases} 1 & |\Omega| \leq B \\ 0 & B < |\Omega| \leq \pi, \end{cases}$ where B is a positive real constant;

(b) $X(\Omega) = \begin{cases} -2j & \Omega \in (-\pi, 0] \\ 2j & \Omega \in (0, \pi]; \end{cases}$ and

(c) $X(\Omega) = \cos(k\Omega)$ for $\Omega \in [-\pi, \pi]$, where k is an integer constant.

11.6 Using a table of Fourier transform pairs and properties of the Fourier transform, find the inverse Fourier transform x of each function X given below.

(a) $X(\Omega) = e^{-j10\Omega}\left[\dfrac{e^{j\Omega}}{e^{j\Omega} - \frac{1}{3}}\right]$;

(b) $X(\Omega) = \dfrac{3}{5 - 4\cos\left(\Omega - \frac{\pi}{3}\right)}$;

(c) $X(\Omega) = \dfrac{e^{j2\Omega}}{\left(e^{j\Omega} - \frac{1}{3}\right)^2}$;

(d) $X(\Omega) = 4\cos^2(\Omega) + 4\sin^2(3\Omega)$ [Hint: Consider the form of X.]; and

(e) $X(\Omega) = \dfrac{-\frac{5}{6}e^{-j\Omega} + 5}{1 + \frac{1}{6}e^{-j\Omega} - \frac{1}{6}e^{-j2\Omega}}$ [Hint: Use a partial-fraction expansion.].

11.7 Using properties of the Fourier transform and the Fourier transform pair

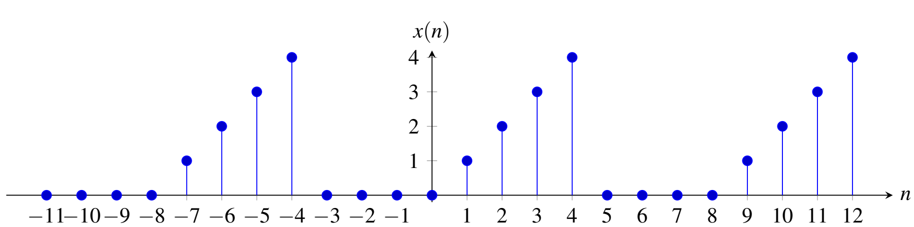
$$a^n u(n) \stackrel{\text{DTFT}}{\longleftrightarrow} \frac{1}{1 - ae^{-j\Omega}},$$

show by induction that the inverse Fourier transform of the function X_k is the sequence x_k, where

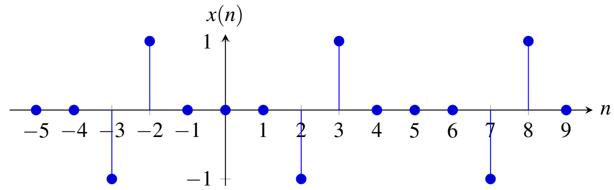
$$X_k(\Omega) = \frac{1}{(1 - ae^{-j\Omega})^k}, \quad x_k(n) = \frac{(n+k-1)!}{n!(k-1)!}a^n u(n),$$

and k is a (strictly) positive integer. [Hint: $x_{k+1}(n) = \frac{n+k}{k}x_k(n)$.]

11.8 Find the Fourier transform X of each periodic sequence x shown below. The number of samples appearing in each plot is an integer multiple of the period.



(a)



(b)

11.9 Determine whether each function X given below is a valid (DT) Fourier transform.
(a) $X(\Omega) = \pi + j$;
(b) $X(\Omega) = \frac{1}{2}\Omega$;
(c) $X(\Omega) = \cos(5\Omega)$;
(d) $X(\Omega) = \cos\left(\frac{1}{5}\Omega\right)$;
(e) $X(\Omega) = \dfrac{\sin(4\Omega)}{\sin\left(\frac{1}{2}\Omega\right)}$; and
(f) $X(\Omega) = \delta(\Omega)$.

11.10 Let x be a sequence with the Fourier transform X. Show that:
(a) if x is even, $X(\Omega) = x(0) + 2\sum_{n=1}^{\infty} x(n)\cos(n\Omega)$; and
(b) if x is odd, $X(\Omega) = -2j\sum_{k=1}^{\infty} x(n)\sin(n\Omega)$.

11.11 Let x be a sequence with even and odd parts x_e and x_o, respectively. Let X, X_e, and X_o denote the Fourier transforms of x, x_e, and x_o, respectively. Show that, for real x, the following relationships hold:
(a) $X_e(\Omega) = \mathrm{Re}[X(\Omega)]$; and
(b) $X_o(\Omega) = j\,\mathrm{Im}[X(\Omega)]$.

11.12 For each pair of sequences x_1 and x_2 given below, use the convolution property of the Fourier transform to compute $x = x_1 * x_2$.
(a) $x_1(n) = \frac{1}{4}\operatorname{sinc}\left(\frac{\pi}{4}n\right)$ and $x_2(n) = \frac{1}{7}\operatorname{sinc}\left(\frac{\pi}{7}n\right)$; and
(b) $x_1(n) = \frac{1}{5}\operatorname{sinc}\left(\frac{\pi}{5}n\right)$ and $x_2(n) = \cos\left(\frac{\pi}{2}n\right)$.

11.13 Compute the energy contained in each sequence x given below.
(a) $x(n) = \operatorname{sinc}\left(\frac{\pi}{11}n\right)$; and
(b) $x(n) = \frac{2}{5}\operatorname{sinc}\left(\frac{\pi}{5}n\right)\cos\left(\frac{\pi}{2}n\right)$.

11.14 Show that, for a real sequence x, the (DT) Fourier-transform synthesis equation can be expressed as

$$x(n) = \frac{1}{\pi}\int_0^{\pi} |X(\Omega)|\cos(\Omega n + \arg[X(\Omega)])d\Omega.$$

11.15 For each sequence x given below, compute the frequency spectrum X of x, and find and plot the corresponding magnitude and phase spectra.
(a) $x(n) = u(n-2) - u(n-7)$;
(b) $x(n) = \frac{1}{2}\delta(n) + \frac{1}{2}\delta(n-1)$;
(c) $x(n) = \frac{1}{2}\delta(n) - \frac{1}{2}\delta(n-1)$; and
(d) $x(n) = (-1)^n[u(n+10) + u(n-11)]$.

11.16 For each difference equation below that defines a LTI system with input x and output y, find the frequency response H of the system.
(a) $y(n) - \frac{1}{2}y(n-1) = x(n)$;
(b) $y(n) - y(n-1) - y(n-2) = x(n-1)$;
(c) $10y(n) + 3y(n-1) - y(n-2) = x(n)$;
(d) $y(n+2) - \frac{1}{4}y(n+1) - \frac{1}{4}y(n) + \frac{1}{16}y(n-1) = x(n+2) - x(n+1)$; and
(e) $y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$.

11.17 For each frequency response H given below for a LTI system with input x and output y, find the difference equation that characterizes the system.
(a) $H(\Omega) = \dfrac{e^{j\Omega} + \frac{8}{25}}{e^{j2\Omega} + e^{j\Omega} + \frac{4}{25}}$;
(b) $H(\Omega) = \dfrac{e^{j\Omega}}{e^{j\Omega} - \frac{1}{2}}$;
(c) $H(\Omega) = -e^{j3\Omega} + 3e^{j2\Omega} + 3e^{j\Omega} - 1$; and
(d) $H(\Omega) = \dfrac{e^{j2\Omega}}{e^{j2\Omega} - \frac{1}{2}e^{j\Omega} + \frac{1}{4}}$.

11.18 Consider a LTI system with the input x, output y, and frequency response H that is characterized by the difference equation

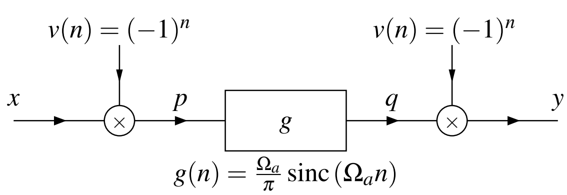
$$y(n) - ay(n-1) = bx(n) + x(n-1),$$

where a and b are real constants and $|a| < 1$. Suppose now that the system is allpass (i.e., $|H(\Omega)| = 1$ for all $\Omega \in \mathbb{R}$). Find an expression for b in terms of a.

11.19 Let h be the impulse response of a LTI system with input x and output y. For each case below, find y.
(a) $h(n) = \left(\frac{1}{2}\right)^n u(n)$ and $x(n) = (-1)^n$;
(b) $h(n) = \frac{1}{2}\operatorname{sinc}\left(\frac{\pi}{2}n\right)$ and $x(n) = \frac{1}{2} + \frac{1}{5}\sin\left(\frac{\pi}{3}n\right) + \frac{1}{6}\cos(\pi n)$; and
(c) $h(n) = \frac{1}{3}\operatorname{sinc}\left(\frac{\pi}{3}n\right)$ and $x(n) = \sum_{k=-\infty}^{\infty}\delta(n-8k)$.

11.20 Let H be the frequency response of a LTI system with input x and output y. For each case below, find y.
(a) $H(\Omega) = e^{-j2\Omega}\operatorname{rect}\left(\frac{2}{\pi}\Omega\right)$ for $\Omega \in (-\pi, \pi]$ and $x(n) = \operatorname{sinc}\left(\frac{\pi}{2}n\right)$.

11.21 Consider the system $\mathcal{H}$ with input x and output y as shown in the figure below. The block labelled by g is a LTI system with impulse response g, where $\Omega_a \in (0, \pi)$. (Note that, the subsystem corresponding to each multiplier is not TI.) Let X, Y, V, P, Q, and G denote the Fourier transforms of x, y, v, p, q, and g, respectively. Find an expression for Y in terms of X. Determine if $\mathcal{H}$ is LTI, and if it is, find its frequency response H.



11.22 Using the Fourier transform pair

$$a^{|n|} \overset{\text{DTFT}}{\longleftrightarrow} \frac{1-a^2}{1-2a\cos\Omega+a^2} \quad \text{for } |a|<1$$

and the duality relationship between the DT Fourier transform and CT Fourier series, find the (CT) Fourier series coefficients of the 1-periodic function

$$x(t) = \frac{1}{5-4\cos(2\pi t)}.$$

11.20.2 Exercises With Answer Key

11.23 Using a table of Fourier transform pairs and and properties of the Fourier transform, find the Fourier transform X of each sequence x given below.
(a) $x(n) = \cos\left(\frac{\pi}{4}n - \frac{\pi}{12}\right)$; and
(b) $x(n) = u(n+10) - u(n+2)$.

Short Answer. (a) $X(\Omega) = \frac{1}{2}e^{-j\pi/12}\delta\left(\Omega-\frac{\pi}{4}\right) + \frac{1}{2}e^{j\pi/12}\delta\left(\Omega+\frac{\pi}{4}\right)$; (b) $X(\Omega) = e^{j(13/2)\Omega}\frac{\sin(2\Omega)}{\sin(\Omega/2)}$

11.21 MATLAB Exercises

Currently, there are no MATLAB exercises.

Chapter 12

z Transform

12.1 Introduction

In this chapter, we introduce an important mathematical tool in the study of discrete-time signals and systems known as the z transform. The z transform can (more or less) be viewed as a generalization of the classical discrete-time Fourier transform. Due to its more general nature, the z transform has a number of advantages over the classical Fourier transform. First, the z transform representation exists for some sequences that do not have Fourier transform representations. So, we can handle some sequences with the z transform that cannot be handled with the Fourier transform. Second, since the z transform is (in some sense) a more general tool, it can provide additional insights beyond those facilitated by the Fourier transform.

12.2 Motivation Behind the z Transform

In Section 9.10, we showed that complex exponentials are eigensequences of discrete-time LTI systems. Suppose now that we have a discrete-time LTI system with impulse response h. Due to the eigensequence property, the response y of the system to the complex exponential input $x(n) = z^n$ (where z is a complex constant) is

$$y(n) = H(z)z^n,$$

where

$$H(z) = \sum_{k=-\infty}^{\infty} h(n)z^{-n}. \tag{12.1}$$

Previously, we referred to H as the system function. In this chapter, we will learn that H is, in fact, the z transform of h. That is, the summation in (12.1) is simply the definition of the z transform. In the case that $z = e^{j\Omega}$ where Ω is real (i.e., z is a point on the unit circle), (12.1) becomes the discrete-time Fourier transform summation (studied in Chapter 11). Since (12.1) includes the Fourier transform as a special case, the z transform can be viewed as a generalization of the (classical) Fourier transform.

12.3 Definition of z Transform

The (bilateral) **z transform** of the sequence x is denoted as $\mathcal{Z}x$ or X and is defined as

$$\mathcal{Z}x(z) = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}. \tag{12.2}$$

Similarly, the **inverse z transform** of X is denoted $\mathcal{Z}^{-1}X$ or x and is given by

$$\mathcal{Z}^{-1}X(n) = x(n) = \tfrac{1}{2\pi j}\oint_{\Gamma} X(z)z^{n-1}dz, \tag{12.3}$$

where Γ is a counterclockwise closed circular contour centered at the origin and with radius r. We refer to x and X as a z transform pair and denote this relationship as

$$x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z).$$

As is evident from (12.3), the computation of the inverse z transform requires a contour integration. In particular, we must integrate along a circular contour in the complex plane. Often, such a contour integration can be tedious to compute. Consequently, in practice, we do not usually compute the inverse z transform using (12.3) directly. Instead, we resort to other means (to be discussed later).

Two different versions of the z transform are commonly used. The first is the bilateral version, as introduced above. The second is the unilateral version. The unilateral z transform is most often used to solve difference equations with nonzero initial conditions. As it turns out, the only difference between the definitions of the unilateral and bilateral z transforms is in the lower limit of summation. In the bilateral case, the lower limit is $-\infty$, while in the unilateral case, the lower limit is zero. In the remainder of this chapter, we will focus our attention primarily on the the bilateral z transform. We will, however, briefly introduce the unilateral z transform as a tool for solving difference equations. Unless otherwise noted, all subsequent references to the z transform should be understood to mean the bilateral z transform.

12.4 Remarks on Notation Involving the z Transform

The z transform operator $\mathcal{Z}$ and inverse z transform operator $\mathcal{Z}^{-1}$ map a sequence to a function and a function to a sequence, respectively. Consequently, the operand for each of these operators must be a function/sequence (not a number). Consider the unnamed sequence that maps n to $e^{-|n/10|}$ as shown in Figure 12.1. Suppose that we would like to write an expression that denotes the z transform of this sequence. At first, we might be inclined to write "$\mathcal{Z}\{e^{-|n/10|}\}$". Strictly speaking, however, this notation is not correct, since the z transform operator requires a sequence as an operand and "$e^{-|n/10|}$" (strictly speaking) denotes a number (i.e., the value of the sequence in the figure evaluated at n). Essentially, the cause of our problems here is that the sequence in question does not have a name (such as "x") by which it can be referred. To resolve this problem, we could define a sequence x using the equation $x(n) = e^{-|n/10|}$ and then write the z transform as "$\mathcal{Z}x$". Unfortunately, introducing a new sequence name just for the sake of strictly correct notation is often undesirable as it frequently leads to highly verbose writing.

One way to avoid overly verbose writing when referring to sequences without names is offered by dot notation, introduced earlier in Section 2.1. Again, consider the sequence from Figure 12.1 that maps n to $e^{-|n/10|}$. Using strictly correct notation, we could write the z transform of this sequence as "$\mathcal{Z}\left\{e^{-|\cdot/10|}\right\}$". In other words, we can indicate that an expression refers to a sequence (as opposed to the value of sequence) by using the interpunct symbol (as discussed in Section 2.1). Some examples of the use of dot notation can be found below in Example 12.1. Dot notation is often extremely beneficial when one wants to employ precise (i.e., strictly correct) notation without being overly verbose.

Example 12.1 (Dot notation)**.** Several examples of the use of dot notation are as follows:

1. To denote the z transform of the sequence x defined by the equation $x(n) = n^2e^{-3n}u(n)$ (without the need to introduce the named sequence x), we can write: $\mathcal{Z}\left\{(\cdot)^2e^{-3(\cdot)}u(\cdot)\right\}$.

2. To denote the z transform of the sequence x defined by the equation $x(n) = n^2e^{-3n}u(n)$ evaluated at $3z$ (without the need to introduce the named sequence x), we can write: $\mathcal{Z}\left\{(\cdot)^2e^{-3(\cdot)}u(\cdot)\right\}(3z)$.

3. To denote the inverse z transform of the function X defined by the equation $X(z) = z^{-1}$ (without the need to introduce the named function X), we can write: $\mathcal{Z}^{-1}\left\{(\cdot)^{-1}\right\}$.

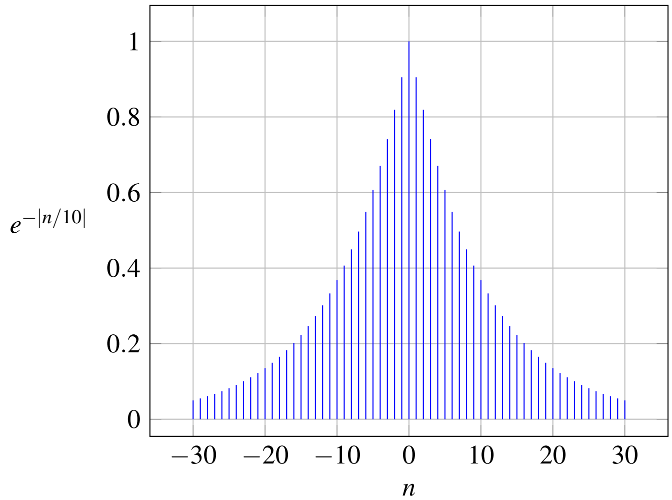


Figure 12.1: A plot of $e^{-|n/10|}$ versus n.

4. To denote the inverse z transform of the function X defined by the equation $X(z) = z^{-1}$ evaluated at $n-1$ (without the need to introduce the named sequence X), we can write: $\mathcal{Z}^{-1}\left\{(\cdot)^{-1}\right\}(n-1)$. ■

If the reader is comfortable with dot notation, the author would encourage the reader to use it when appropriate. Since some readers may find the dot notation to be confusing, however, this book (for the most part) attempts to minimize the use of dot notation. Instead, as a compromise solution, this book adopts the following notational conventions in order to achieve conciseness and a reasonable level of clarity without the need to use dot notation pervasively:

- unless indicated otherwise, in an expression for the operand of the z transform operator $\mathcal{Z}$, the variable "n" is assumed to be the independent variable for the sequence to which the z transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "n" were a "$\cdot$");

- unless indicated otherwise, in an expression for the operand of the inverse z transform operator $\mathcal{Z}^{-1}$, the variable "z" is assumed to be the independent variable for the function to which the inverse z transform is being applied (i.e., in terms of dot notation, the expression is treated as if each "z" were a "$\cdot$").

Some examples of using these book-sanctioned notational conventions can be found below in Example 12.2. Admittedly, these book-sanctioned conventions are not ideal, as they abuse mathematical notation somewhat, but they seem to be the best compromise in order to accommodate those who may prefer not to use dot notation.

Example 12.2 (Book-sanctioned notation)**.** Several examples of using the notational conventions that are employed throughout most of this book (as described above) are as follows:

1. To denote the z transform of the sequence x defined by the equation $x(n) = n^2 e^{-3n} u(n)$ (without the need to introduce the named function x), we can write: $\mathcal{Z}\left\{n^2 e^{-3n} u(n)\right\}$.

2. To denote the z transform of the sequence x defined by the equation $x(n) = n^2 e^{-3n} u(n)$ evaluated at $3z$ (without the need to introduce the named function x), we can write: $\mathcal{Z}\left\{n^2 e^{-3n} u(n)\right\}(3z)$.

3. To denote the inverse z transform of the function X defined by the equation $X(z) = 1 + z^{-1} + z^{-2}$ (without the need to introduce the named function X), we can write: $\mathcal{Z}^{-1}\left\{1 + z^{-1} + z^{-2}\right\}$.

4. To denote the inverse z transform of the function X defined by the equation $X(z) = 1 + z^{-1} + z^{-2}$ evaluated at $n-1$ (without the need to introduce the named function X), we can write: $\mathcal{Z}^{-1}\left\{1 + z^{-1} + z^{-2}\right\}(n-1)$. ■

Since applying the z transform operator to a sequence yields a function, we can evaluate this function at some value. Similarly, since applying the inverse z transform operator to a function yields a sequence, we can evaluate this sequence at some value. Again, consider the sequence from Figure 12.1 that maps n to $e^{-|n/10|}$. To denote the value of the z transform of this sequence evaluated at $3z$, we would write "$\mathcal{Z}\{e^{-|\cdot/10|}\}(3z)$" using dot notation or "$\mathcal{Z}\{e^{-|n/10|}\}(3z)$" using the book-sanctioned notational conventions described above.

12.5 Relationship Between z Transform and Discrete-Time Fourier Transform

In Section 12.3 of this chapter, we introduced the z transform, and in Chapter 11, we studied the (DT) Fourier transform. As it turns out, the z transform and (DT) Fourier transform are very closely related. Recall the definition of the z transform in (12.2). Consider now the special case of (12.2) where $z = e^{j\Omega}$ and Ω is real (i.e., z is on the unit circle). In this case, (12.2) becomes

$$\begin{aligned} X(e^{j\Omega}) &= \left.\left[\sum_{n=-\infty}^{\infty} x(n)z^{-n}\right]\right|_{z=e^{j\Omega}} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} \\ &= \mathcal{F}x(\Omega) \end{aligned}$$

Thus, the Fourier transform is simply the z transform evaluated at $z = e^{j\Omega}$, assuming that this quantity is well defined (i.e., converges). In other words,

$$X(e^{j\Omega}) = \mathcal{F}x(\Omega).$$

Incidentally, it is due to the preceding relationship that the Fourier transform of x is sometimes written as $X(e^{j\Omega})$. When this notation is used, the function X actually corresponds to the z transform of x rather than its Fourier transform (i.e., the expression $X(e^{j\Omega})$ corresponds to the z transform evaluated at points on the unit circle).

Now, consider the general case of an arbitrary complex value for z in (12.2). Let us express z in polar form as $z = re^{j\Omega}$, where $r = |z|$ and $\Omega = \arg z$. Substituting $z = re^{j\Omega}$ into (12.2), we obtain

$$\begin{aligned} X(re^{j\Omega}) &= \sum_{n=-\infty}^{\infty} x(n)(re^{j\Omega})^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} \left[r^{-n}x(n)\right]e^{-j\Omega n} \\ &= \mathcal{F}\{r^{-n}x(n)\}(\Omega). \end{aligned}$$

Thus, we have shown

$$X(re^{j\Omega}) = \mathcal{F}\{r^{-n}x(n)\}(\Omega). \tag{12.4}$$

For this reason, the z transform of x can be viewed as the (DT) Fourier transform of $x'(n) = r^{-n}x(n)$ (i.e., x weighted by a real exponential sequence). As a consequence of multiplying by the real exponential r^{-n}, the z transform of a sequence may exist when the Fourier transform of the same sequence does not.

By using the above relationship, we can derive the formula for the inverse z transform given in (12.3). Let X denote the z transform of x, and let $z = re^{j\Omega}$, where r and Ω are real and $r \geq 0$. From the relationship between the Fourier and z transforms in (12.4), we have

$$X(re^{j\Omega}) = \mathcal{F}\{r^{-n}x(n)\}(\Omega),$$

where r is chosen so that $X(z)$ converges for $z = re^{j\Omega}$. Taking the inverse Fourier transform of both sides of the preceding equation, we obtain

$$r^{-n}x(n) = \mathcal{F}^{-1}\{X(re^{j\Omega})\}(n).$$

Multiplying both sides of this equation by r^n, we have

$$x(n) = r^n\mathcal{F}^{-1}\{X(re^{j\Omega})\}(n).$$

Using the formula for the inverse Fourier transform, we can write

$$x(n) = r^n \tfrac{1}{2\pi}\int_{2\pi} X(re^{j\Omega})e^{j\Omega n}d\Omega.$$

Moving r^n inside integral yields

$$x(n) = \tfrac{1}{2\pi}\int_{2\pi} X(re^{j\Omega})(re^{j\Omega})^n d\Omega.$$

Now, we employ a change of variable. Let $z = re^{j\Omega}$ so that $dz = jre^{j\Omega}d\Omega = jzd\Omega$ and $d\Omega = \frac{1}{j}z^{-1}dz$. Applying the change of variable, we obtain

$$x(n) = \tfrac{1}{2\pi}\oint X(z)z^n\left(\tfrac{1}{j}\right)z^{-1}dz.$$

As Ω goes from 0 to 2π, z traces in a counter-clockwise direction a closed circular contour centered at the origin and with radius r. Thus, we have

$$x(n) = \tfrac{1}{2\pi j}\oint_{\Gamma} X(z)z^{n-1}dz,$$

where Γ denotes a counter-clockwise closed circular contour centered at the origin with radius r. In other words, we have shown that (12.3) holds.

12.6 z Transform Examples

In this section, we calculate the z transform of several relatively simple functions. In the process, we gain some important insights into the z transform.

Example 12.3. Find the z transform X of the sequence

$$x(n) = a^n u(n),$$

where a is a real constant.

Solution. From the definition of the z transform, we can write

$$\begin{aligned}X(z) &= \sum_{n=-\infty}^{\infty} a^n u(n)z^{-n}\\ &= \sum_{n=0}^{\infty} a^n z^{-n}\\ &= \sum_{n=0}^{\infty} (az^{-1})^n.\end{aligned}$$

In order to simplify the preceding summation, we recall the formula for the sum of an infinite geometric sequence (given by (F.3)). The summation converges for $\left|az^{-1}\right| < 1$. Rearranging this inequality, we obtain

$$\begin{aligned} & \left|az^{-1}\right| < 1 \\ \Rightarrow \quad & \left|a^{-1}z\right| > 1 \\ \Rightarrow \quad & \left|a^{-1}\right| |z| > 1 \\ \Rightarrow \quad & |z| > |a|. \end{aligned}$$

From the formula for the sum of an infinite geometric sequence, we can write

$$\begin{aligned} X(z) &= \frac{1}{1-az^{-1}} \\ &= \frac{z}{z-a} \quad \text{for } |z| > |a|. \end{aligned}$$

Thus, we have

$$a^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{1}{1-az^{-1}} \quad \text{for } |z| > |a|.$$

■

Example 12.4. Find the z transform X of the sequence

$$x(n) = -a^n u(-n-1),$$

where a is a real constant.

Solution. From the definition of the z transform, we can write

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n} \\ &= -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n} \\ &= -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} a^{-n} z^n \\ &= -\sum_{n=1}^{\infty} (a^{-1}z)^n. \end{aligned}$$

Recalling the formula for the sum of an infinite geometric sequence (as given by (F.3)), we deduce that $X(z)$ only converges for $\left|a^{-1}z\right| < 1$. Rearranging this inequality, we obtain

$$\begin{aligned} & \left|a^{-1}z\right| < 1 \\ \Rightarrow \quad & \left|az^{-1}\right| > 1 \\ \Rightarrow \quad & |a| \left|z^{-1}\right| > 1 \\ \Rightarrow \quad & |z| < |a|. \end{aligned}$$

From the formula for the sum of an infinite geometric sequence, we can write

$$\begin{aligned} X(z) &= -\frac{a^{-1}z}{1-a^{-1}z} \\ &= \frac{1}{1-az^{-1}} \\ &= \frac{z}{z-a} \quad \text{for } |z| < |a|. \end{aligned}$$

Thus, we have

$$-a^n u(-n-1) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{1}{1-az^{-1}} \quad \text{for } |z| < |a|. \qquad \blacksquare$$

At this point, we compare the results of Examples 12.3 and 12.4, and make an important observation. Notice that the same algebraic expression for X was obtained in both of these examples (i.e., $X(z) = \frac{z}{z-a}$). The only difference is in the convergence properties of X. In one case, $X(z)$ converges for $|z| > |a|$, while in the other it converges for $|z| < |a|$. As it turns out, one must specify both the algebraic expression for X and its region of convergence in order to uniquely determine $x = \mathcal{Z}^{-1}X$ from X.

Example 12.5. Find the z transform X of the sequence

$$x(n) = \delta(n-n_0),$$

where n_0 is an integer constant.

Solution. From the definition of the z transform, we can write

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \delta(n-n_0) z^{-n} \\ &= z^{-n_0}. \end{aligned}$$

If $n_0 > 0$, X has a pole at 0, and therefore the ROC of X is all nonzero (finite) complex numbers as well as ∞. If $n_0 < 0$, X has a pole at ∞, and therefore the ROC of X is all (finite) complex numbers. If $n_0 = 0$, X has no poles, and therefore the ROC of X is all (finite) complex numbers as well as ∞. Thus, the ROC is all nonzero (finite) complex numbers as well as possibly 0 and/or ∞. Thus, we have

$$\delta(n-n_0) \stackrel{\text{ZT}}{\longleftrightarrow} z^{-n_0} \quad \text{for all nonzero (finite) complex } z \text{ as well as possibly } z = 0 \text{ and/or } z = \infty. \qquad \blacksquare$$

Example 12.6. Find the z transform X of the sequence

$$x(n) = \tfrac{1}{n!} a^n u(n),$$

where a is a real constant.

Solution. From the definition of the z transform, we have

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \tfrac{1}{n!} a^n u(n) z^{-n} \\ &= \sum_{n=0}^{\infty} \tfrac{1}{n!} a^n z^{-n} \\ &= \sum_{n=0}^{\infty} \tfrac{1}{n!} (a/z)^n. \end{aligned}$$

Observing from (F.6) that $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ for all complex z, we can rewrite the above equation for X to obtain

$$X(z) = e^{a/z}.$$

(Note that a number of useful series, including the one used above for the exponential function, can be found in Section F.6.) Now, we consider the ROC of X. The only way that the expression for $X(z)$ can fail to converge is if $z = 0$, in which case division by zero occurs. Thus, the ROC of X is $|z| > 0$ (i.e., $z \neq 0$). So, in conclusion, we have

$$X(z) = e^{a/z} \quad \text{for } |z| > 0. \qquad \blacksquare$$

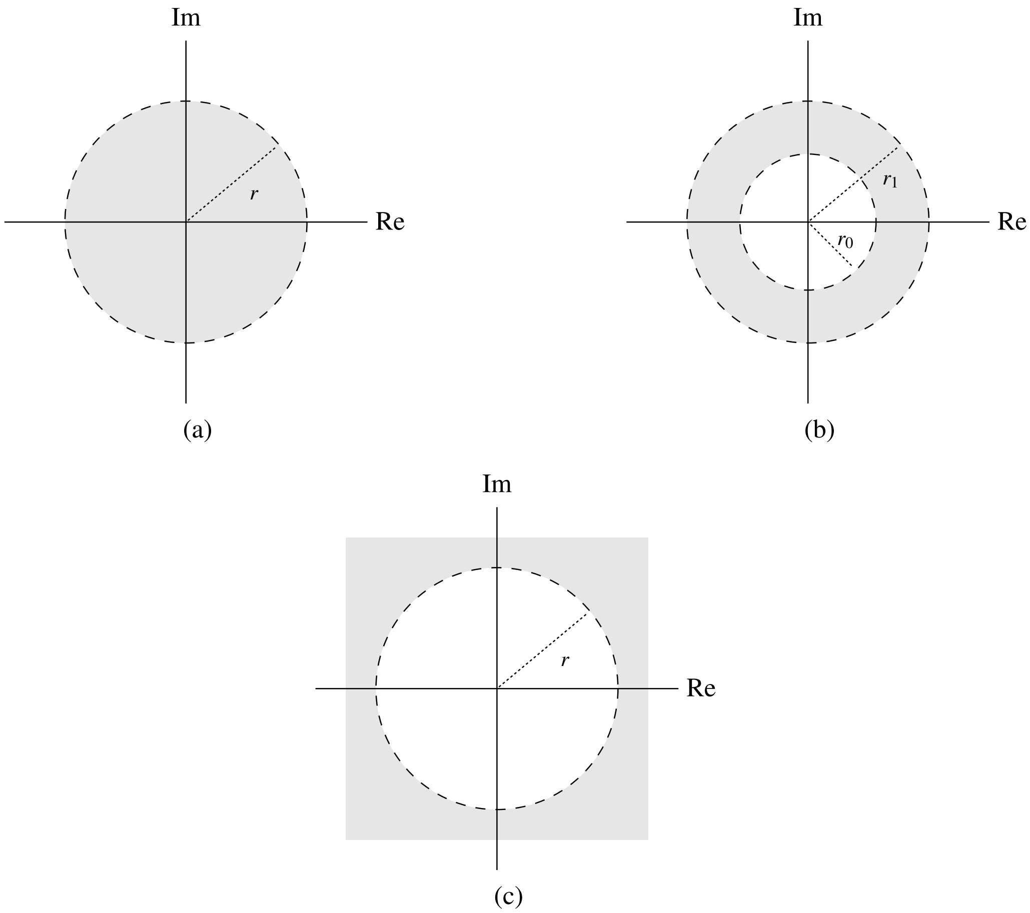


Figure 12.2: Examples of various types of sets. (a) A disk with center 0 and radius r; (b) an annulus with center 0, inner radius r_0, and outer radius r_1; and (c) an exterior of a circle with center 0 and radius r.

12.7 Region of Convergence for the z Transform

Before discussing the region of convergence (ROC) of the z transform in detail, we need to introduce some terminology involving sets in the complex plane. A **disk** (or more specifically an open disk) with center 0 and radius r is the set of all complex numbers z satisfying

$$|z| < r,$$

where r is a real constant and $r > 0$. An **annulus** with center 0, inner radius r_0, and outer radius r_1 is the set of all complex numbers z satisfying

$$r_0 < |z| < r_1,$$

where r_0 and r_1 are real constants and $0 < r_0 < r_1$. The **exterior of a circle** with center 0 and radius r is the set of all complex numbers z satisfying

$$|z| > r,$$

where r is a real constant and $r > 0$. Examples of a disk, an annulus, and an exterior of a circle are given in Figure 12.2.

Since the ROC is a set (of points in the complex plane), we often need to employ some basic set operations when dealing with ROCs. For two sets A and B, the **intersection** of A and B, denoted $A \cap B$, is the set of all points that are in both A and B. An illustrative example of set intersection is shown in Figure 12.3.

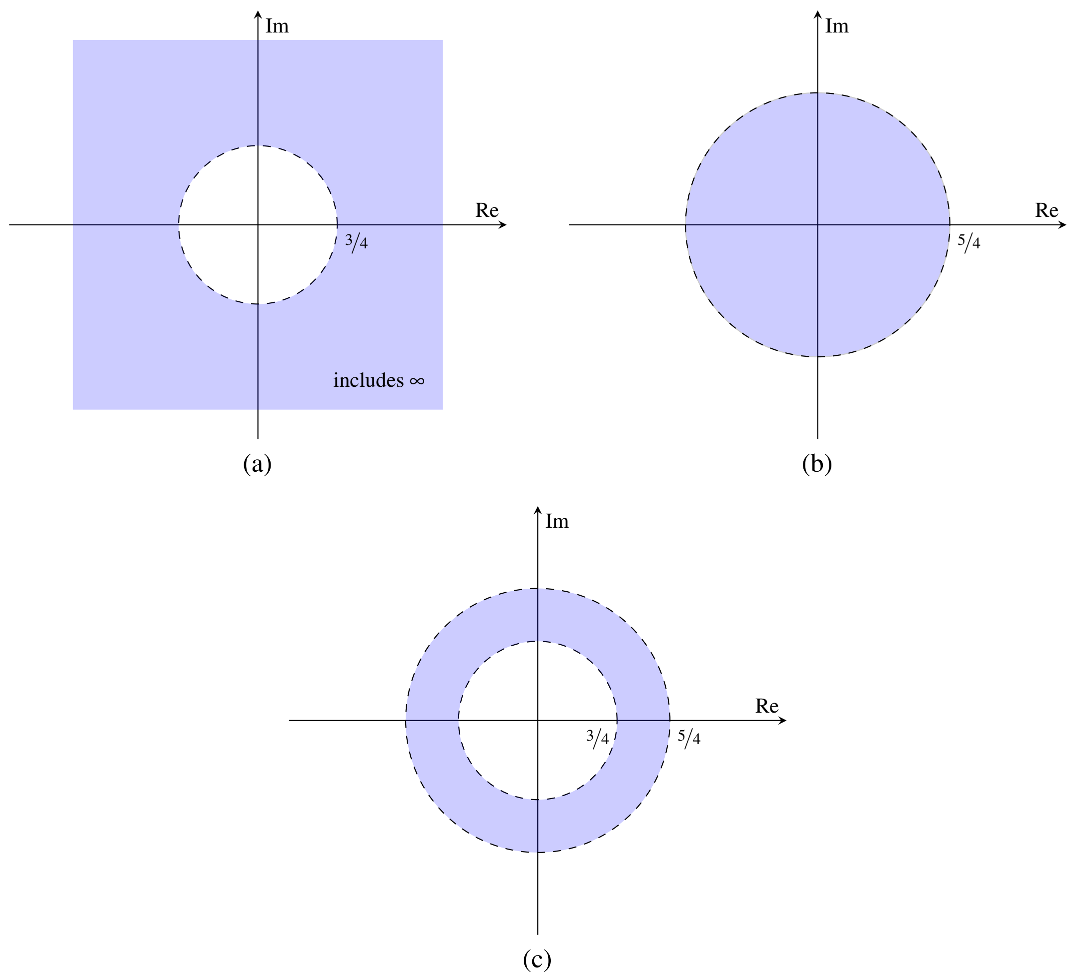


Figure 12.3: Example of set intersection. The sets (a) R_1 and (b) R_2; and (c) their intersection $R_1 \cap R_2$.

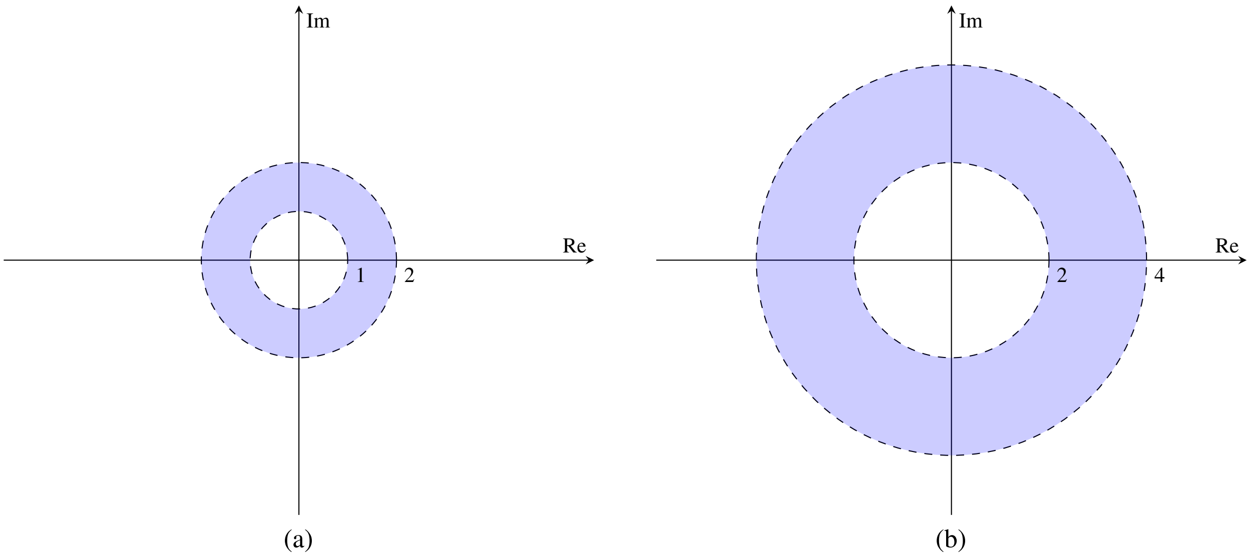


Figure 12.4: Example of multiplying a set by a scalar. (a) The set R. (b) The set $2R$

For a set S and a complex constant a, aS denotes the set given by

$$aS = \{az : z \in S\}.$$

That is, aS denotes the set formed by multiplying each element of S by a. In passing, we note that there is a simple geometric relationship between the sets S and aS. In particular, the region in the complex plane corresponding to the set aS is obtained by rotating the region corresponding to S about the origin by $\arg a$ and then scaling the result by a factor of $|a|$. (The order of the rotation and scaling can be interchanged, since these operations commute.) For example, suppose that R is the set of complex numbers z satisfying

$$1 < |z| < 2,$$

as shown in Figure 12.4(a). Then, $2R$ is the set of complex numbers z satisfying

$$2 < |z| < 4,$$

as shown in Figure 12.4(b). Furthermore, since R is an annulus centered at the origin, $R = aR$ for all complex a satisfying $|a| = 1$. That is, rotating the set R about the origin by any angle yields the same set R.

For a set S, S^{-1} denotes the set given by

$$S^{-1} = \{z^{-1} : z \in S\}.$$

That is, S^{-1} denotes the set formed by taking the reciprocal of each element of S. For example, suppose that R is the set of complex numbers z satisfying

$$|z| > \tfrac{3}{4},$$

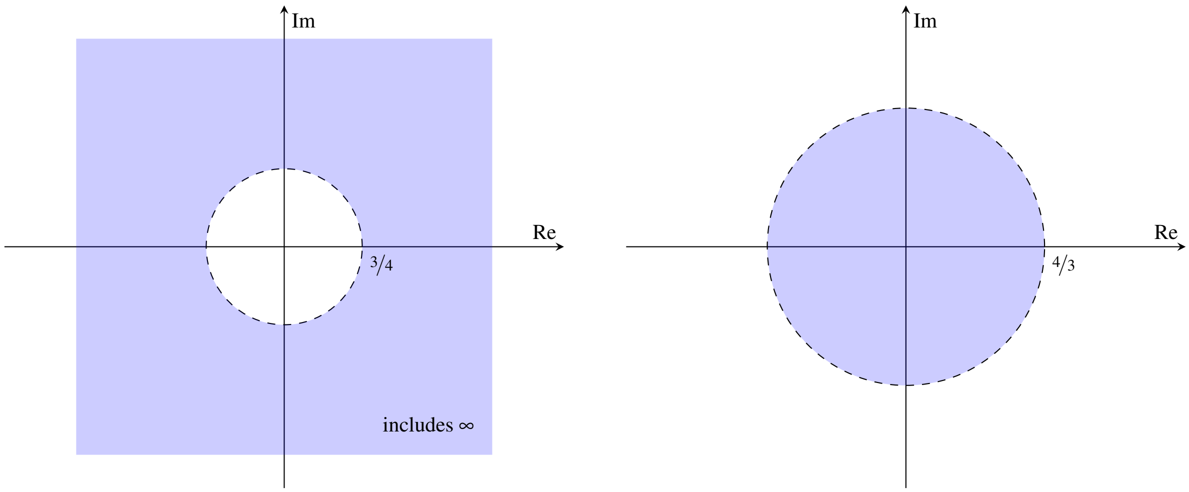
as shown in Figure 12.5(a). Then, R^{-1} is the set of complex numbers z satisfying

$$|z| < \tfrac{4}{3},$$

as shown in Figure 12.5(b).

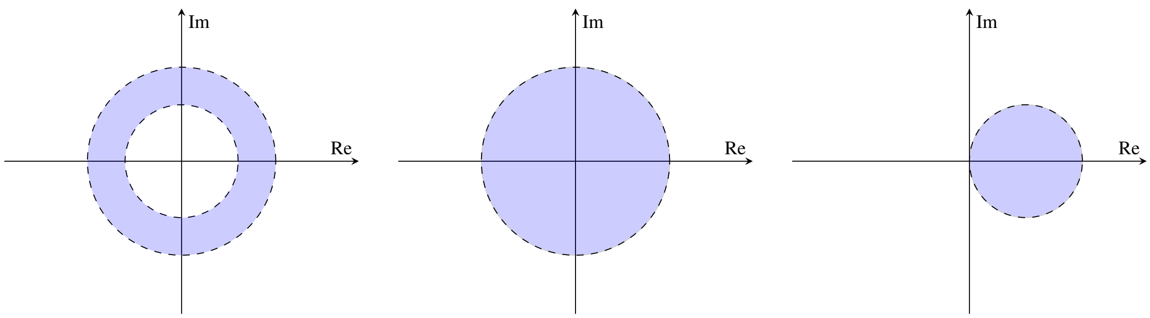
As we saw earlier, for a sequence x, the complete specification of its z transform X requires not only an algebraic expression for X, but also the ROC associated with X. Two distinct sequences can have the same algebraic expression for their z transform. In what follows, we examine some of the constraints on the ROC (of the z transform) for various classes of sequences.

One can show that the ROC of the z transform has the following properties:



(a) (b)

Figure 12.5: Example of the reciprocal of a set. (a) The set R; and its reciprocal R^{-1}.



Valid Valid Invalid

Figure 12.6: Examples of sets that would be either valid or invalid as the ROC of a z transform.

1. The ROC of the z transform X consists of concentric circles centered at the origin in the complex plane. That is, if a point z_0 is in the ROC, then the circle centered at the origin passing through z_0 (i.e., $|z| = |z_0|$) is also in the ROC.

   Justification: The z transform X of the sequence x is simply the (DT) Fourier transform of $x'(n) = |z|^{-n}x(n)$. Thus, X converges whenever this Fourier transform converges. Since the convergence of the Fourier transform only depends on $|z|$, the convergence of the z transform only depends on $|z|$.

   Some examples of sets that would be either valid or invalid as ROCs are shown in Figure 12.6.

2. If a z transform X is rational, the ROC of X does not contain any poles and is bounded by poles or extends to infinity.

   Partial justification: Since X is rational, its value becomes infinite at a pole. So obviously, X does not converge at a pole. Therefore, it follows that the ROC of X cannot contain a pole.

   Some examples of sets that would be either valid or invalid as ROCs of rational z transforms are shown in Figure 12.7.

3. If a sequence x is finite duration and its z transform X converges for at least one point, then X converges for all points in the complex plane, except possibly 0 and/or ∞ (i.e., the ROC is a disk which may exclude 0 or include

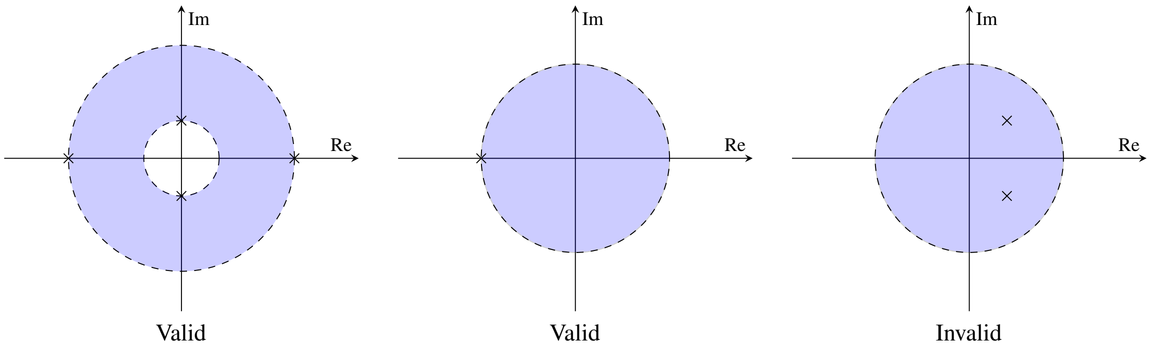


Figure 12.7: Examples of sets that would be either valid or invalid as the ROC of a rational z transform.

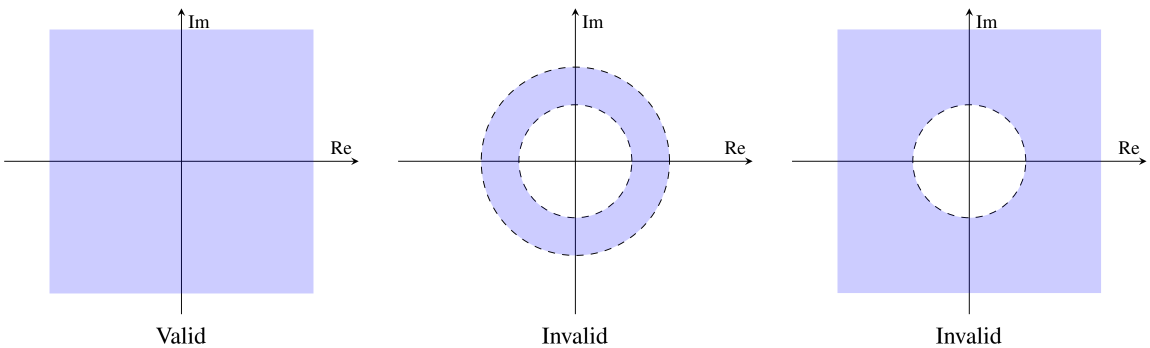


Figure 12.8: Examples of sets that would be either valid or invalid as the ROC of the z transform of a finite-duration sequence.

∞). The ROC includes 0 if $x(n) = 0$ for all $n > 0$ (i.e., $X(z)$ has no negative powers of z). The ROC includes ∞ if x is causal (i.e., $X(z)$ has no positive powers of z). Some examples of sets that would be either valid or invalid as ROCs for X, if x is finite duration, are shown in Figure 12.8.

4. If a sequence x is right sided and the circle $|z| = r_0$ is in the ROC of $X = \mathcal{Z}x$, then all (finite) values of z for which $|z| > r_0$ will also be in the ROC of X (i.e., the ROC contains the exterior of a circle centered at 0, possibly including ∞). The ROC includes ∞ if x is causal (i.e., $X(z)$ has no positive powers of z). Moreover, if x is right sided but not left sided, then the ROC of X is the exterior of a circle centered at 0, possibly including ∞. Examples of sets that would be either valid or invalid as ROCs for X, if x is right sided but not left sided, are shown in Figure 12.9.

5. If a sequence x is left sided and the circle $|z| = r_0$ is in the ROC of $X = \mathcal{Z}x$, then all values of z for which $0 < |z| < r_0$ will also be in the ROC of X (i.e., the ROC contains a disk centered at 0, possibly excluding 0). The ROC includes 0 if $x(n) = 0$ for all $n > 0$ (i.e., $X(z)$ has no negative powers of z). Moreover, if x is left sided but not right sided, the ROC is a disk centered at 0, possibly excluding 0. Examples of sets that would be either valid or invalid as ROCs for X, if x is left sided but not right sided, are shown in Figure 12.10.

6. If a sequence x is two sided and the circle $|z| = r_0$ is in the ROC of $X = \mathcal{Z}x$, then the ROC of X will consist of a ring in the complex plane that contains the circle $|z| = r_0$ (i.e., the ROC is an annulus centered at 0). Examples of sets that would be either valid or invalid as ROCs for X, if x is two sided, are shown in Figure 12.11.

7. If a sequence x has a rational z transform X (with at least one pole), then:

   (a) If x is right sided, the ROC of X is the region in the complex plane outside the circle of radius equal to the largest magnitude of the poles of X (i.e., the region outside the outermost pole), possibly including ∞.

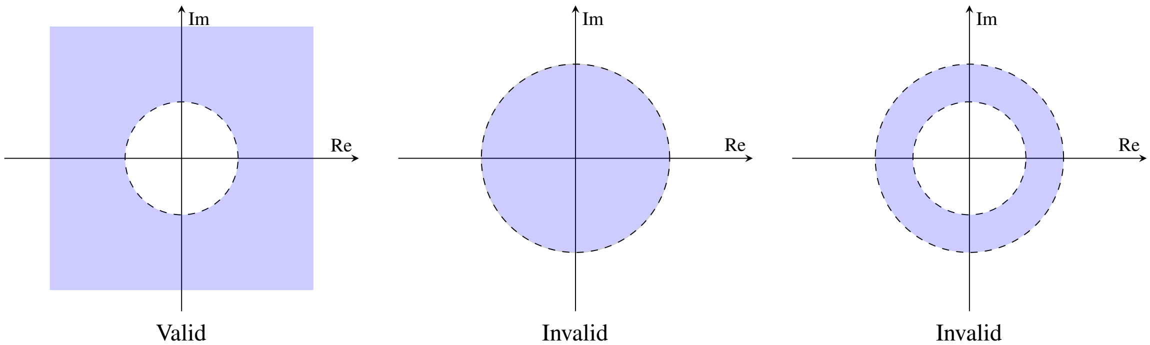


Figure 12.9: Examples of sets that would be either valid or invalid as the ROC of the z transform of a sequence that is right sided but not left sided.

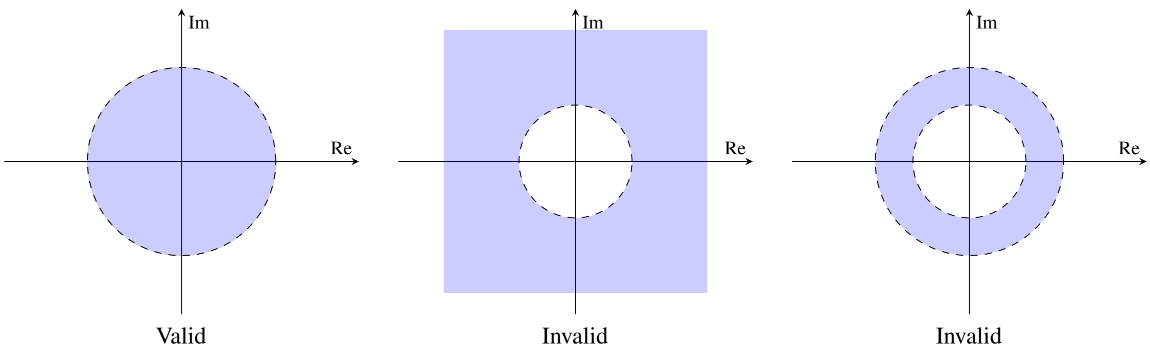


Figure 12.10: Examples of sets that would be either valid or invalid as the ROC of the z transform of a sequence that is left sided but not right sided.

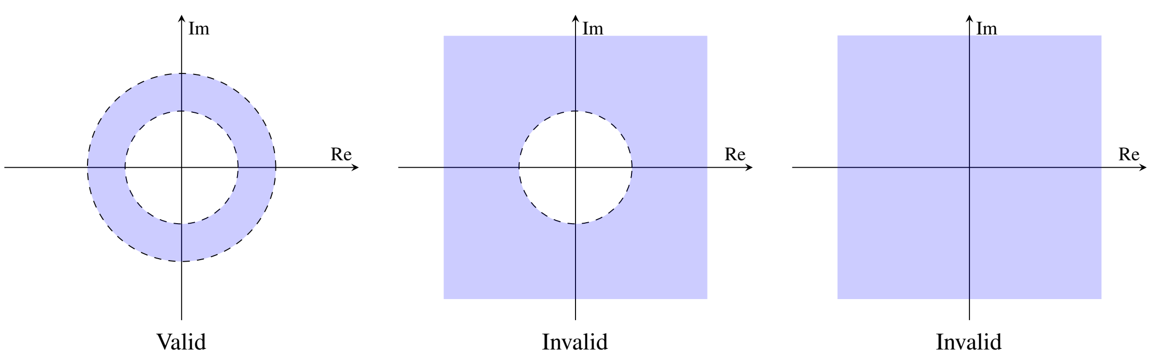


Figure 12.11: Examples of sets that would be either valid or invalid as the ROC of the z transform of a two-sided sequence.

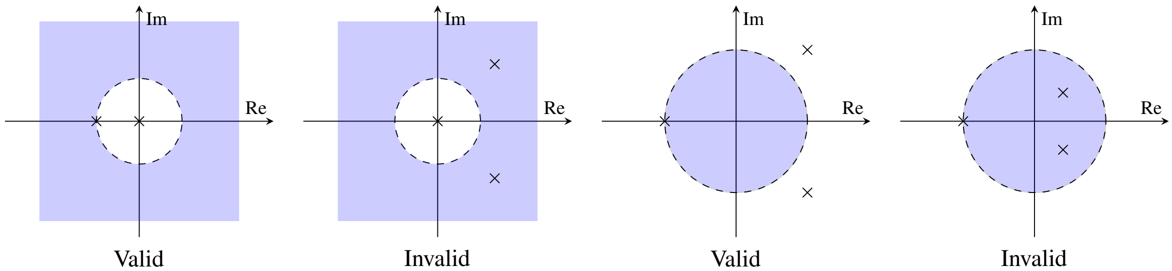


Figure 12.12: Examples of sets that would be either valid or invalid as the ROC of a rational z transform of a left/right-sided sequence.

Table 12.1: Relationship between the sidedness properties of x and the ROC of $X = \mathcal{Z}x$

x		
left sided	right sided	ROC of X
yes	yes	everywhere, except possibly 0 and/or ∞
no	yes	exterior of circle centered at origin, possibly including ∞
yes	no	disk centered at origin, possibly excluding 0
no	no	annulus centered at origin

(b) If x is left sided, the ROC of X is the region in the complex plane inside the circle of radius equal to the smallest magnitude of the nonzero poles of X and extending inward to, and possibly including, 0 (i.e., the region inside the innermost nonzero pole).

Some examples of sets that would be either valid or invalid as ROCs for X, if X is rational and x is left/right sided, are shown in Figure 12.12.

Note that some of the above properties are redundant. For example, properties 1, 2, and 4 imply property 7(a). Also, properties 1, 2, and 5 imply property 7(b). Moreover, since every sequence can be classified as exactly one of left sided but not right sided, right sided but not left sided, two sided (i.e, neither left nor right sided), or finite duration (i.e., both left and right sided), we can infer from properties 3, 4, 5, and 6 that the ROC can only be of the form of

- the entire complex plane, except possibly 0 and/or ∞;
- the exterior of a circle centered at the origin, possibly including ∞;
- a disk centered at the origin, possibly excluding 0;
- an annulus centered at the origin; or
- the empty set.

In particular, the ROC of X depends on the left- and right-sidedness of x as shown in Table 12.1. Thus, the ROC must be a connected set. (A set S is said to be connected, if for every two elements a and b in S, there exists a path from a to b that is contained in S.) For example, the sets shown in Figure 12.13 would not be valid as ROCs.

Example 12.7. The z transform X of the sequence x has the algebraic expression

$$X(z) = \frac{1}{(z^2-1)(z^2+4)}.$$

Identify all of the possible ROCs of X. For each ROC, indicate whether the corresponding sequence x is left sided but not right sided, right sided but not left sided, two sided, or finite duration.

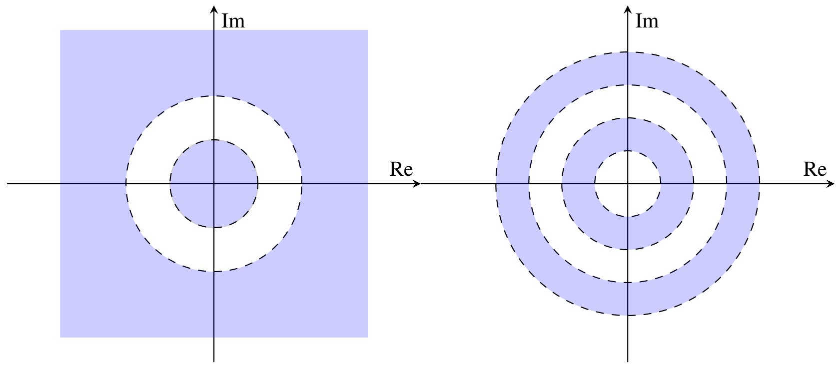


Figure 12.13: Examples of sets that would not be a valid ROC of a z transform.

Solution. The possible ROCs associated with X are determined by the poles of this function. So, we must find the poles of X. Factoring the denominator of X, we obtain

$$X(z) = \frac{1}{(z+1)(z-1)(z+j2)(z-j2)}.$$

Thus, X has poles at -1, 1, $-2j$, and $2j$. Since these poles only have two distinct magnitudes (namely, 1 and 2), there are three possible ROCs:

i) $|z| < 1$,
ii) $1 < |z| < 2$, and
iii) $|z| > 2$.

These ROCs are plotted in Figures 12.14(a), (b), and (c), respectively. The first ROC is a disk, so the corresponding x must be left sided but not right sided. The second ROC is an annulus, so the corresponding x must be two sided. The third ROC is the exterior of a circle, so the corresponding x must be right sided but not left sided. ■

12.8 Properties of the z Transform

The z transform has a number of important properties. In the sections that follow, we introduce several of these properties. For the convenience of the reader, the properties described in the subsequent sections are also summarized in Table 12.2 (on page 533).

12.8.1 Linearity

Arguably, the most important property of the z transform is linearity, as introduced below.

Theorem 12.1 (Linearity)**.** *If $x_1(n) \overset{\text{ZT}}{\longleftrightarrow} X_1(z)$ with ROC R_1 and $x_2(n) \overset{\text{ZT}}{\longleftrightarrow} X_2(z)$ with ROC R_2, then*

$$a_1x_1(n) + a_2x_2(n) \overset{\text{ZT}}{\longleftrightarrow} a_1X_1(z) + a_2X_2(z) \quad \textit{with ROC } R \textit{ containing } R_1 \cap R_2,$$

*where a_1 and a_2 are arbitrary complex constants. This is known as the **linearity property** of the z transform.*

Proof. Let $y(n) = a_1x_1(n) + a_2x_2(n)$, and let Y denote the z transform of y. Using the definition of the z transform

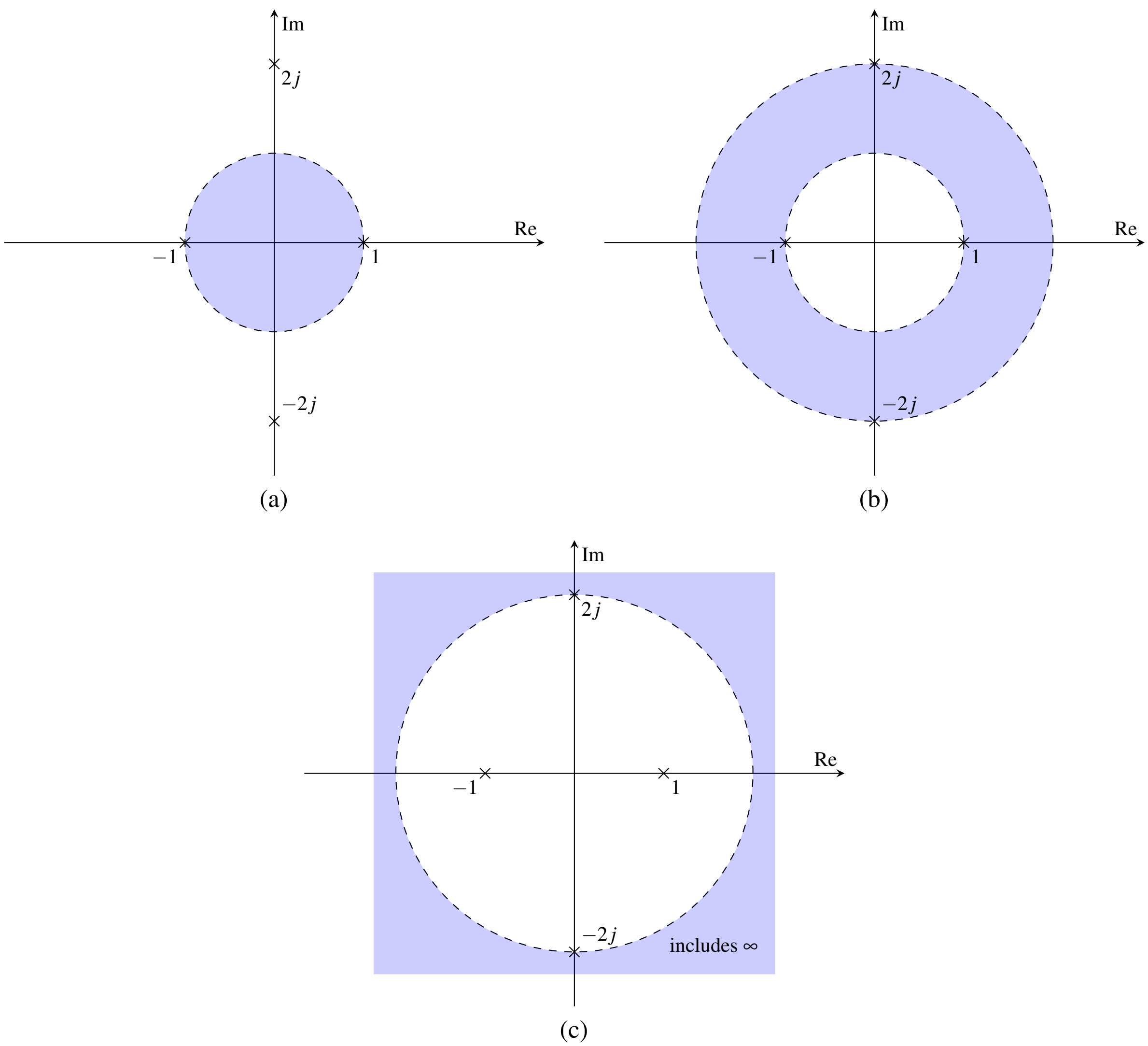


Figure 12.14: ROCs for example. The (a) first, (b) second, and (c) third possible ROCs for X.

and straightforward algebraic manipulation, we have

$$\begin{aligned}Y(z) &= \sum_{n=-\infty}^{\infty} [a_1x_1(n)+a_2x_2(n)]\,z^{-n} \\ &= \sum_{n=-\infty}^{\infty} a_1x_1(n)z^{-n} + \sum_{n=-\infty}^{\infty} a_2x_2(n)z^{-n} \\ &= a_1 \sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n)z^{-n} \\ &= a_1X_1(z)+a_2X_2(z).\end{aligned}$$

The ROC R can be deduced as follows. If X_1 and X_2 both converge at some point z, say $z = \lambda$, then any linear combination of these functions must also converge at $z = \lambda$. Therefore, the ROC R must contain the intersection of R_1 and R_2. Thus, we have shown that the linearity property holds. ■

In the preceding theorem, note that the ROC R can be larger than $R_1 \cap R_2$. When X_1 and X_2 are rational functions, this can only happen if pole-zero cancellation occurs in the expression $a_1X_1(z)+a_2X_2(z)$.

Example 12.8 (Linearity property without pole-zero cancellation)**.** Find the z transform X of the sequence

$$x(n) = a^{|n|},$$

where a is a complex constant satisfying $|a| < 1$.

Solution. To begin, we observe that x can be written as $x = x_1 + x_2$, where

$$x_1(n) = a^{-n}u(-n-1) \quad \text{and} \quad x_2(n) = a^n u(n).$$

From Table 12.3, we know that

$$-a^n u(-n-1) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-a} \text{ for } |z| < |a| \quad \text{and} \quad a^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-a} \text{ for } |z| > |a|.$$

Thus, we have

$$X_1(z) = -\frac{z}{z-a^{-1}} \text{ for } |z| < \left|a^{-1}\right| \quad \text{and} \quad X_2(z) = \frac{z}{z-a} \text{ for } |z| > |a|.$$

From the linearity property of the z transform, we have

$$\begin{aligned}X(z) &= X_1(z)+X_2(z) \\ &= -\frac{z}{z-a^{-1}} + \frac{z}{z-a} \\ &= \frac{z^2-a^{-1}z-(z^2-az)}{(z-a)(z-a^{-1})} \\ &= \frac{z^2-a^{-1}z-z^2+az}{(z-a)(z-a^{-1})} \\ &= \frac{(a-a^{-1})z}{(z-a)(z-a^{-1})}.\end{aligned}$$

Now, we must determine the ROC R of X. Let R_1 and R_2 denote the ROCs of X_1 and X_2, respectively. We know that R must contain $R_1 \cap R_2$, where

$$\begin{aligned}R_1 \cap R_2 &= \{|z| < \left|a^{-1}\right|\} \cap \{|z| > |a|\} \\ &= \{|a| < |z| < \left|a^{-1}\right|\}.\end{aligned}$$

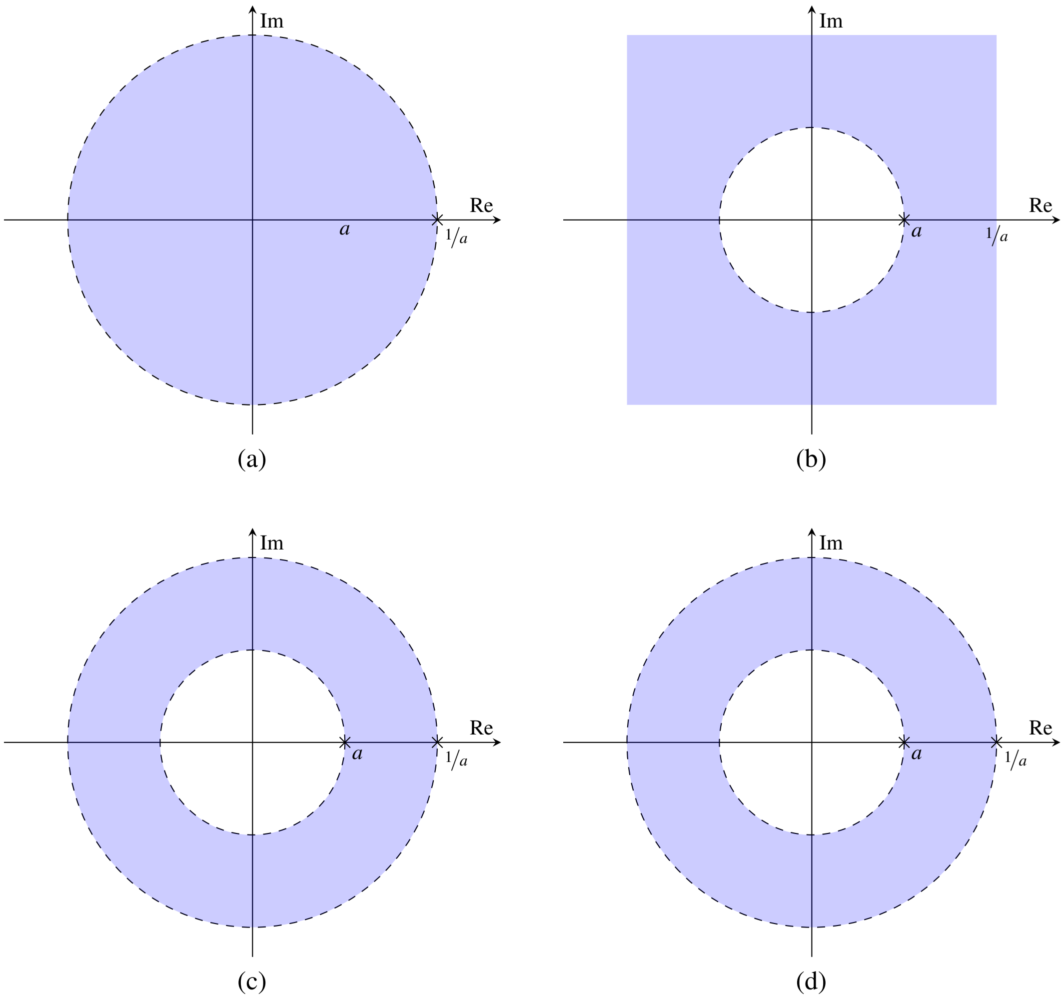


Figure 12.15: ROCs for the linearity example. The (a) ROC of X_1, (b) ROC of X_2, (c) ROC associated with the intersection of the ROCs of X_1 and X_2, and (d) ROC of X.

Furthermore, R cannot be larger than this intersection, since X has a poles at a and a^{-1}. Therefore, $R = R_1 \cap R_2$. The various ROCs are illustrated in Figure 12.15. So, in conclusion, we have

$$X(z) = \frac{(a-a^{-1})z}{(z-a)(z-a^{-1})} \quad \text{for } |a| < |z| < \left|a^{-1}\right|. \qquad \blacksquare$$

Example 12.9 (Linearity property with pole-zero cancellation)**.** Using the linearity property of the z transform and the z transform pairs

$$u(n+1) \overset{\text{ZT}}{\longleftrightarrow} \frac{z^2}{z-1} \quad \text{for } |z| > 1 \quad \text{and} \quad u(n-2) \overset{\text{ZT}}{\longleftrightarrow} \frac{1}{z(z-1)} \quad \text{for } |z| > 1,$$

find the z transform X of the sequence

$$x(n) = u(n+1) - u(n-2).$$

Solution. To begin, let

$$x_1(n) = u(n+1) \quad \text{and} \quad x_2(n) = u(n-2),$$

and let X_1 and X_2 denote the z transforms of x_1 and x_2, respectively. Also, let R_X, R_{X_1}, and R_{X_2} denote the ROCs of X, X_1, and X_2, respectively. From the linearity property of the z transform, we have

$$\begin{aligned} X(z) &= \mathcal{Z}\{x_1 - x_2\}(z) \\ &= X_1(z) - X_2(z). \end{aligned}$$

Using the given z transform pairs, we can rewrite this as

$$\begin{aligned} X(z) &= \frac{z^2}{z-1} - \frac{1}{z(z-1)} \\ &= \frac{z^3-1}{z(z-1)} \\ &= \frac{(z-1)(z^2+z+1)}{z(z-1)} \\ &= \frac{z^2+z+1}{z}. \end{aligned}$$

From the given z transform pairs, we know

$$R_{X_1} = R_{X_2} = \{|z| > 1\}.$$

Now, we must determine R_X. From the linearity property, we know that R_X must at least contain $R_{X_1} \cap R_{X_2}$. Thus, R_X contains

$$R_{X_1} \cap R_{X_2} = R_{X_1} \cap R_{X_1} = R_{X_1} = \{|z| > 1\}.$$

We still must determine, however, if R_X is larger than this intersection. Since the functions X_1 and X_2 are rational, R_X can only be larger than this intersection if pole-zero cancellation occurs. The poles and ROCs of X_1 and X_2 are shown in Figures 12.16(a) and (b), respectively, and the set $R_{X_1} \cap R_{X_2}$ is shown in Figure 12.16(c) along with the poles of X. Clearly, a pole at 1 was cancelled in the computation of X (i.e., X does not have a pole at 1 whereas X_1 and X_2 do have such a pole). Now, we consider whether the ROC could be larger than the one shown in Figure 12.16(c). Since X is rational, we know that the ROC must be bounded by poles or extend outwards towards infinity and inwards towards zero if not bounded by a pole. Clearly, the region in Figure 12.16(c) is not bounded by a pole on its inner boundary. Therefore, R_X must be larger than $R_{X_1} \cap R_{X_2}$. In particular, to obtain R_X, we must extend the region inwards until just before it reaches the pole at the origin. Therefore, we have that

$$R_X = \{|z| > 0\}.$$

The poles of X and R_X are shown in Figure 12.16(d). So, in conclusion, we have

$$X(z) = \frac{z^2+z+1}{z} \quad \text{for } |z| > 0. \qquad \blacksquare$$

12.8.2 Translation (Time Shifting)

The next property of the z transform to be introduced is the translation (i.e., time-domain shifting) property, as given below.

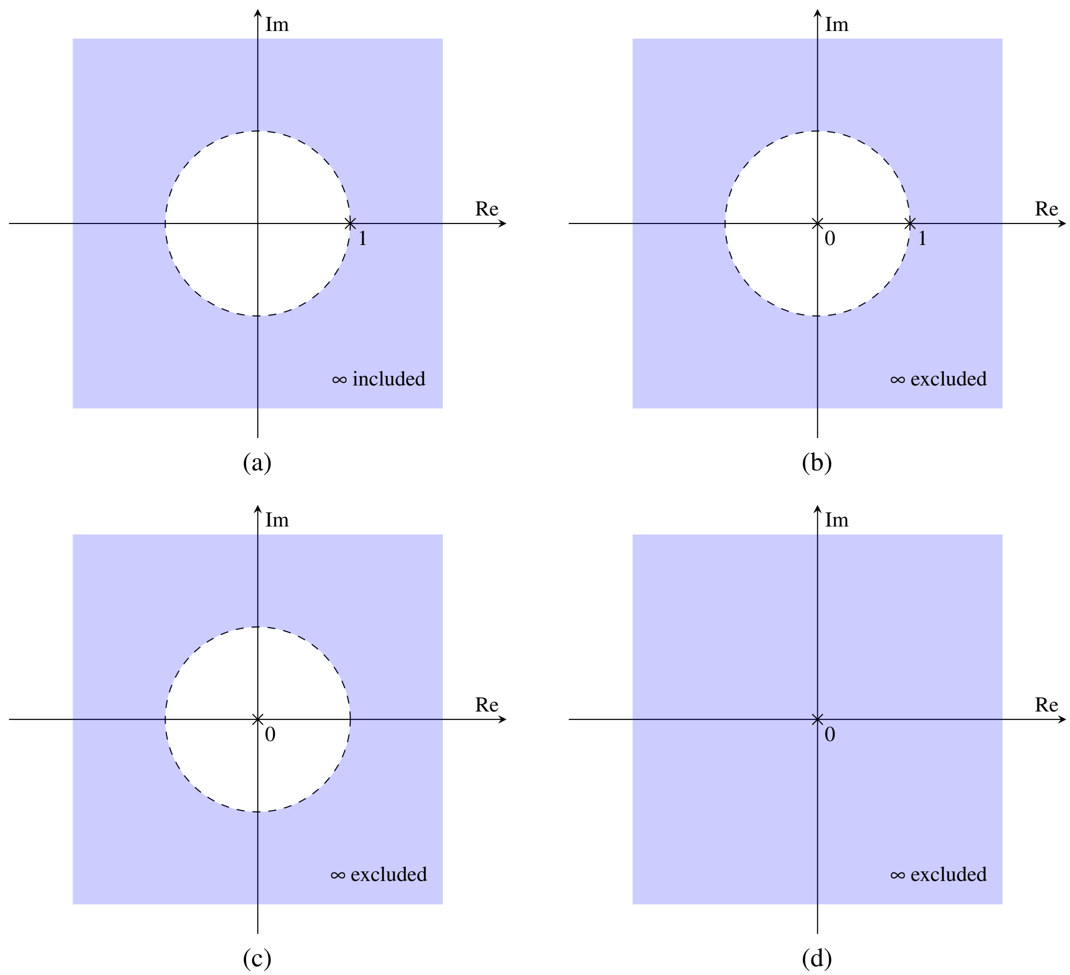


Figure 12.16: ROCs for the linearity example. The (a) ROC of X_1, (b) ROC of X_2, (c) ROC associated with the intersection of the ROCs of X_1 and X_2, and (d) ROC of X.

Theorem 12.2 (Translation (i.e., time shifting))**.** *If* $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ *with ROC R, then*

$$x(n-n_0) \overset{\text{ZT}}{\longleftrightarrow} z^{-n_0} \quad \textit{with ROC R except for the possible addition/deletion of zero or infinity},$$

where n_0 *is an integer constant. This is known as the* ***translation property*** *(or* ***time-shifting property****) of the z transform.*

Proof. Let $y(n) = x(n-n_0)$, and let Y denote the z transform of y. From the definition of the z transform, we have

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n-n_0)z^{-n}.$$

Now, we employ a change of variable. Let $k = n-n_0$ so that $n = k+n_0$. Applying the change of variable, we obtain

$$\begin{aligned} Y(z) &= \sum_{k=-\infty}^{\infty} x(k)z^{-(k+n_0)} \\ &= z^{-n_0} \sum_{k=-\infty}^{\infty} x(k)z^{-k} \\ &= z^{-n_0}X(z). \end{aligned}$$

Now, we must determine the ROC R' of Y. If $X(z)$ converges for some z, then $Y(z)$ can only fail to converge due to the pole associated with z^{-n_0}. This pole is at 0 or ∞ for $n_0 > 0$ and $n_0 < 0$, respectively. So, R' must be the same as R with the possible exception of 0 and ∞. Thus, we have shown that the translation property holds. ∎

Example 12.10 (Translation property)**.** Find the z transform X of the sequence

$$x(n) = u(n-n_0),$$

where n_0 is an integer constant.

Solution. From Table 12.3, we know that $u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-1}$ for $|z| > 1$. Using this fact and the time-shifting property of the z transform, we can write

$$\begin{aligned} X(z) &= z^{-n_0}\mathcal{Z}u(z) \\ &= z^{-n_0}\frac{z}{z-1} \\ &= \frac{z^{1-n_0}}{z-1} \quad \text{for } |z| > 1. \end{aligned}$$

Thus, we have

$$u(n-n_0) \overset{\text{ZT}}{\longleftrightarrow} \frac{z^{1-n_0}}{z-1} \quad \text{for } |z| > 1.$$ ∎

Example 12.11 (Rectangular pulse)**.** Using properties of the z transform and the transform pair

$$u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-1} \quad \text{for } |z| > 1,$$

find the z transform X of the sequence

$$x(n) = \begin{cases} 1 & n \in [n_0 \mathinner{.\,.} n_1) \\ 0 & \text{otherwise}, \end{cases}$$

where n_0 and n_1 are (finite) integer constants and $n_0 < n_1$.

Solution. To begin, we observe that x can be equivalently written as

$$x(n) = u(n-n_0) - u(n-n_1).$$

Let $v_1(n) = u(n-n_0)$ and $v_2(n) = u(n-n_1)$ so that $x(n) = v_1(n) - v_2(n)$. Taking the z transform of v_1 using the translation property of the z transform, we have

$$V_1(z) = z^{-n_0} \mathcal{Z}u(z).$$

Using the given z transform pair, we obtain

$$V_1(z) = z^{-n_0}\left(\frac{z}{z-1}\right).$$

Taking the z transform of v_2 using the translation property of the z transform, we have

$$V_2(z) = z^{-n_1} \mathcal{Z}u(z).$$

Using the given z transform pair, we obtain

$$V_2(z) = z^{-n_1}\left(\frac{z}{z-1}\right).$$

Taking the z transform of x using the linearity property of the z transform, we have

$$X(z) = V_1(z) - V_2(z).$$

Substituting the formulas for V_1 and V_2 into the formula for X, we obtain

$$\begin{aligned} X(z) &= z^{-n_0}\left(\frac{z}{z-1}\right) - z^{-n_1}\left(\frac{z}{z-1}\right) \\ &= (z^{-n_0} - z^{-n_1})\left(\frac{z}{z-1}\right). \end{aligned}$$

Since x is finite duration, the ROC of X is the entire complex plane, except possibly for 0. Whether 0 is included in the ROC depends on the specific values of n_0 and n_1. For example, the ROC includes 0 if $n_1 \leq 1$. ■

12.8.3 Complex Modulation (z-Domain Scaling)

The next property of the z transform to be introduced is the complex modulation (i.e., z-domain scaling) property, as given below.

Theorem 12.3 (Complex modulation (i.e., z-domain scaling))**.** *If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then*

$$a^n x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z/a) \quad \text{with ROC } R' = |a|\, R,$$

where a is a nonzero complex constant. This is known as the ***complex modulation property*** *(or z-domain scaling property) of the z transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = a^n x(n)$ and let Y denote the z transform of y. From the definition of the z transform, we have

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n)(a^{-1}z)^{-n} \\ &= X(z/a). \end{aligned}$$

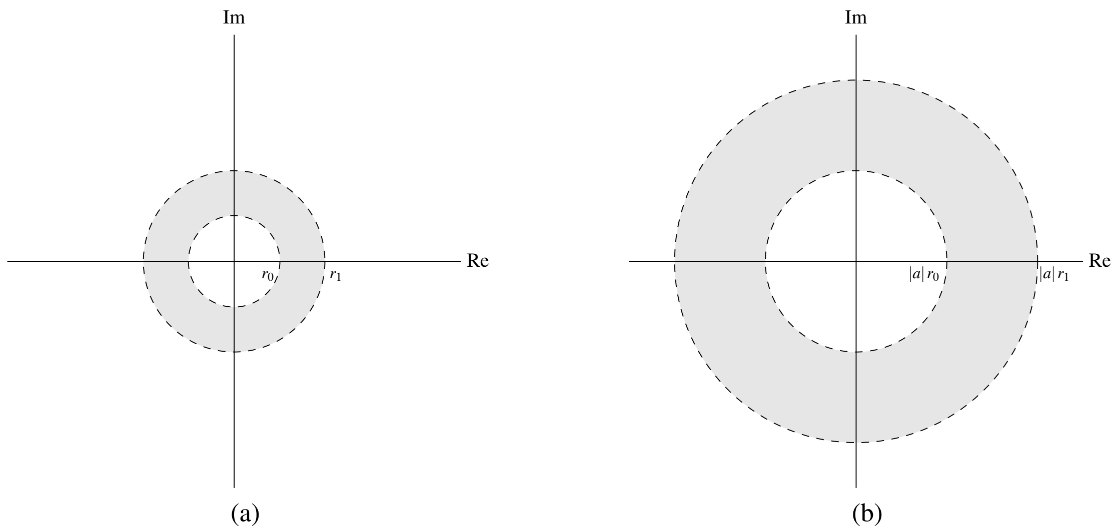


Figure 12.17: ROCs for complex modulation. The ROC of the z transform of the sequence (a) before and (b) after scaling.

Now, we must determine the ROC R' of Y. Since $X(z) = Y(az)$, Y must converge at az if X converges at z. Thus, $R' = aR$. Expressing a in polar form, we have

$$R' = |a|\, e^{j\arg a} R.$$

Due to the fact that R must always consist of concentric circles centered at the origin, $R = e^{j\arg\theta} R$ for any real constant θ. Thus, we can rewrite the above expression for R' as

$$R' = |a|\, R.$$

Thus, we have shown that the complex modulation property holds. ■

The effect of complex modulation on the ROC of the z transform is illustrated in Figure 12.17. Suppose that the ROC of the z transform of a sequence x is as shown in Figure 12.17(a). Then, the ROC of the z transform of the sequence $y(n) = a^n x(n)$ is as shown in Figure 12.17(b).

Example 12.12 (Complex modulation property)**.** Using properties of the z transform and the z transform pair

$$u(n) \overset{\text{ZT}}{\longleftrightarrow} \tfrac{z}{z-1} \;\text{ for } |z| > 1,$$

find the z transform X of the sequence

$$x(n) = a^n u(n).$$

Solution. Using the z-domain scaling property and the given z transform pair, we can write

$$\begin{aligned} X(z) &= \frac{z/a}{z/a - 1} \\ &= \frac{z}{z-a}. \end{aligned}$$

For the ROC R_X of X, we have

$$|z/a| > 1 \Rightarrow |z|\,/\,|a| > 1 \Rightarrow |z| > |a|\,.$$

Thus, we have

$$a^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-a} \quad \text{for } |z| > |a|.$$ ■

12.8.4 Conjugation

The next property of the z transform to be introduced is the conjugation property, as given below.

Theorem 12.4 (Conjugation)**.** *If* $x(n) \xleftrightarrow{\text{ZT}} X(z)$ *with ROC R, then*

$$x^*(n) \xleftrightarrow{\text{ZT}} X^*(z^*) \quad \textit{with ROC } R.$$

This is known as the ***conjugation property*** *of the z transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = x^*(n)$ and let Y denote the z transform of y. From the definition of the z transform, we have

$$\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} x^*(n)z^{-n} \\
&= \left(\sum_{n=-\infty}^{\infty} x^*(n)z^{-n} \right)^{**} \\
&= \left(\sum_{n=-\infty}^{\infty} x(n)(z^{-n})^* \right)^{*} \\
&= \left(\sum_{n=-\infty}^{\infty} x(n)(z^*)^{-n} \right)^{*} \\
&= X^*(z^*).
\end{aligned}$$

Now, we consider the ROC of Y. Since $Y(z) = X^*(z^*)$, the expression $Y(z)$ converges if and only if $X(z^*)$ converges. In turn, $X(z^*)$ converges if and only if $z^* \in R$. Since $z^* \in R$ if and only if $z \in R$, the ROC of Y is R. Thus, we have shown that the conjugation property holds. ■

Example 12.13 (Conjugation property)**.** Let x and y be two sequences related by

$$y(n) = \operatorname{Re}[x(n)].$$

Let X and Y denote the z transforms of x and y, respectively. Find Y in terms of X.

Solution. From properties of complex numbers, we know that $\operatorname{Re}\alpha = \frac{1}{2}(\alpha + \alpha^*)$. So, we have

$$\begin{aligned}
Y(z) &= \mathcal{Z}\left\{\tfrac{1}{2}[x(n) + x^*(n)]\right\}(z) \\
&= \tfrac{1}{2}\mathcal{Z}x(n) + \tfrac{1}{2}\mathcal{Z}\{x^*(n)\}(z) \\
&= \tfrac{1}{2}X(z) + \tfrac{1}{2}X^*(z^*).
\end{aligned}$$

The ROC of Y is the same as the ROC of X. ■

12.8.5 Time Reversal

The next property of the z transform to be introduced is the time-reversal property, as given below.

Theorem 12.5 (Time reversal)**.** *If* $x(n) \xleftrightarrow{\text{ZT}} X(z)$ *with ROC R, then*

$$x(-n) \xleftrightarrow{\text{ZT}} X(z^{-1}) \quad \textit{with ROC } R' = R^{-1}.$$

This is known as the ***time-reversal property*** *of the z transform.*

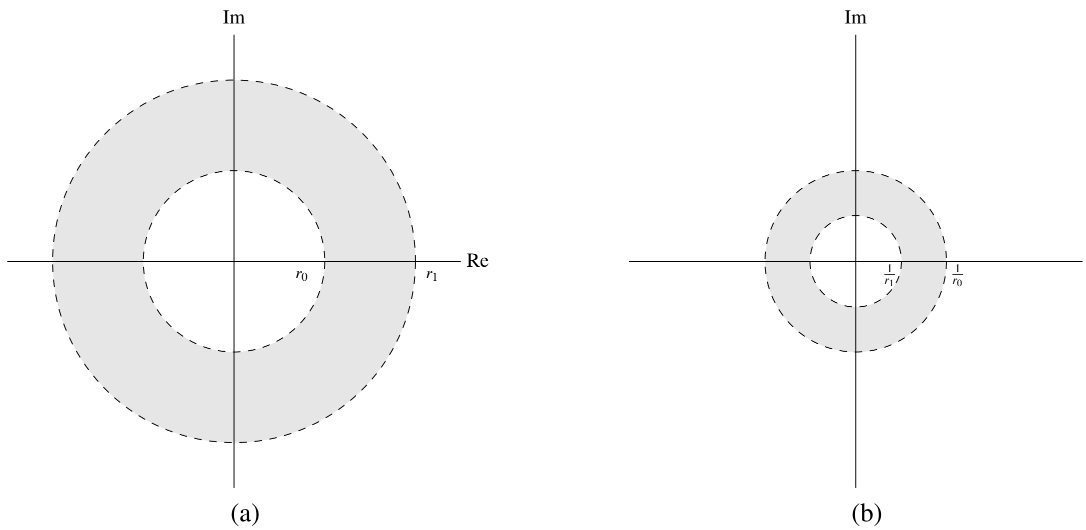


Figure 12.18: ROCs for time reversal. The ROC of the z transform of the sequence (a) before and (b) after time reversal.

Proof. To prove the above property, we proceed as follows. Let $y(n) = x(-n)$ and let Y denote the z transform of y. From the definition of the z transform, we have

$$Y(z) = \sum_{n=-\infty}^{\infty} x(-n)z^{-n}.$$

Now, we employ a change of variable. Let $k = -n$ so that $n = -k$. Applying the change of variable, we obtain

$$\begin{aligned} Y(z) &= \sum_{k=-\infty}^{\infty} x(k)z^{k} \\ &= \sum_{k=-\infty}^{\infty} x(k)(z^{-1})^{-k} \\ &= X(z^{-1}). \end{aligned}$$

Since $Y(z) = X(z^{-1})$, the expression $Y(z)$ must converge if $z^{-1} \in R$, or equivalently, $z \in R^{-1}$. So, the ROC of Y is R^{-1}. Thus, we have shown that the time-reversal property holds. ■

The effect of time reversal on the ROC of the z transform is illustrated in Figure 12.18. Suppose that the ROC of the z transform of a sequence x is as shown in Figure 12.18(a). Then, the ROC of the z transform of the sequence $y(n) = x(-n)$ is as shown in Figure 12.18(b).

Example 12.14 (Time-reversal property)**.** Using properties of the z transform and the z transform pair

$$u(n) \stackrel{\text{ZT}}{\longleftrightarrow} \tfrac{z}{z-1} \quad \text{for } |z| > 1,$$

find the z transform X of the sequence

$$x(n) = u(-n).$$

Solution. Using the given z transform pair and the time-reversal property of the z transform, we can write

$$X(z) = \frac{z^{-1}}{z^{-1}-1} \quad \text{for } \left|z^{-1}\right| > 1.$$

Simplifying the algebraic expression, we obtain

$$X(z) = \frac{z^{-1}}{z^{-1}-1} = \frac{1}{1-z}.$$

Simplifying the expression for the ROC, we obtain

$$\left|z^{-1}\right| > 1 \Rightarrow |z| < 1.$$

Thus, we have

$$X(z) = \frac{1}{1-z} \quad \text{for } |z| < 1. \qquad \blacksquare$$

Example 12.15. Let x and y be two sequences related by

$$y(n) = \operatorname{Even}\{x\}(n) = \tfrac{1}{2}[x(n)+x(-n)].$$

Let X and Y denote the z transforms of x and y, respectively. Find Y in terms of X.

Solution. Let $v(n) = x(-n)$ and let V denote the z transform of v. Let R_X and R_Y denote the ROCs of X and Y, respectively. From the time-reversal property of the z transform, we have

$$V(z) = X\left(z^{-1}\right) \;\text{ for } z \in R_X^{-1}.$$

From the linearity property of the z transform, we have

$$\begin{aligned} Y(z) &= \tfrac{1}{2}[X(z)+V(z)] \\ &= \tfrac{1}{2}[X(z)+X(z^{-1})] \quad \text{and} \\ R_Y &\supset R_X \cap R_X^{-1}. \end{aligned}$$

(Note that "$\supset$" means "contains".) We cannot simplify further without additional knowledge about x. $\blacksquare$

12.8.6 Upsampling (Time Expansion)

In Section 8.2.4, the upsampling operation for sequences was introduced. One might wonder what effect upsampling has on the z transform of a sequence. The answer to this question is given by the theorem below.

Theorem 12.6 (Upsampling (i.e., time expansion))**.** *If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then*

$$(\uparrow M)x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z^M) \quad \text{with ROC } R' = R^{1/M},$$

where $R^{1/M}$ denotes the set formed from the Mth roots of the elements in R. This is known as the ***upsampling property*** *(or* ***time-expansion property****) of the z transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = (\uparrow M)x(n)$. From the definition of the z transform, we have

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n)z^{-n}.$$

Since $y(n) = 0$ if n/M is not an integer, we can rewrite this summation as

$$Y(z) = \sum_{\substack{n\in\mathbb{Z}:\\ M \text{ divides } n}} y(n)z^{-n}.$$

Now, we employ a change of variable. Let $\lambda = n/M$ so that $n = M\lambda$. (Note that λ will always be an integer since the above summation is only taken over terms where n/M is an integer.) Applying the change of variable, we obtain

$$Y(z) = \sum_{\lambda=-\infty}^{\infty} y(M\lambda)z^{-M\lambda}.$$

Using the fact that $x(n) = y(Mn)$, we can rewrite the above equation as

$$\begin{aligned}Y(z) &= \sum_{\lambda=-\infty}^{\infty} x(\lambda)(z^M)^{-\lambda}\\ &= X(z^M).\end{aligned}$$

Now, we consider the ROC R' of Y. Since $Y(z) = X(z^M)$, Y converges at z if and only if X converges at z^M, or equivalently, Y converges at $z^{1/M}$ if and only if X converges at z. This implies that $R' = R^{1/M}$. Thus, we have shown that the upsampling property holds. ■

Example 12.16 (Upsampling property)**.** Find the z transform X of the sequence

$$x(n) = \begin{cases} 1 & n \geq 0 \text{ and } n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Solution. To begin, we observe that $x = (\uparrow 2)u$. From Table 12.3, we know that

$$u(n) \overset{\text{ZT}}{\longleftrightarrow} \tfrac{z}{z-1} \text{ for } |z| > 1.$$

Using the upsampling property of the z transform, we have

$$\begin{aligned}X(z) &= \mathcal{Z}u(z^2)\\ &= \frac{z^2}{z^2-1}.\end{aligned}$$

Let R denote the set $|z| > 1$ (i.e., the ROC of the z transform of u). Since $R^{1/2} = R$, the ROC of X is R. Thus, we have

$$(\uparrow 2)u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z^2}{z^2-1} \text{ for } |z| > 1.$$ ■

12.8.7 Downsampling

In Section 8.2.3, the downsampling operation for sequences was introduced. One might wonder what effect downsampling has on the z transform of a sequence. The answer to this question is given by the theorem below.

Theorem 12.7 (Downsampling)**.** *If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then*

$$(\downarrow M)x(n) \overset{\text{ZT}}{\longleftrightarrow} \tfrac{1}{M}\sum_{k=0}^{M-1} X\left(e^{-j2\pi k/M}z^{1/M}\right) \quad \textit{with ROC } R' = R^M,$$

where R^M denotes the set formed from the Mth powers of the elements in R. This is known as the ***downsampling property*** *of the z transform.*

Proof. To prove this theorem, we proceed as follows. Let $y = (\downarrow M)x$, and let Y denote the z transform of y. The z transform Y can be written as

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y(n)z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(Mn)z^{-n}. \end{aligned}$$

Now, we define the sequence

$$v(n) = \begin{cases} x(n) & \text{if } M \text{ divides } n \text{ (i.e., } n/M \text{ is an integer)} \\ 0 & \text{otherwise.} \end{cases}$$

Using this definition (which implies that $v(Mn) = x(Mn)$ for all integer n), we have

$$Y(z) = \sum_{n=-\infty}^{\infty} v(Mn)z^{-n}.$$

Now, we employ a change of variable. Let $n' = Mn$ so that $n = n'/M$. Applying the change of variable and dropping the primes, we obtain

$$Y(z) = \sum_{\substack{n\in\mathbb{Z}: \\ M \text{ divides } n}} v(n)z^{-n/M}.$$

Since $v(n)$ is zero when M does not divide n, the constraint on the summation index that M divides n can be removed to yield

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} v(n)z^{-n/M} \\ &= V(z^{1/M}). \end{aligned} \tag{12.5}$$

To complete the proof, we express V in terms of X. We observe that v can be written as

$$v(n) = c(n)x(n) \quad \text{where} \quad c(n) = \begin{cases} 1 & \text{if } M \text{ divides } n \\ 0 & \text{otherwise.} \end{cases}$$

The M-periodic sequence c has the Fourier series representation

$$c(n) = \tfrac{1}{M} \sum_{k=0}^{M-1} e^{j2\pi kn/M}.$$

Thus, we can compute the z transform V of v as follows:

$$\begin{aligned} V(z) &= \mathcal{Z}\{c(n)x(n)\}(z) \\ &= \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{M} \sum_{k=0}^{M-1} e^{j2\pi kn/M} x(n) \right) z^{-n} \\ &= \tfrac{1}{M} \sum_{n=-\infty}^{\infty} \sum_{k=0}^{M-1} e^{j2\pi kn/M} x(n) z^{-n} \\ &= \tfrac{1}{M} \sum_{k=0}^{M-1} \sum_{n=-\infty}^{\infty} x(n) \left(ze^{-j2\pi k/M} \right)^{-n} \\ &= \tfrac{1}{M} \sum_{k=0}^{M-1} X(ze^{-j2\pi k/M}). \end{aligned} \tag{12.6}$$

Combining (12.5) and (12.6), we have

$$\begin{aligned} Y(z) &= V(z^{1/M}) \\ &= \tfrac{1}{M} \sum_{k=0}^{M-1} X(z^{1/M} e^{-j2\pi k/M}). \end{aligned}$$

Now, we consider the ROC R' of Y. From the preceding equation, Y converges at z if X converges at $z^{1/M}$, or equivalently, Y converges at z^M if X converges at z. Thus, $R' = R^M$. So, we conclude that

$$Y(z) = \tfrac{1}{M} \sum_{k=0}^{M-1} X(z^{1/M} e^{-j2\pi k/M}) \quad \text{for } z \in R^M.$$

This completes the proof. ■

Example 12.17 (Downsampling property)**.** Let x and y be two sequences related by $y(n) = (\downarrow 2)x(n)$, and let X and Y denote the z transforms of x and y, respectively. Suppose that

$$X(z) = \frac{z^2 - 1}{z^2 - \frac{1}{4}} \quad \text{for } |z| > \tfrac{1}{2}.$$

Find Y.

Solution. From the downsampling property of the z transform, we have

$$\begin{aligned} Y(z) &= \tfrac{1}{2} \sum_{k=0}^{1} X(e^{-j\pi k} z^{1/2}) \\ &= \tfrac{1}{2} \sum_{k=0}^{1} X\left[(-1)^k z^{1/2}\right] \\ &= \tfrac{1}{2}\left[X(z^{1/2}) + X(-z^{1/2})\right] \\ &= \tfrac{1}{2}\left(\frac{(z^{1/2})^2 - 1}{(z^{1/2})^2 - \frac{1}{4}} + \frac{(-z^{1/2})^2 - 1}{(-z^{1/2})^2 - \frac{1}{4}}\right) \\ &= \tfrac{1}{2}\left(\frac{z-1}{z-\frac{1}{4}} + \frac{z-1}{z-\frac{1}{4}}\right) \\ &= \frac{z-1}{z-\frac{1}{4}}. \end{aligned}$$

The ROC of Y is $|z| > \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

In passing, we note that the inverse z transforms of X and Y are given by

$$\begin{aligned} x(n) &= 4\delta(n) - \tfrac{3}{2}\left(\left[1 + (-1)^n\right]\left(\tfrac{1}{2}\right)^n\right) u(n) \quad \text{and} \\ y(n) &= 4\delta(n) - 3\left(\tfrac{1}{4}\right)^n u(n). \end{aligned}$$

As we would expect, $y(n) = x(2n)$. ■

12.8.8 Convolution

The next property of the z transform to be introduced is the convolution property, as given below.

Theorem 12.8 (Convolution)**.** *If* $x_1(n) \overset{\text{ZT}}{\longleftrightarrow} X_1(z)$ *with ROC* R_1 *and* $x_2(n) \overset{\text{ZT}}{\longleftrightarrow} X_2(z)$ *with ROC* R_2*, then*

$$x_1 * x_2(n) \overset{\text{ZT}}{\longleftrightarrow} X_1(z)X_2(z) \quad \text{with ROC } R \text{ containing } R_1 \cap R_2.$$

This is known as the ***time-domain convolution property*** *of the z transform.*

Proof. To prove the above property, we proceed as follows. From the definition of the z transform, we have

$$\mathcal{Z}\{x_1 * x_2\}(z) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)z^{-n}.$$

Now, we employ a change of variable. Let $\lambda = n-k$ so that $n = \lambda + k$. Applying the change of variable, we obtain

$$\begin{aligned}
\mathcal{Z}\{x_1 * x_2\}(z) &= \sum_{\lambda=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)x_2(\lambda)z^{-(\lambda+k)} \\
&= \sum_{k=-\infty}^{\infty} \sum_{\lambda=-\infty}^{\infty} x_1(k)x_2(\lambda)z^{-\lambda}z^{-k} \\
&= \sum_{k=-\infty}^{\infty} x_1(k)z^{-k} \sum_{\lambda=-\infty}^{\infty} x_2(\lambda)z^{-\lambda} \\
&= X_1(z)X_2(z).
\end{aligned}$$

The expression $X_1(z)X_2(z)$ converges if z is such that $X_1(z)$ and $X_2(z)$ both converge (i.e., $z \in R_1 \cap R_2$). So, the ROC R must contain $R_1 \cap R_2$. Thus, we have shown that the convolution property holds. ∎

In the preceding theorem, note that the ROC R can be larger than $R_1 \cap R_2$. When X_1 and X_2 are rational functions, this can only happen if pole-zero cancellation occurs in the expression $X_1(z)X_2(z)$.

The time-domain convolution property of the z transform has important practical implications. Since the z transform effectively converts a convolution into a multiplication, the z transform can be used as a means to avoid directly dealing with convolution operations. This is often extremely helpful when working with (DT) LTI systems, for example, since such systems fundamentally involve convolution.

Example 12.18 (Convolution property)**.** Find the z transform X of the sequence

$$x(n) = u * u(n).$$

Solution. From Table 12.3, we have the z transform pair

$$u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-1} \quad \text{for } |z| > 1.$$

Taking the z transform of x with the help of the above z transform pair, we have

$$\begin{aligned}
X(z) &= \mathcal{Z}\{u * u\}(z) \\
&= \mathcal{Z}u(z)\mathcal{Z}u(z) \\
&= \left(\frac{z}{z-1}\right)\left(\frac{z}{z-1}\right) \\
&= \frac{z^2}{(z-1)^2}.
\end{aligned}$$

Let R_X denote the ROC of X and let $R = \{|z| > 1\}$ (i.e., R is the ROC of the z transform of u). We have that R_X must contain $R \cap R = R$. Since the terms being added to obtain X are rational and no pole-zero cancellation occurs, the ROC cannot be larger than R. Thus, $R_X = \{|z| > 1\}$. Therefore, we conclude

$$X(z) = \frac{z^2}{(z-1)^2} \quad \text{for } |z| > 1.$$

∎

12.8.9 z-Domain Differentiation

The next property of the z transform to be introduced is the z-domain differentiation property, as given below.

Theorem 12.9 (z-domain differentiation)**.** *If* $x(n) \xleftrightarrow{\text{ZT}} X(z)$ *with ROC R, then*

$$nx(n) \xleftrightarrow{\text{ZT}} -z\tfrac{d}{dz}X(z) \quad \text{with ROC } R.$$

This is known as the ***z-domain differentiation property*** *of the z transform.*

Proof. To prove the above property, we proceed as follows. From the definition of the z transform, we have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}.$$

Taking the derivative of both sides of the preceding equation, we obtain

$$\begin{aligned}
\tfrac{d}{dz}X(z) &= \tfrac{d}{dz}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x(n)\tfrac{d}{dz}(z^{-n}) \\
&= \sum_{n=-\infty}^{\infty} x(n)(-n)z^{-n-1} \\
&= -z^{-1}\sum_{n=-\infty}^{\infty} nx(n)z^{-n}.
\end{aligned}$$

Multiplying both sides of the preceding equation by $-z$, we have

$$\begin{aligned}
-z\tfrac{d}{dz}X(z) &= \sum_{n=-\infty}^{\infty} nx(n)z^{-n} \\
&= \mathcal{Z}\{nx(n)\}(z).
\end{aligned}$$

Thus, we have shown that the z-domain differentiation property holds. ■

Example 12.19 (z-domain differentiation property)**.** Using properties of the z transform and the z transform pair

$$\tfrac{1}{n!}a^n u(n) \xleftrightarrow{\text{ZT}} e^{a/z} \text{ for } |z| > 0,$$

find the z transform X of the sequence

$$x(n) = \tfrac{1}{n!}na^n u(n).$$

Solution. Let $v(n) = \frac{1}{n!}a^n u(n)$ (i.e., v is the sequence appearing in the given z transform pair) so that $x(n) = nv(n)$. Taking the z transform of v, we have

$$V(z) = e^{a/z} \text{ for } |z| > 0.$$

Taking the z transform of x using the z-domain differentiation property of the z transform, we obtain

$$X(z) = -z\tfrac{d}{dz}V(z).$$

Substituting the formula for V into the formula for X, we have

$$\begin{aligned}
X(z) &= -z\tfrac{d}{dz}e^{a/z} \\
&= -ze^{a/z}\tfrac{d}{dz}(az^{-1}) \\
&= -ze^{a/z}(-az^{-2}) \\
&= az^{-1}e^{a/z}.
\end{aligned}$$

The ROC of X is the same as the ROC of the given z transform pair. Thus, we have

$$X(z) = az^{-1}e^{a/z} \text{ for } |z| > 0. \qquad \blacksquare$$

12.8.10 Differencing

The next property of the z transform to be introduced is the differencing property, as given below.

Theorem 12.10 (Differencing)**.** *If $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ with ROC R, then*

$$x(n) - x(n-1) \overset{\text{ZT}}{\longleftrightarrow} (1-z^{-1})X(z) \text{ with ROC } R' \text{ containing } R \cap \{|z| > 0\}.$$

This is known as the ***differencing property*** *of the z transform.*

Proof. To prove the above property, we proceed as follows. Let $y(n) = x(n) - x(n-1)$ and let Y denote the z transform of y. From the definition of the z transform, we have

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} [x(n) - x(n-1)]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n)z^{-n} - \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \\ &= X(z) - \sum_{n=-\infty}^{\infty} x(n-1)z^{-n}. \end{aligned}$$

Now, we employ a change of variable. Let $\lambda = n-1$ so that $n = \lambda+1$. Applying the change of variable, we obtain

$$\begin{aligned} Y(z) &= X(z) - \sum_{\lambda=-\infty}^{\infty} x(\lambda)z^{-(\lambda+1)} \\ &= X(z) - z^{-1}\sum_{\lambda=-\infty}^{\infty} x(\lambda)z^{-\lambda} \\ &= X(z) - z^{-1}X(z) \\ &= (1-z^{-1})X(z). \end{aligned}$$

Now, we consider the ROC R' of Y. Suppose that X is rational. Since $Y(z) = (1-z^{-1})X(z) = \frac{z-1}{z}X(z)$, unless the pole at 0 introduced by the $\frac{z-1}{z}$ factor is cancelled, Y has a pole at 0. In this case, the ROC R' cannot contain the origin. Hence, the restriction that $|z| > 0$. Thus, we have shown that the differencing property holds. $\blacksquare$

In the preceding theorem, note that the ROC R' can be larger than $R \cap \{|z| > 0\}$. When X is a rational function, this can only happen if pole-zero cancellation occurs in the expression $(1-z^{-1})X(z) = \frac{z-1}{z}X(z)$.

Example 12.20 (Differencing property)**.** Find the z transform Y of the sequence

$$y(n) = x(n) - x(n-1),$$

where

$$x(n) = a^n u(n)$$

and a is a complex constant.

Solution. From Table 12.3, we have the z transform pair

$$a^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-a} \quad \text{for } |z| > |a|.$$

Using the differencing property of the z transform and the above z transform pair, we have

$$\begin{aligned} Y(z) &= (1-z^{-1})\left(\frac{z}{z-a}\right) \\ &= \left(\frac{z-1}{z}\right)\left(\frac{z}{z-a}\right) \\ &= \frac{z-1}{z-a}. \end{aligned}$$

The ROC R of Y is $|z| > |a|$, unless $a = 1$, in which case R is the entire complex plane. So, we have

$$Y(z) = \frac{z-1}{z-a} \quad \text{for } z \in R \quad \text{where} \quad R = \begin{cases} |z| > |a| & a \neq 1 \\ \mathbb{C} & \text{otherwise.} \end{cases}$$ ■

12.8.11 Accumulation

The next property of the z transform to be introduced is the accumulation property, as given below.

Theorem 12.11 (Accumulation)**.** *If* $x(n) \overset{\text{ZT}}{\longleftrightarrow} X(z)$ *with ROC* R*, then*

$$\sum_{k=-\infty}^{n} x(k) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-1}X(z) \;\; \text{with ROC } R' \text{ containing } R \cap \{|z| > 1\}.$$

This is known as the ***accumulation property*** *of the z transform.*

Proof. To prove this theorem, we proceed as follows. Let $y(n) = \sum_{k=-\infty}^{n} x(k)$ and let Y denote the z transform of y. From the definition of the z transform, we can write

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y(n)z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{n} x(k)z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{n} x(k)u(n-k)z^{-n}. \end{aligned}$$

Now, we employ a change of variable. Let $\lambda = n-k$ so that $n = \lambda + k$. Applying this change of variable, we obtain

$$\begin{aligned} Y(z) &= \sum_{\lambda=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k)u(\lambda)z^{-(\lambda+k)} \\ &= \sum_{k=-\infty}^{\infty} x(k)z^{-k} \sum_{\lambda=-\infty}^{\infty} u(\lambda)z^{-\lambda} \\ &= X(z) \sum_{\lambda=0}^{\infty} z^{-\lambda}. \end{aligned}$$

Assuming that $|z| > 1$, we can simplify the sum of the infinite geometric sequence above to yield

$$Y(z) = \tfrac{z}{z-1}X(z).$$

The ROC R' of Y must contain $|z| > 1$ in order to ensure that infinite series encountered above converges. Suppose that X is rational. Assuming that no pole-zero cancellation occurs, Y has all of the same poles as X plus a new pole added at 1. Therefore, $R' = R \cap \{|z| > 1\}$. Thus, we have shown that the accumulation property holds. ■

In the preceding theorem, note that the ROC R' can be larger than $R \cap \{|z| > 1\}$. When X is a rational function, this can only happen if pole-zero cancellation occurs in the expression $\frac{z}{z-1}X(z)$.

The accumulation property of the z transform has important practical implications. Since the z transform effectively converts accumulation into multiplication (by $\frac{z}{z-1}$), the z transform can be used as a means to avoid directly dealing with accumulation operations. This can often be beneficial when working with equations involving accumulation.

Example 12.21 (Accumulation property)**.** Using the z transform pair

$$\delta(n) \stackrel{\text{ZT}}{\longleftrightarrow} 1$$

and the accumulation property of the z transform, find the z transform X of the sequence

$$x(n) = u(n).$$

Solution. To begin, we observe that

$$x(n) = \sum_{k=-\infty}^{n} \delta(k).$$

Taking the z transform of x using the accumulation property of the z transform, we have

$$X(z) = \tfrac{z}{z-1} \mathcal{Z}\delta(z).$$

Using the given z transform pair, we have

$$\begin{aligned} X(z) &= \tfrac{z}{z-1}(1) \\ &= \tfrac{z}{z-1}. \end{aligned}$$

Since $\mathcal{Z}\delta$ is rational and no pole-cancellation occurs in the multiplication of 1 and $\frac{z}{z-1}$, the ROC R of X is

$$R = \{|z| > 1\}.$$

Thus, we conclude that

$$X(z) = \frac{z}{z-1} \quad \text{for } |z| > 1.$$ ■

12.8.12 Initial and Final Value Theorems

The next properties of the z transform to be introduced are known as the initial and final value theorems, as given below.

Theorem 12.12 (Initial-value theorem)**.** *Let x be a sequence with z transform X. If x is causal, then*

$$x(0) = \lim_{z \to \infty} X(z). \tag{12.7}$$

This result is known as the ***initial-value theorem****.*

Proof. To prove the above result, we proceed as follows. We start from the limit on the right-hand side of (12.7). From the definition of the z transform and using the fact that x is causal, we can write

$$\begin{aligned}\lim_{z\to\infty} X(z) &= \lim_{z\to\infty} \sum_{n=-\infty}^{\infty} x(n)z^{-n} \\ &= \lim_{z\to\infty} \sum_{n=0}^{\infty} x(n)z^{-n}.\end{aligned}$$

Interchanging the order of the summation and limit on the right-hand side, we have

$$\begin{aligned}\lim_{z\to\infty} X(z) &= \sum_{n=0}^{\infty} x(n) \lim_{z\to\infty} z^{-n} \\ &= \sum_{n=0}^{\infty} x(n)\delta(n) \\ &= x(0).\end{aligned}$$

Thus, we have shown that the initial-value theorem holds. ■

Theorem 12.13 (Final-value theorem)**.** *Let x be a sequence with the z transform X. If x is causal and $\lim_{n\to\infty} x(n)$ exists, then*

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1}[(z-1)X(z)]. \tag{12.8}$$

This result is known as the ***final-value theorem****.*

Proof. To prove the above property, we proceed as follows. We consider the right-hand side of (12.8). From the time-domain shifting property, we know that

$$\mathcal{Z}\{x(n+1)-x(n)\}(z) = (z-1)X(z).$$

Moreover, we also have that

$$\mathcal{Z}\{x(n+1)-x(n)\}(z) = \lim_{n\to\infty} \sum_{k=-n}^{n} [x(k+1)-x(k)]z^{-k}.$$

Using this fact and that fact that x is causal, we can write the right-hand side of (12.8) as

$$\begin{aligned}&\lim_{z\to 1}\lim_{n\to\infty} \sum_{k=-n}^{n} [x(k+1)-x(k)]z^{-k} \\ &= \lim_{z\to 1}\lim_{n\to\infty} \big(x(0)z^1 + [x(1)-x(0)]z^0 + [x(2)-x(1)]z^{-1} + \ldots + [x(n)-x(n-1)]z^{-n+1} + \\ &\qquad [x(n+1)-x(n)]z^{-n}\big) \\ &= \lim_{z\to 1}\lim_{n\to\infty} \big(x(0)(z^1-1) + x(1)(1-z^{-1}) + \ldots + x(n)(z^{-n+1}-z^{-n}) + x(n+1)z^{-n}\big) \\ &= \lim_{n\to\infty}\lim_{z\to 1} \big(x(0)(z^1-1) + x(1)(1-z^{-1}) + \ldots + x(n)(z^{-n+1}-z^{-n}) + x(n+1)z^{-n}\big) \\ &= \lim_{n\to\infty}\lim_{z\to 1} \big(x(0)(z-1) + x(1)\tfrac{z-1}{z} + \ldots + x(n)\tfrac{z-1}{z^n} + x(n+1)z^{-n}\big) \\ &= \lim_{n\to\infty} x(n+1) \\ &= \lim_{n\to\infty} x(n).\end{aligned}$$

Thus, we have shown that the final-value theorem holds. ■

Example 12.22 (Initial/final value theorem)**.** A causal sequence x with a (well-defined) limit at ∞ has the z transform

$$X(z) = \frac{4z^2 - 3z}{2z^2 - 3z + 1}.$$

Find $x(0)$ and $\lim_{n\to\infty} x(n)$.

Solution. From the initial value theorem, we have

$$\begin{aligned} x(0) &= \lim_{z\to\infty} \frac{4z^2 - 3z}{2z^2 - 3z + 1} \\ &= \tfrac{4}{2} \\ &= 2. \end{aligned}$$

From the final value theorem, we have

$$\begin{aligned} \lim_{n\to\infty} x(n) &= \lim_{z\to 1} \left[(z-1) \frac{4z^2 - 3z}{2z^2 - 3z + 1} \right] \\ &= \lim_{z\to 1} \left[(z-1) \frac{4z^2 - 3z}{2(z-\frac{1}{2})(z-1)} \right] \\ &= \lim_{z\to 1} \left[\frac{4z^2 - 3z}{2(z-\frac{1}{2})} \right] \\ &= \tfrac{1}{1} \\ &= 1. \end{aligned}$$

In passing, we note that the inverse z transform of X is given by

$$x(n) = \left[\left(\tfrac{1}{2}\right)^n + 1 \right] u(n).$$

As we would expect, the values computed above for $x(0)$ and $x(\infty)$ are consistent with this equation for x. ■

Amongst other things, the initial and final value theorems can be quite useful in checking for errors in z transform calculations. For example, suppose that we are asked to compute the z transform X of the sequence x. If we were to make a mistake in this computation, the values obtained for $x(0)$ and $\lim_{n\to\infty} x(n)$ using X with the initial and final value theorems and using x directly would most likely disagree. In this manner, we can relatively easily detect some types of errors in z transform calculations.

12.9 More z Transform Examples

Earlier in this chapter, we derived a number of z transform pairs. Some of these and other important transform pairs are listed in Table 12.3. Using the various z transform properties listed in Table 12.2 and the z transform pairs listed in Table 12.3, we can more easily determine the z transform of more complicated functions.

Example 12.23. Using properties of the z transform and the z transform pair

$$u(n) \stackrel{\text{ZT}}{\longleftrightarrow} \frac{z}{z-1} \quad \text{for } |z| > 1,$$

find the z transform X of the sequence

$$x(n) = \sum_{k=-\infty}^{n} [u(k+5) - u(k)].$$

Table 12.2: Properties of the (bilateral) z transform

Property	Time Domain	z Domain	ROC
Linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(z)+a_2X_2(z)$	At least $R_1 \cap R_2$
Translation	$x(n-n_0)$	$z^{-n_0}X(z)$	R, except for possible addition/deletion of 0
Complex Modulation	$a^n x(n)$	$X(z/a)$	$\lvert a\rvert R$
Conjugation	$x^*(n)$	$X^*(z^*)$	R
Time Reversal	$x(-n)$	$X(1/z)$	R^{-1}
Upsampling	$(\uparrow M)x(n)$	$X(z^M)$	$R^{1/M}$
Downsampling	$(\downarrow M)x(n)$	$\frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{-j2\pi k/M}z^{1/M}\right)$	R^M
Convolution	$x_1 * x_2(n)$	$X_1(z)X_2(z)$	At least $R_1 \cap R_2$
z-Domain Differentiation	$nx(n)$	$-z\frac{d}{dz}X(z)$	R
Differencing	$x(n)-x(n-1)$	$\frac{z-1}{z}X(z) = (1-z^{-1})X(z)$	At least $R \cap \lvert z\rvert > 0$
Accumulation	$\sum_{k=-\infty}^{n} x(k)$	$\frac{z}{z-1}X(z) = \frac{1}{1-z^{-1}}X(z)$	At least $R \cap \lvert z\rvert > 1$

Property	
Initial Value Theorem	$x(0) = \lim_{z\to\infty} X(z)$
Final Value Theorem	$\lim_{n\to\infty} x(n) = \lim_{z\to 1}[(z-1)X(z)]$

Table 12.3: Transform pairs for the (bilateral) z transform

Pair	$x(n)$	$X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$-u(-n-1)$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4	$nu(n)$	$\frac{z}{(z-1)^2} = \frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
5	$-nu(-n-1)$	$\frac{z}{(z-1)^2} = \frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert < 1$
6	$a^n u(n)$	$\frac{z}{z-a} = \frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
7	$-a^n u(-n-1)$	$\frac{z}{z-a} = \frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
8	$na^n u(n)$	$\frac{az}{(z-a)^2} = \frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
9	$-na^n u(-n-1)$	$\frac{az}{(z-a)^2} = \frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
10	$\frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!} a^n u(n)$	$\frac{z^m}{(z-a)^m} = \frac{1}{(1-az^{-1})^m}$	$\lvert z\rvert > \lvert a\rvert$
11	$-\frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!} a^n u(-n-1)$	$\frac{z^m}{(z-a)^m} = \frac{1}{(1-az^{-1})^m}$	$\lvert z\rvert < \lvert a\rvert$
12	$\cos(\Omega_0 n)u(n)$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1} = \frac{1-(\cos\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
13	$-\cos(\Omega_0 n)u(-n-1)$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1} = \frac{1-(\cos\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert < 1$
14	$\sin(\Omega_0 n)u(n)$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1} = \frac{(\sin\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
15	$-\sin(\Omega_0 n)u(-n-1)$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1} = \frac{(\sin\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert < 1$
16	$a^n\cos(\Omega_0 n)u(n)$	$\frac{z(z-a\cos\Omega_0)}{z^2-2az\cos\Omega_0+a^2} = \frac{1-(a\cos\Omega_0)z^{-1}}{1-(2a\cos\Omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
17	$a^n\sin(\Omega_0 n)u(n)$	$\frac{az\sin\Omega_0}{z^2-2az\cos\Omega_0+a^2} = \frac{(a\sin\Omega_0)z^{-1}}{1-(2a\cos\Omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
18	$u(n)-u(n-M), M>0$	$\frac{z(1-z^{-M})}{z-1} = \frac{1-z^{-M}}{1-z^{-1}}$	$\lvert z\rvert > 0$
19	$a^{\lvert n\rvert}, \lvert a\rvert < 1$	$\frac{(a-a^{-1})z}{(z-a)(z-a^{-1})}$	$\lvert a\rvert < \lvert z\rvert < \lvert a^{-1}\rvert$

Solution. Let $v_1(n) = u(n+5)$ so that

$$x(n) = \sum_{k=-\infty}^{n} [v_1(k) - u(k)].$$

Let $v_2(n) = v_1(n) - u(n)$ so that

$$x(n) = \sum_{k=-\infty}^{n} v_2(k).$$

Taking the z transform of the various sequences, we obtain

$$\begin{aligned} X(z) &= \frac{z}{z-1} V_2(z), \\ V_2(z) &= V_1(z) - \frac{z}{z-1}, \quad \text{and} \\ V_1(z) &= z^5 \left(\frac{z}{z-1}\right). \end{aligned}$$

By substitution, we have

$$\begin{aligned} X(z) &= \frac{z}{z-1} V_2(z) \\ &= \frac{z}{z-1} \left(V_1(z) - \frac{z}{z-1}\right) \\ &= \frac{z}{z-1} \left(z^5 \frac{z}{z-1} - \frac{z}{z-1}\right) \\ &= \frac{z}{z-1} \left(\frac{z^6}{z-1} - \frac{z}{z-1}\right) \\ &= \frac{z}{z-1} \left(\frac{z^6 - z}{z-1}\right) \\ &= \frac{z^2(z^5-1)}{(z-1)^2}. \end{aligned}$$

Since $x(n) = 0$ for all $n \leq -6$, x is right sided. Therefore, the ROC of X must be outside the outermost pole at 1. Thus, we have

$$X(z) = \frac{z^2(z^5-1)}{(z-1)^2} \text{ for } |z| > 1.$$

■

Example 12.24. Find the z transform X of the sequence

$$x(n) = nu(n-1).$$

Solution. Let $v_1(n) = u(n-1)$ so that

$$x(n) = nv_1(n).$$

Taking the z transforms of the various sequences, we have

$$\begin{aligned} V_1(z) &= z^{-1} \left(\frac{z}{z-1}\right) = \frac{1}{z-1} \quad \text{and} \\ X(z) &= -z \frac{d}{dz} V_1(z). \end{aligned}$$

Substituting, we have

$$\begin{aligned} X(z) &= -z\tfrac{d}{dz}\left[(z-1)^{-1}\right] \\ &= -z\left[-(z-1)^{-2}\right] \\ &= \frac{z}{(z-1)^2}. \end{aligned}$$

Since $x(n) = 0$ for all $n < 1$, x is right sided. Therefore, the ROC of X is outside the outermost pole at 1. Thus, we have

$$X(z) = \frac{z}{(z-1)^2} \quad \text{for } |z| > 1. \qquad \blacksquare$$

12.10 Determination of the Inverse z Transform

As suggested earlier, in practice, we rarely use (12.3) directly in order to compute the inverse z transform. This formula requires a contour integration, which is not usually very easy to compute. Instead, we employ a variety of other techniques such as partial fraction expansions, power series expansions, and polynomial long division. We will consider each of these approaches in the sections that follow.

12.10.1 Partial Fraction Expansions

To find the inverse z transform of a rational function, partial fraction expansions are often employed. With this approach, in order to find the inverse z transform of a rational function, we begin by finding a partial fraction expansion of the function. In so doing, we obtain a number of simpler functions (corresponding to the terms in the partial fraction expansion) for which we can usually find the inverse z transforms in a table (e.g., such as Table 12.3). In what follows, we assume that the reader is already familiar with partial fraction expansions. A tutorial on partial fraction expansions is provided in Appendix B for those who might not be acquainted with such expansions.

Consider the computation of the inverse z transform x of a rational function X. Suppose that X has the P (distinct) poles $p_1, p_2, \ldots, p_P$, where the order of p_k is denoted q_k. Assuming that X is strictly proper, it has a partial fraction expansion (in the variable z) of the form

$$X(z) = \sum_{k=1}^{P}\sum_{\ell=1}^{q_k} A_{k,\ell}\frac{1}{(z-p_k)^\ell}.$$

To take the inverse z transform of the right-hand side of the preceding equation, we must be able to take the inverse z transform of a function of the form $\frac{1}{(z-a)^m}$ for some complex constant a and some positive integer m. Unfortunately, Table 12.3 does not directly contain the entries necessary to perform such inverse transform calculations. Although we could use the entries in the table in conjunction with the translation (i.e., time shifting) property of the z transform in order to perform the inverse transform calculation, this would lead to an inverse z transform expression that is complicated by numerous time shifting operations (which can often be undesirable). For this reason, we instead compute a partial fraction expansion of $X(z)/z$. This partial fraction has the form

$$\frac{X(z)}{z} = \sum_{k=1}^{P}\sum_{\ell=1}^{q_k} A'_{k,\ell}\frac{1}{(z-p_k)^\ell}. \tag{12.9}$$

From this partial fraction expansion, we can rewrite X as

$$X(z) = \sum_{k=1}^{P}\sum_{\ell=1}^{q_k} A'_{k,\ell}\frac{z}{(z-p_k)^\ell}.$$

To take the inverse z transform of the right-hand side of the preceding equation, we must be able to take the inverse z transform of a function of the form $\frac{z}{(z-a)^m}$ for some complex constant a and some positive integer m. Fortunately, Table 12.3 does directly contain the entries necessary to handle such inverse z transform calculations (for $m \leq 2$).

Alternatively, we can express X in terms of a partial fraction expansion in the variable z^{-1} (instead of z). Doing this, we obtain a partial fraction expansion for X of the form

$$X(z) = \sum_{k=1}^{P} \sum_{\ell=1}^{q_k} A_{k,\ell} \frac{1}{(1-p_k z^{-1})^\ell}. \tag{12.10}$$

To take the inverse z transform of the right-hand side of the preceding equation, we must be able to take the inverse z transform of a function of the form $\frac{1}{(1-az^{-1})^m}$ for some complex constant a and some positive integer m. Fortunately, Table 12.3 does directly contain the entries necessary to handle such inverse z transform calculations. For example, we can use the following pairs from the table:

$$\frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!} a^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{1}{(1-az^{-1})^m} \quad \text{for } |z| > |a| \quad \text{and}$$
$$-\frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!} a^n u(-n-1) \overset{\text{ZT}}{\longleftrightarrow} \frac{1}{(1-az^{-1})^m} \quad \text{for } |z| < |a|.$$

(A number of other z transform pairs in the table correspond to special cases of the preceding two pairs.)

As seen above, in order to compute an inverse z transform, we can use a partial fraction expansion in the manner shown in (12.9) or in the manner shown in (12.10). The former is likely to be more useful if $X(z)$ is expressed in terms of only positive powers of z, whereas the latter is likely to be more useful if $X(z)$ is expressed in terms of only negative powers of z. Since we tend to write rational z transforms with only positive powers of z herein, we tend to use a partial fraction expansion of the form of (12.9) more often when computing inverse z transforms. In what follows, we will now consider a number of examples of computing inverse z transforms.

Example 12.25. Find the inverse z transform x of the function

$$X(z) = \frac{1}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)} = \frac{z^{-2}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \quad \text{for } |z| > \tfrac{1}{2}.$$

Solution. FIRST APPROACH (USING PARTIAL FRACTION EXPANSION IN THE VARIABLE z). As our starting point, we use $X(z)$ expressed in terms of only positive powers of z:

$$X(z) = \frac{1}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}.$$

Since we cannot directly obtain the inverse z transform of X from Table 12.3, we employ a partial fraction expansion. We find a partial fraction expansion of

$$\frac{X(z)}{z} = \frac{1}{z\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}.$$

(Note that $X(z)/z$ is strictly proper.) This expression has an expansion of the form

$$\frac{X(z)}{z} = \frac{A_1}{z} + \frac{A_2}{z+\frac{1}{2}} + \frac{A_3}{z-\frac{1}{2}}.$$

Calculating the expansion coefficients, we obtain

$$A_1 = \left[z\left(\frac{X(z)}{z}\right)\right]\bigg|_{z=0} = \frac{1}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}\bigg|_{z=0} = \frac{1}{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)} = \frac{1}{\left(-\frac{1}{4}\right)} = -4,$$

$$A_2 = \left[\left(z+\tfrac{1}{2}\right)\left(\frac{X(z)}{z}\right)\right]\bigg|_{z=-1/2} = \frac{1}{z\left(z-\frac{1}{2}\right)}\bigg|_{z=-1/2} = \frac{1}{\left(-\frac{1}{2}\right)(-1)} = \frac{1}{\left(\frac{1}{2}\right)} = 2, \quad \text{and}$$

$$A_3 = \left[\left(z-\tfrac{1}{2}\right)\left(\frac{X(z)}{z}\right)\right]\bigg|_{z=1/2} = \frac{1}{z\left(z+\frac{1}{2}\right)}\bigg|_{z=1/2} = \frac{1}{\left(\frac{1}{2}\right)(1)} = \frac{1}{\left(\frac{1}{2}\right)} = 2.$$

Thus, we have

$$\frac{X(z)}{z} = -4\left(\frac{1}{z}\right) + 2\left(\frac{1}{z+\frac{1}{2}}\right) + 2\left(\frac{1}{z-\frac{1}{2}}\right).$$

So, we can rewrite X as

$$X(z) = -4 + 2\left(\frac{z}{z+\frac{1}{2}}\right) + 2\left(\frac{z}{z-\frac{1}{2}}\right).$$

Taking the inverse z transform, we have

$$x(n) = -4\mathcal{Z}^{-1}\{1\}(n) + 2\mathcal{Z}^{-1}\left\{\frac{z}{z+\frac{1}{2}}\right\}(n) + 2\mathcal{Z}^{-1}\left\{\frac{z}{z-\frac{1}{2}}\right\}(n).$$

Since X converges for $|z| > \frac{1}{2}$, we have

$$\left(-\tfrac{1}{2}\right)^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z+\frac{1}{2}} \ \text{ for } |z| > \tfrac{1}{2} \quad \text{and} \quad \left(\tfrac{1}{2}\right)^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}} \ \text{ for } |z| > \tfrac{1}{2}.$$

Thus, we have

$$\begin{aligned} x(n) &= -4\delta(n) + 2\left[\left(-\tfrac{1}{2}\right)^n u(n)\right] + 2\left[\left(\tfrac{1}{2}\right)^n u(n)\right] \\ &= -4\delta(n) + 2\left(-\tfrac{1}{2}\right)^n u(n) + 2\left(\tfrac{1}{2}\right)^n u(n). \end{aligned}$$ ■

SECOND APPROACH (USING PARTIAL FRACTION EXPANSION IN THE VARIABLE z^{-1}). As our starting point, we use $X(z)$ expressed in terms of only negative powers of z:

$$X(z) = \frac{z^{-2}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}.$$

Since we cannot directly obtain the inverse z transform of X from Table 12.3, we employ a partial fraction expansion. First, we observe that the rational expression in the variable z^{-1} for the function X is not strictly proper. So, we first need to express X as the sum of a polynomial in z^{-1} and a strictly proper rational function in z^{-1}. For convenience in what follows, we observe that $\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right) = 1-\frac{1}{4}z^{-2}$. We have

$$\begin{aligned} X(z) &= \frac{z^{-2} + 4\left(1-\frac{1}{4}z^{-2}\right) - 4\left(1-\frac{1}{4}z^{-2}\right)}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \\ &= -4 + \frac{z^{-2}+4-z^{-2}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \\ &= -4 + \frac{4}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}. \end{aligned}$$

Let $V(z) = \frac{4}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$ so that $X(z) = -4 + V(z)$. Since V is a strictly proper rational function in z^{-1}, it has a partial fraction expansion. This expansion has the form

$$V(z) = \frac{A_1}{1+\frac{1}{2}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}}.$$

Calculating the expansion coefficients, we obtain

$$A_1 = \left(1+\tfrac{1}{2}z^{-1}\right)V(z)\Big|_{z=-1/2} = \left.\frac{4}{1-\frac{1}{2}z^{-1}}\right|_{z=-1/2} = \frac{4}{1-\left(\frac{1}{2}\right)(-2)} = 2 \quad \text{and}$$

$$A_2 = \left(1-\tfrac{1}{2}z^{-1}\right)V(z)\Big|_{z=1/2} = \left.\frac{4}{1+\frac{1}{2}z^{-1}}\right|_{z=1/2} = \frac{4}{1+\left(\frac{1}{2}\right)(2)} = 2.$$

Thus, we have

$$V(z) = 2\left(\frac{1}{1+\frac{1}{2}z^{-1}}\right) + 2\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right).$$

So, we can rewrite X as

$$X(z) = -4 + 2\left(\frac{1}{1+\frac{1}{2}z^{-1}}\right) + 2\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right).$$

Taking the inverse z transform, we have

$$x(n) = -4\mathcal{Z}^{-1}\{1\}(n) + 2\mathcal{Z}^{-1}\left\{\frac{1}{1+\frac{1}{2}z^{-1}}\right\}(n) + 2\mathcal{Z}^{-1}\left\{\frac{1}{1-\frac{1}{2}z^{-1}}\right\}(n).$$

Since X converges for $|z| > \frac{1}{2}$, we have

$$\left(-\tfrac{1}{2}\right)^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{1}{1+\frac{1}{2}z^{-1}} \text{ for } |z| > \tfrac{1}{2} \quad \text{and} \quad \left(\tfrac{1}{2}\right)^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}} \text{ for } |z| > \tfrac{1}{2}.$$

Thus, we have

$$\begin{aligned} x(n) &= -4\delta(n) + 2\left[\left(-\tfrac{1}{2}\right)^n u(n)\right] + 2\left[\left(\tfrac{1}{2}\right)^n u(n)\right] \\ &= -4\delta(n) + 2\left(-\tfrac{1}{2}\right)^n u(n) + 2\left(\tfrac{1}{2}\right)^n u(n). \end{aligned}$$

Example 12.26. Find the inverse z transform x of the function

$$X(z) = \frac{z(z-1)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{5}\right)} \text{ for } |z| > \tfrac{1}{3}.$$

Solution. Since we cannot directly obtain the answer from Table 12.3, we employ a partial fraction expansion of

$$\frac{X(z)}{z} = \frac{z-1}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{5}\right)}.$$

This expansion has the form

$$\frac{X(z)}{z} = \frac{A_1}{z-\frac{1}{3}} + \frac{A_2}{z-\frac{1}{5}}.$$

Calculating the expansion coefficients, we obtain

$$A_1 = \left[\left(z-\tfrac{1}{3}\right)\left(\frac{X(z)}{z}\right)\right]\Bigg|_{z=1/3} = \frac{z-1}{z-\frac{1}{5}}\Bigg|_{z=1/3} = \frac{\left(-\frac{2}{3}\right)}{\left(\frac{2}{15}\right)} = \left(\frac{-2}{3}\right)\left(\frac{15}{2}\right) = -5 \quad \text{and}$$

$$A_2 = \left[\left(z-\tfrac{1}{5}\right)\left(\frac{X(z)}{z}\right)\right]\Bigg|_{z=1/5} = \frac{z-1}{z-\frac{1}{3}}\Bigg|_{z=1/5} = \frac{\left(-\frac{4}{5}\right)}{\left(-\frac{2}{15}\right)} = \left(\frac{-4}{5}\right)\left(\frac{-15}{2}\right) = 6.$$

Thus, we have

$$\frac{X(z)}{z} = \frac{-5}{z-\frac{1}{3}} + \frac{6}{z-\frac{1}{5}}.$$

So, we can rewrite X as

$$X(z) = \frac{-5z}{z-\frac{1}{3}} + \frac{6z}{z-\frac{1}{5}}.$$

Taking the inverse z transform, we have

$$x(n) = -5\mathcal{Z}^{-1}\left\{\frac{z}{z-\frac{1}{3}}\right\}(n) + 6\mathcal{Z}^{-1}\left\{\frac{z}{z-\frac{1}{5}}\right\}(n).$$

Since X converges for $|z| > \frac{1}{3}$, we have

$$\left(\tfrac{1}{3}\right)^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{3}} \;\text{ for } |z| > \tfrac{1}{3} \quad \text{and} \quad \left(\tfrac{1}{5}\right)^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{5}} \;\text{ for } |z| > \tfrac{1}{5}.$$

Thus, we have

$$\begin{aligned} x(n) &= -5\left[\left(\tfrac{1}{3}\right)^n u(n)\right] + 6\left[\left(\tfrac{1}{5}\right)^n u(n)\right] \\ &= \left[-5\left(\tfrac{1}{3}\right)^n + 6\left(\tfrac{1}{5}\right)^n\right] u(n). \end{aligned}$$

■

Example 12.27. Find the inverse z transform x of the function

$$X(z) = \frac{z^2(z-1)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)^2} \;\text{ for } z \in R,$$

for each R given below.
(a) $R = \left\{|z| < \frac{1}{4}\right\}$;
(b) $R = \left\{\frac{1}{4} < |z| < \frac{1}{2}\right\}$; and
(c) $R = \left\{|z| > \frac{1}{2}\right\}$.

Solution. Since we cannot directly obtain the answer from Table 12.3, we employ a partial fraction expansion of

$$\frac{X(z)}{z} = \frac{z(z-1)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)^2}.$$

This expansion has the form

$$\frac{X(z)}{z} = \frac{A_1}{z-\frac{1}{4}} + \frac{A_{2,1}}{z-\frac{1}{2}} + \frac{A_{2,2}}{\left(z-\frac{1}{2}\right)^2}.$$

Calculating the expansion coefficients, we have

$$\begin{aligned}
A_1 &= \left(z-\tfrac{1}{4}\right)\left(\frac{X(z)}{z}\right)\Big|_{z=1/4} = \frac{z(z-1)}{\left(z-\frac{1}{2}\right)^2}\Big|_{z=1/4} = \frac{\left(\frac{1}{4}\right)\left(\frac{-3}{4}\right)}{\left(-\frac{1}{4}\right)^2} = \frac{\left(\frac{-3}{16}\right)}{\left(\frac{1}{16}\right)} \\
&= -3, \\
A_{2,1} &= \frac{1}{(2-1)!}\left[\left(\frac{d}{dz}\right)^{2-1}\left[\left(z-\tfrac{1}{2}\right)^2\left(\frac{X(z)}{z}\right)\right]\right]\Big|_{z=1/2} = \frac{1}{1!}\left[\left(\frac{d}{dz}\right)\left[\left(z-\tfrac{1}{2}\right)^2\left(\frac{X(z)}{z}\right)\right]\right]\Big|_{z=1/2} \\
&= \frac{1}{1!}\left[\left(\frac{d}{dz}\right)\left[\frac{z(z-1)}{z-\frac{1}{4}}\right]\right]\Big|_{z=1/2} = \left[\left(\frac{d}{dz}\right)\left[\left(z^2-z\right)\left(z-\tfrac{1}{4}\right)^{-1}\right]\right]\Big|_{z=1/2} \\
&= \left[(2z-1)\left(z-\tfrac{1}{4}\right)^{-1}+(-1)\left(z-\tfrac{1}{4}\right)^{-2}\left(z^2-z\right)\right]\Big|_{z=1/2} = (-1)\left(\tfrac{1}{4}\right)^{-2}\left(-\tfrac{1}{4}\right) = \left(\tfrac{1}{4}\right)(16) \\
&= 4, \quad \text{and} \\
A_{2,2} &= \frac{1}{(2-2)!}\left[\left(\frac{d}{dz}\right)^{2-2}\left[\left(z-\tfrac{1}{2}\right)^2\left(\frac{X(z)}{z}\right)\right]\right]\Big|_{z=1/2} = \frac{1}{0!}\left[\left[\left(z-\tfrac{1}{2}\right)^2\left(\frac{X(z)}{z}\right)\right]\right]\Big|_{z=1/2} \\
&= \frac{1}{1!}\left[\left(\frac{d}{dz}\right)\left[\frac{z(z-1)}{z-\frac{1}{4}}\right]\right]\Big|_{z=1/2} = \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{\left(-\frac{1}{4}\right)} = \frac{\left(\frac{-1}{4}\right)}{\left(\frac{-1}{4}\right)} \\
&= 1.
\end{aligned}$$

Thus, we have

$$\frac{X(z)}{z} = -\frac{3}{z-\frac{1}{4}} + \frac{4}{z-\frac{1}{2}} + \frac{1}{\left(z-\frac{1}{2}\right)^2}.$$

So, we can rewrite X as

$$X(z) = -\frac{3z}{z-\frac{1}{4}} + \frac{4z}{z-\frac{1}{2}} + \frac{z}{\left(z-\frac{1}{2}\right)^2}.$$

Taking the inverse z transform, we have

$$x(n) = -3\mathcal{Z}^{-1}\left\{\frac{z}{z-\frac{1}{4}}\right\}(n) + 4\mathcal{Z}^{-1}\left\{\frac{z}{z-\frac{1}{2}}\right\}(n) + \mathcal{Z}^{-1}\left\{\frac{z}{\left(z-\frac{1}{2}\right)^2}\right\}(n).$$

To proceed further, we must now consider the ROC R.

(a) Suppose that $R = \left\{|z| < \frac{1}{4}\right\}$. In this case, we have

$$\begin{aligned}
-\left(\tfrac{1}{4}\right)^n u(-n-1) &\xleftrightarrow{\text{ZT}} \frac{z}{z-\frac{1}{4}} \;\text{ for } |z| < \tfrac{1}{4}, \\
-n\left(\tfrac{1}{2}\right)^n u(-n-1) &\xleftrightarrow{\text{ZT}} \frac{z}{z-\frac{1}{2}} \;\text{ for } |z| < \tfrac{1}{2}, \quad \text{and} \\
-\left(\tfrac{1}{2}\right)^n u(-n-1) &\xleftrightarrow{\text{ZT}} \frac{z}{\left(z-\frac{1}{2}\right)^2} \;\text{ for } |z| < \tfrac{1}{2}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
x(n) &= -3\left[-\left(\tfrac{1}{4}\right)^n u(-n-1)\right] + 4\left[-\left(\tfrac{1}{2}\right)^n u(-n-1)\right] + \left[-n\left(\tfrac{1}{2}\right)^n u(-n-1)\right] \\
&= \left[3\left(\tfrac{1}{4}\right)^n - 4\left(\tfrac{1}{2}\right)^n - n\left(\tfrac{1}{2}\right)^n\right] u(-n-1).
\end{aligned}$$

(b) Suppose that $R = \left\{\frac{1}{4} < |z| < \frac{1}{2}\right\}$. In this case, we have

$$\left(\tfrac{1}{4}\right)^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{4}} \text{ for } |z| > \tfrac{1}{4},$$
$$-n\left(\tfrac{1}{2}\right)^n u(-n-1) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}} \text{ for } |z| < \tfrac{1}{2}, \quad \text{and}$$
$$-\left(\tfrac{1}{2}\right)^n u(-n-1) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{\left(z-\frac{1}{2}\right)^2} \text{ for } |z| < \tfrac{1}{2}.$$

Thus, we have

$$\begin{aligned} x(n) &= -3\left[\left(\tfrac{1}{4}\right)^n u(n)\right] + 4\left[-\left(\tfrac{1}{2}\right)^n u(-n-1)\right] + \left[-n\left(\tfrac{1}{2}\right)^n u(-n-1)\right] \\ &= \left[-3\left(\tfrac{1}{4}\right)^n\right] u(n) + \left[-4\left(\tfrac{1}{2}\right)^n - n\left(\tfrac{1}{2}\right)^n\right] u(-n-1). \end{aligned}$$

(c) Suppose that $R = \left\{|z| > \frac{1}{2}\right\}$. In this case, we have

$$\left(\tfrac{1}{4}\right)^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{4}} \text{ for } |z| > \tfrac{1}{4},$$
$$n\left(\tfrac{1}{2}\right)^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}} \text{ for } |z| > \tfrac{1}{2}, \quad \text{and}$$
$$\left(\tfrac{1}{2}\right)^n u(n) \overset{\text{ZT}}{\longleftrightarrow} \frac{z}{\left(z-\frac{1}{2}\right)^2} \text{ for } |z| > \tfrac{1}{2}.$$

Thus, we have

$$\begin{aligned} x(n) &= -3\left[\left(\tfrac{1}{4}\right)^n u(n)\right] + 4\left[\left(\tfrac{1}{2}\right)^n u(n)\right] + \left[n\left(\tfrac{1}{2}\right)^n u(n)\right] \\ &= \left[-3\left(\tfrac{1}{4}\right)^n + 4\left(\tfrac{1}{2}\right)^n + n\left(\tfrac{1}{2}\right)^n\right] u(n). \end{aligned}$$

■

12.10.2 Laurent-Polynomial and Power-Series Expansions

Another approach to finding inverse z transforms is based on Laurent-polynomial or power series expansions. With this approach, in order to find the inverse z transform x of X, we express X as a (Laurent) polynomial or power series. Then, x can be determined by examining the coefficients of this polynomial or power series. A variety of strategies may be used for expressing X as a polynomial or power series. In some cases, X may be a function with a well known power series, such as a trigonometric, exponential, or logarithmic function. In other cases, X may be a rational function, in which case polynomial long division can be used.

Example 12.28. Find the inverse z transform x of the function

$$X(z) = \frac{z^3+2z^2+4z+8}{z^2} \text{ for } |z| > 0.$$

Solution. We can rewrite X as

$$X(z) = z + 2 + 4z^{-1} + 8z^{-2}.$$

From the definition of the z transform, it trivially follows that

$$x(n) = \begin{cases} 1 & n = -1 \\ 2 & n = 0 \\ 4 & n = 1 \\ 8 & n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have that

$$x(n) = \delta(n+1) + 2\delta(n) + 4\delta(n-1) + 8\delta(n-2). \qquad \blacksquare$$

Example 12.29. Find the inverse z transform x of the function

$$X(z) = \cos\left(z^{-1}\right) \quad \text{for } |z| > 0.$$

Solution. To begin, we recall the Maclaurin series for the cos function, which is given by

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{for all complex } z.$$

(Note that a number of useful series, including the preceding one can be found in Section F.6.) Writing X in terms of a Maclaurin series, we have

$$\begin{aligned} X(z) &= \cos\left(z^{-1}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(z^{-1}\right)^{2n}. \end{aligned}$$

(Note that $X(z)$ converges for all nonzero complex z.) Since x must be right sided (due to the ROC of X), we have

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{-(2n)} \\ &= \tfrac{1}{0!}z^0 - \tfrac{1}{2!}z^{-2} + \tfrac{1}{4!}z^{-4} - \tfrac{1}{6!}z^{-6} + \tfrac{1}{8!}z^{-8} - \ldots \\ &= \sum_{\substack{n\in\mathbb{Z}: \\ n \geq 0 \text{ and } n \text{ even}}} \frac{j^n}{n!} z^{-n}. \end{aligned}$$

Now, we observe that

$$\tfrac{1}{2}\left[1 + (-1)^n\right] u(n) = \begin{cases} 1 & n \geq 0 \text{ and } n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Using this observation, we can rewrite the above expression for $X(z)$ as

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \left[\tfrac{1}{2}\left[1+(-1)^n\right] u(n)\right] \frac{j^n}{n!} z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \frac{\left[1+(-1)^n\right] j^n}{2(n!)} u(n) z^{-n}. \end{aligned}$$

Therefore, we conclude that

$$x(n) = \frac{\left[1+(-1)^n\right] j^n}{2(n!)} u(n). \qquad \blacksquare$$

Example 12.30. Use polynomial long division to find the inverse z transform x of the function

$$X(z) = \frac{1}{1 - az^{-1}} \quad \text{for } |z| > |a|.$$

Solution. The ROC of X corresponds to a right-sided sequence. Therefore, we would like for the polynomial long division process to yield decreasing powers of z. This will naturally result from dividing 1 by $1-az^{-1}$. Performing the first few steps of polynomial long division, we obtain

$$
\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + \dots \\
\hline
1-az^{-1} & 1 \\
 & \underline{1 - az^{-1}} \\
 & \quad az^{-1} \\
 & \quad \underline{az^{-1} - a^2z^{-2}} \\
 & \qquad\qquad a^2z^{-2} \\
 & \qquad\qquad\qquad \dots
\end{array}
$$

At this point, we observe that the long division process is yielding the pattern

$$
\begin{aligned}
X(z) &= 1+az^{-1}+a^2z^{-2}+a^3z^{-3}+a^4z^{-4}+\dots \\
&= \sum_{n=0}^{\infty} a^n z^{-n}.
\end{aligned}
$$

Thus, we have that

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}.$$

So, by inspection, we have that

$$x(n) = a^n u(n). \qquad \blacksquare$$

Example 12.31. Use polynomial long division to find the inverse z transform x of the function

$$X(z) = \frac{1}{1-az^{-1}} \quad \text{for } |z| < |a|.$$

Solution. The ROC of X corresponds to a left-sided sequence. Therefore, we would like for the long division process to yield increasing powers of z. So, we rewrite $X(z)$ as

$$X(z) = \frac{z}{z-a}.$$

Performing the first few steps of polynomial long division, we obtain

$$
\begin{array}{r|l}
 & -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \dots \\
\hline
-a+z & z \\
 & \underline{z \qquad - a^{-1}z^2} \\
 & \qquad a^{-1}z^2 \\
 & \qquad \underline{a^{-1}z^2 - a^{-2}z^3} \\
 & \qquad\qquad\qquad a^{-2}z^3 \\
 & \qquad\qquad\qquad\qquad \dots
\end{array}
$$

At this point, we observe that the long division process is yielding the pattern

$$
\begin{aligned}
X(z) &= -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - a^{-4}z^4 - a^{-5}z^5 - \dots \\
&= \sum_{n=-\infty}^{-1} -a^n z^{-n}.
\end{aligned}
$$

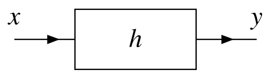


Figure 12.19: Time-domain view of a LTI system with input x, output y, and impulse response h.

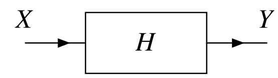


Figure 12.20: z-domain view of a LTI system with input z transform X, output z transform Y, and system function H.

Thus, we have that

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n}.$$

So, by inspection, we have that

$$x(n) = -a^n u(-n-1). \qquad \blacksquare$$

12.11 Characterizing LTI Systems Using the z Transform

Consider a LTI system with input x, output y, and impulse response h, as depicted in Figure 12.19. Such a system is characterized by the equation

$$y(n) = x * h(n).$$

Let X, Y, and H denote the z transforms of x, y, and h, respectively. Taking the z transform of both sides of the above equation and using the time-domain convolution property of the z transform, we have

$$Y(z) = H(z)X(z).$$

The quantity H is known as the **system function** or **transfer function** of the system. If the ROC of H contains the unit circle, then $H(e^{j\Omega})$ is the frequency response of the system. The system can be represented with a block diagram labelled in the z domain as shown in Figure 12.20, where the system is labelled by its system function H.

12.12 Interconnection of LTI Systems

From the properties of the z transform and the definition of the system function, we can derive a number of equivalences involving the system function and series- and parallel-interconnected systems.

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with system functions H_1 and H_2, respectively, that are connected in a series configuration as shown in the left-hand side of Figure 12.21(a). Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(n) = h_1 * h_2(n).$$

Taking the z transform of both sides of this equation yields

$$\begin{aligned} H(z) &= \mathcal{Z}\{h_1 * h_2\}(z) \\ &= \mathcal{Z}h_1(z)\mathcal{Z}h_2(z) \\ &= H_1(z)H_2(z). \end{aligned}$$

Thus, we have the equivalence shown in Figure 12.21.

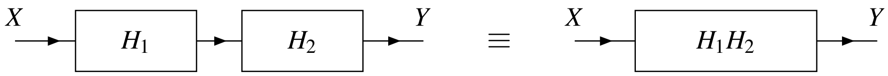


Figure 12.21: Equivalence involving system functions and the series interconnection of LTI systems.

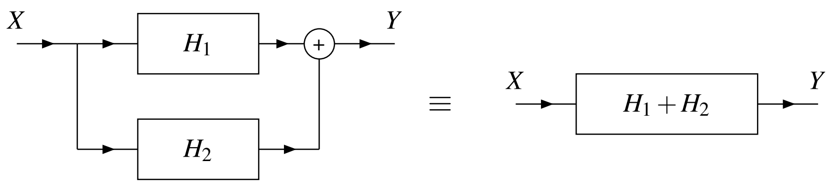


Figure 12.22: Equivalence involving system functions and the parallel interconnection of LTI systems.

Suppose that we have two LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with system functions H_1 and H_2 that are connected in a parallel configuration as shown on the left-hand side of Figure 12.22. Let h_1 and h_2 denote the impulse responses of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The impulse response h of the overall system is given by

$$h(n) = h_1(n) + h_2(n).$$

Taking the z transform of both sides of the equation yields

$$\begin{aligned} H(z) &= \mathcal{Z}\{h_1 + h_2\}(z) \\ &= \mathcal{Z}h_1(s) + \mathcal{Z}h_2(z) \\ &= H_1(z) + H_2(z). \end{aligned}$$

Thus, we have the equivalence shown in Figure 12.22.

12.13 System Function and System Properties

Many properties of a LTI system can be readily determined from the characteristics of its system function, as we shall elaborate upon in the sections that follow.

12.13.1 Causality

From Theorem 9.8, we know that, a LTI system is causal if its impulse response is causal. One might wonder, however, how the causality condition manifests itself in the system function of a LTI system. The answer to this question is given by the theorem below.

Theorem 12.14. *A (DT) LTI system with system function H is causal if and only if the ROC of H is:*

1. *the exterior of a circle, including ∞; or*
2. *the entire complex plane, including ∞ and possibly excluding* 0.

Proof. The proof is left as an exercise for the reader. ■

Theorem 12.15. *A (DT) LTI system with a rational system function H is causal if and only if:*

1. *the ROC of H is the exterior of a (possibly degenerate) circle outside the outermost (finite) pole of H or, if H has no poles, the entire (finite) complex plane; and*
2. *H is proper (i.e., when H is expressed as a ratio of polynomials in z, the order of the numerator polynomial does not exceed the order of the denominator polynomial).*

Proof. The proof is left as an exercise for the reader. ■

Example 12.32. For the LTI system with each system function H below, determine whether the system is causal.
(a) $H(z) = \dfrac{2z^3}{\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}$ for $|z| > \frac{3}{4}$;
(b) $H(z) = \dfrac{10z^2-15z+3}{(z-3)\left(z-\frac{1}{3}\right)}$ for $|z| > 3$;
(c) $H(z) = \dfrac{5z^2-8z+2}{(z-2)\left(z-\frac{1}{2}\right)}$ for $\frac{1}{2} < |z| < 2$; and
(d) $H(z) = \dfrac{1-z^{-9}}{z-1}$ for $|z| > 1$.

Solution. (a) The ROC and poles of H are shown in Figure 12.23(a). The system function H is rational and not proper. Therefore, the ROC cannot contain ∞ and the system is not causal.

In passing, we note that $H(z) = z\left(2 - \frac{1/4}{z-1/4} + \frac{9/4}{z-3/4}\right)$ and $h(n) = 2\delta(n+1) + \frac{1}{4}\left(\frac{1}{4}\right)^n u(n) + \frac{9}{4}\left(\frac{3}{4}\right)^n u(n)$. So, as we would expect, h is not causal.

(b) The ROC and poles of H are shown in Figure 12.23(b). The system function H is rational and proper, and the ROC of H is outside the outermost pole of H (whose magnitude is 3). Therefore, the system is causal.

In passing, we note that $H(z) = z\left(\frac{3}{z} + \frac{1}{z-1/3} + \frac{6}{z-3}\right)$ and $h(n) = 3\delta(n) + \left(\frac{1}{3}\right)^n u(n) + 6(3)^n u(n)$. So, as we would expect, h is causal.

(c) The ROC and poles of H are shown in Figure 12.23(c). The system function H is rational and proper, but the ROC of H is not outside the outermost pole of H (whose magnitude is 2). Therefore, the system is not causal.

In passing, we note that $H(z) = z\left(\frac{2}{z} + \frac{1}{z-1/2} + \frac{2}{z-2}\right)$ and $h(n) = 2\delta(n) + \left(\frac{1}{2}\right)^n u(n) + 2[-2^n u(-n-1)]$. So, as we would expect, h is not causal.

(d) We can rewrite H as

$$H(z) = \frac{z^9-1}{z^9(z-1)}.$$

The ROC and poles of H are shown in Figure 12.23(d). The system function H is rational and proper, and the ROC of H is outside the outermost pole of H (whose magnitude is 1). Therefore, the system is causal.

In passing, we note that $H(z) = z^{-1}\left(\frac{z}{z-1}\right) - z^{-10}\left(\frac{z}{z-1}\right)$ and $h(n) = u(n-1) - u(n-10)$. So, as we would expect, h is causal. ■

12.13.2 BIBO Stability

In this section, we consider the relationship between the system function and BIBO stability. The first important result is given by the theorem below.

Theorem 12.16. *A LTI system is BIBO stable if and only if the ROC of its system function H contains the unit circle (i.e., $|z| = 1$).*

Proof. To begin, we observe that the ROC of H containing the unit circle is equivalent to the condition

$$H(e^{j\Omega}) \text{ converges for all } \Omega. \tag{12.11}$$

Let h denote the inverse z transform of H (i.e., h is the impulse response of the system). From earlier in Theorem 9.11, we know that the system is BIBO stable if and only if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \tag{12.12}$$

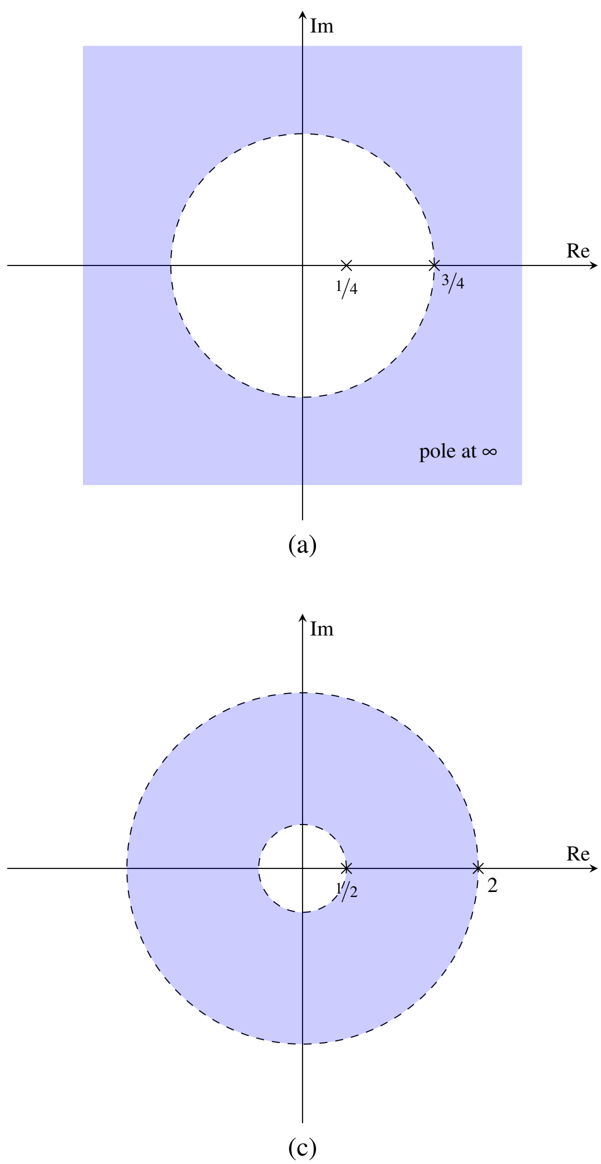


Figure 12.23: Poles and ROCs of the rational system functions in the causality example. The cases of part (a), (b), (c), and (d).

(i.e., h is absolutely summable). Thus, the proof of the theorem is equivalent to showing that (12.12) holds if and only if (12.11) holds.

First, we show that (12.12) being satisfied implies that (12.11) holds. Suppose that (12.12) holds. From the definition of H (evaluated on the unit circle), we have

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega n}.$$

Now, we note the fact that a sum is convergent if it is absolutely convergent. In particular, for any sequence f,

$$\sum_{n=-\infty}^{\infty} f(n) \text{ converges} \quad \text{if } \sum_{n=-\infty}^{\infty} |f(n)| \text{ converges.}$$

Thus, $H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega n}$ converges if $\sum_{n=-\infty}^{\infty} \left|h(n)e^{-j\Omega n}\right|$ converges. We have, however, that

$$\sum_{n=-\infty}^{\infty} \left|h(n)e^{-j\Omega n}\right| = \sum_{n=-\infty}^{\infty} |h(n)| \left|e^{-j\Omega n}\right| = \sum_{n=-\infty}^{\infty} |h(n)|.$$

So, $H(e^{j\Omega})$ converges if (12.12) holds. Thus, (12.12) being satisfied implies that (12.11) holds. The proof that (12.11) being satisfied implies that (12.12) holds is left as an exercise for the reader. ■

In the case that the system is causal, a more specific result can be derived. This result is given by the theorem below.

Theorem 12.17. *A causal LTI system with a (proper) rational system function H is BIBO stable if and only if all of the poles of H are inside the unit circle (i.e., all of the poles have magnitudes less than one).*

Proof. From Theorem 12.16, we know that the system is BIBO stable if and only if the ROC of H contains the unit circle. Since the system is causal and H is rational, the ROC of H is the exterior of the circle outside the outermost pole. Therefore, the ROC contains the unit circle if and only if the outermost pole is inside the unit circle. All of the poles of H being contained inside the unit circle is equivalent to all of the poles having magnitude less than one. ■

Observe from the preceding two theorems (i.e., Theorems 12.16 and 12.17) that, in the case of a LTI system, the characterization of the BIBO stability property is much simpler in the z domain (via the system function) than the time domain (via the impulse response). For this reason, analyzing the stability of LTI systems is typically performed using the z transform.

Example 12.33. A LTI system has the system function

$$H(z) = \frac{z-\frac{1}{3}}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}.$$

Given that the system is BIBO stable, determine the ROC of H.

Solution. Clearly, the system function H is rational with poles at $\frac{1}{4}$ and $\frac{1}{2}$. Therefore, only three possibilities exist for the ROC:

i) $|z| < \frac{1}{4}$,
ii) $\frac{1}{4} < |z| < \frac{1}{2}$, and
iii) $|z| > \frac{1}{2}$.

In order for the system to be BIBO stable, however, the ROC of H must contain the unit circle. Therefore, the ROC must be $|z| > \frac{1}{2}$. This ROC is illustrated in Figure 12.24. ■

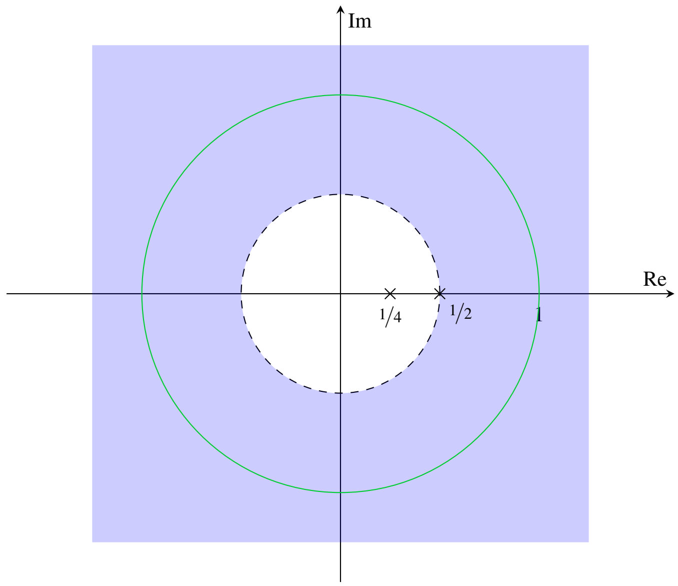


Figure 12.24: ROC for example.

Example 12.34. A LTI system is causal and has the system function

$$H(z) = \frac{z^2-1}{\left(z^2-\frac{1}{9}\right)\left(z^2-\frac{1}{4}\right)}.$$

Determine whether this system is BIBO stable.

Solution. We begin by factoring H to obtain

$$H(z) = \frac{(z+1)(z-1)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{3}\right)\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}.$$

Thus, H has poles at $-\frac{1}{2}$, $-\frac{1}{3}$, $\frac{1}{3}$, and $\frac{1}{2}$. The poles of H are plotted in Figure 12.25. Since the system is causal, the ROC of H must be the exterior of the circle passing through the outermost (finite) pole of H. So, the ROC is $|z| > \frac{1}{2}$. This ROC is shown as the shaded region in Figure 12.25. Since this ROC contains the unit circle, the system is BIBO stable. ■

Example 12.35. For the LTI system with each system function H given below, determine the ROC of H that corresponds to the system being BIBO stable.

(a) $H(z) = \dfrac{z-1}{z^2-\frac{1}{4}}$;

(b) $H(z) = \dfrac{1}{\left(z-\frac{1}{2}e^{j\pi/4}\right)\left(z-\frac{1}{2}e^{-j\pi/4}\right)\left(z-\frac{3}{2}e^{j3\pi/4}\right)\left(z-\frac{3}{2}e^{-j3\pi/4}\right)}$;

(c) $H(z) = \dfrac{z-1}{(z^2+4)(z^2-4)}$; and

(d) $H(z) = \dfrac{z}{z-1}$.

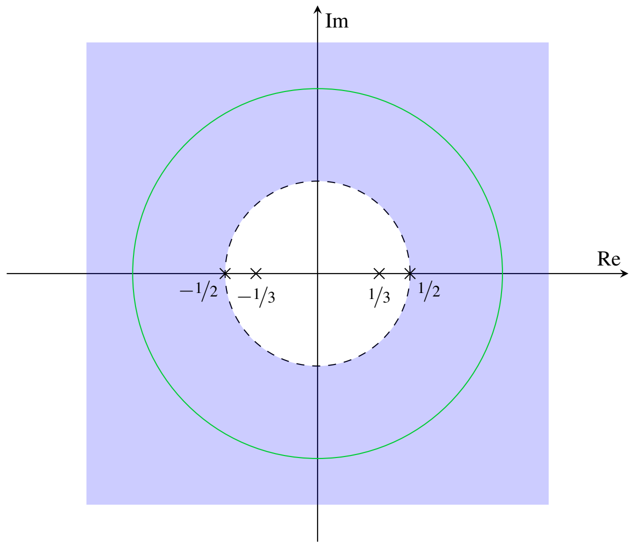


Figure 12.25: The poles and ROC of the system function.

Solution. (a) We can rewrite H as

$$H(z) = \frac{z-1}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}.$$

So, H has poles at $-\frac{1}{2}$ and $\frac{1}{2}$. The poles are shown in Figure 12.26(a). Since H is rational, the ROC must be bounded by poles or extend inwards towards the origin or outwards towards infinity. Since the poles both have the same magnitude (i.e., $\frac{1}{2}$), only two ROCs are possible:

i) $|z| < \frac{1}{2}$ and
ii) $|z| > \frac{1}{2}$.

Since we want a BIBO stable system, the ROC must contain the unit circle. Therefore, the ROC must be $|z| > \frac{1}{2}$. This ROC is shown as the shaded region in Figure 12.26(a).

(b) The function H has poles at $\frac{1}{2}e^{-j\pi/4}$, $\frac{1}{2}e^{j\pi/4}$, $\frac{3}{2}e^{-j3\pi/4}$, and $\frac{3}{2}e^{j3\pi/4}$. The poles are shown in Figure 12.26(b). Since H is rational, the ROC must be bounded by poles or extend inwards towards the origin or outwards towards infinity. Since the poles have only two distinct magnitudes (i.e., $\frac{1}{2}$ and $\frac{3}{2}$), three distinct ROCs are possible:

i) $|z| < \frac{1}{2}$,
ii) $\frac{1}{2} < |z| < \frac{3}{2}$, and
iii) $|z| > \frac{3}{2}$.

Since we want a BIBO stable system, the ROC must contain the unit circle. Therefore, the ROC must be $\frac{1}{2} < |z| < \frac{3}{2}$. This ROC is shown as the shaded region in Figure 12.26(b).

(c) Writing H with its denominator fully factored, we have

$$H(z) = \frac{z-1}{(z+2)(z-2)(z+2j)(z-2j)}.$$

The function H has poles at -2, 2, $-2j$, and $2j$. The poles are shown in Figure 12.26(c). Since H is rational, the ROC must be bounded by poles or extend inwards towards the origin or outwards towards infinity. Since the poles of H have the same magnitude (i.e., 2), two distinct ROCs are possible:

i) $|z| < 2$ and
ii) $|z| > 2$.

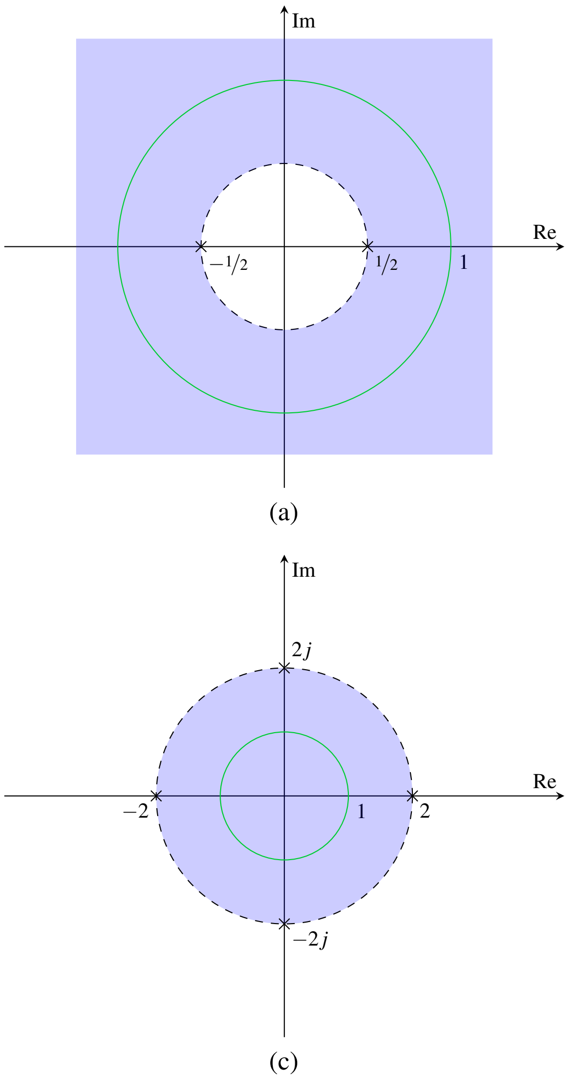


Figure 12.26: Poles and ROCs of the system function H in the (a) first, (b) second, (c) third, and (d) fourth parts of the example.

Since we want a BIBO stable system, the ROC must contain the unit circle. Therefore, the ROC must be $|z| < 2$. This ROC is shown as the shaded region in Figure 12.26(c).

(d) The function H has a pole at 1. The pole is shown in Figure 12.26(d). Since H is rational, it cannot converge at 1 (which is a pole of H). Consequently, the ROC can never contain the unit circle. Therefore, the system function H can never be associated with a BIBO stable system. ■

12.13.3 Invertibility

In this section, we consider the relationship between the system function and invertibility. The first important result is given by the theorem below.

Theorem 12.18 (Inverse of LTI system)**.** *Let $\mathcal{H}$ be a LTI system with system function H. If the inverse $\mathcal{H}^{-1}$ of $\mathcal{H}$ exists, $\mathcal{H}^{-1}$ is LTI and has a system function H_{inv} that satisfies*

$$H(z)H_{\mathsf{inv}}(z) = 1. \tag{12.13}$$

Proof. Let h denote the inverse z transform of H. From Theorem 9.9, we know that the system $\mathcal{H}$ is invertible if and only if there exists another LTI system with impulse response h_{inv} satisfying

$$h * h_{\mathsf{inv}} = \delta.$$

Let H_{inv} denote the z transform of h_{inv}. Taking the z transform of both sides of the above equation, we have

$$\mathcal{Z}\{h * h_{\mathsf{inv}}\} = \mathcal{Z}\delta.$$

From the time-domain convolution property of the z transform and Table 12.3 (i.e., $\mathcal{Z}\delta(z) = 1$), we have

$$H(z)H_{\mathsf{inv}}(z) = 1.$$

■

From the preceding theorem, we have the result below.

Theorem 12.19 (Invertibility of LTI system)**.** *A LTI system $\mathcal{H}$ with system function H is invertible if and only if there exists a function H_{inv} satisfying*

$$H(z)H_{\mathsf{inv}}(z) = 1.$$

Proof. The proof follows immediately from the result of Theorem 12.18 by simply observing that $\mathcal{H}$ being invertible is equivalent to the existence of $\mathcal{H}^{-1}$. ■

From the above theorems, we have that a LTI system $\mathcal{H}$ with system function H has an inverse if and only if a solution for H^{inv} exists in (12.13). Furthermore, if an inverse system exists, its system function is given by

$$H_{\mathsf{inv}}(z) = \frac{1}{H(z)}.$$

Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is not necessarily unique. In practice, however, we often desire a BIBO stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in one specific choice of inverse system (due to these additional constraints of BIBO stability and/or causality).

Example 12.36. Consider the LTI system with system function

$$H(z) = \frac{\left(z - \frac{1}{5}\right)\left(z - \frac{1}{2}\right)}{\left(z + \frac{1}{3}\right)\left(z - \frac{1}{3}\right)} \quad \text{for } |z| > \tfrac{1}{3}.$$

Determine all possible inverses of this system. Comment on the BIBO stability of each of these inverse systems.

Solution. The system function H_{inv} of the inverse system is given by

$$H_{\mathsf{inv}}(z) = \frac{1}{H(z)} = \frac{\left(z + \frac{1}{3}\right)\left(z - \frac{1}{3}\right)}{\left(z - \frac{1}{5}\right)\left(z - \frac{1}{2}\right)}.$$

Since the poles of H_{inv} have two distinct magnitudes (i.e., $\frac{1}{5}$ and $\frac{1}{2}$), three ROCs are possible for H_{inv}:

i) $|z| < \frac{1}{5}$,
ii) $\frac{1}{5} < |z| < \frac{1}{2}$, and
iii) $|z| > \frac{1}{2}$.

Each ROC is associated with a distinct inverse system. The first ROC is associated with a system that is not BIBO stable, since this ROC does not contain the unit circle. The second ROC is associated with a system that is not BIBO stable, since this ROC does not contain the unit circle. The third ROC is associated with a BIBO stable system, since this ROC contains the unit circle. ■

12.14 LTI Systems and Difference Equations

Many LTI systems of practical interest can be described by Nth-order linear difference equations with constant coefficients. Such a system with input x and output y can be characterized by an equation of the form

$$\sum_{k=0}^{N} b_k y(n-k) = \sum_{k=0}^{M} a_k x(n-k), \tag{12.14}$$

where $M \leq N$. Let X and Y denote the z transforms of x and y, respectively. Let H denote the system function of the system. Taking the z transform of both sides of the above equation, we obtain

$$\mathcal{Z}\left\{\sum_{k=0}^{N} b_k y(n-k)\right\}(z) = \mathcal{Z}\left\{\sum_{k=0}^{M} a_k x(n-k)\right\}(z).$$

Using the linearity property of the z transform, we can rewrite this equation as

$$\sum_{k=0}^{N} b_k \mathcal{Z}\{y(n-k)\}(z) = \sum_{k=0}^{M} a_k \mathcal{Z}\{x(n-k)\}(z).$$

Using the time-shifting property of the z transform, we have

$$\sum_{k=0}^{N} b_k z^{-k} Y(z) = \sum_{k=0}^{M} a_k z^{-k} X(z).$$

Factoring, we have

$$Y(z)\sum_{k=0}^{N} b_k z^{-k} = X(z)\sum_{k=0}^{M} a_k z^{-k}.$$

Dividing both sides of this equation by $\sum_{k=0}^{N} b_k z^{-k}$ and $X(z)$, we obtain

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} a_k z^{-k}}{\sum_{k=0}^{N} b_k z^{-k}}.$$

Since $H(z) = \frac{Y(z)}{X(z)}$, we have that H is given by

$$H(z) = \frac{\sum_{k=0}^{M} a_k z^{-k}}{\sum_{k=0}^{N} b_k z^{-k}}.$$

Observe that, for a system of the form considered above (i.e., a system characterized by an equation of the form of (12.14)), the system function is always rational. It is for this reason that rational functions are of particular interest.

Example 12.37 (Difference equation to system function)**.** A causal LTI system with input x and output y is characterized by the difference equation

$$y(n) - ay(n-1) = bx(n),$$

where a and b are real constants and $a \neq 0$. Find the system function H of this system.

Solution. Taking the z transform of the given difference equation, we obtain

$$Y(z) - az^{-1}Y(z) = bX(z).$$

Factoring, we have

$$\left(1 - az^{-1}\right)Y(z) = bX(z).$$

Dividing both sides by $1-az^{-1}$ and $X(z)$, we obtain

$$\frac{Y(z)}{X(z)} = \frac{b}{1-az^{-1}}.$$

Thus, H is given by

$$H(z) = \frac{b}{1-az^{-1}} = \frac{bz}{z-a}.$$

Since the system is causal, the ROC of H must be outside the outermost pole at a. Therefore, we conclude

$$H(z) = \frac{bz}{z-a} \text{ for } |z| > |a|. \qquad \blacksquare$$

Example 12.38 (System function to difference equation)**.** A causal LTI system with input x and output y has the system function

$$H(z) = \frac{b_0 z^2}{z^2 + a_1 z + a_2},$$

where a_1, a_2, and b_0 are real constants and $a_2 \neq 0$. Find the difference equation that characterizes this system.

Solution. Let X and Y denote the z transforms of x and y, respectively. Since $H(z) = \frac{Y(z)}{X(z)}$, we have

$$\frac{Y(z)}{X(z)} = \frac{b_0 z^2}{z^2 + a_1 z + a_2}.$$

Multiplying both sides of this equation by $z^2 + a_1 z + a_2$ and $X(z)$, we obtain

$$z^2 Y(z) + a_1 z Y(z) + a_2 Y(z) = b_0 z^2 X(z).$$

Multiplying both sides of this equation by z^{-2} in order to ensure that the largest power of z is zero, we have

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) = b_0 X(z).$$

Taking the inverse z transform of both sides of this equation (by using the linearity and translation properties of the z transform), we have

$$\begin{aligned} & \mathcal{Z}^{-1}\{Y(z)\}(n) + a_1 \mathcal{Z}^{-1}\{z^{-1}Y(z)\}(n) + a_2 \mathcal{Z}^{-1}\{z^{-2}Y(z)\}(n) = b_0 \mathcal{Z}^{-1}\{X(z)\}(n) \\ \Rightarrow \quad & y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 x(n). \end{aligned}$$

Therefore, the system is characterized by the difference equation

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 x(n). \qquad \blacksquare$$

12.15 Stability Analysis

As mentioned earlier, since BIBO stability is more easily characterized for LTI systems in the z domain than the time domain, the z domain is often used to analyze system stability. In what follows, we will consider this application of the z transform in more detail.

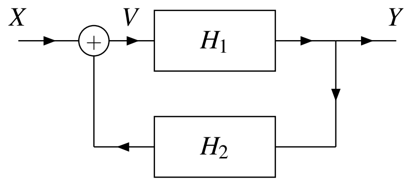


Figure 12.27: Feedback system.

Example 12.39 (Feedback system)**.** Consider the system shown in Figure 12.27 that has input z transform X and output z transform Y, and is formed by the interconnection of two causal LTI systems labelled with their system functions H_1 and H_2. The system functions H_1 and H_2 are given by

$$H_1(z) = \frac{10\beta z}{z-1} \quad \text{and} \quad H_2(z) = 1,$$

where β is a real constant. (a) Find the system function H of the (overall) system. (b) Determine the values of the parameter β for which the system is BIBO stable.

Solution. (a) From the system diagram, we can write

$$V(z) = X(z) + H_2(z)Y(z) \quad \text{and}$$
$$Y(z) = H_1(z)V(z).$$

Combining these two equations and simplifying, we obtain

$$\begin{aligned} & Y(z) = H_1(z)[X(z) + H_2(z)Y(z)] \\ \Rightarrow \quad & Y(z) = H_1(z)X(z) + H_1(z)H_2(z)Y(z) \\ \Rightarrow \quad & Y(z)[1 - H_1(z)H_2(z)] = H_1(z)X(z) \\ \Rightarrow \quad & \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 - H_1(z)H_2(z)}. \end{aligned}$$

Since $H(z) = \frac{Y(z)}{X(z)}$, we have

$$H(z) = \frac{H_1(z)}{1 - H_1(z)H_2(z)}.$$

Substituting the given expressions for H_1 and H_2, and simplifying, we can write

$$\begin{aligned} H(z) &= \frac{\frac{10\beta z}{z-1}}{1 - \frac{10\beta z}{z-1}(1)} \\ &= \frac{\left(\frac{10\beta z}{z-1}\right)}{\left(\frac{z-1-10\beta z}{z-1}\right)} \\ &= \frac{10\beta z}{z - 1 - 10\beta z} \\ &= \frac{10\beta z}{(1-10\beta)z - 1} \\ &= \frac{10\beta}{1-10\beta}\left(\frac{z}{z - \frac{1}{1-10\beta}}\right). \end{aligned}$$

(b) In order to assess the BIBO stability of the system, we need to consider the poles of the system function H. From the expression for H above, we can see that H has a single pole at $\frac{1}{1-10\beta}$. Since the system is causal, the system is BIBO stable if and only if all of the poles are strictly inside the unit circle. Thus, we have that

$$\begin{aligned} & \left|\tfrac{1}{1-10\beta}\right| < 1 \\ \Rightarrow \quad & \tfrac{1}{|1-10\beta|} < 1 \\ \Rightarrow \quad & |1-10\beta| > 1 \\ \Rightarrow \quad & 1-10\beta > 1 \text{ or } 1-10\beta < -1 \\ \Rightarrow \quad & 10\beta < 0 \text{ or } 10\beta > 2 \\ \Rightarrow \quad & \beta < 0 \text{ or } \beta > \tfrac{1}{5} \end{aligned}$$

Therefore, the system is BIBO stable if and only if

$$\beta < 0 \text{ or } \beta > \tfrac{1}{5}. \qquad \blacksquare$$

12.16 Unilateral z Transform

As mentioned earlier, two different versions of the z transform are commonly employed, namely, the bilateral and unilateral versions. So far, we have considered only the bilateral z transform. Now, we turn our attention to the unilateral z transform. The **unilateral z transform** of the sequence x is denoted as $\mathcal{Z}_{\mathsf{u}}x$ or X and is defined as

$$\mathcal{Z}_{\mathsf{u}}x(z) = X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}. \tag{12.15}$$

The **inverse unilateral z transform** has the same definition as in the case of the bilateral transform, namely (12.3).

Comparing the definitions of the unilateral and bilateral z transforms given by (12.15) and (12.2), respectively, we can see that these definitions only differ in the lower limit of summation. Due to the similarity in these definitions, an important relationship exists between these two transforms, as we shall now demonstrate. Consider the bilateral z transform of the sequence xu for an arbitrary sequence x. We have

$$\begin{aligned} \mathcal{Z}\{xu\}(z) &= \sum_{n=-\infty}^{\infty} x(n)u(n)z^{-n} \\ &= \sum_{n=0}^{\infty} x(n)z^{-n} \\ &= \mathcal{Z}_{\mathsf{u}}x(z). \end{aligned}$$

In other words, the unilateral z transform of the sequence x is simply the bilateral z transform of the sequence xu. Since $\mathcal{Z}_{\mathsf{u}}x = \mathcal{Z}\{xu\}$ and xu is always right sided, the ROC associated with $\mathcal{Z}_{\mathsf{u}}x$ is always the one corresponding to a right-sided sequence (i.e., the exterior of a circle or the entire complex plane). For this reason, we often do not explicitly indicate the ROC when working with the unilateral z transform.

From earlier in this chapter, we know that the bilateral z transform is invertible. That is, if the sequence x has the bilateral z transform $X = \mathcal{Z}x$, then $\mathcal{Z}^{-1}X = x$. Now, let us consider the invertibility of the unilateral z transform. To do this, we must consider the quantity $\mathcal{Z}_{\mathsf{u}}^{-1}\mathcal{Z}_{\mathsf{u}}x$. Since $\mathcal{Z}_{\mathsf{u}}x = \mathcal{Z}\{xu\}$ and the inverse equations for the unilateral and bilateral z transforms are identical, we can write

$$\begin{aligned}
\mathcal{Z}_u^{-1}\mathcal{Z}_u x(n) &= \mathcal{Z}_u^{-1}\{\mathcal{Z}\{xu\}\}(n) \\
&= \mathcal{Z}^{-1}\{\mathcal{Z}\{xu\}\}(n) \\
&= x(n)u(n) \\
&= \begin{cases} x(n) & n \geq 0 \\ 0 & n < 0. \end{cases}
\end{aligned}$$

Thus, we have that $\mathcal{Z}_u^{-1}\mathcal{Z}_u x = x$ only if x is causal. In other words, the unilateral z transform is invertible only for causal sequences. For noncausal sequences, we can only recover $x(n)$ for $n \geq 0$. In essence, the unilateral z transform discards all information about the value of the sequence x at n for $n < 0$. Since this information is discarded, it cannot be recovered by an inverse unilateral z transform operation.

Due to the close relationship between the unilateral and bilateral z transforms, these two transforms have some similarities in their properties. Since these two transforms are not identical, however, their properties differ in some cases, often in subtle ways. The properties of the unilateral z transform are summarized in Table 12.4.

By comparing the properties of the unilateral and bilateral z transforms listed in Tables 12.4 and 12.2, respectively, we can see that the unilateral z transform has some of the same properties as its bilateral counterpart, namely, the linearity, modulation, conjugation, upsampling, downsampling, and z-domain differentiation properties. The initial-value and final-value theorems also apply in the case of the unilateral z transform.

Since the unilateral and bilateral z transforms are defined differently, their properties also differ in some cases. These differences can be seen by comparing the bilateral z transform properties listed in Table 12.2 with the unilateral z transform properties listed in Table 12.4. In the unilateral case, we can see that:

1. the translation property has been replaced by the time-delay and time-advance properties;
2. the time-reversal property has been dropped;
3. the convolution property has the additional requirement that the sequences being convolved must be causal;
4. the differencing property has an extra term in the expression for $\mathcal{Z}_u\{x(n) - x(n-1)\}$; and
5. the accumulation property has a different lower limit in the summation (namely, 0 instead of $-\infty$).

Since $\mathcal{Z}_u x = \mathcal{Z}\{xu\}$, we can easily generate a table of unilateral z transform pairs from a table of bilateral transform pairs. Using the bilateral z transform pairs from Table 12.3, and the preceding relationship between the unilateral and bilateral z transforms, we can trivially deduce the unilateral z transform pairs in Table 12.5. Since, in the unilateral case, the ROC is always the exterior of a circle (or the entire complex plane), we do not explicitly indicate the ROC in the table. That is, the ROC is implicitly assumed to be the exterior of a circle (or the entire complex plane).

The inverse unilateral z transform is computed through the same means used in the bilateral case (e.g., partial fraction expansions). The only difference is that the ROC is always assumed to correspond to a right-sided sequence.

12.17 Solving Difference Equations Using the Unilateral z Transform

Many systems of interest in engineering applications can be characterized by constant-coefficient linear difference equations. As it turns out, a system that is described by such an equation need not be linear. In particular, the system will be linear only if the initial conditions for the difference equation are all zero. If one or more of the initial conditions is nonzero, then the system is what we refer to as **incrementally linear**. For our purposes here, incrementally linear systems can be thought of as a generalization of linear systems. The unilateral z transform is sometimes quite useful due to its ability to easily handle nonzero initial conditions. For example, one common use of the unilateral z transform is in solving constant-coefficient linear difference equations with nonzero initial conditions. In what follows, we consider some examples that exploit the unilateral z transform to this end.

Table 12.4: Properties of the unilateral z transform

Property	Time Domain	z Domain
Linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(z)+a_2X_2(z)$
Time Delay	$x(n-1)$	$z^{-1}X(z)+x(-1)$
Time Advance	$x(n+1)$	$zX(z)-zx(0)$
Complex Modulation	$a^nx(n)$	$X(z/a)$
Conjugation	$x^*(n)$	$X^*(z^*)$
Upsampling	$(\uparrow M)x(n)$	$X(z^M)$
Downsampling	$(\downarrow M)x(n)$	$\frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{-j2\pi k/M}z^{1/M}\right)$
Convolution	$x_1 * x_2(n), x_1$ and x_2 are causal	$X_1(z)X_2(z)$
Z-Domain Differentiation	$nx(n)$	$-z\frac{d}{dz}X(z)$
Differencing	$x(n)-x(n-1)$	$\frac{z-1}{z}X(z)-x(-1)=(1-z^{-1})X(z)-x(-1)$
Accumulation	$\sum_{k=0}^{n} x(k)$	$\frac{z}{z-1}X(z)=\frac{1}{1-z^{-1}}X(z)$

Property	
Initial Value Theorem	$x(0)=\lim_{z\to\infty} X(z)$
Final Value Theorem	$\lim_{n\to\infty} x(n)=\lim_{z\to 1}[(z-1)X(z)]$

Table 12.5: Transform pairs for the unilateral z transform

Pair	$x(n)$, $n \geq 0$	$X(z)$
1	$\delta(n)$	1
2	1	$\dfrac{z}{z-1} = \dfrac{1}{1-z^{-1}}$
3	n	$\dfrac{z}{(z-1)^2} = \dfrac{z^{-1}}{(1-z^{-1})^2}$
4	a^n	$\dfrac{z}{z-a} = \dfrac{1}{1-az^{-1}}$
5	$a^n n$	$\dfrac{az}{(z-a)^2} = \dfrac{az^{-1}}{(1-az^{-1})^2}$
6	$\cos(\Omega_0 n)$	$\dfrac{z(z-\cos\Omega_0)}{z^2-(2\cos\Omega_0)z+1} = \dfrac{1-(\cos\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$
7	$\sin(\Omega_0 n)$	$\dfrac{z\sin\Omega_0}{z^2-(2\cos\Omega_0)z+1} = \dfrac{(\sin\Omega_0)z^{-1}}{1-(2\cos\Omega_0)z^{-1}+z^{-2}}$
8	$a^n\cos(\Omega_0 n)$	$\dfrac{z(z-a\cos\Omega_0)}{z^2-(2a\cos\Omega_0)z+a^2} = \dfrac{1-(a\cos\Omega_0)z^{-1}}{1-(2a\cos\Omega_0)z^{-1}+a^2z^{-2}}$
9	$a^n\sin(\Omega_0 n)$	$\dfrac{za\sin\Omega_0}{z^2-(2a\cos\Omega_0)z+a^2} = \dfrac{(a\sin\Omega_0)z^{-1}}{1-(2a\cos\Omega_0)z^{-1}+a^2z^{-2}}$

Example 12.40 (Unilateral z transform of two-unit delay)**.** Let x and y be sequences related by

$$y(n) = x(n-2).$$

Find the unilateral z transform Y of y in terms of the unilateral z transform X of x.

Solution. Define the sequence

$$v(n) = x(n-1) \tag{12.16}$$

so that

$$y(n) = v(n-1). \tag{12.17}$$

Let V denote the unilateral z transform of v. Taking the unilateral z transform of (12.16) (using the delay property), we have

$$V(z) = z^{-1}X(z) + x(-1). \tag{12.18}$$

Taking the unilateral z transform of (12.17) (using the delay property), we have

$$Y(z) = z^{-1}V(z) + v(-1). \tag{12.19}$$

Substituting the expression for V in (12.18) into (12.19) and using the fact that $v(-1) = x((-1)-1) = x(-2)$, we have

$$\begin{aligned} Y(z) &= z^{-1}V(z) + v(-1) \\ &= z^{-1}\left[z^{-1}X(z) + x(-1)\right] + x(-2) \\ &= z^{-2}X(z) + z^{-1}x(-1) + x(-2). \end{aligned}$$

Thus, we have that

$$Y(z) = z^{-2}X(z) + z^{-1}x(-1) + x(-2).$$

■

Example 12.41 (Investment earning compound interest)**.** An investment earns compound interest at a fixed rate of r percent annually with the fraction of a year that constitutes the compounding period being α. (For example, $\alpha = \frac{1}{12}$ corresponds to monthly compounding.) Let y be a sequence such that $y(n)$ is the value of the investment at the start of the nth compounding period. Then, y satisfies the difference equation

$$y(n) = \left(1 + \tfrac{\alpha r}{100}\right) y(n-1).$$

Consider an investment with an initial value of \$1000 that earns interest at a fixed rate of 6% annually compounded monthly. Use the unilateral z transform to find the value of the investment after 10 years.

Solution. Since $r = 6$ and $\alpha = \frac{1}{12}$, y is the solution to the difference equation

$$\begin{aligned} y(n) &= \left(1 + \tfrac{6}{12(100)}\right) y(n-1) \\ &= \tfrac{201}{200} y(n-1), \end{aligned}$$

subject to the initial condition $y(-1) = 1000$. Taking the unilateral z transform of this equation (with the help of the delay property) and solving for Y, we obtain

$$\begin{aligned} & Y(z) = \tfrac{201}{200}\left[z^{-1}Y(z) + y(-1)\right] \\ \Rightarrow\quad & Y(z) = \tfrac{201}{200} z^{-1} Y(z) + \tfrac{201}{200} y(-1) \\ \Rightarrow\quad & Y(z) = \tfrac{201}{200} z^{-1} Y(z) + 1005 \\ \Rightarrow\quad & \left[1 - \tfrac{201}{200} z^{-1}\right] Y(z) = 1005 \\ \Rightarrow\quad & Y(z) = \frac{1005}{1 - \frac{201}{200} z^{-1}}. \end{aligned}$$

Taking the inverse z transform of y, we obtain

$$y(n) = 1005 \left(\tfrac{201}{200}\right)^n \quad \text{for } n \geq 0.$$

The value of n corresponding to 10 years is given by

$$n = -1 + 12(10) = 119.$$

Thus, the value of the investment after 10 years is

$$y(119) = 1005 \left(\tfrac{201}{200}\right)^{119} \approx 1819.39.$$

■

Example 12.42 (Savings withdrawal schedule)**.** Consider a savings account that earns compounding interest at a fixed rate of 6% annually and is compounded monthly. The balance of the account is initially \$10000. Starting in the first month of the second year, monthly withdrawals of \$100 are started. Determine how long it will take in order for the account to become overdrawn (i.e., have a negative balance).

Solution. Let $y(n)$ denote the balance of the account at the start of the nth compounding period, and let $x(n)$ denote the amount withdrawn at the start of the same compounding period. Let y be a sequence such that $y(n)$ is the account balance at the start of the nth compounding period. Then, y satisfies the difference equation

$$y(n) = \left(1 + \tfrac{\alpha r}{100}\right) y(n-1) + x(n),$$

where $x(n)$ is the amount added to the account at the start of the nth compounding period, r is the annual interest rate in percent, and α is the fraction of a year in the compounding period (e.g., $\alpha = \frac{1}{12}$ for monthly compounding). Substituting the given values of $r = 6$ and $\alpha = \frac{1}{12}$, we obtain

$$y(n) = \left(\tfrac{201}{200}\right) y(n-1) + x(n), \tag{12.20}$$

The initial condition $y(-1)$ corresponds to the initial account balance. Thus, we have

$$y(-1) = 10000. \tag{12.21}$$

The time index n is in units of compounding periods, with $n = -1$ corresponding to the starting time (i.e., time "zero"). Withdrawals commence at the beginning of the second year, which corresponds to $n = -1 + 12 = 11$. So, we have

$$x(n) = -100u(n-11). \tag{12.22}$$

Taking the unilateral z transform of the causal sequence x (in (12.22)), we have

$$X(z) = \frac{-100z^{-11}}{1-z^{-1}}.$$

Taking the unilateral z transform of the difference equation (12.20), we have

$$\begin{aligned}
& Y(z) = \tfrac{201}{200}\left(z^{-1}Y(z) + y(-1)\right) + X(z) \\
\Rightarrow\quad & Y(z) = \tfrac{201}{200}z^{-1}Y(z) + \tfrac{201}{200}y(-1) + X(z) \\
\Rightarrow\quad & Y(z) = \tfrac{201}{200}z^{-1}Y(z) + 10050 - \frac{100z^{-11}}{1-z^{-1}}.
\end{aligned}$$

Rearranging the above equation to solve for Y, we have

$$\begin{aligned}
& \left(1 - \tfrac{201}{200}z^{-1}\right)Y(z) = 10050 - \frac{100z^{-11}}{1-z^{-1}} \\
\Rightarrow\quad & Y(z) = \frac{10050}{1 - \frac{201}{200}z^{-1}} - \frac{100z^{-11}}{(1-z^{-1})\left(1 - \frac{201}{200}z^{-1}\right)}.
\end{aligned}$$

Now, we find a partial fraction expansion of the second term in the preceding equation for Y excluding the z^{-11} factor. This expansion has the form

$$\frac{100}{(1-z^{-1})\left(1-\frac{201}{200}z^{-1}\right)} = \frac{A_1}{1-z^{-1}} + \frac{A_2}{1-\frac{201}{200}z^{-1}}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1 &= \left(1-z^{-1}\right)Y(z)\big|_{z=1} \\
&= \left.\frac{1}{1-\frac{201}{200}z^{-1}}\right|_{z=1} \\
&= \frac{1}{-1/200} = -20000 \quad \text{and} \\
A_2 &= \left(1-\tfrac{201}{200}z^{-1}\right)Y(z)\big|_{z=201/200} \\
&= \left.\frac{100}{1-z^{-1}}\right|_{z=201/200} \\
&= \frac{100}{1/201} = 20100.
\end{aligned}$$

Substituting the above partial fraction expansion into the above expression for $Y(z)$, we have

$$\begin{aligned}
Y(z) &= \frac{10050}{1-\frac{201}{200}z^{-1}} - z^{-11}\left[\frac{-20000}{1-z^{-1}} + \frac{20100}{1-\frac{201}{200}z^{-1}}\right] \\
&= 10050\frac{1}{1-\frac{201}{200}z^{-1}} + 20000z^{-11}\frac{1}{1-z^{-1}} - 20100z^{-11}\frac{1}{1-\frac{201}{200}z^{-1}}.
\end{aligned}$$

Taking the inverse z transform of Y, we have

$$y(n) = 10050\left(\tfrac{201}{200}\right)^n u(n) + 20000u(n-11) - 20100\left(\tfrac{201}{200}\right)^{n-11} u(n-11) \quad \text{for } n \geq 0.$$

The sequence y starts monotonically decreasing for $n > 11$ and $y(160) \approx 61.20 \geq 0$ and $y(161) \approx -38.50 < 0$. Therefore, the account becomes overdrawn at $n = 161$, which corresponds to $161-(-1) = 162$ months from the time of the initial balance. ■

Example 12.43 (Fibonacci sequence)**.** The Fibonacci sequence f is defined as

$$f_n = \begin{cases} f_{n-1} + f_{n-2} & n \geq 2 \\ 1 & n = 1 \\ 0 & n = 0, \end{cases}$$

where n is a nonnegative integer. The first few elements of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots.$$

Use the z transform to find a closed-form expression for the nth Fibonacci number f_n.

Solution. The Fibonacci sequence can be expressed in terms of a linear difference equation with constant coefficients. In particular, the Fibonacci sequence is given by the solution x to the difference equation

$$x(n) = x(n-1) + x(n-2) + \delta(n-1), \tag{12.23a}$$

with the initial conditions

$$x(-1) = x(-2) = 0. \tag{12.23b}$$

We can easily confirm that this equation is correct. Substituting $n = 0$ and $n = 1$ into (12.23a) yields

$$\begin{aligned}
x(0) &= x(-1) + x(-2) + \delta(-1) \\
&= 0 + 0 + 0 \\
&= 0 \\
&= f_0 \quad \text{and} \\
x(1) &= x(0) + x(-1) + \delta(0) \\
&= 0 + 0 + 1 \\
&= 1 \\
&= f_1.
\end{aligned}$$

Substituting $n \geq 2$ into (12.23a) yields

$$\begin{aligned}
x(n) &= x(n-1) + x(n-2) + \delta(n-1) \\
&= x(n-1) + x(n-2) + 0 \\
&= x(n-1) + x(n-2) \\
&= f_n.
\end{aligned}$$

Thus, the Fibonacci sequence is, in fact, the solution of (12.23a).

Now, we employ the z transform to solve the above difference equation. Taking the unilateral z transform of the difference equation, we obtain

$$\begin{aligned}
&\mathcal{Z}_{\mathsf{u}}x(z) = z^{-1}\mathcal{Z}_{\mathsf{u}}x(z) + x(-1) + z^{-1}\left[z^{-1}\mathcal{Z}_{\mathsf{u}}x(z) + x(-1)\right] + x(-2) + z^{-1}\mathcal{Z}_{\mathsf{u}}\delta(z) + \delta(-1).\\
\Rightarrow\quad &X(z) = z^{-1}X(z) + x(-1) + z^{-2}X(z) + z^{-1}x(-1) + x(-2) + z^{-1}\mathcal{Z}_{\mathsf{u}}\delta(z) + \delta(-1).\\
\Rightarrow\quad &X(z) = z^{-1}X(z) + z^{-2}X(z) + z^{-1}.\\
\Rightarrow\quad &(1 - z^{-1} - z^{-2})X(z) = z^{-1}\\
\Rightarrow\quad &X(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}.
\end{aligned}$$

(Note that, since all of the initial conditions are zero, we could have simply employed the bilateral z transform.) So, we have

$$X(z) = \frac{z}{z^2 - z - 1}.$$

We factor the denominator polynomial in order to determine the poles of X. Using the quadratic formula, we find the roots of $z^2 - z - 1$ to be $\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)} = \frac{1\pm\sqrt{5}}{2}$. Thus, we have

$$X(z) = \frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)}.$$

Now, we find a partial fraction expansion of

$$\frac{X(z)}{z} = \frac{1}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)}.$$

The expansion has the form

$$\frac{X(z)}{z} = \frac{A_1}{z - \frac{1+\sqrt{5}}{2}} + \frac{A_2}{z - \frac{1-\sqrt{5}}{2}}.$$

Calculating the expansion coefficients, we have

$$\begin{aligned}
A_1 &= \left(z - \tfrac{1+\sqrt{5}}{2}\right)\left(\tfrac{X(z)}{z}\right)\Big|_{z=\frac{1+\sqrt{5}}{2}}\\
&= \left(\frac{1}{z - \frac{1-\sqrt{5}}{2}}\right)\Bigg|_{z=\frac{1+\sqrt{5}}{2}}\\
&= \tfrac{1}{\sqrt{5}} \quad \text{and}\\
A_2 &= \left(z - \tfrac{1-\sqrt{5}}{2}\right)\left(\tfrac{X(z)}{z}\right)\Big|_{z=\frac{1-\sqrt{5}}{2}}\\
&= \left(\frac{1}{z - \frac{1+\sqrt{5}}{2}}\right)\Bigg|_{z=\frac{1-\sqrt{5}}{2}}\\
&= -\tfrac{1}{\sqrt{5}}.
\end{aligned}$$

Thus, we have

$$X(z) = \tfrac{1}{\sqrt{5}}\left(\frac{z}{z - \frac{1+\sqrt{5}}{2}}\right) - \tfrac{1}{\sqrt{5}}\left(\frac{z}{z - \frac{1-\sqrt{5}}{2}}\right).$$

Taking the inverse z transform, we obtain

$$x(n) = \tfrac{1}{\sqrt{5}}\left(\tfrac{1+\sqrt{5}}{2}\right)^n - \tfrac{1}{\sqrt{5}}\left(\tfrac{1-\sqrt{5}}{2}\right)^n \quad \text{for } n \geq 0.$$

■

Example 12.44 (First-order difference equation)**.** Consider the causal system with input x and output y characterized by the difference equation

$$2y(n) + y(n-1) = x(n).$$

If $x(n) = \left(\tfrac{1}{4}\right)^n u(n)$ and $y(-1) = 2$, find y.

Solution. Taking the unilateral z transform of x, we have

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}.$$

Taking the unilateral z transform of both sides of the given difference equation yields

$$2Y(z) + z^{-1}Y(z) + y(-1) = X(z).$$

Substituting the above expression for X and the given initial condition into this equation yields

$$\begin{aligned}
& 2Y(z) + z^{-1}Y(z) + 2 = \frac{1}{1-\frac{1}{4}z^{-1}} \\
\Leftrightarrow \quad & \left(2 + z^{-1}\right)Y(z) = \frac{1}{1-\frac{1}{4}z^{-1}} - 2 \\
\Leftrightarrow \quad & 2\left(1 + \tfrac{1}{2}z^{-1}\right)Y(z) = \frac{1 - 2 + \frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-1}} \\
\Leftrightarrow \quad & 2\left(1 + \tfrac{1}{2}z^{-1}\right)Y(z) = \frac{-1 + \frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-1}} \\
\Leftrightarrow \quad & Y(z) = \frac{-\frac{1}{2} + \frac{1}{4}z^{-1}}{(1-\frac{1}{4}z^{-1})(1+\frac{1}{2}z^{-1})}.
\end{aligned}$$

Now, we must find a partial fraction expansion of Y. Such an expansion is of the form

$$Y(z) = \frac{A_1}{1-\frac{1}{4}z^{-1}} + \frac{A_2}{1+\frac{1}{2}z^{-1}}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1 &= \left(1 - \tfrac{1}{4}z^{-1}\right)Y(z)\Big|_{z=1/4} \\
&= \left.\frac{-\frac{1}{2} + \frac{1}{4}z^{-1}}{1 + \frac{1}{2}z^{-1}}\right|_{z=1/4} = \tfrac{1/2}{3} \\
&= \tfrac{1}{6} \quad \text{and} \\
A_2 &= (1 + \tfrac{1}{2}z^{-1})Y(z)\Big|_{z=-1/2} \\
&= \left.\frac{-\frac{1}{2} + \frac{1}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}}\right|_{z=-1/2} = \tfrac{-1}{3/2} \\
&= -\tfrac{2}{3}.
\end{aligned}$$

So, we can rewrite Y as

$$Y(z) = \tfrac{1}{6}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right) - \tfrac{2}{3}\left(\frac{1}{1+\frac{1}{2}z^{-1}}\right).$$

Taking the inverse unilateral z transform of Y yields

$$y(n) = \tfrac{1}{6}\left(\tfrac{1}{4}\right)^n - \tfrac{2}{3}\left(-\tfrac{1}{2}\right)^n \quad \text{for } n \geq 0.$$

■

Example 12.45 (Second-order difference equation)**.** Consider the causal system with input x and output y characterized by the difference equation

$$10y(n) + 3y(n-1) - y(n-2) = x(n).$$

If $x(n) = 6u(n)$, $y(-1) = 3$, and $y(-2) = 6$, find y.

Solution. Taking the unilateral z transform of x, we obtain

$$X(z) = \frac{6}{1-z^{-1}}.$$

Taking the unilateral z transform of both sides of the given difference equation yields

$$\begin{aligned}
&10Y(z) + 3\left[z^{-1}Y(z) + y(-1)\right] - \left(z^{-1}\left[z^{-1}Y(z) + y(-1)\right] + y(-2)\right) = X(z) \\
\Rightarrow\quad &10Y(z) + 3z^{-1}Y(z) + 3y(-1) - \left(z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right) = X(z) \\
\Rightarrow\quad &10Y(z) + 3z^{-1}Y(z) + 3y(-1) - z^{-2}Y(z) - z^{-1}y(-1) - y(-2) = X(z) \\
\Rightarrow\quad &10Y(z) + 3z^{-1}Y(z) - z^{-2}Y(z) + 3y(-1) - y(-2) - z^{-1}y(-1) = X(z).
\end{aligned}$$

Rearranging in order to solve for Y, we have

$$\begin{aligned}
&\left(10 + 3z^{-1} - z^{-2}\right)Y(z) = -3y(-1) + y(-2) + z^{-1}y(-1) + X(z) \\
\Rightarrow\quad &Y(z) = \frac{-3y(-1) + y(-2) + z^{-1}y(-1) + X(z)}{10 + 3z^{-1} - z^{-2}}.
\end{aligned}$$

Now, we factor the denominator of Y. Solving for the roots of $10 + 3z^{-1} - z^{-2} = z^{-2}(10z^2 + 3z - 1) = 0$, we obtain $\frac{-3\pm\sqrt{3^2-4(10)(-1)}}{2(10)} = \frac{3\pm7}{20}$ So, the roots of $10 + 3z^{-1} - z^{-2} = 0$ are $\frac{1}{5}$ and $-\frac{1}{2}$. Thus, we have

$$Y(z) = \frac{-3y(-1) + y(-2) + z^{-1}y(-1) + X(z)}{10\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)}.$$

Substituting the above expression for X and the given initial conditions into the above equation for Y, we obtain

$$\begin{aligned}
Y(z) &= \frac{-3(3) + 6 + 3z^{-1} + \frac{6}{1-z^{-1}}}{10\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)} \\
&= \frac{-3 + 3z^{-1} + \frac{6}{1-z^{-1}}}{10\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)} \\
&= \frac{(-3 + 3z^{-1})(1-z^{-1}) + 6}{10(1-z^{-1})\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)} \\
&= \frac{-3 + 6z^{-1} - 3z^{-2} + 6}{10(1-z^{-1})\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)} \\
&= \frac{3 + 6z^{-1} - 3z^{-2}}{10(1-z^{-1})\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)}.
\end{aligned}$$

Now, we find a partial fraction expansion for Y. Such an expansion has the form

$$Y(z) = \frac{A_1}{1-z^{-1}} + \frac{A_2}{1+\frac{1}{2}z^{-1}} + \frac{A_3}{1-\frac{1}{5}z^{-1}}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1 &= \left(1-z^{-1}\right)Y(z)\big|_{z=1} \\
&= \left.\frac{3+6z^{-1}-3z^{-2}}{10\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)}\right|_{z=1} = \frac{6}{10(\frac{3}{2})(\frac{4}{5})} \\
&= \tfrac{1}{2}, \\
A_2 &= \left(1+\tfrac{1}{2}z^{-1}\right)Y(z)\big|_{z=-1/2} \\
&= \left.\frac{3+6z^{-1}-3z^{-2}}{10\left(1-z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)}\right|_{z=-1/2} = \frac{3-12-12}{10(3)\left(\frac{7}{5}\right)} = \tfrac{-21}{42} \\
&= -\tfrac{1}{2}, \quad \text{and} \\
A_3 &= \left(1-\tfrac{1}{5}z^{-1}\right)Y(z)\big|_{z=1/5} \\
&= \left.\frac{3+6z^{-1}-3z^{-2}}{10\left(1-z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}\right|_{z=1/5} = \tfrac{3+30-75}{10(-4)(7/2)} = \tfrac{42}{140} \\
&= \tfrac{3}{10}.
\end{aligned}$$

So, we can rewrite Y as

$$Y(z) = \tfrac{1}{2}\left(\frac{1}{1-z^{-1}}\right) - \tfrac{1}{2}\left(\frac{1}{1+\frac{1}{2}z^{-1}}\right) + \tfrac{3}{10}\left(\frac{1}{1-\frac{1}{5}z^{-1}}\right).$$

Taking the inverse z transform of Y, we obtain

$$y(n) = \tfrac{1}{2} - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)^n + \tfrac{3}{10}\left(\tfrac{1}{5}\right)^n \quad \text{for } n \geq 0.$$

■

12.18 Exercises

12.18.1 Exercises Without Answer Key

12.1 Using the definition of the z transform, find the z transform X of each sequence x given below.
(a) $x(n) = nu(n)$;
(b) $x(n) = na^n u(n)$, where a is a complex constant; and
(c) $x(n) = \cos(an)u(n)$, where a is a nonzero real constant.

12.2 Using properties of the z transform and a table of z transform pairs, find the z transform X of each sequence x given below.
(a) $x(n) = n\left(\frac{1}{2}\right)^{|n|}$;
(b) $x(n) = n\left(\frac{1}{3}\right)^{|n-1|}$;
(c) $x(n) = \left(\frac{2}{3}\right)^{n-1} u(n-1)$;
(d) $x(n) = 2^n u(-n)$;
(e) $x(n) = \delta(n+5) - \delta(n-5)$;
(f) $x(n) = 3^n u(-n-2)$;
(g) $x(n) = x_1 * x_2(n)$, where $x_1(n) = 2^{n-1}u(n)$ and $x_2(n) = \cos(\frac{\pi}{6}n + \frac{\pi}{3})u(n)$;
(h) $x(n) = n\sin\left(\frac{\pi}{2}n\right)u(n)$;
(i) $x(n) = u * x_1(n)$, where $x_1(n) = u(n-1)$; and
(j) $x(n) = c^n \cos(an+b)u(-n-1)$, where $a, b \in \mathbb{R}$, $c \in \mathbb{C}$, $a \neq 0$, and $|c| > 1$.

12.3 For each z transform algebraic expression X given below, determine all possible ROCs of X.
(a) $X(z) = \dfrac{3}{1+\frac{1}{3}z^{-1}}$;
(b) $X(z) = \dfrac{z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+3z^{-1}\right)}$;
(c) $X(z) = 1 + z^{-1}$;
(d) $X(z) = \dfrac{z}{(z^2+\frac{1}{4})(z^2-\frac{1}{4})}$; and
(e) $X(z) = \dfrac{z}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{5}\right)}$.

12.4 Determine whether the sequence x with each z transform X given below is finite duration, right sided but not left sided, left sided but not right sided, or two sided.
(a) $X(z) = \dfrac{5}{1+\frac{1}{3}z^{-1}}$ for $|z| > \frac{1}{3}$;
(b) $X(z) = \dfrac{z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$ for $\frac{1}{4} < |z| < \frac{1}{2}$;
(c) $X(z) = \dfrac{z^{-1}}{1-z^{-1}}$ for $|z| < 1$;
(d) $X(z) = \frac{1}{3}(1+z^{-1}+z^{-2})$ for $|z| > 0$; and
(e) $X(z) = \frac{1}{3}(z^2+z+1)$ for all finite z.

12.5 Find the inverse z transform x of each function X given below. If a partial fraction expansion is used, it must be chosen so as to avoid unnecessary time shifts in the final answer for x.
(a) $X(z) = \dfrac{10z^2-15z+3}{(z-3)\left(z-\frac{1}{3}\right)}$ for $|z| > 3$;

(b) $X(z) = \dfrac{5z^2-8z+2}{(z-2)\left(z-\frac{1}{2}\right)}$ for $\frac{1}{2} < |z| < 2$;

(c) $X(z) = \dfrac{1-z^{-9}}{z-1}$ for $|z| > 1$ [Hint: A PFE is not necessary.];

(d) $X(z) = \dfrac{2z^3}{\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}$ for $|z| > \frac{3}{4}$;

(e) $X(z) = \dfrac{1}{(z-1)\left(z+\frac{1}{2}\right)}$ for $|z| > 1$;

(f) $X(z) = \dfrac{2z}{1-z} + 1 + z^{-1}$ for $|z| > 1$; and

(g) $X(z) = \dfrac{\frac{5}{6}z^{-1}}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{2}{9}z^{-1}\right)}$ for $|z| < \frac{2}{9}$.

12.6 For each algebraic expression X given below for the z transform of x, find all possible x.

(a) $X(z) = \dfrac{2-z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{3}{2}z^{-1}\right)}$;

(b) $X(z) = \dfrac{5}{1+\frac{1}{3}z^{-1}}$; and

(c) $X(z) = \dfrac{-z^2+\frac{1}{4}z}{(z-1)\left(z-\frac{3}{4}\right)}$.

12.7 Find the inverse z transform x of each function X given below.

(a) $X(z) = e^{a/z}$ for $|z| > 0$, where a is a complex constant [Hint: Use (F.6).];

(b) $X(z) = -\ln(1-az)$ for $|z| < |a|^{-1}$ [Hint: Use (F.7).];

(c) $X(z) = -\ln(1-a^{-1}z^{-1})$ for $|z| > |a|^{-1}$ [Hint: Use (F.7).]; and

(d) $X(z) = \sin\left(z^{-1}\right)$ for $|z| > 0$. [Hint: Use (F.4).].

12.8 Determine whether the LTI system having each system function H given below is causal.

(a) $H(z) = \dfrac{z^2+3z+2}{z-1}$ for $|z| > 1$;

(b) $H(z) = \dfrac{1+3z^{-1}}{1-z^{-1}}$ for $|z| < 1$; and

(c) $H(z) = \dfrac{1+\frac{1}{2}z^{-1}}{z-\frac{1}{2}}$ for $|z| > \frac{1}{2}$.

12.9 For the LTI system $\mathcal{H}$ with each system function H given below, find all inverses of $\mathcal{H}$ (specified in terms of system functions). Comment on the causality and BIBO stability of each inverse system.

(a) $H(z) = \dfrac{z-\frac{1}{4}}{z-\frac{1}{2}}$ for $|z| > \frac{1}{2}$;

(b) $H(z) = \dfrac{1}{(z^2-\frac{1}{4})(z^2-\frac{1}{9})}$ for $|z| > \frac{1}{2}$; and

(c) $H(z) = \dfrac{\left(z^2-\frac{1}{4}\right)\left(z^2+4\right)}{\left(z^2-\frac{1}{9}\right)\left(z^2+\frac{1}{9}\right)}$ for $|z| > \frac{1}{3}$.

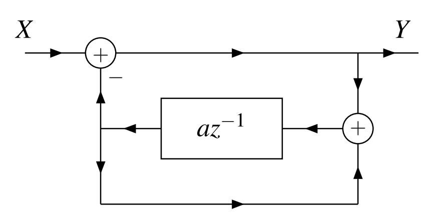
12.10 For each LTI system whose system function H is given below, determine the ROC of H if the system is BIBO stable (if such a ROC exists).
(a) $H(z) = \dfrac{1+z^{-1}}{1+z^{-1}+\frac{1}{2}z^{-2}}$;
(b) $H(z) = \dfrac{1-\frac{1}{3}z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}$;
(c) $H(z) = \dfrac{1+z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$;
(d) $H(z) = \dfrac{1}{1-5z^{-1}+6z^{-2}}$; and
(e) $H(z) = \dfrac{z^2-1}{z^2+1}$.

12.11 For the causal LTI system with input x and output y characterized by each difference equation given below, find the system function H of the system.
(a) $y(n) - \frac{2}{3}y(n-1) + \frac{1}{9}y(n-2) = x(n) + \frac{1}{2}x(n-1)$; and
(b) $y(n+2) - \frac{1}{4}y(n+1) - \frac{1}{4}y(n) + \frac{1}{16}y(n-1) = x(n+2) - x(n+1)$.

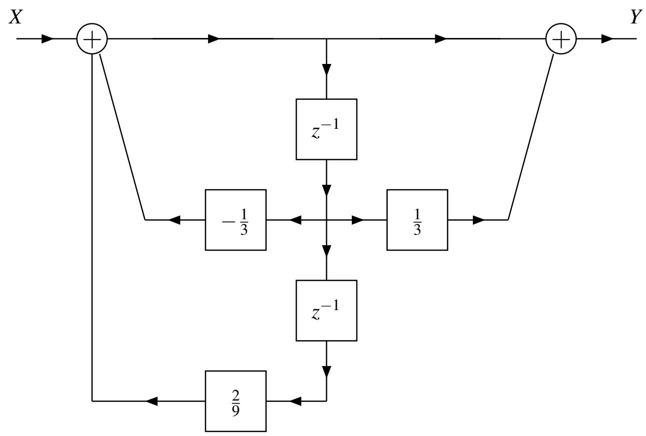
12.12 For each LTI system whose system function H is given below, find the difference equation that characterizes the system.
(a) $H(z) = \dfrac{1-z^{-1}-\frac{2}{9}z^{-2}}{1-\frac{1}{4}z^{-1}-\frac{1}{4}z^{-2}+\frac{1}{16}z^{-3}}$ for $|z| > \frac{1}{2}$; and
(b) $H(z) = \dfrac{z^2+\frac{1}{4}}{z^3-\frac{17}{12}z^2+\frac{5}{8}z-\frac{1}{12}}$ for $|z| > \frac{2}{3}$.

12.13 For the system with input x and output y characterized by each difference equation given below, find y for the given x and initial conditions.
(a) $y(n) - \frac{1}{2}y(n-1) = x(n)$, where $x(n) = 2u(n)$ and $y(-1) = 2$;
(b) $y(n) + \frac{1}{4}y(n-2) = x(n)$, where $x(n) = 0$, $y(-1) = 0$, and $y(-2) = 4$; and
(c) $y(n) + 3y(n-1) = x(n)$, where $x(n) = \left(\frac{1}{2}\right)^n u(n)$ and $y(-1) = 1$.

12.14 Consider the system with input z transform X and output z transform Y as shown in the figure, where each causal LTI subsystem is labelled with its system function and a is an arbitrary real constant. (a) Find the system function H of the overall system. (b) Determine for what values of a the system is BIBO stable.



12.15 Consider the system $\mathcal{H}$ with input z transform X and output z transform Y as shown in the figure. In the figure, each subsystem is LTI and causal and labelled with its system function. (a) Find the system function H of the system $\mathcal{H}$. (b) Determine whether the system $\mathcal{H}$ is BIBO stable.



12.18.2 Exercises With Answer Key

12.16 Find the inverse z transform x of each function X given below.

(a) $X(z) = \dfrac{z\left(10z^2 - \frac{27}{2}z + \frac{15}{4}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)(z-1)}$ for $|z| > 1$.

Short Answer. (a) $x(n) = \left[\frac{16}{3}\left(\frac{1}{4}\right)^n + 4\left(\frac{1}{2}\right)^n + \frac{2}{3}\right]u(n)$

12.19 MATLAB Exercises

Currently, there are no MATLAB exercises.

Part III

Appendices

Appendix A

Complex Analysis

A.1 Introduction

Complex analysis is an essential tool in the study of signals and systems. For this reason, a brief review of complex analysis is provided in this appendix.

A.2 Complex Numbers

A **complex number** is a number of the form

$$z = x + jy,$$

where x and y are real and j is the constant defined by

$$j^2 = -1$$

(i.e., $j = \sqrt{-1}$). The **real part**, **imaginary part**, **magnitude**, and **argument** of the complex number z are denoted as $\operatorname{Re} z$ and $\operatorname{Im} z$, $|z|$, and $\arg z$, respectively, and defined as

$$\operatorname{Re} z = x, \quad \operatorname{Im} z = y,$$
$$|z| = \sqrt{x^2 + y^2}, \quad \text{and} \quad \arg z = \operatorname{atan2}(y, x) + 2\pi k,$$

where k is an arbitrary integer, and

$$\operatorname{atan2}(y,x) \triangleq \begin{cases} \arctan(y/x) & x > 0 \\ \pi/2 & x = 0 \text{ and } y > 0 \\ -\pi/2 & x = 0 \text{ and } y < 0 \\ \arctan(y/x) + \pi & x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - \pi & x < 0 \text{ and } y < 0. \end{cases} \tag{A.1}$$

(The notation $\angle z$ is sometimes also used to denote the quantity $\arg z$.) The complex number z can be represented by a point (x, y) in the complex plane, as illustrated in Figure A.1. This figure also shows the relationship between the real part, imaginary part, magnitude, and argument of a complex number.

For any given complex number z, the quantity $\arg z$ is not unique. This follows from the fact that, for any integer k, the quantities θ and $\theta + 2\pi k$ physically represent the same overall angular displacement. The value θ of $\arg z$ that lies in the range $-\pi < \theta \leq \pi$ is called the **principal argument** of z and is denoted as $\operatorname{Arg} z$. For a particular nonzero z, this quantity is uniquely specified. In particular, $\operatorname{Arg} z = \operatorname{atan2}(y, x)$.

As an aside, we note that the function $\operatorname{atan2}(y, x)$ computes the angle that the directed line segment from $(0, 0)$ to (x, y) forms with the real axis, and is defined such that $-\pi < \operatorname{atan2}(y, x) \leq \pi$.

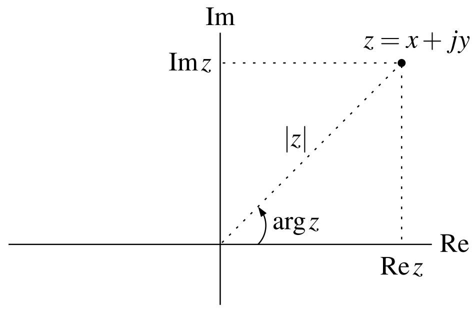


Figure A.1: Graphical representation of a complex number.

Example A.1. Compute the real part, imaginary part, magnitude, and principal argument of each complex number z below.
(a) $z = \frac{\sqrt{3}}{2} + j\frac{1}{2}$; and
(b) $z = 1 - j$.

Solution. (a) We have

$$\begin{gathered} \operatorname{Re} z = \tfrac{\sqrt{3}}{2}, \quad \operatorname{Im} z = \tfrac{1}{2}, \\ |z| = \sqrt{\left(\tfrac{\sqrt{3}}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} = 1, \quad \text{and} \\ \operatorname{Arg} z = \arctan\left(\tfrac{1}{\sqrt{3}}\right) = \tfrac{\pi}{6}. \end{gathered}$$

(b) We have

$$\begin{gathered} \operatorname{Re} z = 1, \quad \operatorname{Im} z = -1, \\ |z| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \quad \text{and} \\ \operatorname{Arg} z = \arctan\left(\tfrac{-1}{1}\right) = -\tfrac{\pi}{4}. \end{gathered}$$

■

A.3 Representations of Complex Numbers

Two different representations are commonly used for complex numbers, namely, the Cartesian and polar forms. The Cartesian form is also sometimes referred to as rectangular form. Depending on the particular situation, one form may be more convenient to use than the other.

In the case of the **Cartesian form**, a complex number z is represented as

$$z = x + jy,$$

where x and y are real. That is, z is expressed directly in terms of its real and imaginary parts. The quantity z can also be treated as a point (x, y) in a Cartesian coordinate system as shown in Figure A.2(a).

In the case of the **polar form**, a complex number z is represented as

$$z = r(\cos\theta + j\sin\theta),$$

where r and θ are real and $r \geq 0$. One can show through simple geometry that $r = |z|$ and $\theta = \arg z$. Thus, in the polar case, a complex number is expressed directly in terms of its magnitude and argument. In this way, we

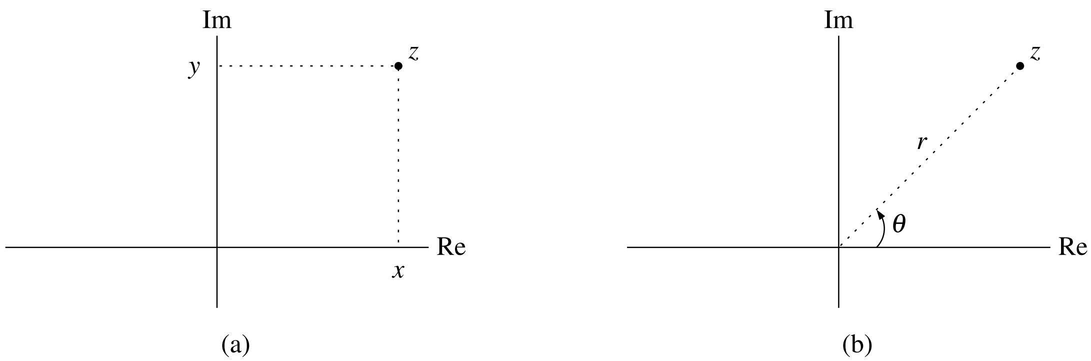


Figure A.2: Representations of complex numbers. The (a) Cartesian and (b) polar forms.

can treat the quantity z as a point (r, θ) in a polar coordinate system, as shown in Figure A.2(b). As we note later, $\cos\theta + j\sin\theta = e^{j\theta}$. Therefore, the polar form can equivalently be expressed as

$$z = re^{j\theta}.$$

This exponential notation is often used, due to its relative compactness.

A.4 Arithmetic Operations

In what follows, we consider a few basic arithmetic operations on complex numbers.

A.4.1 Conjugation

The **conjugate** of the complex number $z = x + jy$ (where x and y are real) is denoted as z^* and defined as

$$z^* = x - jy.$$

Geometrically, the conjugation operation reflects a point in the complex plane about the real axis, as illustrated in Figure A.3. One can easily verify that for any complex numbers z, z_1, and z_2, the following identities hold:

$$\begin{aligned} zz^* &= |z|^2; \\ \operatorname{Re} z &= \tfrac{1}{2}(z + z^*); \\ \operatorname{Im} z &= \tfrac{1}{2j}(z - z^*); \\ (z_1 + z_2)^* &= z_1^* + z_2^*; \\ (z_1 z_2)^* &= z_1^* z_2^*; \quad \text{and} \\ (z_1/z_2)^* &= z_1^*/z_2^* \text{ for } z_2 \neq 0. \end{aligned}$$

Trivially, we also have that

$$\begin{aligned} |z^*| &= |z| \quad \text{and} \\ \arg z^* &= -\arg z. \end{aligned}$$

A.4.2 Addition

Consider the addition of the complex numbers z_1 and z_2. Suppose that z_1 and z_2 are expressed in Cartesian form as

$$z_1 = x_1 + jy_1 \quad \text{and} \quad z_2 = x_2 + jy_2.$$

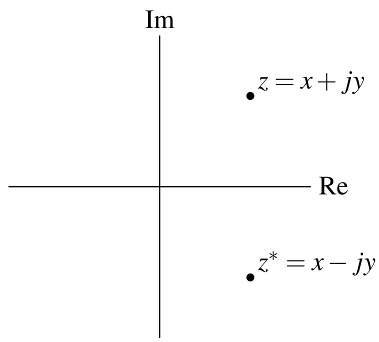


Figure A.3: Conjugate of complex number.

Then, the sum of z_1 and z_2 can be computed as

$$\begin{aligned} z_1 + z_2 &= (x_1 + jy_1) + (x_2 + jy_2) \\ &= (x_1 + x_2) + j(y_1 + y_2). \end{aligned}$$

Suppose that z_1 and z_2 are expressed in polar form as

$$z_1 = r_1 e^{j\theta_1} \quad \text{and} \quad z_2 = r_2 e^{j\theta_2}.$$

Then, the sum of z_1 and z_2 can be computed as

$$\begin{aligned} z_1 + z_2 &= r_1 e^{j\theta_1} + r_2 e^{j\theta_2} \\ &= (r_1 \cos\theta_1 + r_2 \cos\theta_2) + j(r_1 \sin\theta_1 + r_2 \sin\theta_2). \end{aligned}$$

Example A.2 (Addition with the Cartesian form)**.** Given that $z_1 = 3 + j4$ and $z_2 = -2 - j3$, compute $z_1 + z_2$.

Solution. We have

$$z_1 + z_2 = (3 + j4) + (-2 - j3) = 1 + j.$$ ■

Example A.3 (Addition with the polar form)**.** Given that $z_1 = 3 + j4$ and $z_2 = \sqrt{2}e^{j5\pi/4}$, compute $z_1 + z_2$.

Solution. In order to compute this sum, we first convert z_2 to Cartesian form to obtain

$$\operatorname{Re} z_2 = \sqrt{2}\cos\left(\tfrac{5\pi}{4}\right) = -1 \quad \text{and}$$
$$\operatorname{Im} z_2 = \sqrt{2}\sin\left(\tfrac{5\pi}{4}\right) = -1.$$

Then, we have

$$z_1 + z_2 = (3 + j4) + (-1 - j) = 2 + j3.$$ ■

A.4.3 Multiplication

Consider the multiplication of the complex numbers z_1 and z_2. Suppose that z_1 and z_2 are represented in Cartesian form as

$$z_1 = x_1 + jy_1 \quad \text{and} \quad z_2 = x_2 + jy_2.$$

Then, the product of z_1 and z_2 can be computed as

$$\begin{aligned} z_1 z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= x_1 x_2 + jx_1 y_2 + jx_2 y_1 - y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1). \end{aligned}$$

Suppose that z_1 and z_2 are represented in polar form as

$$z_1 = r_1 e^{j\theta_1} \quad \text{and} \quad z_2 = r_2 e^{j\theta_2}.$$

Then, the product of z_1 and z_2 can be computed as

$$\begin{aligned} z_1 z_2 &= \left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) \\ &= r_1 r_2 e^{j(\theta_1+\theta_2)}. \end{aligned}$$

Example A.4 (Multiplication with the Cartesian form)**.** Given that $z_1 = 2 + j3$ and $z_2 = 3 - j4$, compute $z_1 z_2$.

Solution. Using straightforward algebraic manipulation, we have

$$z_1 z_2 = (2 + j3)(3 - j4) = 6 - j8 + j9 + 12 = 18 + j. \qquad \blacksquare$$

Example A.5 (Multiplication with the polar form)**.** Given that $z_1 = \sqrt{2}e^{j\pi/4}$ and $z_2 = 3e^{j\pi/6}$, compute $z_1 z_2$.

Solution. Using straightforward algebraic manipulation, we obtain

$$z_1 z_2 = (\sqrt{2}e^{j\pi/4})(3e^{j\pi/6}) = 3\sqrt{2}e^{j5\pi/12}. \qquad \blacksquare$$

A.4.4 Division

Consider the division of the complex numbers z_1 and z_2. Suppose that z_1 and z_2 are represented in Cartesian form as

$$z_1 = x_1 + jy_1 \quad \text{and} \quad z_2 = x_2 + jy_2.$$

Then, the quotient of z_1 and z_2 can be computed as

$$\begin{aligned} \frac{z_1}{z_2} &= \left(\frac{z_1}{z_2}\right)\left(\frac{z_2^*}{z_2^*}\right) = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} \\ &= \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 - jx_1 y_2 + jx_2 y_1 + y_1 y_2}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 + y_1 y_2 + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}. \end{aligned}$$

Suppose that z_1 and z_2 are represented in polar form as

$$z_1 = r_1 e^{j\theta_1} \quad \text{and} \quad z_2 = r_2 e^{j\theta_2}.$$

Then, the quotient of z_1 and z_2 can be computed as

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} \\ &= \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}. \end{aligned}$$

Example A.6 (Division with the Cartesian form)**.** Given that $z_1 = 1+j$ and $z_2 = 2-j$, compute z_1/z_2.

Solution. Using straightforward algebraic manipulation, we have

$$\frac{z_1}{z_2} = \left(\frac{1+j}{2-j}\right)\left(\frac{2+j}{2+j}\right) = \frac{2+2j+j-1}{2^2+1^2} = \frac{1+3j}{5}. \qquad \blacksquare$$

Example A.7 (Division with the polar form)**.** Given that $z_1 = 2e^{j\pi/3}$ and $z_2 = 3e^{j\pi/4}$, compute z_1/z_2.

Solution. Using straightforward algebraic manipulation, we have

$$\frac{z_1}{z_2} = \frac{2e^{j\pi/3}}{3e^{j\pi/4}} = \tfrac{2}{3}e^{j\left(\frac{\pi}{3}-\frac{\pi}{4}\right)} = \tfrac{2}{3}e^{j\pi/12}. \qquad \blacksquare$$

A.4.5 Miscellany

For arbitrary complex numbers z_1 and z_2, the following identities hold:

$$\begin{aligned} |z_1z_2| &= |z_1|\,|z_2|; \\ \left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|} \quad \text{for } z_2 \neq 0; \\ \arg z_1z_2 &= \arg z_1 + \arg z_2; \quad \text{and} \\ \arg\left(\frac{z_1}{z_2}\right) &= \arg z_1 - \arg z_2 \quad \text{for } z_2 \neq 0. \end{aligned}$$

A.5 Arithmetic Properties of Complex Numbers

In what follows, we consider some of the properties of arithmetic over the complex numbers.

A.5.1 Commutative Property

For complex numbers, addition and multiplication are commutative. That is, for any two complex numbers z_1 and z_2, the following identities hold:

$$\begin{aligned} z_1 + z_2 &= z_2 + z_1 \quad \text{and} \\ z_1z_2 &= z_2z_1. \end{aligned}$$

A.5.2 Associative Property

For complex numbers, addition and multiplication are associative. That is, for any three complex numbers z_1, z_2, and z_3, the following identities hold:

$$\begin{aligned} (z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3) \quad \text{and} \\ (z_1z_2)z_3 &= z_1(z_2z_3). \end{aligned}$$

A.5.3 Distributive Property

The distributive property also holds for complex numbers. That is, for any three complex numbers z_1, z_2, and z_3, the following identity holds:

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3.$$

A.6 Roots of Complex Numbers

Every complex number z has n distinct nth roots in the complex plane. In particular, the nth roots of $z = re^{j\theta}$, where $r = |z|$ and $\theta = \arg z$, are given by

$$\sqrt[n]{r}e^{j(\theta+2\pi k)/n} \quad \text{for } k \in [0\,..\,n-1]. \tag{A.2}$$

Example A.8. Find the four fourth roots of 2.

Solution. Let z_k for $k \in [0\,..\,3]$ denote each of the four fourth roots of 2. From (A.2), we have

$$z_k = \sqrt[4]{2}e^{j(0+2\pi k)/4} = \sqrt[4]{2}e^{jk\pi/2}.$$

Thus, we obtain

$$\begin{aligned}
z_0 &= \sqrt[4]{2}e^{j0} = \sqrt[4]{2}, \\
z_1 &= \sqrt[4]{2}e^{j\pi/2} = j\sqrt[4]{2}, \\
z_2 &= \sqrt[4]{2}e^{j\pi} = -\sqrt[4]{2}, \quad \text{and} \\
z_3 &= \sqrt[4]{2}e^{j3\pi/2} = -j\sqrt[4]{2}.
\end{aligned}$$

So, we conclude that the four fourth roots of 2 are

$$\sqrt[4]{2}, \quad j\sqrt[4]{2}, \quad -\sqrt[4]{2}, \quad \text{and} \quad -j\sqrt[4]{2}.$$ ■

A.7 Euler's Relation and De Moivre's Theorem

An important relationship exists between exponentials and sinusoids as given by the theorem below.

Theorem A.1 (Euler's relation)**.** *For any real θ, the following identity holds:*

$$e^{j\theta} = \cos\theta + j\sin\theta. \tag{A.3}$$

*This identity is known as **Euler's relation**.*

Proof. The preceding theorem can be proven as follows. Recall that the Maclaurin series expansions of e^x, $\cos x$, and $\sin x$ are given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \tag{A.4}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \quad \text{and} \tag{A.5}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \tag{A.6}$$

Using (A.4), we can write $e^{j\theta}$ as the series

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \cdots.$$

By regrouping terms and using (A.5) and (A.6), we obtain

$$\begin{aligned}
e^{j\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\
&= \cos\theta + j\sin\theta.
\end{aligned}$$ ■

From Euler's relation, we can deduce the following additional identities:

$$\cos\theta = \tfrac{1}{2}(e^{j\theta} + e^{-j\theta}) \quad \text{and} \tag{A.7a}$$

$$\sin\theta = \tfrac{1}{2j}(e^{j\theta} - e^{-j\theta}). \tag{A.7b}$$

Another important result involving exponential functions is given by the theorem below.

Theorem A.2 (De Moivre's theorem)**.** *For all real θ and all integer n, the following identity holds:*

$$e^{jn\theta} = \left(e^{j\theta}\right)^n. \tag{A.8}$$

This result is known as ***De Moivre's theorem****.*

Proof. The result of the above theorem can be proven by induction, and is left as an exercise for the reader. (See Exercise A.10.) ■

Note that, in the preceding theorem, n must be an integer. The identity (A.8) does not necessarily hold if n is not an integer. For example, consider $\theta = -\pi$ and $n = \frac{1}{2}$ (where n is clearly not an integer). We have that

$$e^{jn\theta} = e^{j(1/2)(-\pi)} = e^{-j\pi/2} = -j \quad \text{and}$$

$$\left(e^{j\theta}\right)^n = \left(e^{-j\pi}\right)^{1/2} = (-1)^{1/2} = j.$$

Clearly, in this case, $e^{jn\theta} \neq \left(e^{j\theta}\right)^n$.

A.8 Conversion Between Cartesian and Polar Form

Suppose that we have a complex number $z = x + jy = re^{j\theta}$. Using Euler's relation, we can derive the following expressions for converting from polar to Cartesian form:

$$x = r\cos\theta \quad \text{and} \tag{A.9a}$$

$$y = r\sin\theta. \tag{A.9b}$$

Similarly, we can deduce the following expressions for converting from Cartesian to polar form:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \tag{A.10a}$$

$$\theta = \operatorname{atan2}(y, x), \tag{A.10b}$$

where the atan2 function is as defined in (A.1).

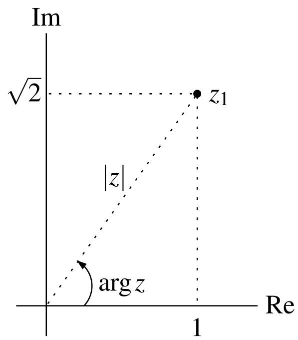
If we choose to use the arctan function directly in order to compute θ (instead of using the atan2 function), we must be careful to consider the quadrant in which the point (x, y) lies. This complication is due to the fact that the arctan function is defined such that $-\frac{\pi}{2} < \arctan\theta < \frac{\pi}{2}$. Consequently, if the point does not lie in the first or fourth quadrant of the complex plane, the arctan function will not yield the desired angle.

Example A.9. Express each complex number z given below in polar form.
(a) $z = 1 + j\sqrt{2}$; and
(b) $z = -1 - j$.

Solution. (a) The magnitude and argument of z are given by

$$|z| = \sqrt{1^2 + \left(\sqrt{2}\right)^2} = \sqrt{3} \quad \text{and}$$

$$\arg z = \arctan\left(\tfrac{\sqrt{2}}{1}\right) = \arctan\sqrt{2}.$$



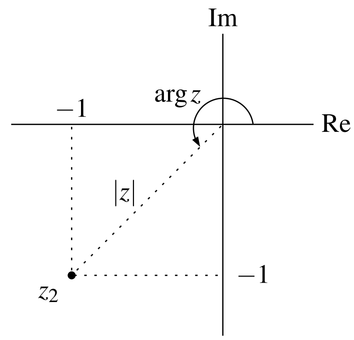


Figure A.4: Example of converting complex numbers from Cartesian to polar form. The case of the (a) first and (b) second part of the example.

Thus, we have

$$z = \sqrt{3}e^{j(\arctan\sqrt{2})}.$$

The result is plotted in Figure A.4(a).

(b) The magnitude and argument of z are given by

$$|z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} \quad \text{and}$$
$$\arg z = \arctan(\tfrac{-1}{-1}) - \pi = \arctan(1) - \pi = -\tfrac{3\pi}{4}.$$

Thus, we have

$$z = \sqrt{2}e^{-j3\pi/4}.$$

The result is plotted in Figure A.4(b). ■

A.9 Complex Functions

A **complex function** maps complex numbers to complex numbers. For example, the function

$$f(z) = z^2 + 2z + 1,$$

where z is complex, is a complex function.

A **complex polynomial function** is mapping of the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

where a_0, a_1, ..., a_n, and z are complex. A **complex rational function** is a mapping of the form

$$f(z) = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m},$$

where $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m$ and z are complex. In the context of systems theory, polynomial and rational functions play a particularly important role.

Given any complex function f, we can always write f in the form $f(z) = \mathrm{Re}[f(z)] + j\,\mathrm{Im}[f(z)]$. Writing z in Cartesian form as $z = x + jy$, we have that $f(z) = \mathrm{Re}[f(x+jy)] + j\,\mathrm{Im}[f(x+jy)]$. Now, we can express $\mathrm{Re}[f(x+jy)]$ as a real-valued function v of the two real variables x and y. Similarly, we can express $\mathrm{Im}[f(x+jy)]$ as a real-valued function w of the two real variables x and y. Thus, we can always express a complex function f in the form

$$f(z) = f(x+jy) = v(x,y) + jw(x,y), \tag{A.11}$$

where v and w are each real-valued functions of the two real variables x and y (and $z = x + jy$).

A.10 Circles, Disks, and Annuli

A **circle** in the complex plane with center z_0 and radius r is the set of points z satisfying

$$|z - z_0| = r,$$

where r is a strictly positive real constant. A plot of a circle is shown in Figure A.5.

A **disk** is the set of points inside of a circle, possibly including the points on the circle itself. If the points on the circle are not included in the set, the disk is said to be open; otherwise, the disk is said to be closed. More formally, an **open disk** with center z_0 and radius r is the set of points z satisfying

$$|z - z_0| < r,$$

where r is a strictly positive real constant. A plot of an open disk is shown in Figure A.6.

An **annulus** (i.e., a ring) is the set of points between two concentric circles, possibly including the points on one or both circles. If an annulus does not include the points on its two defining circles, it is said to be open. More formally, an **open annulus** with center z_0, inner radius r_1, and outer radius r_2 is the set of points z satisfying

$$r_1 < |z - z_0| < r_2,$$

where r_1 and r_2 are strictly positive real constants. A plot of an annulus is shown in Figure A.7.

A.11 Limit

Let f be a complex function and z_0 a complex number. We want to define the limit of $f(z)$ as z approaches z_0. Unlike in the case of real functions, the value z_0 can be approached from infinitely many directions in the complex plane. In order for the limit to be useful, however, we want it to be defined in such a way that it is independent of the direction from which z_0 is approached. With this in mind, we define the notion of a limit below.

A function f evaluated at z is said to have the limit L as z approaches z_0 if

1. f is defined in some open disk about z_0, except possibly at the point z_0; and
2. for every positive real number ε, there exists a positive real number δ such that $|f(z) - L| < \varepsilon$ for all values of z in the disk $|z - z_0| < \delta$ except $z = z_0$.

This limit can be expressed as

$$\lim_{z \to z_0} f(z) = L.$$

A.12 Continuity

A function f is said to be **continuous at a point** z_0 if $f(z_0)$ is defined and given by

$$f(z_0) = \lim_{z \to z_0} f(z).$$

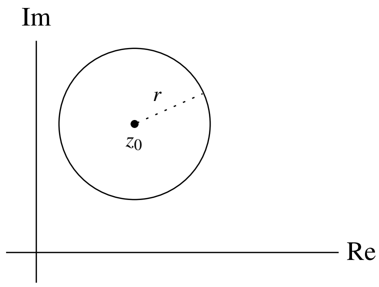


Figure A.5: Circle about z_0 with radius r.

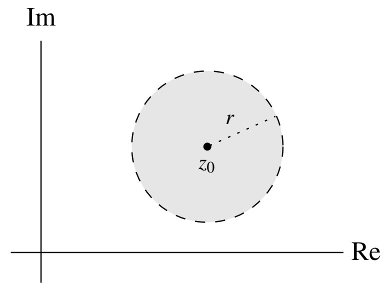


Figure A.6: Open disk of radius r about z_0.

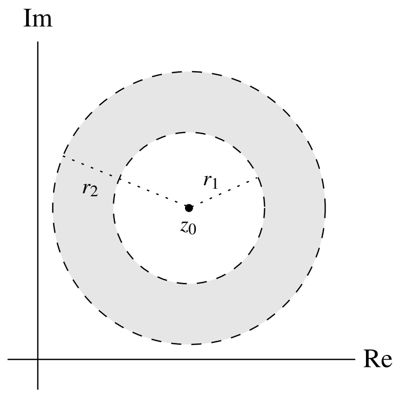


Figure A.7: Open annulus about z_0 with inner radius r_1 and outer radius r_2.

A function is said to be **continuous** if it is continuous at every point in its domain. Polynomial functions are continuous everywhere. For example, the function $f(z) = 3z^3 + z^2 - z + 1$ is continuous for all complex z. Rational functions (i.e., quotients of polynomials) are continuous everywhere except at points where the denominator polynomial becomes zero. For example, the function $f(z) = \frac{(z+j)(z-j)}{(z-1)(z+1)}$ is continuous for all complex z except $z = 1$ and $z = -1$.

A.13 Differentiability

A function f is said to be **differentiable at a point** $z = z_0$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. This limit is called the **derivative** of f at the point z_0. A function is said to be **differentiable** if it is differentiable at every point in its domain.

In general, the rules for differentiating sums, products, and quotients are the same for complex functions as for real functions. Let f and g be functions and let a be a scalar constant. Let the prime symbol denote a derivative. If $f'(z_0)$ and $g'(z_0)$ exist, then

1. $(af)'(z_0) = af'(z_0)$ for any complex constant a;
2. $(f+g)'(z_0) = f'(z_0) + g'(z_0)$;
3. $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$;
4. $(f/g)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$; and

5. if $z_0 = g(w_0)$ and $g'(w_0)$ exists, then the derivative of $f(g(z))$ at w_0 is $f'(z_0)g'(w_0)$ (i.e., the chain rule).

Polynomial functions are differentiable everywhere. Rational functions are differentiable everywhere except at points where the denominator polynomial becomes zero.

A.14 Analyticity

A function f is said to be **analytic at a point** z_0 if it is differentiable at every point in some open disk about z_0. A function f is said to be **analytic** if it is analytic at every point in its domain.

One extremely useful test for the analyticity of a function is given by the theorem below.

Theorem A.3 (Cauchy-Riemann equations)**.** *Let f be a complex function expressed in the form of (A.11). So, we have*

$$f(z) = v(x,y) + jw(x,y),$$

where $z = x + jy$. The function f is analytic in S if and only if v and w satisfy the following conditions at all points in S:

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial x}.$$

*These equations are known as the **Cauchy-Riemann equations**.*

Proof. A proof of this theorem is somewhat tedious and therefore omitted here. ■

Polynomial functions are both continuous and differentiable everywhere. Therefore, such functions are analytic everywhere. Rational functions are both continuous and differentiable everywhere, except at points where the denominator polynomial becomes zero. Consequently, rational functions are analytic at all but these points.

Example A.10. Determine for what values of z the function $f(z) = z^2$ is analytic.

Solution. First, we observe that f is a polynomial function. Then, we recall that polynomial functions are analytic everywhere. Therefore, f is analytic everywhere.

Alternate Solution. We can demonstrate the analyticity of f using Theorem A.3. We express z in Cartesian form as $z = x + jy$. We rewrite f in the form of $f(x,y) = v(x,y) + jw(x,y)$ as follows:

$$f(z) = f(x+jy) = (x+jy)^2 = x^2 + j2xy - y^2 = (x^2 - y^2) + j(2xy).$$

Thus, we have that $f(z) = v(x,y) + jw(x,y)$, where

$$v(x,y) = x^2 - y^2 \quad \text{and} \quad w(x,y) = 2xy.$$

Now, computing the partial derivatives, we obtain

$$\frac{\partial v}{\partial x} = 2x, \quad \frac{\partial w}{\partial y} = 2x, \quad \frac{\partial v}{\partial y} = -2y, \quad \text{and} \quad \frac{\partial w}{\partial x} = 2y.$$

From this, we can see that

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial x}.$$

Therefore, the Cauchy-Riemann equations are satisfied for all complex $z = x + jy$. Therefore, f is analytic everywhere. ■

Example A.11. Determine for what values of z the function $f(z) = 1/z$ is analytic.

Solution. We can deduce the analyticity properties of f as follows. First, we observe that f is a rational function. Then, we recall that a rational function is analytic everywhere except at points where its denominator polynomial becomes zero. Since the denominator polynomial of f only becomes zero at 0, f is analytic everywhere except at 0.

Alternate Solution. To study the analyticity of f, we use Theorem A.3. We express z in Cartesian form as $z = x + jy$. We rewrite f in the form $f(x,y) = v(x,y) + jw(x,y)$ as follows:

$$f(z) = f(x+jy) = \frac{1}{x+jy} = \left(\frac{1}{x+jy}\right)\left(\frac{x-jy}{x-jy}\right) = \frac{x-jy}{x^2+y^2}.$$

Thus, we have that $f(x,y) = v(x,y) + jw(x,y)$, where

$$v(x,y) = \frac{x}{x^2+y^2} = x(x^2+y^2)^{-1} \quad \text{and}$$
$$w(x,y) = \frac{-y}{x^2+y^2} = -y(x^2+y^2)^{-1}.$$

Now, computing the partial derivatives, we obtain

$$\frac{\partial v}{\partial x} = (x^2+y^2)^{-1} + (-1)(x^2+y^2)^{-2}(2x^2) = \frac{-2x^2+(x^2+y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$
$$\frac{\partial w}{\partial y} = (-1)(x^2+y^2)^{-1} + (-1)(x^2+y^2)^{-2}(2y)(-y) = \frac{2y^2-(x^2+y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$
$$\frac{\partial v}{\partial y} = (-1)(x^2+y^2)^{-2}(2y)x = \frac{-2xy}{(x^2+y^2)^2}, \quad \text{and}$$
$$\frac{\partial w}{\partial x} = (-1)(x^2+y^2)^{-2}(2x)(-y) = \frac{2xy}{(x^2+y^2)^2}.$$

So, we have that, for $z \neq 0$ (i.e., x and y not both zero),

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial x}$$

(i.e., the Cauchy-Riemann equations are satisfied). Therefore, f is analytic everywhere except at 0. ■

A.15 Zeros and Singularities

If a function f is analytic in a domain D and is zero at a point z_0 in D, f is said to have a **zero** at z_0. If, in addition, the first $n-1$ derivatives of f are also zero at z_0 (i.e., $f^{(1)}(z_0) = f^{(2)}(z_0) = \ldots = f^{(n-1)}(z_0) = 0$), f is said to have an ***n*th-order zero** at z_0. An analytic function f is said to have an nth order zero at infinity if the function $g(z) = f(1/z)$ has an nth order zero at 0.

A point at which a function fails to be analytic is called a **singularity**. A singularity may be isolated or nonisolated. If a function f is analytic for z in an annulus $0 < |z - z_0| < r$ but not at z_0, f is said to have an **isolated singularity** at z_0. There are three types of isolated singularities: a removable singularity, an essential singularity, and a pole.

Herein, we are often interested in polynomial and rational functions. Polynomial functions do not have singularities, since such functions are analytic everywhere. In contrast, rational functions can have singularities. In the case of rational functions, we are normally interested in poles (since rational functions cannot have essential singularities and removable singularities are not very interesting).

Consider a rational function f. We can always express such a function in factored form as

$$f(z) = \frac{K(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\cdots(z-a_M)^{\alpha_M}}{(z-b_1)^{\beta_1}(z-b_2)^{\beta_2}\cdots(z-b_N)^{\beta_N}},$$

where K is complex, $a_1,a_2,\ldots,a_M,b_1,b_2,\ldots,b_N$ are distinct complex constants, and $\alpha_1,\alpha_2,\ldots,\alpha_M$ and $\beta_1,\beta_2,\ldots,\beta_N$ are positive integers. One can show that f has poles at $b_1,b_2,\ldots,b_N$ and zeros at $a_1,a_2,\ldots,a_M$. Furthermore, the kth pole (i.e., b_k) is of order β_k, and the kth zero (i.e., a_k) is of order α_k. A pole of first order is said to be **simple**, whereas a pole of order two or greater is said to be **repeated**. A similar terminology can also be applied to zeros (i.e., **simple zero** and **repeated zero**).

Example A.12 (Poles and zeros of a rational function)**.** Find and plot the poles and (finite) zeros of the function

$$f(z) = \frac{z^2(z^2+1)(z-1)}{(z+1)(z^2+3z+2)(z^2+2z+2)}.$$

Solution. We observe that f is a rational function, so we can easily determine the poles and zeros of f from its factored form. We now proceed to factor f. First, we factor z^2+3z+2. To do this, we solve for the roots of $z^2+3z+2=0$ to obtain

$$z = \frac{-3\pm\sqrt{3^2-4(1)(2)}}{2(1)} = -\tfrac{3}{2}\pm\tfrac{1}{2} = \{-1,-2\}.$$

(For additional information on how to find the roots of a quadratic equation, see Section A.16.) So, we have

$$z^2+3z+2 = (z+1)(z+2).$$

Second, we factor z^2+2z+2. To do this, we solve for the roots of $z^2+2z+2=0$ to obtain

$$z = \frac{-2\pm\sqrt{2^2-4(1)(2)}}{2(1)} = -1\pm j = \{-1+j,-1-j\}.$$

So, we have

$$z^2+2z+2 = (z+1-j)(z+1+j).$$

Lastly, we factor z^2+1. Using the well-known factorization for a sum of squares, we obtain

$$z^2+1 = (z+j)(z-j).$$

Combining the above results, we can rewrite f as

$$f(z) = \frac{z^2(z+j)(z-j)(z-1)}{(z+1)^2(z+2)(z+1-j)(z+1+j)}.$$

From this expression, we can trivially deduce that f has:

- first order zeros at 1, j, and $-j$,
- a second order zero at 0,
- first order poles at $-1+j$, $-1-j$, -2, and
- a second order pole at -1.

The zeros and poles of this function are plotted in Figure A.8. In such plots, the poles and zeros are typically denoted by the symbols “x” and “o”, respectively. ■

Example A.13. Find the zeros and poles of the function

$$f(z) = \frac{z^3-2}{z^5+4}.$$

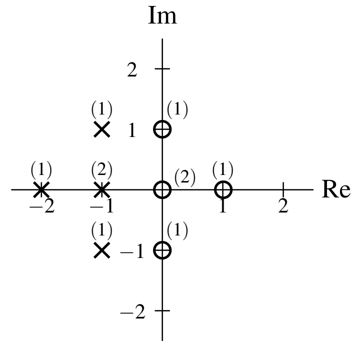


Figure A.8: Plot of the poles and zeros of f (with their orders indicated in parentheses).

Solution. The zeros of f are given by the roots of $z^3-2=0$, which is equivalent to $z^3=2=2e^{j0}$. This equation has three distinct solutions (i.e., the three third roots of 2), namely,

$$\sqrt[3]{2}, \quad \sqrt[3]{2}e^{j2\pi/3}, \quad \text{and} \quad \sqrt[3]{2}e^{j4\pi/3}.$$

The poles of f are given by the roots of $z^5+4=0$, which is equivalent to $z^5=-4=4e^{-j\pi}$. This equation has five distinct solutions (i.e., the five fifth roots of -4), namely,

$$\begin{aligned}
\sqrt[5]{4}e^{j(-\pi+0)/5} &= \sqrt[5]{4}e^{-j\pi/5}, \\
\sqrt[5]{4}e^{j(-\pi+2\pi)/5} &= \sqrt[5]{4}e^{j\pi/5}, \\
\sqrt[5]{4}e^{j(-\pi+4\pi)/5} &= \sqrt[5]{4}e^{j3\pi/5}, \\
\sqrt[5]{4}e^{j(-\pi+6\pi)/5} &= \sqrt[5]{4}e^{j\pi} = -\sqrt[5]{4}, \quad \text{and} \\
\sqrt[5]{4}e^{j(-\pi+8\pi)/5} &= \sqrt[5]{4}e^{j7\pi/5}.
\end{aligned}$$

■

A.16 Quadratic Formula

Consider the equation $az^2+bz+c=0$, where a, b, c, and z are complex, and $a \neq 0$. The roots of this equation are given by

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}. \tag{A.12}$$

This formula is often useful in factoring quadratic polynomials. For example, from the quadratic formula, we can conclude that the general quadratic az^2+bz+c has the factorization

$$az^2+bz+c = a(z-z_0)(z-z_1),$$

where

$$z_0 = \frac{-b-\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad z_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}.$$

A.17 Exercises

A.17.1 Exercises Without Answer Key

A.1 Express each complex number given below in Cartesian form.
(a) $2e^{j2\pi/3}$;
(b) $\sqrt{2}e^{j\pi/4}$;
(c) $2e^{j7\pi/6}$; and
(d) $3e^{j\pi/2}$.

A.2 Express each complex number given below in polar form. In each case, plot the value in the complex plane, clearly indicating its magnitude and argument. State the principal value for the argument.
(a) $-\sqrt{3}+j$;
(b) $-\frac{1}{2}-j\frac{\sqrt{3}}{2}$;
(c) $\sqrt{2}-j\sqrt{2}$;
(d) $1+j\sqrt{3}$;
(e) $-1-j\sqrt{3}$; and
(f) $-3+4j$.

A.3 Evaluate each of the expressions below, stating the final result in the specified form. When giving a final result in polar form, state the principal value of the argument.
(a) $2\left(\frac{\sqrt{3}}{2}-j\frac{1}{2}\right)+j\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$ in Cartesian form;
(b) $\left(\frac{\sqrt{3}}{2}-j\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$ in polar form;
(c) $\left(\frac{\sqrt{3}}{2}-j\frac{1}{2}\right)/(1+j)$ in polar form;
(d) $e^{1+j\pi/4}$ in Cartesian form;
(e) $\left[\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)^*\right]^8$ in polar form;
(f) $(1+j)^{10}$ in Cartesian form;
(g) $\dfrac{1+j}{1-j}$ in polar form;
(h) $\dfrac{1}{1+re^{j\theta}}$ in Cartesian form, where r and θ are real constants and $r \geq 0$; and
(i) $\dfrac{1}{1-re^{j\theta}}$ in Cartesian form, where r and θ are real constants and $r \geq 0$.

A.4 Show that each of the identities below holds, where z, z_1, and z_2 are arbitrary complex numbers.
(a) $|z_1/z_2| = |z_1|/|z_2|$ for $z_2 \neq 0$;
(b) $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ for $z_2 \neq 0$;
(c) $z+z^* = 2\operatorname{Re}\{z\}$;
(d) $zz^* = |z|^2$; and
(e) $(z_1z_2)^* = z_1^*z_2^*$.

A.5 Use Euler's relation to show that each of the identities below holds, where θ is an arbitrary real constant.
(a) $\cos\theta = \frac{1}{2}\left[e^{j\theta}+e^{-j\theta}\right]$;
(b) $\sin\theta = \frac{1}{2j}\left[e^{j\theta}-e^{-j\theta}\right]$; and
(c) $\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)]$.

A.6 For each rational function f of a complex variable given below, find the poles and zeros of f and their orders. Also, plot the poles and zeros of f in the complex plane.
(a) $f(z) = z^2 + jz + 3$;
(b) $f(z) = z + 3 + 2z^{-1}$;
(c) $f(z) = \dfrac{(z^2+2z+5)(z^2+1)}{(z^2+2z+2)(z^2+3z+2)}$;
(d) $f(z) = \dfrac{z^3-z}{z^2-4}$;
(e) $f(z) = \dfrac{z+\frac{1}{2}}{(z^2+2z+2)(z^2-1)}$; and
(f) $f(z) = \dfrac{z^2(z^2-1)}{(z^2+4z+\frac{17}{4})^2(z^2+2z+2)}$.

A.7 Determine the points at which each function f given below is: i) continuous, ii) differentiable, and iii) analytic. To deduce the answer, use your knowledge about polynomial and rational functions. Simply state the final answer along with a short justification (i.e., two or three sentences). (In other words, it is not necessary to use the Cauchy-Riemann equations for this problem.)
(a) $f(z) = 3z^3 - jz^2 + z - \pi$;
(b) $f(z) = \dfrac{z-1}{(z^2+3)(z^2+z+1)}$;
(c) $f(z) = \dfrac{z}{z^4-16}$; and
(d) $f(z) = z + 2 + z^{-1}$.

A.8 Use the Cauchy-Riemann equations to show that the function $f(z) = e^{az}$ is analytic for all z, where a is a real constant and z is complex.

A.9 For each function f of a real variable given below, find an expression for $|f(\omega)|$ and $\arg f(\omega)$.
(a) $f(\omega) = \dfrac{1}{(1+j\omega)^{10}}$;
(b) $f(\omega) = \dfrac{-2-j\omega}{(3+j\omega)^2}$;
(c) $f(\omega) = \dfrac{2e^{j11\omega}}{(3+j5\omega)^7}$;
(d) $f(\omega) = \dfrac{-5}{(-1-j\omega)^4}$;
(e) $f(\omega) = \dfrac{j\omega^2}{(j\omega-1)^{10}}$; and
(f) $f(\omega) = \dfrac{j\omega-1}{j\omega+1}$.

A.10 Use induction to prove De Moivre's theorem (i.e., Theorem A.2), which states that $(e^{j\theta})^n = e^{j\theta n}$ for all real θ and all integer n.

A.11 Show that each of the following identities hold:
(a) $\displaystyle\sum_{n=\langle N\rangle} e^{j(2\pi/N)kn} = \begin{cases} N & k/N \in \mathbb{Z} \\ 0 & \text{otherwise,} \end{cases}$
where $\sum_{n=\langle N\rangle}$ denotes a summation over a single period of the N-periodic summand (i.e., the expression being

summed) [Hint: Use the formula for the sum of a geometric sequence.]; and

(b) $\displaystyle\int_T e^{j(2\pi/T)kt} = \begin{cases} T & k=0 \\ 0 & \text{otherwise,} \end{cases}$

where $\int_T$ denotes integration over a single period of the T-periodic integrand (i.e., the expression being integrated).

A.12 Let $k, N, N_1, N_2 \in \mathbb{Z}$, where $N > 0$ and $N_1 \leq N_2$. Show that

$$\sum_{n=N_1}^{N_2} e^{-j(2\pi/N)kn} = \begin{cases} e^{-j\pi(N_1+N_2)k/N}\left[\frac{\sin[\pi(N_2-N_1+1)k/N]}{\sin(\pi k/N)}\right] & \frac{k}{N} \notin \mathbb{Z} \\ N_2 - N_1 + 1 & \frac{k}{N} \in \mathbb{Z}. \end{cases}$$

A.17.2 Exercises With Answer Key

A.13 A rational function f has a first-order pole at -1, a second-order pole at -2, and a first-order zero at 0. The function is known not to have any other poles or zeros. If $f(1) = 1$, find f.

Short Answer. $f(z) = \frac{18z}{(z+1)(z+2)^2}$

A.14 For each function f given below, find the poles and zeros of f and their orders.

(a) $f(z) = \dfrac{z-3}{z^5+7z}$.

Short Answer. (a) first-order zero at 3; first-order poles at 0, $\sqrt[4]{7}e^{j\pi/4}$, $\sqrt[4]{7}e^{j3\pi/4}$, $\sqrt[4]{7}e^{-j3\pi/4}$, and $\sqrt[4]{7}e^{-j\pi/4}$.

A.18 MATLAB Exercises

A.101 Consider the rational function

$$f(z) = \frac{64z^4 - 48z^3 - 36z^2 + 27z}{64z^6 - 128z^5 - 112z^4 + 320z^3 - 84z^2 - 72z + 27}.$$

Use the Symbolic Math Toolbox in order to find the poles and zeros of f and their orders. (Hint: Some of the following functions may be useful: `sym`, `solve`, `factor`, and `pretty`.)

A.102 Use the `roots` function to find the poles and zeros of the rational function

$$f(z) = \frac{z^4 + 6z^3 + 10z^2 + 8z}{z^9 + 21z^8 + 199z^7 + 1111z^6 + 4007z^5 + 9639z^4 + 15401z^3 + 15689z^2 + 9192z + 2340}.$$

Plot these poles and zeros using the `plot` function.

Appendix B

Partial Fraction Expansions

B.1 Introduction

Sometimes we find it beneficial to be able to express a rational function as a sum of lower-order rational functions. This type of decomposition is known as a partial fraction expansion. Partial fraction expansions are often useful in the calculation of inverse Laplace transforms, inverse z transforms, and inverse (continuous-time and discrete-time) Fourier transforms.

B.2 Partial Fraction Expansions

Suppose that we have a rational function

$$f(z) = \frac{\alpha_m z^m + \alpha_{m-1} z^{m-1} + \ldots + \alpha_1 z + \alpha_0}{\beta_n z^n + \beta_{n-1} z^{n-1} + \ldots + \beta_1 z + \beta_0},$$

where m and n are nonnegative integers. Such a function is said to be **strictly proper** if $m < n$ (i.e., the order of the numerator polynomial is strictly less than the order of the denominator polynomial). We can always write a rational function as the sum of a polynomial function and a strictly proper rational function. This can be accomplished through polynomial long division. In what follows, we consider partial fraction expansions of strictly proper rational functions.

Consider a rational function f of the form

$$f(z) = \frac{a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0}{z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0},$$

where $a_0, a_1, \ldots, a_{m-1}, b_0, b_1, \ldots, b_{n-1}$ are complex constants, m and n are nonnegative integers, and $m < n$ (i.e., f is strictly proper). Let us denote the polynomial in the denominator of the above expression for f as d. We can always factor d as

$$d(z) = (z - p_1)(z - p_2) \cdots (z - p_n),$$

where the p_k are the roots of d.

First, let us suppose that the roots of d are distinct (i.e., the p_k are distinct). In this case, f can be expanded as

$$f(z) = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \ldots + \frac{A_{n-1}}{z - p_{n-1}} + \frac{A_n}{z - p_n},$$

where

$$A_k = \left. (z - p_k) f(z) \right|_{z = p_k}.$$

To see why the preceding formula for A_k is correct, we simply evaluate $(z-p_k)f(z)|_{z=p_k}$. We have that

$$\begin{aligned}(z-p_k)f(z)|_{z=p_k} &= \left[\frac{A_1(z-p_k)}{z-p_1}+\frac{A_2(z-p_k)}{z-p_2}+\ldots+\frac{A_{k-1}(z-p_k)}{z-p_{k-1}}+A_k+\right.\\&\qquad\left.\left.\frac{A_{k+1}(z-p_k)}{z-p_{k+1}}+\ldots+\frac{A_n(z-p_k)}{z-p_n}\right]\right|_{z=p_k}\\&= A_k\end{aligned}$$

Now, let us suppose that the roots of d are not distinct. In this case, we can factor d as

$$d(z)=(z-p_1)^{q_1}(z-p_2)^{q_2}\cdots(z-p_P)^{q_P}.$$

One can show that, in this case, f has a partial fraction expansion of the form

$$\begin{aligned}f(z) &= \left[\frac{A_{1,1}}{z-p_1}+\frac{A_{1,2}}{(z-p_1)^2}+\ldots+\frac{A_{1,q_1}}{(z-p_1)^{q_1}}\right]\\&\quad+\left[\frac{A_{2,1}}{z-p_2}+\ldots+\frac{A_{2,q_2}}{(z-p_2)^{q_2}}\right]\\&\quad+\ldots+\left[\frac{A_{P,1}}{z-p_P}+\ldots+\frac{A_{P,q_P}}{(z-p_P)^{q_P}}\right],\end{aligned}$$

where

$$A_{k,\ell}=\frac{1}{(q_k-\ell)!}\left.\left[\left[\tfrac{d}{dz}\right]^{q_k-\ell}\left[(z-p_k)^{q_k}f(z)\right]\right]\right|_{z=p_k}.$$

Note that the q_kth-order pole p_k contributes q_k terms to the partial fraction expansion.

Example B.1 (Simple pole)**.** Find the partial fraction expansion of the function

$$f(z)=\frac{3}{z^2+3z+2}.$$

Solution. First, we rewrite f with the denominator polynomial factored to obtain

$$f(z)=\frac{3}{(z+1)(z+2)}.$$

From this, we know that f has a partial fraction expansion of the form

$$f(z)=\frac{A_1}{z+1}+\frac{A_2}{z+2},$$

where A_1 and A_2 are constants to be determined. Now, we calculate A_1 and A_2 as follows:

$$\begin{aligned}A_1 &= (z+1)f(z)|_{z=-1}\\&= \left.\frac{3}{z+2}\right|_{z=-1}\\&= 3 \quad \text{and}\\A_2 &= (z+2)f(z)|_{z=-2}\\&= \left.\frac{3}{z+1}\right|_{z=-2}\\&= -3.\end{aligned}$$

Thus, the partial fraction expansion of f is given by

$$f(z)=\frac{3}{z+1}-\frac{3}{z+2}.$$

■

Example B.2 (Repeated pole)**.** Find the partial fraction expansion of the function

$$f(z) = \frac{4z+8}{(z+1)^2(z+3)}.$$

Solution. Since f has a repeated pole, we know that f has a partial fraction expansion of the form

$$f(z) = \frac{A_{1,1}}{z+1} + \frac{A_{1,2}}{(z+1)^2} + \frac{A_{2,1}}{z+3}.$$

where $A_{1,1}$, $A_{1,2}$, and $A_{2,1}$ are constants to be determined. To calculate these constants, we proceed as follows:

$$\begin{aligned}
A_{1,1} &= \frac{1}{(2-1)!}\left[\left(\tfrac{d}{dz}\right)^{2-1}\left[(z+1)^2 f(z)\right]\right]\Big|_{z=-1} \\
&= \frac{1}{1!}\left[\tfrac{d}{dz}\left[(z+1)^2 f(z)\right]\right]\Big|_{z=-1} \\
&= \left[\tfrac{d}{dz}\left(\frac{4z+8}{z+3}\right)\right]\Big|_{z=-1} \\
&= \left[4(z+3)^{-1} + (-1)(z+3)^{-2}(4z+8)\right]\Big|_{z=-1} \\
&= \left[\frac{4}{(z+3)^2}\right]\Big|_{z=-1} \\
&= \tfrac{4}{4} \\
&= 1, \\
A_{1,2} &= \frac{1}{(2-2)!}\left[\left(\tfrac{d}{dz}\right)^{2-2}\left[(z+1)^2 f(z)\right]\right]\Big|_{z=-1} \\
&= \frac{1}{0!}\left[(z+1)^2 f(z)\right]\Big|_{z=-1} \\
&= \left[\frac{4z+8}{z+3}\right]\Big|_{z=-1} \\
&= \tfrac{4}{2} \\
&= 2, \quad \text{and} \\
A_{2,1} &= (z+3)f(z)\big|_{z=-3} \\
&= \frac{4z+8}{(z+1)^2}\Big|_{z=-3} \\
&= \tfrac{-4}{4} \\
&= -1.
\end{aligned}$$

Thus, the partial fraction expansion of f is given by

$$f(z) = \frac{1}{z+1} + \frac{2}{(z+1)^2} - \frac{1}{z+3}. \qquad \blacksquare$$

Example B.3 (Improper rational function)**.** Find the partial fraction expansion of the function

$$f(z) = \frac{2z^3+9z^2-z+2}{z^2+3z+2}.$$

Solution. Since f is not strictly proper, we must rewrite f as the sum of a polynomial function and a strictly proper rational function. Using polynomial long division, we have

$$\begin{array}{r|l} & 2z + 3 \\ \hline z^2+3z+2 & 2z^3 + 9z^2 - z + 2 \\ & \underline{2z^3 + 6z^2 + 4z} \\ & \quad 3z^2 - 5z + 2 \\ & \quad \underline{3z^2 + 9z + 6} \\ & \quad\quad -14z - 4. \end{array}$$

Thus, we have

$$f(z) = 2z+3+g(z),$$

where

$$g(z) = \frac{-14z-4}{z^2+3z+2} = \frac{-14z-4}{(z+2)(z+1)}.$$

Now, we find a partial fraction expansion of g. Such an expansion is of the form

$$g(z) = \frac{A_1}{z+1} + \frac{A_2}{z+2}.$$

Solving for the expansion coefficients, we have

$$\begin{aligned} A_1 &= (z+1)g(z)|_{z=-1} \\ &= \left.\frac{-14z-4}{z+2}\right|_{z=-1} \\ &= 10 \quad \text{and} \\ A_2 &= (z+2)g(z)|_{z=-2} \\ &= \left.\frac{-14z-4}{z+1}\right|_{z=-2} \\ &= -24. \end{aligned}$$

Thus, g has the expansion

$$g(z) = \frac{10}{z+1} - \frac{24}{z+2}.$$

Thus, we can decompose f using a partial fraction expansion as

$$f(z) = 2z+3+\frac{10}{z+1} - \frac{24}{z+2}.$$ ■

B.3 Exercises

B.3.1 Exercises Without Answer Key

B.1 Find the partial fraction expansion of each function f given below.

(a) $f(z) = \dfrac{-z^2+2z+7}{4z^3+24z^2+44z+24}$;

(b) $f(z) = \dfrac{-16z-10}{8z^2+6z+1}$;

(c) $f(z) = \dfrac{7z+26}{z^2+7z+10}$;

(d) $f(z) = \dfrac{-2z^2+5}{z^3+4z^2+5z+2}$;

(e) $f(z) = \dfrac{2z^2+15z+21}{z^2+4z+3}$; and

(f) $f(z) = \dfrac{4z^3+36z^2+103z+95}{(z+1)(z+3)^3}$.

B.3.2 Exercises With Answer Key

B.2 Find the partial fraction expansion of each function f given below.

(a) $f(z) = \dfrac{2}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$; and

(b) $f(z) = \dfrac{\frac{1}{10}}{\left(1-\frac{9}{10}z^{-1}\right)\left(1-z^{-1}\right)}$.

Short Answer. (a) $f(z) = \frac{-2}{1-\frac{1}{4}z^{-1}} + \frac{4}{1-\frac{1}{2}z^{-1}}$; (b) $f(z) = \frac{-9/10}{1-\frac{9}{10}z^{-1}} + \frac{1}{1-z^{-1}}$

B.4 MATLAB Exercises

B.101 Use MATLAB to find a partial fraction expansion for each function f given below. [Hint: The `partfrac` function may be helpful.]

(a) $f(z) = \dfrac{1}{z^2+3z+1}$.

Appendix C

Solution of Constant-Coefficient Linear Differential Equations

C.1 Overview

Many systems of practical interest can be represented using linear differential equations with constant coefficients. For this reason, we are interested in solution techniques for such equations. This appendix briefly introduces time-domain methods for solving constant-coefficient linear differential equations.

C.2 Constant-Coefficient Linear Differential Equations

An Nth-order linear differential equation with constant coefficients has the general form

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = \sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t),$$

where the a_k and b_k are constants. If the right-hand side of the above equation is identically equal to zero, the equation is said to be **homogeneous**. Otherwise, the equation is said to be **nonhomogeneous**. Depending on whether the above equation is homogeneous or nonhomogeneous, the solution method differs slightly.

C.3 Solution of Homogeneous Equations

First, we consider the solution of homogeneous equations. In this case, we have an equation of the form

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = 0. \tag{C.1}$$

Let us define the quantity

$$\phi(s) \triangleq \sum_{k=0}^{N} b_k s^k.$$

Then, we refer to

$$\phi(s) = 0$$

as the characteristic (or auxiliary) equation of (C.1). The solution of (C.1) depends on the roots of the characteristic equation, as specified by the theorem below.

Theorem C.1. *Suppose that $\phi(s) = 0$ is the characteristic equation associated with the homogeneous linear differential equation*

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = 0.$$

If $\phi(s) = 0$ has a real root p of multiplicity k, then a solution of the differential equation is

$$(a_0 + a_1 t + \ldots + a_{k-1} t^{k-1}) e^{pt}.$$

If $\phi(s) = 0$ has a pair of complex conjugate roots $\sigma \pm j\omega$, each root being of multiplicity k, then a solution of the differential equation is

$$e^{\sigma t}\left[(a_0 + a_1 t + \ldots + a_{k-1} t^{k-1}) \cos(\omega t) + (b_0 + b_1 t + \ldots + b_{k-1} t^{k-1}) \sin(\omega t)\right].$$

A general solution of the differential equation is obtained by taking a linear combination of the solutions obtained by considering all roots of $\phi(s) = 0$.

From the above theorem, we can see that, in order to solve an equation of the form (C.1), we begin by finding the roots of the corresponding characteristic equation. Then, we find a solution associated with each distinct root (or pair of complex conjugate roots). Finally, the general solution is formed by taking a linear combination of all of these individual solutions.

Example C.1. Find the general solution to the differential equation

$$y''(t) + 4y'(t) + 5y(t) = 0.$$

Solution. The given differential equation has the characteristic equation

$$s^2 + 4s + 5 = 0.$$

Solving for the roots of the characteristic equation yields

$$\begin{aligned} s &= \frac{-4 \pm \sqrt{4^2 - 4(5)}}{2} \\ &= -2 \pm \tfrac{1}{2}\sqrt{-4} \\ &= -2 \pm j. \end{aligned}$$

Thus, we have one pair of complex conjugate roots (i.e., $-2 \pm j$), each root being of multiplicity 1. Therefore, the general solution to the given equation is of the form

$$y(t) = e^{-2t} (a_1 \cos t + b_1 \sin t). \qquad \blacksquare$$

Example C.2. Find the general solution to the differential equation

$$y''(t) + 5y'(t) + 6y(t) = 0.$$

Solution. The given differential equation has the characteristic equation

$$s^2 + 5s + 6 = 0,$$

which can be factored as

$$(s+2)(s+3) = 0.$$

Clearly, the characteristic equation has the roots -2 and -3, each of multiplicity 1. Therefore, the general solution of the given equation is of the form

$$y(t) = a_1 e^{-3t} + a_2 e^{-2t}. \qquad \blacksquare$$

Example C.3. Find the general solution to the differential equation

$$y''(t) + 2y'(t) + y(t) = 0.$$

Solution. The given differential equation has the characteristic equation

$$s^2 + 2s + 1 = 0,$$

which can be factored as

$$(s+1)^2 = 0.$$

Clearly, the characteristic equation has the root -1 of multiplicity 2. Therefore, the general solution to the given equation is of the form

$$y(t) = (a_0 + a_1 t)e^{-t}. \qquad \blacksquare$$

C.4 Particular Solution of Nonhomogeneous Equations

So far, we have only considered the solution of homogeneous equations. Now, we consider the nonhomogeneous case. In the nonhomogeneous case, we have an equation of the form

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = f(t). \tag{C.2}$$

As it turns out, in order to find a general solution to the above equation, we must first find a particular solution.

To find a particular solution, we must consider the form of the function f. Suppose that a particular solution to (C.2) is given by the function y_p. Since y_p is a solution to (C.2), when we substitute y_p into (C.2), the left-hand side of (C.2) must equal f. Therefore, y_p and its derivatives must be comprised of terms that resemble the terms of f. Thus, by examining f, we can deduce a general expression for y_p containing one or more unknown coefficients. We, then, solve for these unknown coefficients. This solution technique is sometimes referred to as the **method of undetermined coefficients**.

Table C.1 shows the general form of y_p that should be nominally used in the case of several frequently encountered forms for f. There is, however, one caveat. The general expression chosen for y_p must not include any terms that are linearly dependent on terms in the solution to the corresponding complementary equation. If linearly dependent terms exist in our preliminary choice of y_p, we must replace each term $v(t)$ with $t^m v(t)$ where m is the smallest positive integer eliminating this linear dependence. To illustrate how this solution process works in more detail, we will now consider a few example problems.

Example C.4. Find a particular solution of the differential equation

$$y''(t) + 4y'(t) + 5y(t) = 5t^2 + 3t + 8.$$

Solution. We begin by considering the form of the function on the right-hand side of the given differential equation. Since terms in t^2, t^1, and t^0 yield terms in t^2, t^1, and t^0 when substituted into the left-hand side of the given equation, we deduce that a particular solution is of the form

$$y_\mathsf{p}(t) = a_2 t^2 + a_1 t + a_0. \tag{C.3}$$

Table C.1: Forms for the particular solution

$f(t)$	$y_\mathsf{p}(t)$
$c_0 + c_1 t + \ldots + c_n t^n$	$p_0 + p_1 t + \ldots + p_n t^n$
ce^{at}	pe^{at}
$c \sin bt$ or $c \cos bt$	$p \sin bt + q \cos bt$
$(c_0 + c_1 t + \ldots + c_n t^n) e^{at}$	$(p_0 + p_1 t + \ldots + p_n t^n) e^{at}$
$(c_0 + c_1 t + \ldots + c_n t^n) \cos bt$ or $(c_0 + c_1 t + \ldots + c_n t^n) \sin bt$	$(p_0 + p_1 t + \ldots + p_n t^n) \cos bt + (q_0 + q_1 t + \ldots + q_n t^n) \sin bt$
$(c_0 + c_1 t + \ldots + c_n t^n) e^{at} \cos bt$ or $(c_0 + c_1 t + \ldots + c_n t^n) e^{at} \sin bt$	$(p_0 + p_1 t + \ldots + p_n t^n) e^{at} \cos bt + (q_0 + q_1 t + \ldots + q_n t^n) e^{at} \sin bt$

Differentiating y_p, we obtain

$$\begin{aligned} y_\mathsf{p}'(t) &= 2a_2 t + a_1 \quad \text{and} \\ y_\mathsf{p}''(t) &= 2a_2. \end{aligned}$$

Substituting y_p and its derivatives into the given differential equation yields

$$\begin{aligned} & 2a_2 + 4[2a_2 t + a_1] + 5\left[a_2 t^2 + a_1 t + a_0\right] = 5t^2 + 3t + 8 \\ \Rightarrow \quad & 2a_2 + 8a_2 t + 4a_1 + 5a_2 t^2 + 5a_1 t + 5a_0 = 5t^2 + 3t + 8 \\ \Rightarrow \quad & [5a_2]t^2 + [8a_2 + 5a_1]t + [2a_2 + 4a_1 + 5a_0] = 5t^2 + 3t + 8. \end{aligned}$$

Comparing the left- and right-hand sides of the above equation, we see that

$$\begin{aligned} 5a_2 &= 5, \\ 8a_2 + 5a_1 &= 3, \quad \text{and} \\ 2a_2 + 4a_1 + 5a_0 &= 8. \end{aligned}$$

Solving this system of equations yields $a_0 = 2$, $a_1 = -1$, and $a_2 = 1$. Therefore, from (C.3), the given differential equation has the particular solution

$$y_\mathsf{p}(t) = t^2 - t + 2. \qquad \blacksquare$$

Example C.5. Find a particular solution of the differential equation

$$y''(t) + 4y'(t) + 5y(t) = e^{-2t}.$$

Solution. We begin by considering the form of the function on the right-hand side of the given differential equation. Since terms in e^{-2t} yield terms in e^{-2t} when substituted into the left-hand side of the equation, we deduce that the particular solution y_p is of the form

$$y_\mathsf{p}(t) = ae^{-2t}.$$

Differentiating y_p, we obtain

$$\begin{aligned} y_\mathsf{p}'(t) &= -2ae^{-2t} \quad \text{and} \\ y_\mathsf{p}''(t) &= 4ae^{-2t}. \end{aligned}$$

Substituting y_p and its derivatives into the given differential equation yields

$$\begin{aligned} & 4ae^{-2t} + 4\left[-2ae^{-2t}\right] + 5\left[ae^{-2t}\right] = e^{-2t} \\ \Rightarrow \quad & 4ae^{-2t} - 8ae^{-2t} + 5ae^{-2t} = e^{-2t} \\ \Rightarrow \quad & ae^{-2t} = e^{-2t}. \end{aligned}$$

Comparing the left- and right-hand sides of the above equation, we have that $a = 1$. Therefore, the given differential equation has the particular solution

$$y_p(t) = e^{-2t}. \qquad \blacksquare$$

Example C.6. Find a particular solution of the differential equation

$$y''(t) + 4y'(t) + 5y(t) = \sin t.$$

Solution. To begin, we examine the form of the function of the right-hand side of the given differential equation. Since terms in $\sin t$ yield terms in $\sin t$ and $\cos t$ when substituted into the left-hand side of the given equation, we deduce that the particular solution y_p is of the form

$$y_p(t) = a_1 \cos t + a_2 \sin t.$$

Differentiating y_p, we obtain

$$\begin{aligned} y_p'(t) &= -a_1 \sin t + a_2 \cos t \quad \text{and} \\ y_p''(t) &= -a_1 \cos t - a_2 \sin t. \end{aligned}$$

Substituting y_p and its derivatives into the given differential equation yields

$$\begin{aligned} & [-a_1 \cos t - a_2 \sin t] + 4[-a_1 \sin t + a_2 \cos t] + 5[a_1 \cos t + a_2 \sin t] = \sin t \\ \Rightarrow \quad & [-a_1 + 4a_2 + 5a_1] \cos t + [-a_2 - 4a_1 + 5a_2] \sin t = \sin t \\ \Rightarrow \quad & [4a_1 + 4a_2] \cos t + [4a_2 - 4a_1] \sin t = \sin t. \end{aligned}$$

Comparing the left- and right-hand sides of the above equation, we have that

$$\begin{aligned} 4a_1 + 4a_2 &= 0 \quad \text{and} \\ 4a_2 - 4a_1 &= 1. \end{aligned}$$

Solving this system of equations yields $a_1 = -\frac{1}{8}$ and $a_2 = \frac{1}{8}$. Therefore, the given differential equation has the particular solution

$$y_p(t) = -\tfrac{1}{8} \cos t + \tfrac{1}{8} \sin t. \qquad \blacksquare$$

C.5 General Solution of Nonhomogeneous Equations

With every nonhomogeneous constant-coefficient linear differential equation

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = \sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t),$$

we can associate a homogeneous equation

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = 0$$

called the **complementary equation**. The complementary equation is formed by simply setting the function x (and its derivatives) to zero in the original equation.

As it turns out, in order to find the solution of a nonhomogeneous equation, we must find a particular solution to the given equation and also a general solution to its complementary equation. This process is more precisely specified by the theorem below.

Theorem C.2. *A general solution of the linear differential equation*

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = \sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t)$$

has the form

$$y(t) = y_{\mathsf{c}}(t) + y_{\mathsf{p}}(t),$$

where y_{c} is a general solution of the associated complementary equation and y_{p} is any particular solution of the given equation.

Example C.7. Consider the differential equation

$$y''(t) + 2y'(t) + 2y(t) = -2t + 4.$$

(a) Find the general solution of this equation.
(b) Find the solution if $y(0) = 1$ and $y'(0) = 0$.

Solution. (a) First, we need to find the general solution y_{c} of the corresponding complementary equation

$$y''(t) + 2y'(t) + 2y(t) = 0.$$

This equation has the characteristic equation

$$s^2 + 2s + 2 = 0.$$

Solving for the roots of this equation, we have

$$\begin{aligned} s &= \frac{-2 \pm \sqrt{2^2 - 4(2)}}{2} \\ &= -1 \pm j. \end{aligned}$$

Therefore, the characteristic equation has a pair of complex conjugate roots $-1 \pm j$, each root being of multiplicity 1. From this, we know that the complementary equation has a general solution of the form

$$y_{\mathsf{c}}(t) = e^{-t}(a_1 \cos t + b_1 \sin t). \tag{C.4}$$

Now, we must find a particular solution y_{p} of the given differential equation. We consider the form of the function on the right-hand side of the given equation. Since terms in t^1 and t^0 yield terms in t^1 and t^0 when substituted into the left-hand side of the equation, we deduce that y_{p} is of the form

$$y_{\mathsf{p}}(t) = c_1 t + c_0.$$

Differentiating y_p, we obtain

$$y_p'(t) = c_1 \quad \text{and}$$
$$y_p''(t) = 0.$$

Substituting y_p and its derivatives into the given differential equation yields

$$2c_1 + 2[c_1 t + c_0] = -2t + 4$$
$$\Rightarrow \quad [2c_1]t + [2c_1 + 2c_0] = -2t + 4.$$

Comparing the left- and right-hand sides of the above equation, we have

$$2c_1 = -2 \quad \text{and}$$
$$2c_1 + 2c_0 = 4.$$

Solving this system of equations yields $c_0 = 3$ and $c_1 = -1$. Therefore, the particular solution is given by

$$y_p(t) = -t + 3. \tag{C.5}$$

Combining the results of (C.4) and (C.5), we conclude that the given equation has the general solution

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= e^{-t}(a_1 \cos t + b_1 \sin t) - t + 3. \end{aligned}$$

(b) We compute the derivative of y as

$$y'(t) = -e^{-t}(a_1 \cos t + b_1 \sin t) + e^{-t}(-a_1 \sin t + b_1 \cos t) - 1.$$

From the given initial conditions, we have

$$0 = -a_1 + b_1 - 1 \quad \text{and}$$
$$1 = a_1 + 3.$$

Solving for a_1 and b_1 yields $a_1 = -2$ and $b_1 = -1$. Therefore, we have

$$\begin{aligned} y(t) &= e^{-t}(-2\cos t - \sin t) - t + 3 \\ &= -e^{-t}(2\cos t + \sin t) - t + 3. \end{aligned}$$

■

Example C.8. Consider the differential equation

$$y''(t) + 3y'(t) + 2y(t) = e^{-t}.$$

(a) Find the general solution of this equation.
(b) Determine the solution if $y(0) = -1$ and $y'(0) = 1$.

Solution. First, we need to find the general solution y_c of the corresponding complementary equation

$$y''(t) + 3y'(t) + 2y(t) = 0.$$

This equation has the characteristic equation

$$s^2 + 3s + 2 = 0,$$

which can be factored as

$$(s+2)(s+1) = 0.$$

Thus, the characteristic equation has roots at -2 and -1, each of multiplicity 1. From this, we can deduce that

$$y_{\mathsf{c}}(t) = a_1 e^{-t} + a_2 e^{-2t}. \tag{C.6}$$

Now, we need to find a particular solution y_{p} of the given differential equation. Since y_{c} contains a term with e^{-t}, we deduce that y_{p} is of the form

$$y_{\mathsf{p}}(t) = cte^{-t}.$$

Differentiating y_{p}, we obtain

$$\begin{aligned} y'_{\mathsf{p}}(t) &= ce^{-t} - cte^{-t} \quad \text{and} \\ y''_{\mathsf{p}}(t) &= -ce^{-t} - c(e^{-t} - te^{-t}) = -2ce^{-t} + cte^{-t}. \end{aligned}$$

Substituting y_{p} and its derivatives into the given differential equation yields

$$\begin{aligned} & (-2ce^{-t} + cte^{-t}) + 3(ce^{-t} - cte^{-t}) + 2cte^{-t} = e^{-t} \\ \Rightarrow \quad & (-2c + 3c)e^{-t} + (c - 3c + 2c)te^{-t} = e^{-t} \\ \Rightarrow \quad & ce^{-t} = e^{-t}. \end{aligned}$$

Comparing the left- and right-hand sides of this equation, we conclude $c = 1$. Therefore, we have

$$y_{\mathsf{p}}(t) = te^{-t}. \tag{C.7}$$

Combining the results from (C.6) and (C.7), we have

$$\begin{aligned} y(t) &= y_{\mathsf{c}}(t) + y_{\mathsf{p}}(t) \\ &= a_1 e^{-t} + a_2 e^{-2t} + te^{-t}. \end{aligned}$$

(b) We compute the derivative of y as

$$y'(t) = -a_1 e^{-t} - 2a_2 e^{-2t} + e^{-t} - te^{-t}.$$

From the given initial conditions, we have

$$\begin{aligned} 1 &= -a_1 - 2a_2 + 1 \quad \text{and} \\ -1 &= a_1 + a_2. \end{aligned}$$

Solving for a_1 and a_2 yields

$$\begin{aligned} a_1 &= -2 \quad \text{and} \\ a_2 &= 1. \end{aligned}$$

Therefore, we have that

$$y(t) = -2e^{-t} + e^{-2t} + te^{-t}. \qquad \blacksquare$$

C.6 Exercises

C.6.1 Exercises Without Answer Key

C.1 Find the general solution to each of the differential equations below.
(a) $8y''(t)+6y'(t)+y(t)=0$;
(b) $y'''(t)+5y''(t)+17y'(t)+13y(t)=0$ [Hint: One root of the characteristic equation is -1.];
(c) $y''(t)+9y'(t)+20y(t)=0$; and
(d) $y''(t)+2y'(t)+y(t)=0$.

C.2 Find a particular solution to each of the differential equations below. In each case, the solution y_c of the corresponding complementary equation is given.
(a) $y''(t)+3y'(t)+2y(t)=t^2$ with $y_c(t)=c_1e^{-t}+c_2e^{-2t}$;
(b) $y''(t)+3y'(t)+2y(t)=e^{-3t}+t$ with $y_c(t)=c_1e^{-t}+c_2e^{-2t}$;
(c) $y''(t)+4y'(t)+3y(t)=e^{-t}$ with $y_c(t)=c_1e^{-t}+c_2e^{-3t}$; and
(d) $y''(t)+2y'(t)+y(t)=\sin t$ with $y_c(t)=c_0e^{-t}+c_1te^{-t}$.

C.3 Consider the differential equation

$$y''(t)+3y'(t)+2y(t)=t+1.$$

(a) Find the general solution to this equation.
(b) Determine the solution in the case that $y(0)=-\frac{1}{4}$ and $y'(0)=-\frac{1}{2}$.

C.4 Consider the differential equation

$$y''(t)+5y'(t)+6y(t)=2e^{-3t}.$$

(a) Find the general solution of this equation.
(b) Determine the solution in the case that $y(0)=0$ and $y'(0)=1$.

C.5 Find the general solution to each of the differential equations below.
(a) $y''(t)+7y'(t)+12y(t)=6t^2-5t+18$;
(b) $y''(t)+7y'(t)+12y(t)=e^{-3t}$;
(c) $y''(t)+4y'(t)+8y(t)=e^{-t}$; and
(d) $y''(t)+2y'(y)+5y(t)=1+e^{-t}$.

C.6.2 Exercises With Answer Key

Currently, there are no exercises available with an answer key.

C.7 MATLAB Exercises

C.101 Use the `dsolve` function in MATLAB to solve each of the differential equations in Exercise C.1.

C.102 Use the `dsolve` function in MATLAB to solve Exercise C.4.

Appendix D

MATLAB

D.1 Introduction

MATLAB is a software tool that is useful for solving a wide variety of problems arising in engineering applications. The MATLAB software is a product of a company called MathWorks. Extensive information on this software (including detailed guides and manuals) is available, free of charge, from the company's web site (https://www.mathworks.com). A number of helpful books on MATLAB are also available, such as [1, 2]. In this appendix, a reasonably detailed introduction to MATLAB is also provided.

D.2 Octave

Although MATLAB is very powerful, it is a commercial software product. Therefore, MATLAB is not free. Fortunately, an open-source MATLAB-like software package is available called Octave. Octave is available for download from its official web site https://www.octave.org. This software is included in several major Linux distributions (e.g., Fedora and Ubuntu). As well, Octave is also available for the Cygwin environment under Microsoft Windows. (For more details about Cygwin, see https://www.cygwin.com.)

D.3 Invoking MATLAB

On most UNIX systems, the MATLAB software is started by invoking the `matlab` command.

D.3.1 UNIX

The MATLAB software is invoked using a command line of the form:

`matlab` *[options]*

The `matlab` command supports a number of options including the following:

`-help` or `-h`
: Display information on MATLAB options.

`-nodisplay`
: Disable all graphical output. The MATLAB desktop will not be started.

`-nojvm`
: Disable all Java support by not starting the Java virtual machine. In particular, the MATLAB desktop will not be started.

Table D.1: Keys for command-line editing

Key	Action
Up arrow	Recall previous line
Down arrow	Recall next line
Left arrow	Move left one character
Right arrow	Move right one character
Home	Move to beginning of line
End	Move to end of line
Ctrl-C	Cancel current line

`-nodesktop`
Disable the MATLAB desktop. Use the current terminal for commands.

`-display` *displayname*
Specify the X display to use for graphics output.

Like most UNIX programs, MATLAB uses the X Windows System for rendering graphics output. The `DISPLAY` environment variable provides the default display setting for graphics output. If necessary, one can override the default display setting by explicitly specifying the display to use via the `-display` option.

When running MATLAB remotely, it may be necessary to disable the desktop (with the `-nodesktop` option). This is due to the fact that, when the desktop is enabled, MATLAB tends to require a relatively large amount of network bandwidth, which can be problematic over lower-speed network connections.

D.3.2 Microsoft Windows

Unfortunately, the author has not used MATLAB under Microsoft Windows. So, he cannot comment on the specifics of running MATLAB under this operating system.

D.4 Command Line Editor

In MATLAB, several keys are quite useful for editing purposes, as listed in Table D.1. For example, the arrow keys can be used to perform editing in the usual way.

D.5 MATLAB Basics

Arguably, one of the most helpful commands in MATLAB is the `help` command. This command can be used to obtain information on many of the operators, functions, and commands available in MATLAB. For example, to find information on the `help` command, one can type:

```
help help
```

In a similar vein, the `doc` command can be used to obtain detailed documentation on many of the functions and commands in MATLAB. For example, to display documentation on the `doc` command, one can type:

```
doc doc
```

D.5.1 Identifiers

Identifiers (i.e., variable/function names) are case sensitive and may consist of uppercase and lowercase letters, underscores, and digits, but the first character cannot be a digit or an underscore. Although an identifier can be arbitrarily long, only the first *n* characters are significant, where *n* depends on the particular version of MATLAB being used. Any characters after the first *n* are simply ignored. (The `namelengthmax` function can be used to query the precise

Table D.2: Predefined variables

Variable	Description
`pi`	π
`i`	$\sqrt{-1}$
`j`	$\sqrt{-1}$
`nan`	not-a-number (NaN)
`inf`	infinity
`ans`	last expression evaluated that was not assigned to variable
`date`	date
`clock`	wall clock
`realmin`	smallest usable positive real number
`realmax`	largest usable positive real number

Table D.3: Operators

Symbol	Description
`+`	unary plus (i.e., identity) and binary plus (i.e., addition)
`-`	unary minus (i.e., negation) and binary minus (i.e., subtraction)
`*`	multiplication
`/`	right division
`\`	left division
`^`	exponentiation
`'`	conjugate transpose
`.*`	element-wise multiplication
`./`	element-wise division
`.^`	element-wise exponentiation
`.'`	transpose

value of n.) Several variables are automatically predefined by MATLAB as listed in Table D.2. A new value can be assigned to a predefined variable, causing its original value to be lost.

D.5.2 Basic Functionality

Generally, it is desirable to include comments in code to explain how the code works. In MATLAB, comments begin with a percent sign character and continue to the end of line.

Some of the operators supported by MATLAB are listed in Table D.3. As a matter of terminology, an operator that takes one operand is said to be **unary**, while an operator that takes two operands said to be **binary**. Note that the plus (`+`) and minus (`-`) operators have both unary and binary forms. For example, the expression "`-x`" employs the unary minus (i.e., negation) operator (where the minus operator has the single operand `x`), while the expression "`x - y`" employs the binary minus (i.e., subtraction) operator (where the operator has two operands `x` and `y`).

Some math functions provided by MATLAB are listed in Tables D.4, D.5, D.6, D.7, D.8, and D.9. Note that the `sinc` function in MATLAB does not compute the sinc function as defined (by (3.20)) herein. Instead, this MATLAB function computes the normalized sinc function (as defined by (3.21)).

Example D.1. Some examples of very basic calculations performed using MATLAB are as follows:

```
a = [1 2 3; 4 5 6; 7 8 9] % 3 x 3 array
b = [1 2 3
     4 5 6
     7 8 9] % 3 x 3 array
a - b
x = [1; 3; -1] % 3-dimensional column vector
```


Table D.4: Elementary math functions

Name	Description
abs	magnitude of complex number
angle	principal argument of complex number
imag	imaginary part of complex number
real	real part of complex number
conj	conjugate of complex number
round	round to nearest integer
fix	round towards zero
floor	round towards $-\infty$
ceil	round towards ∞
sign	signum function
rem	remainder (with same sign as dividend)
mod	remainder (with same sign as divisor)

Table D.5: Other math-related functions

Name	Description
min	minimum value
max	maximum value
mean	mean value
std	standard deviation
median	median value
sum	sum of elements
prod	product of elements
cumsum	cumulative sum of elements
cumprod	cumulative product of elements
polyval	evaluate polynomial
cart2pol	Cartesian-to-polar coordinate conversion
pol2cart	polar-to-Cartesian coordinate conversion

Table D.6: Exponential and logarithmic functions

Name	Description
exp	exponential function
log	natural logarithmic function
log10	base-10 logarithmic function
sqrt	square root function

Table D.7: Trigonometric functions

Name	Description
sin	sine function
cos	cosine function
tan	tangent function
asin	arcsine function
acos	arccosine function
atan	arctangent function
atan2	two-argument form of arctangent function

Table D.8: Other math functions

Name	Description
sinc	normalized sinc function (as defined in (3.21))

Table D.9: Radix conversion functions

Name	Description
`dec2bin`	convert decimal to binary
`bin2dec`	convert binary to decimal
`dec2hex`	convert decimal to hexadecimal
`hex2dec`	convert hexadecimal to decimal
`dec2base`	convert decimal to arbitrary radix
`base2dec`	convert arbitrary radix to decimal

```
y = x .* x + 3 * x + 2
y = a * x

t = 5;
s = t ^ 2 + 3 * t - 7;

z = 3 + 4 * j; % complex number in Cartesian form
z = 20 * exp(j * 10); % complex number in polar form
```

■

The `disp` function prints a single string. For example, the following code fragment prints "Hello, world" (followed by a newline character):

```
disp('Hello, world');
```

The `sprintf` function provides very sophisticated string formatting capabilities, and is often useful in conjunction with the `disp` function. The use of the `sprintf` function is illustrated by the following code fragment:

```
name = 'Jane Doe';
id = '06020997';
mark = 91.345678912;
disp(sprintf('The student %s (ID %s) received a grade of %4.2f%%.', ...
  name, id, mark));
```

The `sprintf` function is very similar in spirit to the function of the same name used in the C programming language.

D.6 Arrays

Frequently, it is necessary to determine the dimensions of an array (i.e., matrix or vector). For this purpose, MATLAB provides two very useful functions as listed in Table D.10. The function `size` can be used to determine the number of rows and/or columns in an array:

- `size(a)` returns a row vector with the number of rows and columns in `a` as elements (in that order);
- `size(a, 1)` returns the number of rows in `a`; and
- `size(a, 2)` returns the number of columns in `a`.

The function `length` is used to find the maximum of the two array dimensions. That is, `length(a)` is equivalent to `max(size(a))`. Usually, the `length` function is used in conjunction with arrays that are known to be row/column vectors.

Example D.2. Suppose that `a = [1 2 3 4; 5 6 7 8]`. Then, `size(a)` returns `[2 4]`, `size(a, 1)` returns `2`, `size(a, 2)` returns `4`, and `length(a)` returns `4`. ■

Table D.10: Array size functions

Name	Description
`size`	query array dimensions
`length`	query vector/array dimension

Table D.11: Examples of abbreviated forms of vectors

Abbreviated Form	Long Form
`1 : 4`	`[1 2 3 4]`
`0 : 0.2 : 1`	`[0 0.2 0.4 0.6 0.8 1]`
`1 : -1 : -2`	`[1 0 -1 -2]`
`0 : 10 : 25`	`[0 10 20]`
`-1.5 : -1 : -4`	`[-1.5 -2.5 -3.5]`

D.6.1 Arrays with Equally-Spaced Elements

Often, it is necessary to specify a vector with equally-spaced elements. As a result, MATLAB provides a compact means for specifying such a vector. In particular, an expression of the following form is employed:

start : *step* : *end*

The above expression is equivalent to a row vector with its first element equal to *start* and each of the subsequent elements increasing in value by *step* until the value would exceed *end*. Note that *step* may be negative.

Example D.3. In Table D.11, some examples of abbreviated forms of vectors are given. ■

D.6.2 Array Subscripting

Suppose that we have an array `a`. We can access elements of the array by specifying the rows and columns in which the elements are contained. In particular, `a(`*rowspec*`,` *colspec*`)` is the array consisting of the elements of `a` that are in the rows specified by *rowspec* and columns specified by *colspec*. Here, *rowspec* is either a vector containing row indices or the special value ":" which means "all rows". Similarly, *colspec* is either a vector containing column indices or the special value ":" which means "all columns". We can also access elements of the array `a` by specifying a 1-dimensional element index, where elements in the array are numbered in column-major order. That is, `a(`*indspec*`)` is the vector of elements of `a` that have the indices specified by *indspec*. Here, *indspec* is either a vector containing element indices or the special value ":" which means "all elements".

Example D.4. Suppose that `a` is a 10×10 matrix and `x` is 10×1 vector. Some examples of array subscripting involving `a` and `x` are given in Table D.12. ■

D.6.3 Other Array Functions

Certain types of matrices tend to be used frequently in code. For this reason, MATLAB provides functions for generating some common forms of matrices. These functions are listed in Table D.13.

MATLAB provides functions for performing some common operations to matrices. These are listed in Table D.14.

D.7 Scripts

Instead of interactively entering MATLAB code for immediate execution, code can be placed in a file and then executed. Normally, MATLAB code is placed in what are called M-files. The term "M file" originates from the fact that these files use a name ending in the suffix ".m". To create an M-file script, one simply creates a file with a name ending in the suffix ".m". Then, the code in the M-file can be invoked by using a command with the same name as the M-file but without the ".m" extension. For reasons that will become apparent shortly, the base name of the M-file

Table D.12: Array subscripting examples

Expression	Meaning
`a(1, :)`	first row of `a`
`a(:, 1)`	first column of `a`
`a(1 : 50)`	first 50 elements of `a` arranged in a row vector
`a(1 : 10)`	first 10 elements of `a` arranged in a row vector (i.e., the first column of `a`)
`a(1 : 2 : 10, :)`	odd-indexed rows of `a`
`a(:, 2 : 2 : 10)`	even-indexed columns of `a`
`a(1 : 5, :)`	rows 1 to 5 of `a`
`a(:, 6 : 10)`	columns 6 to 10 of `a`
`a(1 : 2, 9 : 10)`	submatrix consisting of elements that are in rows 1,2 and also in columns 9,10
`x(1 : 3)`	first three elements of `x` (arranged as a row or column vector to match `x`)

Table D.13: Special matrix/vector functions

Name	Description
`eye`	identity matrix
`ones`	matrix of ones
`zeros`	matrix of zeros
`diag`	diagonal matrix
`rand`	random matrix
`linspace`	vector with linearly spaced elements
`logspace`	vector with logarithmically spaced elements

Table D.14: Basic array manipulation functions

Name	Description
`rot90`	rotate array by 90 degrees
`fliplr`	flip array horizontally
`flipud`	flip array vertically
`reshape`	change array dimensions

(i.e., the name without the ".m" extension) must be a valid MATLAB identifier. For example, `2foobar.m` is not a valid M-file name since "2foobar" is not a valid MATLAB identifier. (Recall that MATLAB identifiers cannot start with a digit such as "2".) Also, in order for MATLAB to find an M-file, the file must be in one of the directories listed in the MATLAB search path. We will explain how to query and change the MATLAB search path later in this section. Before doing so, however, we provide a few examples of M-file scripts below.

Example D.5. Suppose that we have an M-file script called `hello.m` with the following contents:

```
% Print a greeting.
disp('Hello, World.');
```

Then, the code in this file can be executed by simply typing the following in MATLAB:

```
hello
```

That is, we invoke the code in the M-file by using the base name of the file. (It is tacitly assumed that the file `hello.m` has been placed in one of the directories listed in the MATLAB search path.) ■

Example D.6. In order to save some typing, we can create a file called `main.m` containing the following:

```
a = [
  0.9501 0.8913 0.8214 0.9218;
  0.2311 0.7621 0.4447 0.7382;
  0.6068 0.4565 0.6154 0.1763;
  0.4860 0.0185 0.7919 0.4057;
];
y0 = a * [1 2 3 4].'
y1 = a * [-1 2.5 3 4].'
y3 = a * [41 -22 3 4].'
```

Then, to invoke the code in the above file, we simply type the following in our MATLAB session:

```
main
```

(Again, it is tacitly assumed that the file `main.m` has been placed in one of the directories listed in the MATLAB search path.) ■

Generally, one should avoid giving a script file a name that is associated with a previously defined variable or function, as this leads to the potential for naming conflicts. For example, it would be a bad idea to name a script file as `sin.m`, since the `sin` function is already built into MATLAB.

Clearly, MATLAB needs a means to locate M-file scripts, since there are usually many directories in which a user might choose to place a script. For this purpose, the MATLAB search path is used. The MATLAB search path is a list of directories in which MATLAB looks for M-files. In order for the code in an M-file to be successfully located by MATLAB and executed, the M-file must be stored in a directory listed in the MATLAB search path. The MATLAB search path can be queried with the `path` command:

```
path
```

This command will output all of the directories in the MATLAB search path (i.e., all of the directories in which MATLAB will look for M-file scripts).

A new directory can be added to the MATLAB search path with the `addpath` command:

`addpath` *dirname*

This adds the directory *dirname* to the MATLAB search path.

A few other commands are also sometimes useful in the context of M-file scripts. These commands are described below.

The working directory for MATLAB can be changed using the `cd` command:

`cd` *dirname*

Table D.15: Examples of expressions involving relational operators

Expression	Value
`a > b`	`[0 0 0 1 1]`
`a == b`	`[0 0 1 0 0]`
`a < b`	`[1 1 0 0 0]`
`a >= 2 & a <= 3`	`[0 1 1 0 0]`
`a < 2 | a > 4`	`[1 0 0 0 1]`
`~a`	`[0 0 0 0 0]`

Table D.16: Relational operators

Symbol	Description
`<`	less than
`<=`	less than or equal
`>`	greater than
`>=`	greater than or equal
`==`	equal
`~=`	not equal

Table D.17: Logical operators

Symbol	Description
`&`	element-wise and
`|`	element-wise or
`~`	not
`&&`	short-circuit and
`||`	short-circuit or

The working directory is changed to the directory named *dirname*.

The current working directory can be queried with the `pwd` command:

```
pwd
```

This command will display the current working directory.

D.8 Relational and Logical Operators

The relational and logical operators provided by MATLAB are listed in Tables D.16 and D.17, respectively. Some functions that are also quite useful in relational and logical expressions are listed in Table D.18. As far as boolean expressions are concerned, MATLAB considers any nonzero number to be true and zero to be false.

Example D.7. Suppose that `a = [1 2 3 4 5]` and `b = [5 4 3 2 1]`. Some examples of expressions using `a` and `b` in conjunction with relational operators are given in Table D.15. ■

Example D.8. The following code fragment illustrates how one might use some of the relational/logical functions such as `all`, `any`, and `find`:

```
a = [1 2 3; 4 5 6; 7 8 9];
if all(a > 0)
    disp('All matrix elements are positive.');
end
if all(a < 0)
    disp('All matrix elements are negative.');
end
```


Table D.18: Relational and logical functions

Name	Description
`any`	any element nonzero
`all`	all elements nonzero
`find`	find nonzero elements
`exist`	check if variables exist
`isfinite`	detect finite values
`isinf`	detect infinities
`isnan`	detect NaNs
`isempty`	detect empty matrices
`isstr`	detect strings
`strcmp`	compare strings

Table D.19: Operator precedence

Operators	Precedence Level
`()`	highest
`.' ' .^ ^`	
unary+ unary- `~`	
`.* * ./ .\ / \`	
binary+ binary-	
`:`	
`< <= > >= == ~=`	
`&`	
`|`	
`&&`	
`||`	lowest

In older versions of MATLAB, `&` and `|` have the same precedence.

```
if ~any(a == 0)
    disp('All matrix elements are nonzero.');
end
if all(real(a) == a)
    disp('All matrix elements are real.');
end
i = find(a >= 8);
disp('The following matrix elements are greater than or equal to 8:');
disp(a(i));
```

■

D.9 Operator Precedence

When an expression involves more than one operator (such as "`-x ^ 2 + 2 * x - 1`"), the order in which operators are applied becomes important. This ordering is more formally referred to as operator precedence. The precedence of operators in MATLAB is shown in Table D.19. Note that the unary and binary minus operators have different precedence, and the unary and binary plus operators have different precedence. As one would probably expect, the multiplication operator (`*`) has higher precedence than the addition operator (binary `+`), and the exponentiation operator (`^`) has higher precedence than both the addition and multiplication operators.

D.10 Control Flow

In the sections that follow, we introduce the conditional execution and looping constructs available in MATLAB.

D.10.1 If-Elseif-Else

The **if**-**elseif**-**else** construct allows groups of statements to be conditionally executed, and has a number of variants. The simplest variant (i.e., the **if** variant) has the form:

if *expression*
    statements
end

If the expression *expression* is true, then the statements *statements* are executed. The next simplest variant (i.e., the **if**-**else** variant) has the form:

if $expression_1$
    $statements_1$
else
    $statements_2$
end

If the expression $expression_1$ is true, then the statements $statements_1$ are executed. Otherwise, the statements $statements_2$ are executed. Finally, the most general variant has the form:

if $expression_1$
    $statements_1$
elseif $expression_2$
    $statements_2$
        ⋮
elseif $expression_{n-1}$
    $statements_{n-1}$
else
    $statements_n$
end

Note that the **elseif** and **else** clauses are optional.

Example D.9. The following code fragment tests the sign of the variable x and prints an appropriate message:

```
if x > 0
    disp('x is positive');
elseif x < 0
    disp('x is negative');
else
    disp('x is neither positive nor negative');
end
```
■

D.10.2 Switch

The **switch** construct provides another means to conditionally execute groups of statements. The general form of this construct is as follows:

switch *expression*
case $test\_expression_1$
    $statements_1$
case $test\_expression_2$
    $statements_2$
        ⋮
case $test\_expression_{n-1}$

statements$_{n-1}$
otherwise
statements$_n$
end

The switch expression *expression* is compared to each of the test expressions *test_expression*$_1$, *test_expression*$_2$, ..., *test_expression*$_{n-1}$ in order. The first test expression, say *test_expression*$_k$, matching the expression *expression* has its corresponding statements *statements*$_k$ executed. If none of the test expressions match the switch expression and an **otherwise** clause is present, the statements *statements*$_n$ in this clause are executed. The switch expression must be either a scalar or string.

Example D.10. The following code fragment examines the real variable n and prints some information concerning its value:

```
n = 5;
switch mod(n, 2)
case 0
    disp('number is even integer');
case 1
    disp('number is odd integer');
case {0.5, 1.5}
    disp('number is half integer');
otherwise
    disp('number is not an integer');
end
```
■

Example D.11. The following code fragment converts a mass specified in a variety of units to kilograms:

```
x = 100; % input mass
units = 'lb'; % units for input mass
switch units
case {'g'}
    y = 0.001 * x;
case {'kg'}
    y = x;
case {'lb'}
    y = 0.4536 * x;
otherwise
    error('unknown units');
end
disp(sprintf('%f %s converts to %f kg', x, units, y));
```
■

D.10.3 For

The **for** construct allows a group of statements to be repeated a fixed number of times. This construct has the general form:

for *variable* = *array*
statements
end

The statements *statements* are executed once for each value in the array *array*, where the variable *variable* is set to the corresponding array value each time.

Example D.12 (Degree to radian conversion)**.** The following code fragment outputs a table for converting angles from units of degrees to radians:

```
disp('Degrees    Radians');
for theta_degrees = -5 : 0.5 : 5
    theta_radians = theta_degrees * pi / 180;
    disp(sprintf('%7.1f    %7.4f', theta_degrees, theta_radians));
end
```
■

Example D.13. The following code fragment applies a linear transformation (represented by the matrix a) to each column of the matrix [0 2 4 6; 1 3 5 7]:

```
a = [1 0; 1 -1];
for v = [0 2 4 6; 1 3 5 7]
    disp(a * v);
end
```
■

D.10.4 While

The **while** construct allows a group of statements to be executed an indefinite number of times. This construct has the form:

while *expression*
    statements
end

The statements *statements* are executed repeatedly as long as the condition *expression* is true.

Example D.14. Suppose that we would like to compute the smallest machine-representable positive real number that, when added to one, is still greater than one. This quantity is sometimes referred to as **machine epsilon**. Due to finite-precision effects, there is a lower bound on this quantity. Although machine epsilon is available via the built-in eps variable in MATLAB, we can compute its value explicitly using the following code:

```
epsilon = 1;
while (1 + epsilon / 2) > 1
    epsilon = epsilon / 2;
end
disp(epsilon);
```
■

D.10.5 Break and Continue

Sometimes, it may necessary to prematurely break out of a loop or prematurely continue with its next iteration. This is accomplished via **break** and **continue**.

Example D.15. The following two code fragments are equivalent, where the first employs a **break** statement and the second does not:

```
% Code fragment 1
done = 0;
while 1
    if done
        break % Terminate (i.e., break out of) loop.
    end
    % Do something here.
    % If we are finished, set done to one.
end
```



```
% Code fragment 2
done = 0;
while ~done
    % Do something here.
    % If we are finished, set done to one.
end
```

■

Example D.16. The following code fragment gives an example of the use of the **`continue`** statement:

```
a = [1 0 3 2 0];
for i = a
    if i == 0
        % Skip over the processing of a zero element in the array.
        continue
    end
    % Process the nonzero array element.
    disp(i);
end
```

The above code will print only the nonzero elements of the array `a`. ■

D.11 Functions

MATLAB supports user-defined functions. To create a user-defined function, the code for the function is placed in an M-file. In this sense, user-defined functions are very similar to script files. For this reason, most of the material on script files in Section D.7 is also applicable here. There is, however, one key difference between a script and function file. A function file must include a **`function`** directive (whereas a script file must not). This directive is primarily used to indicate the number of input and output arguments for a function.

The first (non-comment) line in function file must contain the **`function`** directive. This directive has the form:

`function` `[` $\mathit{argout}_1, \mathit{argout}_2, \ldots, \mathit{argout}_n$ `]` `=` *funcname* `(`$\mathit{argin}_1, \mathit{argin}_2, \ldots, \mathit{argin}_m$`)`

This indicates that the function has the name *funcname*, the m input arguments $\mathit{argin}_1, \mathit{argin}_2, \ldots, \mathit{argin}_m$, and the n output arguments $\mathit{argout}_1, \mathit{argout}_2, \ldots, \mathit{argout}_n$. The function name *funcname* should always be the same as the base name of the file in which the function is stored (i.e., the file name without the ".m" suffix). Immediately following the line containing the **`function`** directive, one should provide comments to be printed in response to a `help` inquiry for the function. The body of a function extends from the **`function`** directive to a corresponding **`end`** directive or, if no such **`end`** directive is present, the end of the file. The code in a function executes until either the end of the function is reached or a **`return`** statement is encountered.

In MATLAB all input arguments to a function are passed by value. For this reason, changes to the input arguments made inside of a function will not propagate to the caller. Also, any variables accessed/manipulated inside of a function are local in scope to that function.

Example D.17 (Sinc function)**.** As noted earlier, the `sinc` function provided by MATLAB computes the normalized sinc function (as defined by (3.21)), not the sinc function (as defined by (3.20)). In Listing D.1, we define a function `mysinc` that computes the sinc function. The command `help mysinc` will result in MATLAB printing the first block of comments from the above function file.

Listing D.1: `mysinc.m`

```
function y = mysinc(x)
    % mysinc - Compute the sinc function.
    % mysinc(x) returns a matrix whose elements are the sinc of the
    % elements of x

    % Initialize the output array to all ones.
```

Table D.20: Special predefined function variables

Name	Description
`nargin`	number of input arguments
`nargout`	number of output arguments
`varargin`	variable-length input argument
`varargout`	variable-length output argument

```
    y = ones(size(x));
    % Determine the indices of all nonzero elements in the input array.
    i = find(x);
    % Compute the sinc function for all nonzero elements.
    % The zero elements are already covered, since the output
    % array was initialized to all ones above.
    y(i) = sin(x(i)) ./ (x(i));
end
```

■

Example D.18 (Factorial function)**.** Suppose that we would like to write a function called `myfactorial` that takes a single integer argument n and returns $n!$. We can achieve this functionality with the code in Listing D.2, which is placed in a file named `myfactorial.m`. The code is invoked by calling the `myfactorial` function. For example, `myfactorial(4)` returns the value 24.

Listing D.2: `myfactorial.m`

```
function y = myfactorial(x)
    % myfactorial - compute factorial

    y = 1;
    for n = 2 : x
        y = y * n;
    end
end
```

■

In MATLAB, functions may take a variable number of input arguments and may return a variable number of output arguments. In order for a function to determine the number of input and output arguments and access these arguments, several variables are automatically defined upon entry to a function. These variables are listed in Table D.20. In what follows, we give some examples of functions that take a variable number of input arguments.

Example D.19 (Function with variable number of input arguments)**.** Suppose that we would like to write a function called `mysum` that takes one, two, or three arguments and returns their sum. We can achieve this functionality with the code in Listing D.3, which is placed in a file named `mysum.m`. The code is invoked by calling the `mysum` function. For example, `mysum(1)` returns the value 1, `mysum(1, 2)` returns the value 3, and `mysum(1, 2, 3)` returns the value 6.

Listing D.3: `mysum.m`

```
function y = mysum(a, b, c)
    % mysum - calculate the sum (of one to three quantities)

    if nargin == 1
        % function called with one argument
        y = a;
    elseif nargin == 2
        % function called with two arguments
        y = a + b;
    elseif nargin == 3
        % function called with three arguments
```


Table D.21: Basic plotting functions

Name	Description
plot	linear x-y plot
loglog	log log x-y plot
semilogx	semi-log x-y plot (x-axis logarithmic)
semilogy	semi-log x-y plot (y-axis logarithmic)
polar	polar plot
bar	bar chart
stem	stem plot
pcolor	pseudocolor (checkerboard) plot

```
        y = a + b + c;
    else
        error('invalid number of arguments');
    end
end
```
■

Perhaps, we would like to write a function similar to the one in the previous example, except we would like to be able to handle an arbitrary number of input arguments (possibly many more than three). This can be accomplished by using the special predefined `varargin` variable.

Example D.20 (Variable number of input arguments)**.** Suppose that we would like to write a function that returns the sum of its input arguments, but allows the number of input arguments to be arbitrary. We can achieve this functionality with the code in Listing D.4, which is placed in a file named `mysum2.m`. The code is invoked by calling the `mysum2` function. For example, `mysum2(1)` returns the value 1, `mysum2(1, 2, 3)` returns the value 6, and `mysum2(1, 1, 1, 1, 1, 1, 1, 1)` returns the value 8.

Listing D.4: `mysum2.m`

```
function y = mysum2(varargin)
    % mysum2 - Compute the sum of the input arguments

    if nargin == 0
        y = [];
        return
    end
    y = varargin{1};
    for i = 2 : nargin
        y = y + varargin{i};
    end
end
```
■

D.12 Graphing

MATLAB has a very rich set of graphing capabilities. Herein, we will try to illustrate some of these capabilities by way of examples.

Some of the basic plotting functions are listed in Table D.21 and several other graphing-related functions/commands are given in Table D.22.

When generating plots, it is sometimes desirable to be able to specify line styles, line colors, and marker styles. The supported line styles, line colors, and marker styles are listed in Tables D.23, D.24, and D.25, respectively.

Example D.21 (Simple plot)**.** Suppose that we want to plot the function $x(t) = \sin t$ over the interval $[-4\pi, 4\pi]$. This can be accomplished with the following code:

Table D.22: Other graphing functions/commands

Name	Description
`axis`	control axis scaling and appearance
`hold`	hold current plot
`subplot`	multiple axes in single figure
`figure`	create figure

Table D.23: Line styles

Symbol	Line Style
`-`	solid
`:`	dotted
`-.`	dash dot
`--`	dashed

Table D.24: Line colors

Symbol	Line Color
`b`	blue
`g`	green
`r`	red
`c`	cyan
`m`	magenta
`y`	yellow
`k`	black
`w`	white

Table D.25: Marker styles

Symbol	Marker Style
`.`	point
`o`	circle
`x`	cross
`+`	plus sign
`*`	asterisk
`s`	square
`d`	diamond
`v`	triangle (down)
`^`	triangle (up)
`<`	triangle (left)
`>`	triangle (right)
`p`	pentagram
`h`	hexagram

Table D.26: Graph annotation functions

Name	Description
`title`	graph title
`xlabel`	x-axis label
`ylabel`	y-axis label
`grid`	grid lines
`text`	arbitrarily-positioned text
`gtext`	mouse-positioned text

```
% Select the sample points for the function to be plotted.
t = linspace(-4 * pi, 4 * pi, 500);

% Sample the function to be plotted at the sample points.
y = sin(t);

% Clear the current figure.
clf

% Plot the data.
plot(t, y);
```

The resulting plot is shown in Figure D.1. ∎

Often, we need to add annotations to plots (e.g., a title, axis labels, etc.). This is accomplished using the functions listed in Table D.26. Sometimes, we wish to use special symbols (such as Greek letters) in annotations. Fortunately, numerous symbols are available in MATLAB as listed in Table D.27.

Example D.22 (Annotated plot)**.** Suppose that we would like to plot the function $\alpha(\omega) = |\omega|^2 \sin \omega$, using special symbols in the axis labels to display Greek letters. This can be accomplished with the following code:

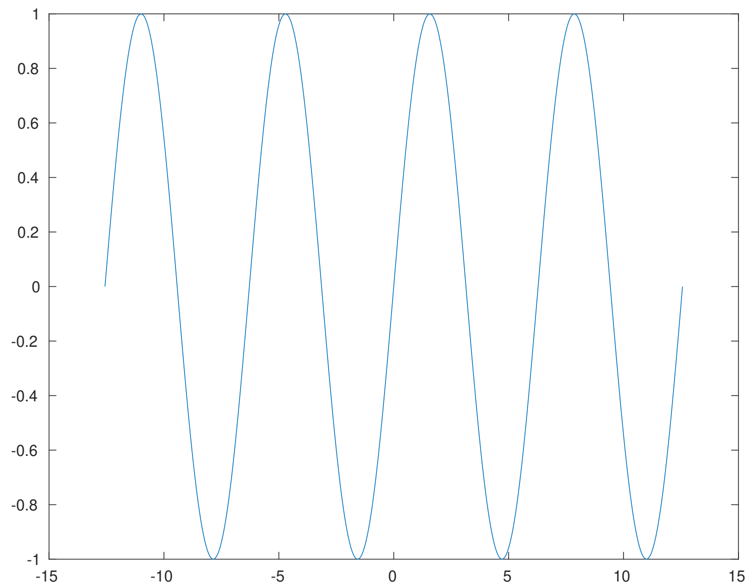


Figure D.1: Plot from example.

Table D.27: Special symbols for annotation text

String	Symbol
`\alpha`	α
`\beta`	β
`\delta`	δ
`\gamma`	γ
`\omega`	ω
`\theta`	θ
`\Delta`	Δ
`\Omega`	Ω

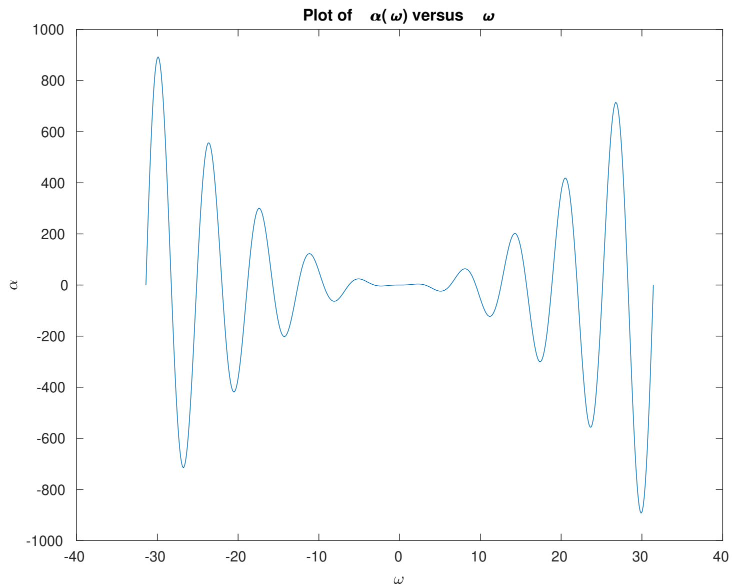


Figure D.2: Plot from example.

```
w = linspace(-10 * pi, 10 * pi, 500);
a = abs(w) .^ 2 .* sin(w);

% Clear the current figure.
clf
% Plot the data.
plot(w, a);

% Set the title, x-axis label, and y-axis label for the plot.
title('Plot of \alpha(\omega) versus \omega');
xlabel('\omega');
ylabel('\alpha');
```

The resulting plot is shown in Figure D.2. ■

Example D.23 (Multiple plots on single axes)**.** Suppose that we want to plot both $\sin t$ and $\cos t$ on the same axes for t in the interval $[-2\pi, 2\pi]$. This can be accomplished with the following code:

```
t = linspace(-2 * pi, 2 * pi, 500);

% Clear the current figure.
clf

% Plot sin(t) versus t using a dashed red line.
plot(t, sin(t), 'r--');

% Prevent subsequent plots from erasing the current one.
hold on

% Plot cos(t) versus t using a solid blue line.
plot(t, cos(t), 'b-');

```

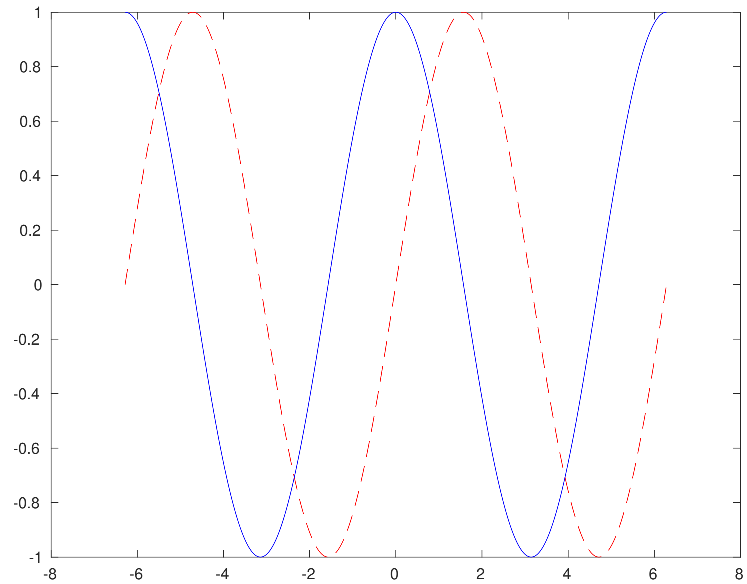



Figure D.3: Plot from example.

```
% Allow subsequent plots to erase the current one.
hold off
```

The resulting plot is shown in Figure D.3. ■

Example D.24 (Multiple axes on same figure)**.** Suppose that we would like to plot the functions $\cos t$, $\sin t$, $\arccos t$, and $\arcsin t$ using four separate plots in the same figure. This can be done using the following code:

```
% Select the sample points for the subsequent plots.
t = linspace(-pi, pi, 500);

% Clear the current figure.
clf

% Select the first subplot in a 2-by-2 layout.
subplot(2, 2, 1);
% Plot cos(t) versus t.
plot(t, cos(t));
title('cos(t)')

% Select the second subplot in a 2-by-2 layout.
subplot(2, 2, 2);
% Plot sin(t) versus t.
plot(t, sin(t));
title('sin(t)');

% Select the sample points for the subsequent plots.
t = linspace(-1, 1, 500);

% Select the third subplot in a 2-by-2 layout.
subplot(2, 2, 3);
% Plot acos(t) versus t.
```

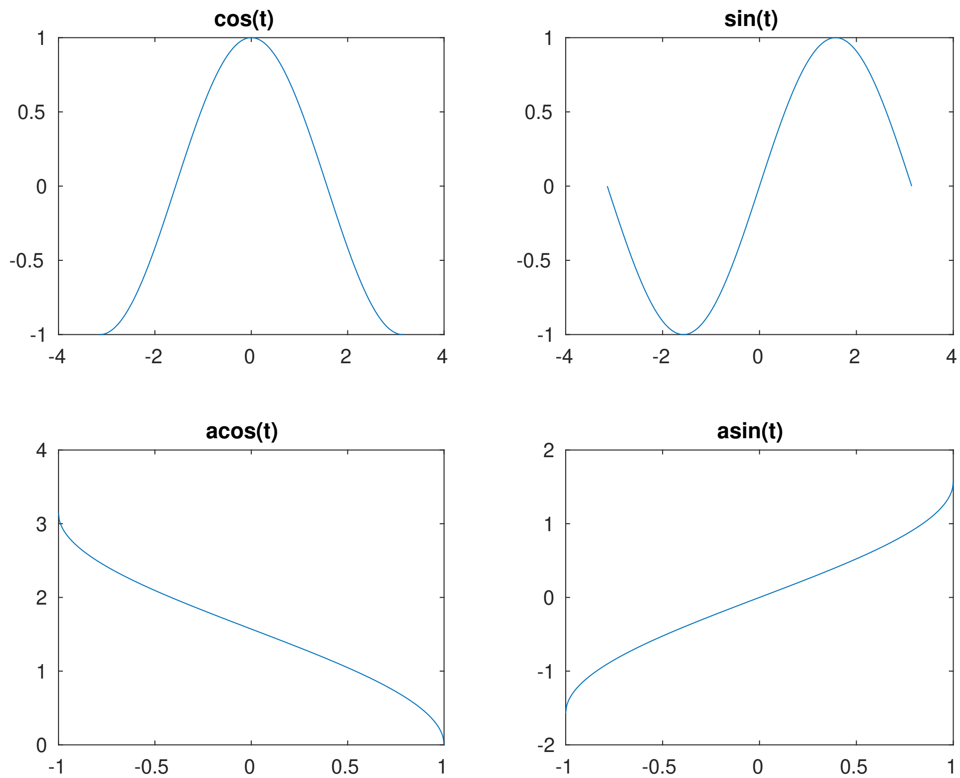


Figure D.4: Plot from example.

```
plot(t, acos(t));
title('acos(t)')

% Select the fourth subplot in a 2-by-2 layout.
subplot(2, 2, 4);
% Plot asin(t) versus t.
plot(t, asin(t));
title('asin(t)')
```

The resulting graphs are shown in Figure D.4. ■

D.13 Printing

To print copies of figures, the `print` command is employed. This command has various forms, but one of the simplest is as follows:

`print -d`*device filename*

Here, *device* specifies the output format and *filename* is the name of the file in which to save the output. For example, for a PostScript printer, the `ps` format should be used. For more details on additional options, type `help print`.

D.14 Symbolic Math Toolbox

Symbolic computation is sometimes quite helpful in solving engineering problems. For example, a very complicated formula or equation involving several variables might need to be simplified without assuming specific values for the variables in the formula/equation. The Symbolic Math Toolbox provides MATLAB with such symbolic computation capabilities.

D.14.1 Symbolic Objects

The Symbolic Math Toolbox defines a new data type called a **symbolic object**. The toolbox uses symbolic objects to represent symbolic variables, constants, and expressions. In essence, a symbolic object can have as its value any valid mathematical expression.

Symbolic objects can be used in many of the same ways as non-symbolic objects. One must, however, keep in mind that performing a computation symbolically is quite different than performing it non-symbolically. Generally speaking, symbolic computation is much slower than non-symbolic computation.

D.14.2 Creating Symbolic Objects

A symbolic variable is created using either the `sym` function or `syms` directive. For example, a symbolic variable `x` (whose value is simply itself) can be created using the `sym` function with the code:

```
x = sym('x');
```

The same result can be achieved in a less verbose manner using the `syms` directive as follows:

```
syms x;
```

Also, the `syms` directive allows multiple variables to be specified. So, a single invocation of this directive can be used to create multiple variables. For example, symbolic variables named `x`, `y`, and `z` can be created with the code:

```
syms x y z;
```

The `sym` function can also be used to create symbolic constants. For example, we can create a symbolic constant `p` that has the value π with the code:

```
p = sym('pi');
```

Note that the character string argument passed to the `sym` function can only contain a variable name or constant. It cannot be an arbitrary expression. For example, code like the following is not allowed:

```
f = sym('a * x ^ 2 + b * x + c');
  % ERROR: cannot pass arbitrary expression to sym function
```

From symbolic variables and constants, more complicated symbolic expressions can be constructed. For example, a symbolic expression `f` with the value `a * x ^ 2 + b * x + c` can be created with the code:

```
syms a b c x;
f = a * x ^ 2 + b * x + c;
```

As another example, a symbolic expression `f` with the value `pi * r ^ 2` can be created with the code:

```
syms r;
f = sym('pi') * r ^ 2;
```

D.14.3 Manipulating Symbolic Objects

Symbolic objects can often be used in the same way as non-symbolic objects. For example, we can do things like:

```
syms t;
f = t + 1;
g0 = f ^ 3 - 2 * f - 21;
g1 = cos(f) * sin(f / 3);
```

Symbolic objects are quite distinctive from other types of objects in MATLAB. For example, the following two lines of code have very different effects:

```
x = pi;
x = sym('pi');
```

It is important to understand the difference in what these two lines of code do.

To substitute some expression/variable for another variable, use the `subs` function. For example, to substitute `t + 1` for `t` in the expression `t ^ 2`, we can use the following code:

```
syms t;
f = t ^ 2;
g = subs(f, t, t + 1)
```

After executing the preceding code, `g` is associated with the expression `(t + 1) ^ 2`.

To factor a symbolic expression, use the `factor` function. For example, suppose that we want to factor the polynomial t^2+3t+2. This could be accomplished with the following code:

```
syms t;
f = t ^ 2 + 3 * t + 2;
g = factor(f)
```

After executing the preceding code, `g` is associated with the (row-vector) expression `[t + 2, t + 1]`. Note that the `factor` function will only produce factors with real roots.

To simplify a symbolic expression, use the `simplify` function. For example, suppose that we want to substitute `2 * t + 1` for `t` in the expression `t ^ 2 - 1` and simplify the result. This can be accomplished with the following code:

```
syms t;
f = t ^ 2 - 1;
g = simplify(subs(f, t, 2 * t + 1))
```

After executing the preceding code, `g` is associated with the expression `(2 * t + 1) ^ 2 - 1`.

To expand an expression, use the `expand` function. For example, to compute $(t+1)^5$, we can use the following code:

```
syms t;
f = (t + 1) ^ 5;
g = expand(f)
```

After executing the preceding code, `g` is associated with the expression:

```
t ^ 5 + 5 *t ^ 4 + 10 * t ^ 3 + 10 * t ^ 2 + 5 * t + 1
```

To display an expression in a human-friendly format, use the `pretty` function. For example, to compute $\left[\frac{1}{2}t^2+\frac{1}{3}(t+1)\right]^{-4}$ in an expanded and beautified format, we can use the following code:

```
syms t;
f = ((1/2) * t^2 + (1/3) * (t+1))^(-4);
pretty(expand(f))
```

The output of the `pretty` function in this case might look something like:

```
                                        1
-----------------------------------------------------------------------
 8     7     6        5        4        3        2
t     t     t     11 t     65 t     22 t     4 t     4 t     1
-- + -- + -- + ----- + ----- + ----- + ---- + --- + --
16    6     3      27      162       81      27      81    81
```

Example D.25 (Sum of an arithmetic sequence)**.** Consider a sum of the form

$$S(a,d,n) \triangleq \sum_{k=0}^{n-1} (a+kd)$$

(i.e., the sum of an arithmetic sequence). Suppose that we did not happen to know that this sum can be calculated as

$$S(a,d,n) = \tfrac{1}{2}n\left(2a+d(n-1)\right).$$

We could determine a general formula for the sum using the following code:

```
syms a d k n;
simplify(symsum(a + k * d, k, 0, n - 1))
```

The code yields the result `a * n + (d * n * (n - 1)) / 2`. Clearly, this result is equivalent to the known expression for $S(a,d,n)$ given above. ■

Example D.26 (Derivatives)**.** Use MATLAB to compute each of the derivatives given below.
(a) $\frac{d}{dt}\left[e^t \cos(3t)\right]$; and
(b) $\frac{d}{d\omega}\left(\frac{\cos\omega}{\omega^2+1}\right)$.

Solution. (a) The derivative $\frac{d}{dt}\left[e^t \cos(3t)\right]$ can be evaluated with the code:

```
clear
syms t
pretty(simplify(diff(exp(t) * cos(3 * t), t)))
```

The result produced should resemble:

```
exp(t) (cos(3 t) - sin(3 t) 3)
```

(b) The derivative $\frac{d}{d\omega}\left(\frac{\cos\omega}{\omega^2+1}\right)$ can be evaluated with the code:

```
clear
syms omega
pretty(simplify(diff(cos(omega) / (omega ^ 2 + 1), omega)))
```

The result produced should resemble:

```
  sin(omega)    2 omega cos(omega)
- ---------- - ------------------
      2                2     2
  omega  + 1      (omega  + 1)
```

■

Example D.27 (Definite and indefinite integrals)**.** Use MATLAB to compute each of the integrals given below.
(a) $\int_{t-1}^{t+1} (\tau+1)^2 d\tau$;
(b) $\int t\cos(2t)dt$;
(c) $\int_0^{\pi/3} \cos^2(t)dt$; and
(d) $\int \left(t^{-1}+t\right) dt$.

Solution. (a) The integral $\int_{t-1}^{t+1} (\tau+1)^2 d\tau$ can be evaluated with the code:

```
clear
syms t tau
pretty(simplify(int((tau + 1) ^ 2, tau, [(t - 1) (t + 1)])))
```

The result produced should resemble:

```
   2          8
2 t  + 4 t + -
             3
```

(b) The integral $\int t\cos(2t)dt$ can be evaluated with the code:

```
clear
syms t
pretty(simplify(int(t * cos(2 * t), t)))
```

The result produced should resemble:

```
cos(2 t)   t sin(2 t)
-------- + ----------
    4          2
```

(c) The integral $\int_0^{\pi/3}\cos^2(t)dt$ can be evaluated with the code:

```
clear
syms t pi
pretty(simplify(int(cos(t) ^ 2, t, [0 (pi / 3)])))
```

The result produced should resemble:

```
pi   sqrt(3)
-- + -------
 6      8
```

(d) The integral $\int \left(t^{-1}+t\right)dt$ can be evaluated with the code:

```
clear
syms t
pretty(simplify(int(t ^ (-1) + t, t)))
```

The result produced should resemble:

```
          2
         t
log(t) + --
          2
```

■

D.14.4 Plotting Symbolic Expressions

To plot a symbolic expression in one variable, the `ezplot` function can be used. For example, to plot the function $f(t) = 3t^2 - 4t + 2$, we can use the code:

```
syms t;
ezplot(3 * t ^ 2 - 4 * t + 2);
```

The range of the independent variable may optionally be specified. For example, to plot the function $f(t) = 3t^2 - 4t + 2$ over the interval $[-1, 1]$, we can use the code:

```
syms t;
ezplot(3 * t ^ 2 - 4 * t + 2, [-1 1]);
```

Table D.28: Some functions related to signal processing

Name	Description
`besself`	Bessel CT filter design
`bilinear`	bilinear transformation with optional frequency prewarping
`bode`	Bode frequency response of system
`butter`	Butterworth CT/DT filter design
`cfirpm`	complex and nonlinear-phase equiripple FIR filter design
`cheby1`	Chebyshev type-I CT/DT filter design
`cheby2`	Chebyshev type-II CT/DT filter design
`fir1`	linear-phase FIR DT filter design using windowing method
`fir2`	arbitrary-shape DT filter design using frequency-sampling method
`fircls`	linear-phase FIR filter design using constrained least-squares method
`firls`	linear-phase FIR filter design using least-squares error minimization
`firpm`	Parks-McClellan optimal equiripple FIR filter design
`freqs`	compute and optionally plot frequency response of CT system
`freqz`	compute and optionally plot frequency response of DT system
`fft`	compute forward DFT
`ellip`	elliptic or Cauer CT/DT filter design
`ifft`	compute inverse DFT
`impulse`	compute and plot impulse response of CT/DT system
`impz`	compute and plot impulse response of DT system
`lsim`	simulate time response of CT/DT system to arbitrary input
`maxflat`	generalized-Butterworth (i.e., maximally-flat) DT filter design
`mag2db`	convert magnitude to decibels
`step`	compute step response of CT/DT system
`stepz`	compute step response of DT system
`tf`	construct system model for CT/DT system

D.15 Signal Processing

MATLAB provides many functions that are helpful in the analysis of continuous-time (CT) and discrete-time (DT) signals and systems. Some of the more useful functions are listed in Table D.28. In what follows, we provide several examples to illustrate how some of these functions can be used.

D.15.1 Continuous-Time Signal Processing

In the sections that follow, we introduce some of the functionality in MATLAB that is useful for continuous-time signals and systems.

Most continuous-time LTI systems of practical interest are causal with rational transfer functions. For this reason, MATLAB has considerable functionality for working with such systems. Consider the rational transfer function

$$H(s) = \frac{\sum_{k=1}^{n} b_k s^{n-k-1}}{\sum_{k=1}^{m} a_k s^{n-k-1}} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{a_1 s^{m-1} + a_2 s^{m-2} + \ldots + a_m}. \tag{D.1}$$

Typically, MATLAB represents such a transfer function using two vectors of coefficients, one for the $\{b_k\}$ and one for the $\{a_k\}$.

D.15.1.1 Frequency Responses

For a LTI system with a transfer function of the form of (D.1), the `freqs` function can be used to compute and optionally plot the frequency response of the system.

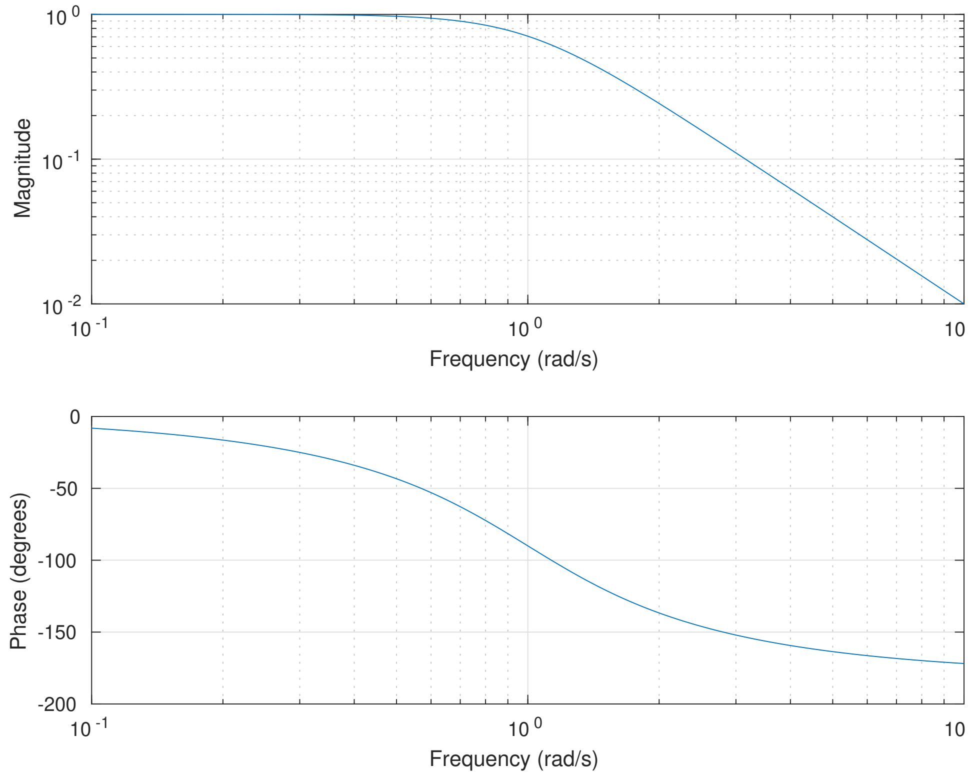


Figure D.5: The frequency response of the filter as produced by the `freqs` function.

Example D.28 (Computing and plotting frequency responses with freqs)**.** Consider the LTI system with transfer function

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}.$$

This system is a second-order Butterworth lowpass filter with a cutoff frequency of 1 rad/s. Suppose that we would like to evaluate the frequency response of this system. This is equivalent to evaluating the transfer function H at points on the imaginary axis. To this end, we can employ the `freqs` function in MATLAB. More specifically, we can calculate and plot the magnitude and phase responses of the above system with the following code:

```
% Initialize the numerator and denominator coefficients of the transfer
% function.
tf_num = [1];
tf_denom = [1 sqrt(2) 1];

% Plot the magnitude and phase responses.
freqs(tf_num, tf_denom);
```

The plot produced by the `freqs` function is shown in Figure D.5. ■

Example D.29 (Plotting frequency responses)**.** Suppose that we would like to have a function that behaves in a similar manner as the MATLAB `freqs` function, but with a few differences in how plotting is performed. In particular, we would like the magnitude response plotted with a linear (instead of logarithmic) scale and the phase response plotted in unwrapped form. This can be accomplished with the code given in Listing D.5.

Listing D.5: `myfreqs.m`


```
function [freq_resp, omega] = myfreqs(tf_num, tf_denom, omega)
    % The myfreqs function has essentially the same interface as the
    % MATLAB freqs function, but performs plotting slightly differently.
    % The magnitude response is plotted as a unitless quantity (not in
    % decibels).
    % The phase response is plotted with the phase unwrapped.

    % If the frequencies have been specified as an input argument, then simply
    % pass them through to the real freqs function.
    if nargin >= 3
        [freq_resp, omega] = freqs(tf_num, tf_denom, omega);
    else
        [freq_resp, omega] = freqs(tf_num, tf_denom);
    end

    % If no output arguments were specified, plot the frequency response.
    if nargout == 0

        % Compute the magnitude response as a unitless quantity.
        mag_resp = abs(freq_resp);

        % Compute the phase response with the phase unwrapped.
        phase_resp = unwrap(angle(freq_resp)) / pi * 180;

        % On the first of two graphs, plot the magnitude response.
        subplot(2, 1, 1);
        plot(omega, mag_resp);
        title('Magnitude Response');
        xlabel('Frequency (rad/s)');
        ylabel('Magnitude (unitless)');

        % On the second of two graphs, plot the phase response.
        subplot(2, 1, 2);
        plot(omega, phase_resp);
        title('Phase Response');
        xlabel('Frequency (rad/s)');
        ylabel('Angle (degrees)');

    end
end
```

For the filter in Example D.28, the `myfreqs` function produces the frequency-response plots shown in Figure D.6. ■

D.15.1.2 Impulse and Step Responses

Sometimes, we need to determine the response of a LTI system to a specific input. Two inputs of particular interest are the unit-impulse function δ and unit-step function u. Fortunately, it is quite easy to compute impulse and step responses using the `impulse` and `step` functions in MATLAB, as illustrated by the example below.

Example D.30 (Computing impulse and step responses)**.** Consider the LTI system with the transfer function

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}.$$

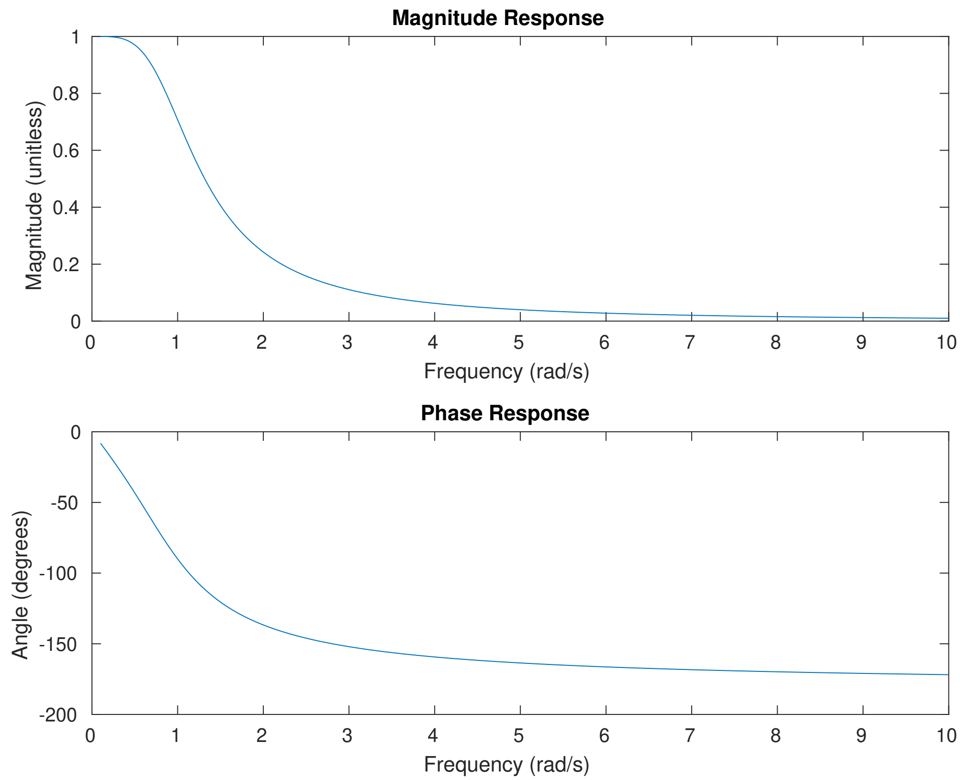


Figure D.6: The frequency response of the filter as produced by the `myfreqs` function.

Suppose that we wish to calculate and plot the impulse and step responses of this system. This can be accomplished with the code given in Listing D.6. Executing this code produces the plots shown in Figure D.7.

Listing D.6: Computing and plotting the impulse and step responses

```
% Initialize the numerator and denominator coefficients of the transfer
% function.
tf_num = [1];
tf_denom = [1 sqrt(2) 1];

% Determine the system model associated with the given transfer function.
sys = tf(tf_num, tf_denom);

% Plot the impulse response.
subplot(2, 1, 1);
impulse(sys);

% Plot the step response.
subplot(2, 1, 2);
step(sys);
```

■

D.15.1.3 Filter Design

A number of functions are provided in MATLAB to assist in the design of various types of filters. In what follows, we consider a few examples of using such functions.

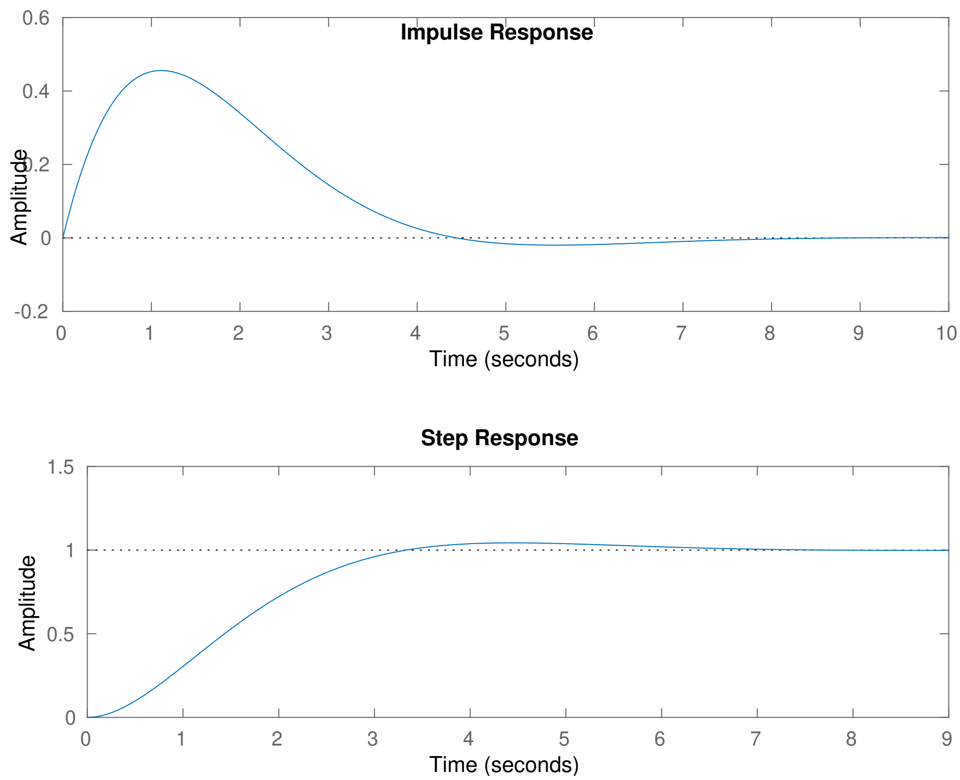


Figure D.7: The impulse and step responses of the system obtained from the code example.

Example D.31 (Butterworth lowpass filter design)**.** Suppose that we want to design a tenth-order Butterworth lowpass filter with a cutoff frequency of 100 rad/s. We can design such a filter as well as plot its frequency response using the code given in Listing D.7 below. The frequency response of the filter obtained from this code is shown in Figure D.8.

Listing D.7: Butterworth lowpass filter design

```
% Calculate the transfer function coefficients for a tenth-order Butterworth
% lowpass filter with a cutoff frequency of 100 rad/s.
[tf_num, tf_denom] = butter(10, 100, 's');

% Plot the frequency response of the filter.
freqs(tf_num, tf_denom);
```

■

Example D.32 (Bessel lowpass filter design)**.** Suppose that we want to design a tenth-order Bessel lowpass filter with a cutoff frequency of 100 rad/s. We can design such a filter as well as plot its frequency response using the code given in Listing D.8 below. The frequency response of the filter obtained from this code is shown in Figure D.9.

Listing D.8: Bessel lowpass filter design

```
% Calculate the transfer function coefficients for a tenth-order Bessel
% lowpass filter with a cutoff frequency of 100 rad/s.
[tf_num, tf_denom] = besself(10, 100);

% Plot the frequency response of the filter.
freqs(tf_num, tf_denom);
```

■

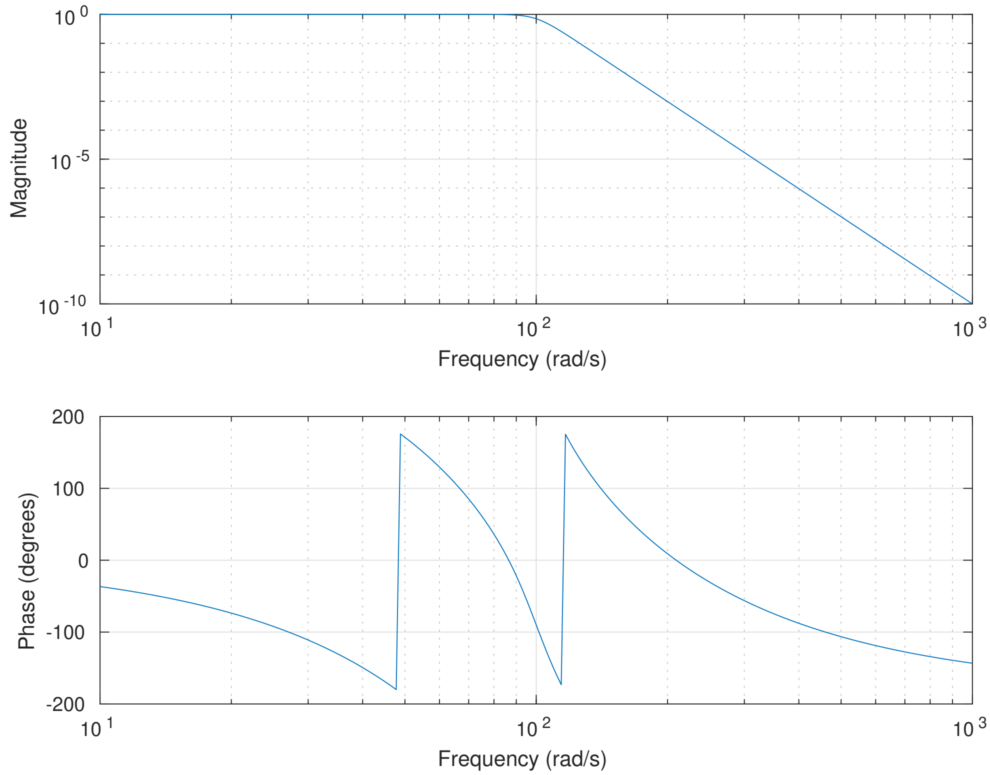


Figure D.8: The frequency response of the Butterworth lowpass filter obtained from the code example.

D.15.2 Discrete-Time Signal Processing

In the sections that follow, we introduce some of the functionality in MATLAB that is useful for discrete-time signals and systems.

Most discrete-time LTI systems of practical interest are causal with rational transfer functions. For this reason, MATLAB has considerable functionality for working with such systems. Consider the rational transfer function

$$H(z) = \frac{\sum_{k=1}^{n} b_k z^{-(k-1)}}{\sum_{k=1}^{m} a_k z^{-(k-1)}} = \frac{b_1 + b_2 z^{-1} + \ldots + b_n z^{-(n-1)}}{a_1 + a_2 z^{-1} + \ldots + a_m z^{-(m-1)}}. \tag{D.2}$$

Typically, MATLAB represents such a transfer function using two vectors of coefficients, one for the $\{b_k\}$ and one for the $\{a_k\}$.

D.15.2.1 Frequency Responses

For a LTI system with a transfer function of the form of (D.2), the `freqz` function can be used to compute and optionally plot the frequency response of the system.

Example D.33 (Computing and plotting frequency responses with freqz)**.** Consider the LTI system with transfer function

$$H(z) = \frac{-0.2037z^4 + 0.5925z^2 - 0.2037}{z^4} = -0.2037 + 0.5925z^{-2} - 0.2037z^{-4}.$$

Suppose that we would like to evaluate the frequency response of this system. This is equivalent to evaluating the transfer function H at points on the unit circle. To this end, we can employ the `freqz` function in MATLAB. More specifically, we can calculate and plot the magnitude and phase responses of the above system with the following code:

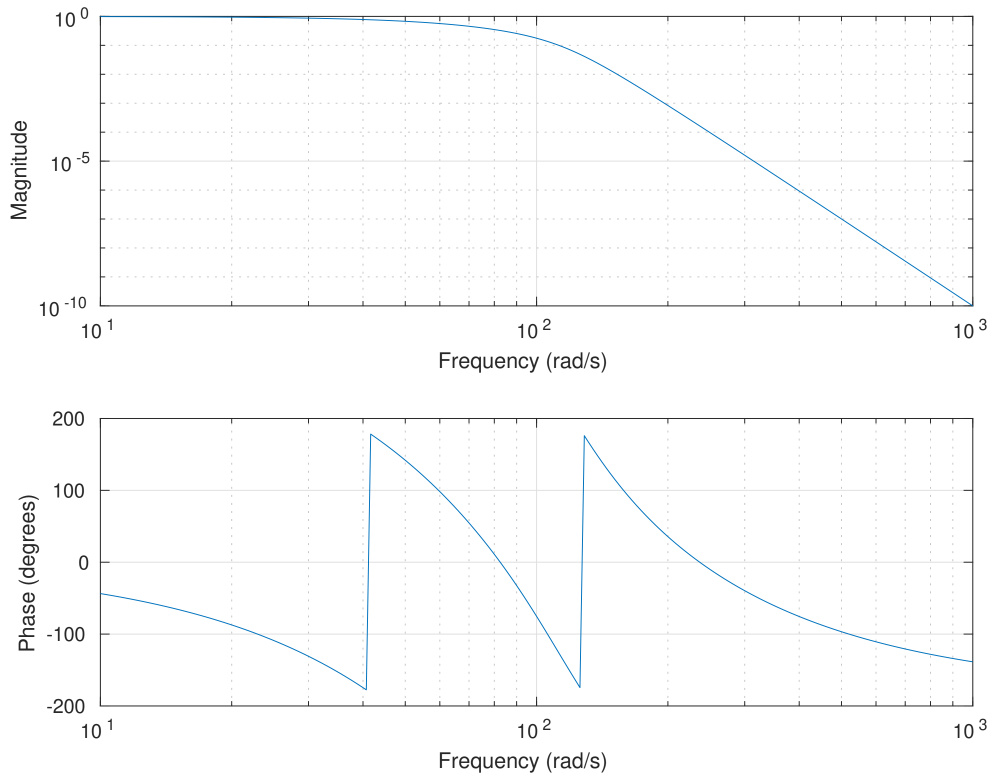


Figure D.9: The frequency response of the Bessel lowpass filter obtained from the code example.

```
% Initialize the numerator and denominator coefficients of the transfer
% function.
tf_num = [
  -0.2037
   0.0000
   0.5925
   0.0000
  -0.2037
];
tf_denom = [1];

% Plot the magnitude and phase responses.
freqz(tf_num, tf_denom);
```

The plot produced by the `freqz` function is shown in Figure D.10. ■

Example D.34 (Plotting frequency responses)**.** Suppose that we would like to have a function that behaves in a similar way to the MATLAB `freqz` function, but with a few differences in how plotting is performed. In particular, we would like the plots generated with the magnitude response as a unitless quantity (not in decibels) and the phase response in unwrapped form. This can be accomplished with the code given in Listing D.9.

Listing D.9: `myfreqz.m`

```
function [freq_resp, omega] = myfreqz(tf_num, tf_denom, omega)
    % The myfreqz function has essentially the same interface as the
    % MATLAB freqz function, but performs plotting slightly differently.
    % The magnitude response is plotted as a unitless quantity (not in
```

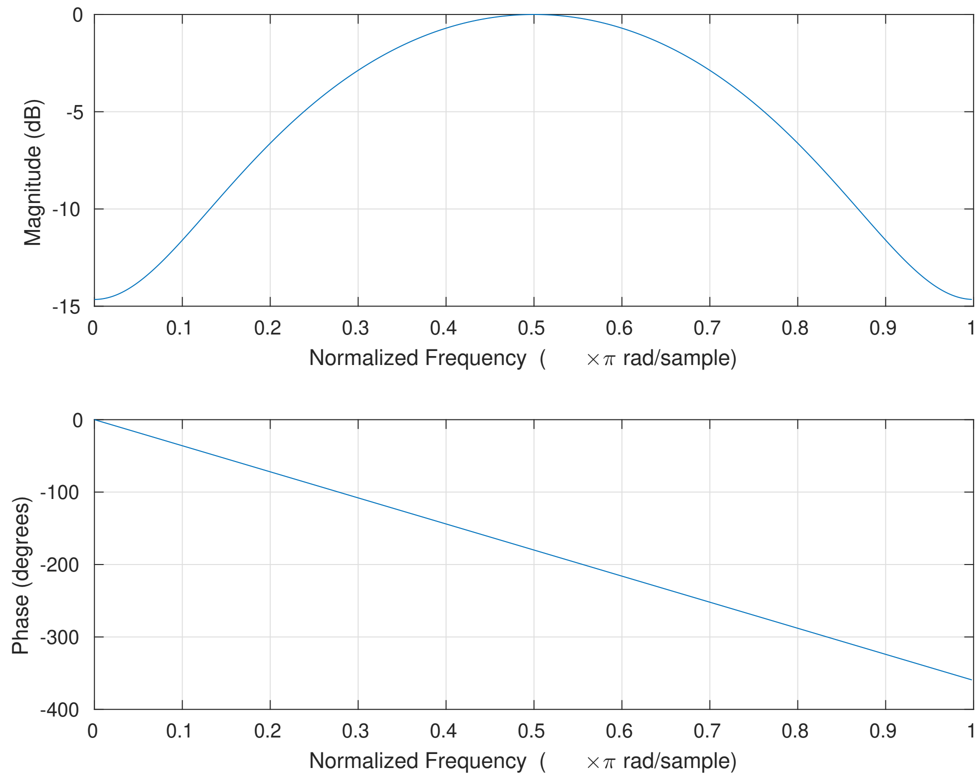


Figure D.10: The frequency response of the filter as produced by the `freqz` function.

```
% decibels).
% The phase response is plotted with the phase unwrapped.

% If the frequencies have been specified as an input argument, then simply
% pass them through to the real freqz function.
if nargin >= 3
    [freq_resp, omega] = freqz(tf_num, tf_denom, omega);
else
    [freq_resp, omega] = freqz(tf_num, tf_denom);
end

% If no output arguments were specified, plot the frequency response.
if nargout == 0

    % Compute the magnitude response as a unitless quantity.
    mag_resp = abs(freq_resp);

    % Compute the phase response with the phase unwrapped.
    phase_resp = unwrap(angle(freq_resp)) * 180 / pi;

    % On the first of two graphs, plot the magnitude response.
    subplot(2, 1, 1);
    plot(omega / pi, mag_resp);
    title('Magnitude Response');
```

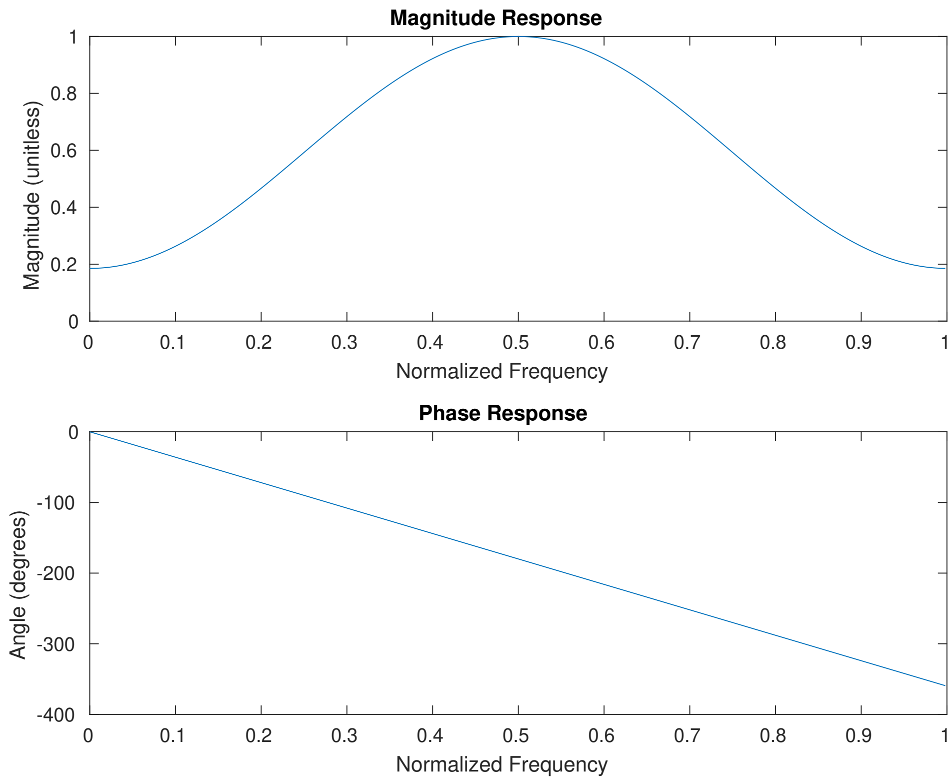



Figure D.11: The frequency response of the filter as produced by the `myfreqz` function.

```
        xlabel('Normalized Frequency');
        ylabel('Magnitude (unitless)');

        % On the second of two graphs, plot the phase response.
        subplot(2, 1, 2);
        plot(omega / pi, phase_resp);
        title('Phase Response');
        xlabel('Normalized Frequency');
        ylabel('Angle (degrees)');

    end
end
```

For the filter in Example D.33, the `myfreqz` function produces the frequency-response plots shown in Figure D.11. ■

D.15.2.2 Impulse and Step Responses

Sometimes, we need to determine the response of a LTI system to a specific input. Two inputs of particular interest are the unit-impulse sequence δ and unit-step sequence u. Fortunately, it is quite easy to compute impulse and step responses using the `impz` and `stepz` functions in MATLAB, as illustrated by the example below.

Example D.35 (Computing impulse and step responses)**.** Consider the LTI system with the transfer function

$$H(z) = \frac{0.571 - 0.591z^{-1} + 0.503z^{-2} + 0.503z^{-3} - 0.591z^{-4} + 0.571z^{-5}}{1.000 - 3.588z^{-1} + 5.303z^{-2} - 4.003z^{-3} + 1.538z^{-4} - 0.239z^{-5}}.$$

Suppose that we wish to calculate and plot the impulse and step responses of this system. This can be accomplished with the code given in Listing D.10. Executing this code produces the plots shown in Figure D.12.

Listing D.10: Computing and plotting the impulse and step responses

```
% Initialize the numerator and denominator coefficients of the transfer
% function.
tf_num = [
   0.571
  -0.591
   0.503
   0.503
  -0.591
   0.571
];
tf_denom = [
   1.000
  -3.588
   5.303
  -4.003
   1.538
  -0.239
];

% Plot the impulse response.
subplot(2, 1, 1);
impz(tf_num, tf_denom);

% Plot the step response.
subplot(2, 1, 2);
stepz(tf_num, tf_denom);
```

■

D.15.2.3 Filter Design

A number of functions are provided in MATLAB to assist in the design of various types of discrete-time filters. In what follows, we consider a few examples of using such functions.

Example D.36 (Chebyshev type-II lowpass filter design)**.** Suppose that the sampling rate is 2000 rad/s, and we want to design a fifth-order Chebyshev type-II lowpass filter with a cutoff frequency of 250 rad/s and stopband ripple of 20 dB. The cutoff frequency in normalized units is $250/(\frac{2000}{2}) = 0.25$. We can design the desired filter as well as plot its frequency response using the code given in Listing D.11 below. The frequency response of the filter obtained from this code is shown in Figure D.13.

Listing D.11: Chebyshev type-II lowpass filter design

```
% Calculate the transfer function coefficients for a
% fifth-order Chebyshev type-II lowpass filter with a
% (normalized) cutoff frequency of 0.25 and a
% stopband ripple of 20 dB.
[tf_num, tf_denom] = cheby2(5, 20, 0.25, 'low');

% Plot the frequency response of the filter.
freqz(tf_num, tf_denom);
```

■

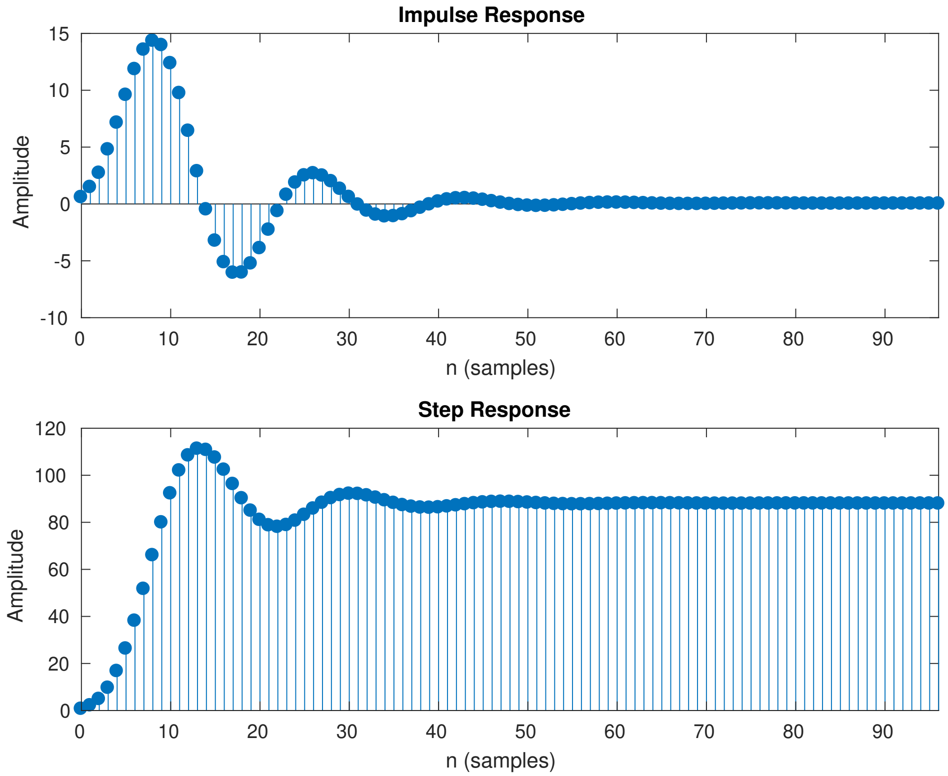


Figure D.12: The impulse and step responses of the system obtained from the code example.

Example D.37 (Linear-phase FIR bandpass filter design)**.** Suppose that the sampling rate is 2000 rad/s, and we want to design an order-64 linear-phase FIR bandpass filter with cutoff frequencies of 250 and 750 rad/s using the `fir1` function in MATLAB. The cutoff frequencies in normalized units are $250/(\frac{2000}{2}) = 0.25$ and $750/(\frac{2000}{2}) = 0.75$. We can design the desired filter as well as plot its frequency response using the code given in Listing D.12 below. The frequency response of the filter obtained from this code is shown in Figure D.14.

Listing D.12: Linear-phase FIR bandpass filter design

```
% Calculate the transfer function coefficients for a
% 64th-order FIR bandpass filter with
% (normalized) cutoff frequencies of 0.25 and 0.75.
tf_num = fir1(64, [0.25 0.75], 'bandpass');
tf_denom = [1];

% Plot the frequency response of the filter.
freqz(tf_num, tf_denom);
```

■

D.16 Miscellany

Some other functions that might be useful are listed in Table D.29.

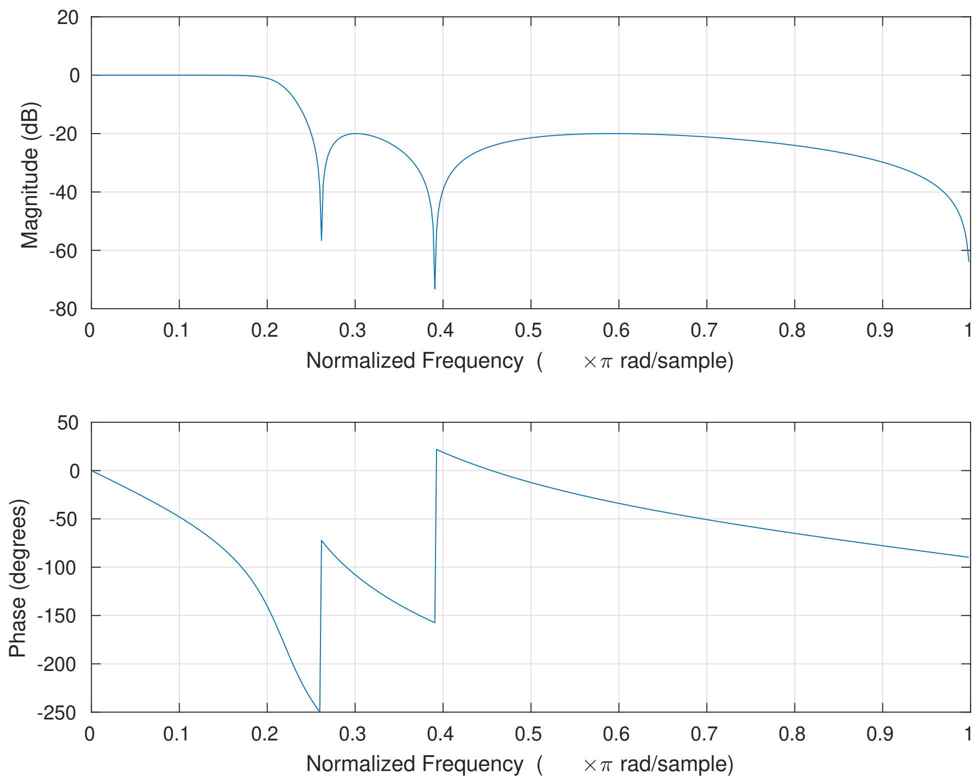


Figure D.13: The frequency response of the Chebyshev type-II lowpass filter obtained from the code example.

Table D.29: Miscellaneous functions/commands

Name	Description
`roots`	find roots of polynomial
`clear`	clear a variable
`diary`	log MATLAB session
`echo`	echo commands on execution (for debugging)
`quit`	quit MATLAB
`format`	output format for numbers

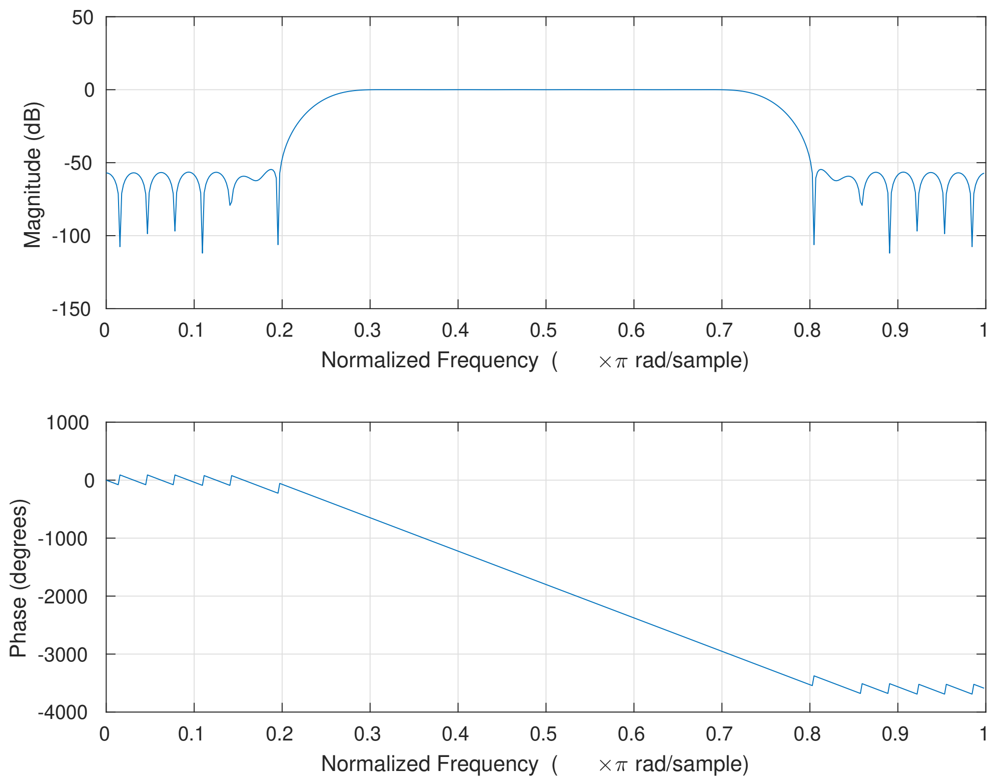


Figure D.14: The frequency response of the linear-phase FIR bandpass filter obtained from the code example.

D.17 Exercises

D.101 Indicate whether each of the following is a valid MATLAB identifier (i.e., variable/function name):
(a) `4ever`
(b) `$rich$`
(c) `foobar`
(d) `foo_bar`
(e) `_foobar`

D.102 Let T_C, T_F, and T_K denote the temperature measured in units of Celsius, Fahrenheit, and Kelvin, respectively. Then, these quantities are related by

$$T_F = \tfrac{9}{5}T_C + 32 \quad \text{and}$$
$$T_K = T_C + 273.15.$$

Write a program that generates a temperature conversion table. The first column of the table should contain the temperature in Celsius. The second and third columns should contain the corresponding temperatures in units of Fahrenheit and Kelvin, respectively. The table should have entries for temperatures in Celsius from -50 to 50 in steps of 10.

D.103 Let F denote the complex-valued function of a real variable given by

$$F(\omega) = \frac{1}{j\omega + 1}.$$

Write a program to plot $|F(\omega)|$ and $\arg F(\omega)$ for ω in the interval $[-10, 10]$. Use `subplot` to place both plots on the same figure.

D.104 In what follows, we consider a simple algorithm for generating a set of 2^d points in the complex plane, where d is a positive integer. For $n = 0, 1, \ldots, 2^d - 1$, this algorithm computes the nth point p_n as

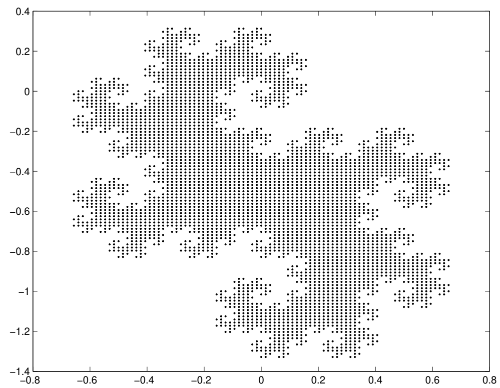
$$p_n = \sum_{k=0}^{d-1} a_k \beta^{k+1},$$

where $\beta = \frac{1}{\sqrt{2}} e^{-j\pi/4}$, $a_k \in \{0, 1\}$, and $n = \sum_{k=0}^{d-1} a_k 2^k$. Note that the binary sequence $a_{d-1}, \ldots, a_1, a_0$ is simply the d-bit binary representation of the integer n, where a_0 corresponds to the least-significant bit. For example, if $d = 3$, then the relationship between n and a_2, a_1, a_0 is as shown in the following table:

n	a_2	a_1	a_0
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1

(a) Write a function called `twindragon` that calculates and plots the point set obtained by using the above algorithm for a specified value of the parameter d. The value of d should be passed as an input argument to the function. [Hint: It is possible to implement this function in about 15 to 20 lines of code. The `polyval` and

dec2bin functions should be quite helpful.]
(b) Using the function developed in part (a), plot the set of points obtained with the parameter d equal to 12. In the limit as d approaches infinity, the resulting set of points converges to the well-known twin-dragon fractal set. Although choosing $d = 12$ should be sufficient to see the approximate shape of the point set in the limiting case, larger values of d will yield better approximations, albeit at the expense of significantly more computation. You should obtain a plot resembling that shown below.



D.105 In what follows, let $\min(a,b)$ denote the minimum of a and b. For a complex number z, we define an iterative process that generates the complex-number sequence $v_z(0), v_z(1), v_z(2), \ldots$, where

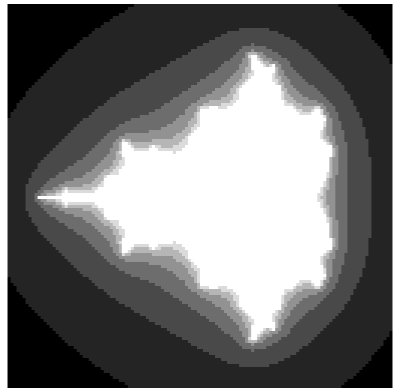
$$v_z(n) = \begin{cases} v_z^2(n-1)+z & n \geq 1 \\ 0 & n = 0. \end{cases}$$

Let $g(z)$ denote the smallest value of n for which $|v_z(n)| > 10$. In the case that $|v_z(n)| > 10$ is not satisfied for any n, $g(z)$ is simply defined to be ∞. Let $f(z) = \min(g(z), 10)$. For example, we have the following:

z	First few elements of v_z	g	f
0	$0,0,0,0$	∞	10
j	$0, j, -1+j, -j, -1+j, -j$	∞	10
2	$0,2,6,38$	3	3
$2j$	$0, 2j, -4+2j, 12-14j$	3	3

Write a function called mandelbrotfunc to compute the value $f(z)$ for any (complex) value of z. Using the function mandelbrotfunc, evaluate f on an evenly-spaced rectangular grid (in the complex plane) having width 128, height 128, and with the bottom-left and top-right corners of the grid corresponding to the points $-2.25-1.5j$ and $1+1.5j$, respectively. Store these computed values into a 128×128 matrix. Then, using the pcolor function in MATLAB, plot the contents of this matrix. After the call to pcolor, use the command "shading interp" to eliminate the grid lines from the plot (which causes the plot to look somewhat less attractive).

A complete solution to this exercise requires less than 25 lines of code (excluding comments). A correct solution should yield a plot resembling the one shown below. Incidentally, the innermost region in the plot is an approximation to the famous Mandelbrot (fractal) set.



D.106 Consider the vector v defined by the following line of code:

```
v = [0 1 2 3 4 5]
```

Write an expression in terms of v that yields a new vector of the same dimensions as v, where each element t of the original vector v has been replaced by the given quantity below. In each case, the expression should be as short as possible.
(a) $2t-3$;
(b) $1/(t+1)$;
(c) t^5-3; and
(d) $|t|+t^4$.

D.107 (a) Write a function called `unitstep` that takes a single real argument t and returns $u(t)$, where

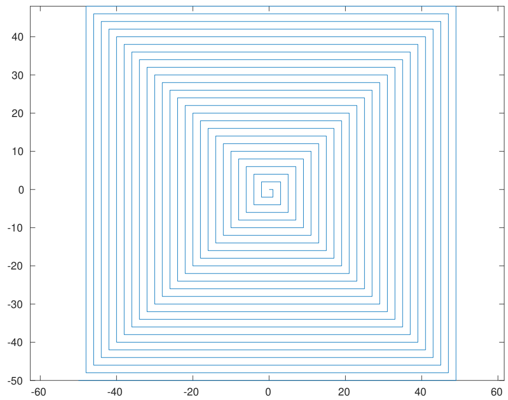
$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Modify the function from part (a) so that it takes a single vector argument $t = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}^T$ (where $n \geq 1$ and $t_1, t_2, \ldots, t_n$ are real) and returns the vector $\begin{bmatrix} u(t_1) & u(t_2) & \cdots & u(t_n) \end{bmatrix}^T$. Your solution must employ a looping construct (e.g., a for loop).
(c) With some ingenuity, part (b) of this exercise can be solved using only two lines of code, without the need for any looping construct. Find such a solution. [Hint: In MATLAB, to what value does an expression like "`[-2 -1 0 1 2] >= 0`" evaluate?]

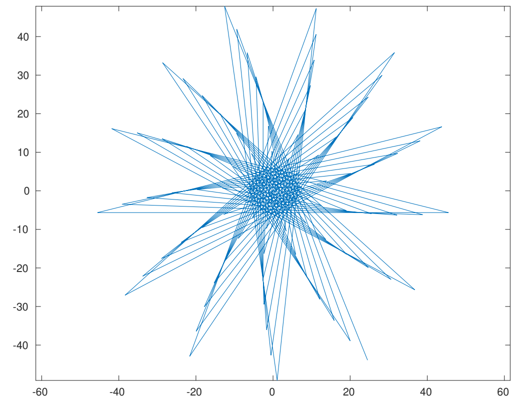
D.108 In this exercise, we consider an algorithm for generating a sequence p of n points in the plane (i.e., $p_0, p_1, \ldots, p_{n-1}$). The first point p_0 is chosen as the origin (i.e., $p_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$), with the remaining points being given by the formula

$$p_i = p_{i-1} + \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{i-1} \begin{bmatrix} i \\ 0 \end{bmatrix}.$$

(a) Using MATLAB, write a function called `drawpattern` that takes n and θ as input arguments (in that order) with θ being specified in degrees, and then computes and plots the points $p_0, p_1, \ldots, p_{n-1}$ connected by straight lines (i.e., draw a line from p_0 to p_1, p_1 to p_2, p_2 to p_3, and so on). When performing the plotting, be sure to use `axis('equal')` in order to maintain the correct aspect ratio for the plot. For illustrative purposes, the plots produced for two sets of θ and n values are shown in Figures (a) and (b) below.
(b) Generate the plots obtained by invoking `drawpattern` with $n = 100$ and θ set to each of the following values: 89°, 144°, and 154°. [Note: In MATLAB, the `sin` and `cos` functions take values in radians, not degrees.]



(a) $\theta = 90°$ and $n = 100$



(b) $\theta = 166°$ and $n = 100$

Appendix E

Additional Exercises

E.1 Overview

This appendix contains numerous additional exercises, which may be useful for practice purposes. These exercises have been deliberately left uncategorized, so that the reader can gain practice in identifying the general approach required to solve a problem. In the case of each exercise for which a short answer is possible, an answer key is provided.

E.2 Continuous-Time Signals and Systems

E.1 A communication channel heavily distorts high frequencies but does not significantly affect very low frequencies. Determine which of the functions x_1, x_2, x_3, and x_4 would be least distorted by the communication channel, where

$$x_1(t) = \delta(t), \quad x_2(t) = 5, \quad x_3(t) = 10e^{j1000t}, \quad \text{and} \quad x_4(t) = 1/t.$$

Short Answer. x_2

E.2 Consider a system consisting of a communication channel with input x and output y. Since the channel is not ideal, y is typically a distorted version of x. Suppose that the channel can be modelled as a causal LTI system with impulse response $h(t) = e^{-t}u(t) + \delta(t)$. Determine whether we can devise a physically-realizable BIBO-stable system that recovers x from y. If such a system exists, find its impulse response g.

Short Answer. $g(t) = \delta(t) - e^{-2t}u(t)$

E.3 A causal LTI system has impulse response h, system function H, and the following characteristics: 1) H is rational with one pole at -2 and no zeros; and 2) $h(0^+) = 4$. Find h.

Short Answer. $h(t) = 4e^{-2t}u(t)$

E.4 A causal LTI system with input x and output y is characterized by the differential equation

$$y'(t) + 3y(t) = 2x(t),$$

where the prime symbol denotes derivative. Find the impulse response h of the system.

Short Answer. $h(t) = 2e^{-3t}u(t)$

E.5 A causal LTI system $\mathcal{H}$ is such that $y = \mathcal{H}x$, where $x(t) = e^{-2t}u(t)$ and $y(t) = e^{-3t}u(t)$. Find the unit-step response s of the system.

Short Answer. $s(t) = \frac{2}{3}u(t) + \frac{1}{3}e^{-3t}u(t)$

E.6 A causal BIBO-stable LTI system $\mathcal{H}$ has impulse response h and system function H. The function H is rational, contains a pole at -2 and does not have a zero at the origin. The other poles and zeros of H are unknown. Determine whether each of the statements below is true, false, or uncertain (i.e., insufficient information to determine).
(a) $\mathcal{F}\{e^{3t}h(t)\}$ converges;
(b) $\int_{-\infty}^{\infty} h(t)dt = 0$;
(c) $th(t)$ is the impulse response of a causal and BIBO-stable system;
(d) $\mathcal{D}h$ has at least one pole in its Laplace transform expression, where $\mathcal{D}$ denotes the derivative operator.

Short Answer. (a) false; (b) false; (c) true; (d) true

E.7 A communication channel can be well approximated by a LTI system with impulse response $h(t) = \frac{1000}{\pi}\operatorname{sinc}(1000t)$. Determine which of the functions x_1, x_2, and x_3 would be least distorted by this channel, where

$$x_1(t) = \delta(t), \quad x_2(t) = u(t), \quad \text{and} \quad x_3(t) = \cos(100t).$$

Short Answer. x_3

E.8 A common problem in real-world instrumentation systems is electromagnetic interference caused by 60 Hz power lines. In particular, 60 Hz power lines can often introduce a significant amount of interference (i.e., noise) at 60 Hz and its higher harmonics (i.e., 120 Hz, 180 Hz, 240 Hz, and so on). Consider a causal system with an impulse response of the form $h(t) = a[u(t) - u(t-b)]$, where a and b are nonzero real constants. With an appropriate choice of a and b, such a system can be made to reject interference at 60 Hz and all of its higher harmonics. Find a and b.

Short Answer. $a \neq 0$ and $b = \frac{1}{60}$

E.9 For the causal LTI system with input x, output y, and impulse response h that is characterized by each differential equation given below, find h. [Note: The prime symbol denotes derivative.]
(a) $4y''(t) = 2x(t) - x'(t)$.

Short Answer. (a) $(\frac{1}{2}t - \frac{1}{4})u(t)$

E.3 Discrete-Time Signals and Systems

No additional exercises for discrete-time signals and systems are currently available.

Appendix F

Miscellaneous Information

F.1 Overview

This appendix contains numerous mathematical formulas and tables that are likely to be helpful in relation to the material covered in this book.

F.2 Combinatorial Formulas

For a nonnegative integer n, the **factorial** of n, denoted $n!$, is defined as

$$n! = \begin{cases} n(n-1)(n-2)\cdots(1) & n \geq 1 \\ 1 & n = 0. \end{cases}$$

For two integers n and k such that $0 \leq k \leq n$, the $\binom{n}{k}$ **binomial coefficient** is defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

F.3 Derivatives

Some basic derivatives include:

$$\begin{aligned} \tfrac{d}{dx} x^n &= n x^{n-1} \quad \text{for } n \in \mathbb{Z}; \\ \tfrac{d}{dx} \cos x &= -\sin x; \\ \tfrac{d}{dx} \sin x &= \cos x; \quad \text{and} \\ \tfrac{d}{dx} e^x &= e^x. \end{aligned}$$

Some formulas for the derivatives of products and quotients are as follows:

$$\begin{aligned} \tfrac{d}{dx}(uv) &= u\tfrac{dv}{dx} + v\tfrac{du}{dx}; \quad \text{and} \\ \tfrac{d}{dx}\left(\frac{u}{v}\right) &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}. \end{aligned}$$

F.4 Integrals

Some basic integrals include:

$$\begin{aligned}
\int x^n dx &= \begin{cases} \frac{1}{n+1}x^{n+1}+C & n \neq -1 \\ \ln|x|+C & n=-1; \end{cases} \\
\int \cos x dx &= \sin x + C; \\
\int \sin x dx &= -\cos x + C; \quad \text{and} \\
\int e^x dx &= e^x + C.
\end{aligned}$$

Some additional integrals include:

$$\begin{aligned}
\int xe^x dx &= e^x(x-1)+C; \\
\int xe^{ax} dx &= \tfrac{1}{a^2}e^{ax}(ax-1)+C; \\
\int x^2 e^{ax} dx &= \tfrac{1}{a^3}e^{ax}(a^2x^2-2ax+2)+C; \\
\int e^{ax}\cos(bx)dx &= \frac{e^{ax}[a\cos(bx)+b\sin(bx)]}{a^2+b^2}+C; \\
\int e^{ax}\sin(bx)dx &= \frac{e^{ax}[a\sin(bx)-b\cos(bx)]}{a^2+b^2}+C; \\
\int x\cos x dx &= \cos x + x\sin x + C; \quad \text{and} \\
\int x\sin x dx &= \sin x - x\cos x + C.
\end{aligned}$$

The formula for integration by parts is as follows:

$$\int u dv = uv - \int v du.$$

F.5 Arithmetic and Geometric Sequences

The sum of the arithmetic sequence $a, a+d, a+2d, \ldots, a+(n-1)d$ is given by

$$\sum_{k=0}^{n-1}(a+kd) = \frac{n[2a+d(n-1)]}{2}. \tag{F.1}$$

The sum of the geometric sequence $a, ra, r^2a, \ldots, r^{n-1}a$ is given by

$$\sum_{k=0}^{n-1} r^k a = a\frac{r^n-1}{r-1} \quad \text{for } r \neq 1. \tag{F.2}$$

The sum of the infinite geometric sequence $a, ra, r^2a, \ldots$ is given by

$$\sum_{k=0}^{\infty} r^k a = \frac{a}{1-r} \quad \text{for } |r| < 1. \tag{F.3}$$

F.6 Taylor/Maclaurin Series

Some series for trigonometric functions are as follows:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for all } x \in \mathbb{C}; \quad \text{and} \tag{F.4}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{for all } x \in \mathbb{C}. \tag{F.5}$$

Some series for exponential and logarithmic functions are as follows:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x \in \mathbb{C}; \tag{F.6}$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for all } x \in \mathbb{C} \text{ satisfying } |x| < 1 \text{ or } x = -1; \quad \text{and} \tag{F.7}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad \text{for all } x \in \mathbb{C} \text{ satisfying } |x| < 1 \text{ or } x = 1.$$

Some series for hyperbolic functions are as follows:

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x \in \mathbb{C}; \quad \text{and}$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \text{for all } x \in \mathbb{C}.$$

F.7 Other Formulas for Sums

Formulas for various sums are as follows:

$$\sum_{k=1}^{n} kx^k = \begin{cases} \dfrac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2} & x \in \mathbb{C}, x \neq 1 \\ \dfrac{n(n+1)}{2} & x = 1; \end{cases} \tag{F.8}$$

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(x-1)^2} \quad \text{for all } x \in \mathbb{C} \text{ satisfying } |x| < 1; \tag{F.9}$$

$$\sum_{k=1}^{n} k^2 x^k = \begin{cases} \dfrac{n^2 x^{n+3} - (2n^2 + 2n - 1)x^{n+2} + (n^2 + 2n + 1)x^{n+1} - x(x+1)}{(x-1)^3} & x \in \mathbb{C}, x \neq 1 \\ \dfrac{n\,(2n+1)\,(n+1)}{6} & x = 1; \end{cases} \quad \text{and}$$

$$\sum_{k=1}^{\infty} k^2 x^k = \frac{x(1+x)}{(1-x)^3} \quad \text{for all } x \in \mathbb{C} \text{ satisfying } |x| < 1.$$

F.8 Trigonometric Identities

A key Pythagorean identity is as follows:

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

Some angle-sum and angle-difference identities are as follows:

$$\begin{aligned}
\sin(a+b) &= \sin(a)\cos(b)+\cos(a)\sin(b); \\
\sin(a-b) &= \sin(a)\cos(b)-\cos(a)\sin(b); \\
\cos(a+b) &= \cos(a)\cos(b)-\sin(a)\sin(b); \\
\cos(a-b) &= \cos(a)\cos(b)+\sin(a)\sin(b); \\
\tan(a+b) &= \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}; \quad \text{and} \\
\tan(a-b) &= \frac{\tan(a)-\tan(b)}{1+\tan(a)\tan(b)}.
\end{aligned}$$

Some double-angle identities are as follows:

$$\begin{aligned}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta); \\
\cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta); \quad \text{and} \\
\tan(2\theta) &= \frac{2\tan(\theta)}{1-\tan^2(\theta)}.
\end{aligned}$$

Some product-to-sum identities are as follows:

$$\begin{aligned}
\sin(a)\sin(b) &= \tfrac{1}{2}[-\cos(a+b)+\cos(a-b)]; \\
\sin(a)\cos(b) &= \tfrac{1}{2}[\sin(a+b)+\sin(a-b)]; \quad \text{and} \\
\cos(a)\cos(b) &= \tfrac{1}{2}[\cos(a+b)+\cos(a-b)].
\end{aligned}$$

Some sum-to-product identities are as follows:

$$\begin{aligned}
\sin(a)+\sin(b) &= 2\sin\left[\tfrac{1}{2}(a+b)\right]\cos\left[\tfrac{1}{2}(a-b)\right]; \\
\sin(a)-\sin(b) &= 2\cos\left[\tfrac{1}{2}(a+b)\right]\sin\left[\tfrac{1}{2}(a-b)\right]; \\
\cos(a)+\cos(b) &= 2\cos\left[\tfrac{1}{2}(a+b)\right]\cos\left[\tfrac{1}{2}(a-b)\right]; \quad \text{and} \\
\cos(a)-\cos(b) &= -2\sin\left[\tfrac{1}{2}(a+b)\right]\sin\left[\tfrac{1}{2}(a-b)\right].
\end{aligned}$$

F.9 Exact Trigonometric Function Values

The exact values of various trigonometric functions for certain special angles can be found in Table F.1.

Table F.1: Exact values of various trigonometric functions for certain special angles

θ				
Degrees	Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	0	1	0
15	$\frac{\pi}{12}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$2-\sqrt{3}$
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
75	$\frac{3\pi}{4}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$2+\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	undefined
120	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
135	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
150	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
180	π	0	-1	0
210	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
225	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
240	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
270	$\frac{3\pi}{2}$	-1	0	undefined
300	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
315	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
330	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$

Appendix G

Video Lectures

G.1 Introduction

The author has prepared video lectures for some of the material covered in this textbook. All of the videos are hosted by YouTube and available through the author's YouTube channel:

- https://www.youtube.com/iamcanadian1867

The most up-to-date information about this video-lecture content can be found at:

- https://www.ece.uvic.ca/~mdadams/sigsysbook/#video_lectures

For the convenience of the reader, some information on the video-lecture content available at the time of this writing is provided in the remainder of this appendix.

G.2 2020-05 ECE 260 Video Lectures

The author prepared video lectures for all of the continuous-time material covered in this textbook in order to teach the 2020-05 offering of the course ECE 260 (titled "Continuous-Time Signals and Systems") in the Department of Electrical and Computer Engineering at the University of Victoria, Victoria, Canada. All of these videos are available from the author's YouTube channel. Although these video lectures are based on a modified version of Edition 2.0 of the textbook and lecture slides, these video lectures were prepared so as to be independent of the textbook edition used. For this reason, these video lectures are likely to be an extremely valuable resource to the reader, regardless of the particular edition of the textbook that they are using. The video lectures for the above course can be found in the following YouTube playlist:

- https://www.youtube.com/playlist?list=PLbHYdvrWBMxYGMvQ3QG6paNu7CuIRL5dX

An information package for the video lectures is available that includes:

- a copy of the edition of the lecture slides used in the video lectures (in PDF format);
- a copy of all of the fully-annotated worked-through examples used in the video lectures (in PDF format); and
- a fully-cataloged list of the slides covered in the video lectures, where each slide in the list has a link to the corresponding time offset in the YouTube video where the slide is covered.

This information package is available from the video lecture section of the web site for the textbook:

- https://www.ece.uvic.ca/~mdadams/sigsysbook/#video_lectures

For the convenience of the reader, the catalog of the video lectures is also included in what follows.

G.2.1 Video-Lecture Catalog

To allow the content in the video lectures to be more easily located and navigated, a catalog of the video lectures is included below. This catalog contains a list of all slides covered in the lectures, where each slide in the list has a link to the corresponding time offset in the YouTube video where the slide is covered. By using this catalog, it is a trivial exercise to jump to the exact point in the video lectures where a specific slide/topic is covered (i.e., simply click on the appropriate hyperlink).

G.2.1.1 Introduction

The following is a link to the full video:
- ◇ https://youtu.be/jApBXeFQTMk [duration: 00:23:12]

The following are links to particular offsets within the video:
- ◇ 00:00: [intro] Unit: Introduction
- ◇ 00:18: [intro] Signals
- ◇ 01:41: [intro] Classification of Signals
- ◇ 06:01: [intro] Graphical Representation of Signals
- ◇ 07:09: [intro] Systems
- ◇ 07:44: [intro] Classification of Systems
- ◇ 09:19: [intro] Signal Processing Systems
- ◇ 11:50: [intro] Communication Systems
- ◇ 14:01: [intro] Control Systems
- ◇ 17:52: [intro] Why Study Signals and Systems?
- ◇ 20:03: [intro] System Failure Example: Tacoma Narrows Bridge
- ◇ 22:41: [intro] System Failure Example: Tacoma Narrows Bridge (Continued)

G.2.1.2 Complex Analysis

The following is a link to the full video:
- ◇ https://youtu.be/8_KKaTEUZB0 [duration: 00:45:42]

The following are links to particular offsets within the video:
- ◇ 00:00: [complex] Unit: Complex Analysis
- ◇ 00:26: [complex] Complex Numbers
- ◇ 01:45: [complex] Complex Numbers (Continued)
- ◇ 03:07: [complex] Geometric Interpretation of Cartesian and Polar Forms
- ◇ 03:58: [complex] The arctan Function
- ◇ 07:14: [complex] The atan2 Function
- ◇ 08:32: [complex] Conversion Between Cartesian and Polar Form
- ◇ 09:46: [complex] Properties of Complex Numbers
- ◇ 11:02: [complex] Conjugation
- ◇ 12:08: [complex] Properties of Conjugation
- ◇ 13:37: [complex] Addition
- ◇ 14:38: [complex] Multiplication
- ◇ 15:44: [complex] Division
- ◇ 17:47: [complex] Properties of the Magnitude and Argument
- ◇ 18:56: [complex] Euler's Relation and De Moivre's Theorem
- ◇ 20:06: [complex] Roots of Complex Numbers
- ◇ 21:05: [complex] Quadratic Formula
- ◇ 22:04: [complex] Complex Functions
- ◇ 23:35: [complex] Continuity
- ◇ 25:11: [complex] Differentiability
- ◇ 26:40: [complex] Open Disks

- ◇ 27:53: [complex] Analyticity
- ◇ 29:51: [complex] Example A.10
- ◇ 30:27: [complex] Example A.11
- ◇ 31:25: [complex] Zeros and Singularities
- ◇ 35:20: [complex] Zeros and Poles of a Rational Function
- ◇ 39:00: [complex] Example A.12

G.2.1.3 Preliminaries — Introduction

The following is a link to the full video:
- ◇ `https://youtu.be/0950-nR1KqQ` [duration: 00:00:26]

The following are links to particular offsets within the video:
- ◇ 00:00: [prelim] Unit: Preliminaries

G.2.1.4 Preliminaries — Functions, Sequences, System Operators, and Transforms

The following is a link to the full video:
- ◇ `https://youtu.be/LREmWSf5v3k` [duration: 00:33:33]

The following are links to particular offsets within the video:
- ◇ 00:00: [prelim] Section: Functions, Sequences, System Operators, and Transforms
- ◇ 00:11: [prelim] Sets
- ◇ 01:07: [prelim] Notation for Intervals on the Real Line
- ◇ 03:41: [prelim] Mappings
- ◇ 06:19: [prelim] Functions
- ◇ 09:59: [prelim] Example 2.2
- ◇ 15:58: [prelim] Sequences
- ◇ 19:42: [prelim] System Operators
- ◇ 22:48: [prelim] Remarks on Operator Notation for CT Systems
- ◇ 28:21: [prelim] Example 2.6
- ◇ 30:08: [prelim] Example 2.7
- ◇ 31:38: [prelim] Transforms
- ◇ 32:49: [prelim] Examples of Transforms

G.2.1.5 Preliminaries — Signal Properties

The following is a link to the full video:
- ◇ `https://youtu.be/iWDh_nhVEII` [duration: 00:08:59]

The following are links to particular offsets within the video:
- ◇ 00:00: [prelim] Section: Properties of Signals
- ◇ 00:10: [prelim] Even Symmetry
- ◇ 01:53: [prelim] Odd Symmetry
- ◇ 03:59: [prelim] Conjugate Symmetry
- ◇ 05:04: [prelim] Periodicity
- ◇ 06:46: [prelim] Periodicity (Continued 1)
- ◇ 07:49: [prelim] Periodicity (Continued 2)

G.2.1.6 CT Signals and Systems — Introduction

The following is a link to the full video:
- ◇ `https://youtu.be/9wJGqOaEbWg` [duration: 00:00:23]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctsigsys] Unit: Continuous-Time (CT) Signals and Systems

G.2.1.7 CT Signals and Systems — Independent/Dependent-Variable Transformations

The following is a link to the full video:

- ◇ https://youtu.be/dtfMWDJPqEs [duration: 00:38:00]

The following are links to particular offsets within the video:

- ◇ 00:00: [ctsigsys] Section: Independent- and Dependent-Variable Transformations
- ◇ 00:18: [ctsigsys] Time Shifting (Translation)
- ◇ 01:24: [ctsigsys] Time Shifting (Translation): Example
- ◇ 03:17: [ctsigsys] Time Reversal (Reflection)
- ◇ 04:45: [ctsigsys] Time Compression/Expansion (Dilation)
- ◇ 06:02: [ctsigsys] Time Compression/Expansion (Dilation): Example
- ◇ 08:04: [ctsigsys] Time Scaling (Dilation/Reflection)
- ◇ 10:21: [ctsigsys] Time Scaling (Dilation/Reflection): Example
- ◇ 11:57: [ctsigsys] Combined Time Scaling and Time Shifting
- ◇ 19:36: [ctsigsys] Exercise 3.3
- ◇ 25:55: [ctsigsys] Combined Time Scaling and Time Shifting: Example
- ◇ 28:26: [ctsigsys] Two Perspectives on Independent-Variable Transformations
- ◇ 31:14: [ctsigsys] Demonstration: Two Views of Time-Shifting Transformations
- ◇ 33:26: [ctsigsys] Amplitude Scaling
- ◇ 35:19: [ctsigsys] Amplitude Shifting
- ◇ 36:11: [ctsigsys] Combined Amplitude Scaling and Amplitude Shifting

G.2.1.8 CT Signals and Systems — Function Properties

The following is a link to the full video:

- ◇ https://youtu.be/PRD3WoWrxa4 [duration: 00:26:30]

The following are links to particular offsets within the video:

- ◇ 00:00: [ctsigsys] Section: Properties of Functions
- ◇ 00:11: [ctsigsys] Symmetry and Addition/Multiplication
- ◇ 02:28: [ctsigsys] Decomposition of a Function into Even and Odd Parts
- ◇ 04:48: [ctsigsys] Theorem 3.1
- ◇ 08:17: [ctsigsys] Sum of Periodic Functions
- ◇ 11:50: [ctsigsys] Example 3.2
- ◇ 14:00: [ctsigsys] Example 3.4
- ◇ 17:17: [ctsigsys] Right-Sided Functions
- ◇ 19:49: [ctsigsys] Left-Sided Functions
- ◇ 21:54: [ctsigsys] Finite-Duration and Two-Sided Functions
- ◇ 23:32: [ctsigsys] Bounded Functions
- ◇ 25:14: [ctsigsys] Energy and Power of a Function

G.2.1.9 CT Signals and Systems — Elementary Functions

The following is a link to the full video:

- ◇ https://youtu.be/qVurOxA8oYM [duration: 00:51:59]

The following are links to particular offsets within the video:

- ◇ 00:00: [ctsigsys] Section: Elementary Functions
- ◇ 00:17: [ctsigsys] Real Sinusoidal Functions
- ◇ 00:49: [ctsigsys] Complex Exponential Functions
- ◇ 01:23: [ctsigsys] Real Exponential Functions
- ◇ 02:35: [ctsigsys] Complex Sinusoidal Functions
- ◇ 03:49: [ctsigsys] Complex Sinusoidal Functions (Continued)
- ◇ 04:15: [ctsigsys] Plots of Complex Sinusoidal Functions
- ◇ 05:14: [ctsigsys] General Complex Exponential Functions

- ◇ 07:00: [ctsigsys] General Complex Exponential Functions (Continued)
- ◇ 07:44: [ctsigsys] Unit-Step Function
- ◇ 09:08: [ctsigsys] Signum Function
- ◇ 09:50: [ctsigsys] Rectangular Function
- ◇ 10:59: [ctsigsys] Cardinal Sine Function
- ◇ 12:55: [ctsigsys] Unit-Impulse Function
- ◇ 17:25: [ctsigsys] Unit-Impulse Function as a Limit
- ◇ 19:55: [ctsigsys] Properties of the Unit-Impulse Function
- ◇ 22:17: [ctsigsys] Figure: Graphical Interpretation of Equivalence Property
- ◇ 24:23: [ctsigsys] Example 3.8
- ◇ 25:55: [ctsigsys] Example 3.9
- ◇ 29:34: [ctsigsys] Example 3.10
- ◇ 36:30: [ctsigsys] Representing a Rectangular Pulse (Using Unit-Step Functions)
- ◇ 39:08: [ctsigsys] Example 3.11
- ◇ 43:06: [ctsigsys] Representing Functions Using Unit-Step Functions
- ◇ 44:21: [ctsigsys] Example 3.12

G.2.1.10 CT Signals and Systems — Systems

The following is a link to the full video:
- ◇ `https://youtu.be/InFEzaTvC1o` [duration: 00:05:40]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctsigsys] Section: Continuous-Time (CT) Systems
- ◇ 00:13: [ctsigsys] CT Systems
- ◇ 02:36: [ctsigsys] Block Diagram Representations
- ◇ 02:57: [ctsigsys] Interconnection of Systems

G.2.1.11 CT Signals and Systems — System Properties

The following is a link to the full video:
- ◇ `https://youtu.be/C53uS3lkeJQ` [duration: 01:14:19]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctsigsys] Section: Properties of (CT) Systems
- ◇ 00:11: [ctsigsys] Memory
- ◇ 02:45: [ctsigsys] Memory (Continued)
- ◇ 04:42: [ctsigsys] Example 3.15
- ◇ 06:09: [ctsigsys] Example 3.16
- ◇ 08:02: [ctsigsys] Causality
- ◇ 11:14: [ctsigsys] Causality (Continued)
- ◇ 12:58: [ctsigsys] Example 3.19
- ◇ 14:36: [ctsigsys] Example 3.20
- ◇ 16:28: [ctsigsys] Invertibility
- ◇ 20:53: [ctsigsys] Invertibility (Continued)
- ◇ 22:39: [ctsigsys] Example 3.23
- ◇ 27:28: [ctsigsys] Example 3.24
- ◇ 31:22: [ctsigsys] Bounded-Input Bounded-Output (BIBO) Stability
- ◇ 34:57: [ctsigsys] Example 3.27
- ◇ 37:46: [ctsigsys] Example 3.28
- ◇ 40:37: [ctsigsys] Time Invariance (TI)
- ◇ 43:30: [ctsigsys] Time Invariance (Continued)
- ◇ 44:59: [ctsigsys] Example 3.32
- ◇ 47:28: [ctsigsys] Example 3.33

- ◇ 51:27: [ctsigsys] Additivity, Homogeneity, and Linearity
- ◇ 56:03: [ctsigsys] Additivity, Homogeneity, and Linearity (Continued 1)
- ◇ 58:38: [ctsigsys] Additivity, Homogeneity, and Linearity (Continued 2)
- ◇ 01:00:21: [ctsigsys] Example 3.35
- ◇ 01:03:58: [ctsigsys] Example 3.36
- ◇ 01:08:59: [ctsigsys] Eigenfunctions of Systems
- ◇ 01:10:44: [ctsigsys] Example 3.41

G.2.1.12 CT LTI Systems — Introduction

The following is a link to the full video:
- ◇ `https://youtu.be/oOl2SX3Fzw8` [duration: 00:02:24]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctltisys] Unit: Continuous-Time Linear Time-Invariant (LTI) Systems
- ◇ 00:52: [ctltisys] Why Linear Time-Invariant (LTI) Systems?

G.2.1.13 CT LTI Systems — Convolution

The following is a link to the full video:
- ◇ `https://youtu.be/q2n6l3-gi_c` [duration: 00:57:49]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctltisys] Section: Convolution
- ◇ 00:15: [ctltisys] CT Convolution
- ◇ 03:32: [ctltisys] Example X.4.1
- ◇ 10:46: [ctltisys] Practical Convolution Computation
- ◇ 14:25: [ctltisys] Example 4.1
- ◇ 34:26: [ctltisys] Exercise 4.18(u)
- ◇ 46:21: [ctltisys] Properties of Convolution
- ◇ 49:03: [ctltisys] Theorem 4.1
- ◇ 56:06: [ctltisys] Representation of Functions Using Impulses

G.2.1.14 CT LTI Systems — Convolution and LTI Systems

The following is a link to the full video:
- ◇ `https://youtu.be/fnH51-gRiqg` [duration: 00:25:29]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctltisys] Section: Convolution and LTI Systems
- ◇ 00:19: [ctltisys] Impulse Response
- ◇ 03:03: [ctltisys] Theorem 4.5
- ◇ 07:14: [ctltisys] Example 4.5
- ◇ 10:52: [ctltisys] Step Response
- ◇ 12:52: [ctltisys] Block Diagram Representation of LTI Systems
- ◇ 13:31: [ctltisys] Interconnection of LTI Systems
- ◇ 16:31: [ctltisys] Example 4.7

G.2.1.15 CT LTI Systems — Properties of LTI Systems

The following is a link to the full video:
- ◇ `https://youtu.be/cOhpZyxyDW8` [duration: 00:46:04]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctltisys] Section: Properties of LTI Systems
- ◇ 00:35: [ctltisys] Memory
- ◇ 04:03: [ctltisys] Example 4.8

- ◇ 06:21: [ctltisys] Example 4.9
- ◇ 07:14: [ctltisys] Causality
- ◇ 09:32: [ctltisys] Example 4.10
- ◇ 12:09: [ctltisys] Example 4.11
- ◇ 14:29: [ctltisys] Invertibility
- ◇ 16:39: [ctltisys] Example 4.12
- ◇ 20:12: [ctltisys] BIBO Stability
- ◇ 21:56: [ctltisys] Example 4.14
- ◇ 27:32: [ctltisys] Example 4.15
- ◇ 32:22: [ctltisys] Eigenfunctions of LTI Systems
- ◇ 35:10: [ctltisys] Representations of Functions Using Eigenfunctions
- ◇ 38:07: [ctltisys] Example: Corollary of Theorem 4.12
- ◇ 41:29: [ctltisys] Example 4.16

G.2.1.16 Interlude

The following is a link to the full video:
- ◇ `https://youtu.be/5pliTVUox0I` [duration: 00:04:54]

The following are links to particular offsets within the video:
- ◇ 00:00: Interlude

G.2.1.17 CT Fourier Series — Introduction

The following is a link to the full video:
- ◇ `https://youtu.be/YxQ2Bi3Z4Iw` [duration: 00:01:33]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctfs] Unit: Continuous-Time Fourier Series (CTFS)
- ◇ 00:31: [ctfs] Introduction

G.2.1.18 CT Fourier Series — Fourier Series

The following is a link to the full video:
- ◇ `https://youtu.be/u55IQ5kSGoM` [duration: 00:19:23]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctfs] Section: Fourier Series
- ◇ 00:16: [ctfs] Harmonically-Related Complex Sinusoids
- ◇ 01:42: [ctfs] CT Fourier Series
- ◇ 03:51: [ctfs] CT Fourier Series (Continued)
- ◇ 06:37: [ctfs] Example 5.1
- ◇ 13:28: [ctfs] Example 5.3

G.2.1.19 CT Fourier Series — Convergence Properties of Fourier Series

The following is a link to the full video:
- ◇ `https://youtu.be/Vrf1Q93-JFM` [duration: 00:28:16]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctfs] Section: Convergence Properties of Fourier Series
- ◇ 00:32: [ctfs] Remarks on Equality of Functions
- ◇ 06:06: [ctfs] Convergence of Fourier Series
- ◇ 08:48: [ctfs] Convergence of Fourier Series (Continued)
- ◇ 10:38: [ctfs] Convergence of Fourier Series: Continuous Case
- ◇ 11:40: [ctfs] Convergence of Fourier Series: Finite-Energy Case
- ◇ 12:55: [ctfs] Dirichlet Conditions

◇ 17:21: [ctfs] Convergence of Fourier Series: Dirichlet Case
◇ 18:50: [ctfs] Example 5.6
◇ 22:14: [ctfs] Gibbs Phenomenon
◇ 23:28: [ctfs] Gibbs Phenomenon: Periodic Square Wave Example
◇ 23:41: [ctfs] Gibbs Phenomenon: Periodic Square Wave Example [Annotated]

G.2.1.20 CT Fourier Series — Properties of Fourier Series

The following is a link to the full video:
◇ `https://youtu.be/WRCjY_pPAZE` [duration: 00:10:15]
The following are links to particular offsets within the video:
◇ 00:00: [ctfs] Section: Properties of Fourier Series
◇ 00:14: [ctfs] Properties of (CT) Fourier Series
◇ 00:33: [ctfs] Linearity
◇ 01:25: [ctfs] Even and Odd Symmetry
◇ 02:07: [ctfs] Real Functions
◇ 04:36: [ctfs] Trigonometric Forms of a Fourier Series
◇ 08:07: [ctfs] Other Properties of Fourier Series
◇ 09:20: [ctfs] Zeroth Coefficient of Fourier Series

G.2.1.21 CT Fourier Series — Fourier Series and Frequency Spectra

The following is a link to the full video:
◇ `https://youtu.be/FDyi1EUAC9M` [duration: 00:22:08]
The following are links to particular offsets within the video:
◇ 00:00: [ctfs] Section: Fourier Series and Frequency Spectra
◇ 00:20: [ctfs] A New Perspective on Functions: The Frequency Domain
◇ 01:50: [ctfs] Motivating Example
◇ 04:53: [ctfs] Motivating Example (Continued)
◇ 07:01: [ctfs] Fourier Series and Frequency Spectra
◇ 10:10: [ctfs] Fourier Series and Frequency Spectra (Continued)
◇ 12:20: [ctfs] Example 5.7
◇ 20:20: [ctfs] Frequency Spectra of Real Functions

G.2.1.22 CT Fourier Series — Fourier Series and LTI Systems

The following is a link to the full video:
◇ `https://youtu.be/Vhwaw0NdCDM` [duration: 00:25:47]
The following are links to particular offsets within the video:
◇ 00:00: [ctfs] Section: Fourier Series and LTI Systems
◇ 00:22: [ctfs] Frequency Response
◇ 03:26: [ctfs] Fourier Series and LTI Systems
◇ 05:24: [ctfs] Example 5.9
◇ 09:27: [ctfs] Filtering
◇ 10:42: [ctfs] Ideal Lowpass Filter
◇ 12:05: [ctfs] Ideal Highpass Filter
◇ 13:26: [ctfs] Ideal Bandpass Filter
◇ 15:20: [ctfs] Example 5.10

G.2.1.23 CT Fourier Transform — Introduction

The following is a link to the full video:
◇ `https://youtu.be/uqnNXbisNgA` [duration: 00:01:40]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctft] Unit: CT Fourier Transform
- ◇ 00:38: [ctft] Motivation for the Fourier Transform

G.2.1.24 CT Fourier Transform — Fourier Transform

The following is a link to the full video:
- ◇ https://youtu.be/3ghUBR8AHxg [duration: 00:12:45]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctft] Section: Fourier Transform
- ◇ 00:13: [ctft] Development of the Fourier Transform [Aperiodic Case]
- ◇ 01:44: [ctft] Development of the Fourier Transform [Aperiodic Case] (Continued)
- ◇ 03:58: [ctft] Generalized Fourier Transform
- ◇ 05:33: [ctft] CT Fourier Transform (CTFT)
- ◇ 07:12: [ctft] Example 6.1
- ◇ 09:43: [ctft] Example 6.3

G.2.1.25 CT Fourier Transform — Convergence Properties

The following is a link to the full video:
- ◇ https://youtu.be/hWc6Gkscx0A [duration: 00:13:45]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctft] Section: Convergence Properties of the Fourier Transform
- ◇ 00:32: [ctft] Convergence of the Fourier Transform
- ◇ 02:26: [ctft] Convergence of the Fourier Transform: Finite-Energy Case
- ◇ 04:14: [ctft] Dirichlet Conditions
- ◇ 08:45: [ctft] Convergence of the Fourier Transform: Dirichlet Case
- ◇ 10:45: [ctft] Example 6.6

G.2.1.26 CT Fourier Transform — Properties of the Fourier Transform

The following is a link to the full video:
- ◇ https://youtu.be/laIz9zjetzc [duration: 01:22:02]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctft] Section: Properties of the Fourier Transform
- ◇ 00:19: [ctft] Properties of the (CT) Fourier Transform
- ◇ 00:36: [ctft] Properties of the (CT) Fourier Transform (Continued)
- ◇ 00:52: [ctft] (CT) Fourier Transform Pairs
- ◇ 02:08: [ctft] Linearity
- ◇ 03:09: [ctft] Example 6.7
- ◇ 05:33: [ctft] Time-Domain Shifting (Translation)
- ◇ 06:24: [ctft] Example 6.9
- ◇ 11:37: [ctft] Frequency-Domain Shifting (Modulation)
- ◇ 12:24: [ctft] Example 6.10
- ◇ 17:40: [ctft] Time- and Frequency-Domain Scaling (Dilation)
- ◇ 18:42: [ctft] Example 6.11
- ◇ 23:01: [ctft] Conjugation
- ◇ 23:53: [ctft] Example 6.12
- ◇ 24:56: [ctft] Duality
- ◇ 30:35: [ctft] Example 6.13
- ◇ 33:15: [ctft] Time-Domain Convolution
- ◇ 34:35: [ctft] Example 6.14
- ◇ 37:34: [ctft] Time-Domain Multiplication

- ◇ 39:58: [ctft] Example 6.15
- ◇ 45:09: [ctft] Time-Domain Differentiation
- ◇ 46:48: [ctft] Example 6.16
- ◇ 48:26: [ctft] Frequency-Domain Differentiation
- ◇ 49:15: [ctft] Example 6.17
- ◇ 51:20: [ctft] Time-Domain Integration
- ◇ 53:00: [ctft] Example 6.18
- ◇ 55:22: [ctft] Parseval's Relation
- ◇ 56:54: [ctft] Example 6.19
- ◇ 59:12: [ctft] Even/Odd Symmetry
- ◇ 59:53: [ctft] Real Functions
- ◇ 01:01:33: [ctft] More Fourier Transforms
- ◇ 01:01:47: [ctft] Example 6.26
- ◇ 01:07:21: [ctft] Exercise 6.5(g)
- ◇ 01:16:00: [ctft] Exercise 6.2(j)

G.2.1.27 CT Fourier Transform — Fourier Transform of Periodic Functions

The following is a link to the full video:
- ◇ `https://youtu.be/oh1MA8axvtI` [duration: 00:12:55]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctft] Section: Fourier Transform of Periodic Functions
- ◇ 00:25: [ctft] Fourier Transform of Periodic Functions
- ◇ 03:21: [ctft] Fourier Transform of Periodic Functions (Continued)
- ◇ 05:08: [ctft] Example 6.20
- ◇ 07:03: [ctft] Example 6.21
- ◇ 10:00: [ctft] Example 6.24

G.2.1.28 CT Fourier Transform — Fourier Transform and Frequency Spectra of Functions

The following is a link to the full video:
- ◇ `https://youtu.be/1JI9Qs3vbJA` [duration: 00:19:03]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctft] Section: Fourier Transform and Frequency Spectra of Functions
- ◇ 00:21: [ctft] The Frequency-Domain Perspective on Functions
- ◇ 02:25: [ctft] Fourier Transform and Frequency Spectra
- ◇ 04:40: [ctft] Fourier Transform and Frequency Spectra (Continued 1)
- ◇ 05:55: [ctft] Fourier Transform and Frequency Spectra (Continued 2)
- ◇ 08:26: [ctft] Example 6.30
- ◇ 13:31: [ctft] Frequency Spectra of Real Functions
- ◇ 15:03: [ctft] Bandwidth

G.2.1.29 CT Fourier Transform — Fourier Transform and LTI Systems

The following is a link to the full video:
- ◇ `https://youtu.be/zf9uo2wk8pw` [duration: 00:15:42]

The following are links to particular offsets within the video:
- ◇ 00:00: [ctft] Section: Fourier Transform and LTI Systems
- ◇ 00:35: [ctft] Frequency Response of LTI Systems
- ◇ 02:35: [ctft] Frequency Response of LTI Systems (Continued 1)
- ◇ 04:08: [ctft] Frequency Response of LTI Systems (Continued 2)
- ◇ 04:53: [ctft] Block Diagram Representations of LTI Systems
- ◇ 05:49: [ctft] Interconnection of LTI Systems

◇ 07:37: [ctft] LTI Systems and Differential Equations
◇ 09:19: [ctft] Example 6.34
◇ 12:44: [ctft] Example 6.35

G.2.1.30 CT Fourier Transform — Application: Filtering

The following is a link to the full video:
◇ https://youtu.be/tfEhqrCDeJ0 [duration: 00:06:26]
The following are links to particular offsets within the video:
◇ 00:00: [ctft] Section: Application: Filtering
◇ 00:19: [ctft] Filtering
◇ 01:31: [ctft] Ideal Lowpass Filter
◇ 01:33: [ctft] Ideal Highpass Filter
◇ 01:35: [ctft] Ideal Bandpass Filter
◇ 01:48: [ctft] Example 6.38

G.2.1.31 CT Fourier Transform — Application: Circuit Analysis

The following is a link to the full video:
◇ https://youtu.be/LTs-O4k9OpQ [duration: 00:17:50]
The following are links to particular offsets within the video:
◇ 00:00: [ctft] Section: Application: Circuit Analysis
◇ 00:19: [ctft] Electronic Circuits
◇ 02:00: [ctft] Resistors
◇ 03:05: [ctft] Inductors
◇ 04:24: [ctft] Capacitors
◇ 05:52: [ctft] Circuit Analysis with the Fourier Transform
◇ 07:46: [ctft] Example 6.40

G.2.1.32 CT Fourier Transform — Application: Amplitude Modulation

The following is a link to the full video:
◇ https://youtu.be/Ua_H1OiZL-c [duration: 00:28:55]
The following are links to particular offsets within the video:
◇ 00:00: [ctft] Section: Application: Amplitude Modulation (AM)
◇ 00:46: [ctft] Motivation for Amplitude Modulation (AM)
◇ 04:16: [ctft] Trivial Amplitude Modulation (AM) System
◇ 09:21: [ctft] Trivial Amplitude Modulation (AM) System: Example
◇ 10:06: [ctft] Double-Sideband Suppressed-Carrier (DSB-SC) AM
◇ 12:58: [ctft] Example: Analysis of DSB-SC AM — Transmitter
◇ 16:14: [ctft] Example: Analysis of DSB-SC AM — Receiver
◇ 21:13: [ctft] Example: Analysis of DSB-SC AM — Complete System
◇ 24:19: [ctft] Example: Analysis of DSB-SC AM — Spectra
◇ 26:46: [ctft] Single-Sideband Suppressed-Carrier (SSB-SC) AM
◇ 27:48: [ctft] SSB-SC AM: Example

G.2.1.33 CT Fourier Transform — Application: Sampling and Interpolation

The following is a link to the full video:
◇ https://youtu.be/GkOrtV2BkZ8 [duration: 00:33:16]
The following are links to particular offsets within the video:
◇ 00:00: [ctft] Section: Application: Sampling and Interpolation
◇ 00:31: [ctft] Sampling and Interpolation

- ◇ 02:19: [ctft] Periodic Sampling
- ◇ 03:35: [ctft] Invertibility of Sampling
- ◇ 06:49: [ctft] Model of Sampling
- ◇ 09:16: [ctft] Model of Sampling: Various Signals
- ◇ 10:41: [ctft] Model of Sampling: Invertibility of Sampling Revisited
- ◇ 13:12: [ctft] Model of Sampling: Characterization
- ◇ 15:15: [ctft] Analysis of Sampling — Multiplication by a Periodic Impulse Train (Part 1)
- ◇ 16:37: [ctft] Analysis of Sampling — Fourier Series for a Periodic Impulse Train
- ◇ 18:59: [ctft] Analysis of Sampling — Multiplication by a Periodic Impulse Train (Part 2)
- ◇ 20:15: [ctft] Model of Sampling: Aliasing
- ◇ 22:34: [ctft] Model of Sampling: Aliasing (Continued)
- ◇ 26:38: [ctft] Model of Interpolation
- ◇ 28:43: [ctft] Sampling Theorem
- ◇ 30:33: [ctft] Example 6.41

G.2.1.34 Partial Fraction Expansions (PFEs)

The following is a link to the full video:
- ◇ `https://youtu.be/wgTXbvhSgnk` [duration: 00:12:49]

The following are links to particular offsets within the video:
- ◇ 00:00: [pfe] Unit: Partial Fraction Expansions (PFEs)
- ◇ 00:10: [pfe] Motivation for PFEs
- ◇ 00:55: [pfe] Strictly-Proper Rational Functions
- ◇ 01:53: [pfe] Partial Fraction Expansions (PFEs) [CT and DT Contexts]
- ◇ 03:28: [pfe] Simple-Pole Case [CT and DT Contexts]
- ◇ 04:39: [pfe] Example B.1
- ◇ 07:13: [pfe] Repeated-Pole Case [CT and DT Contexts]
- ◇ 09:24: [pfe] Example B.2

G.2.1.35 Laplace Transform — Introduction

The following is a link to the full video:
- ◇ `https://youtu.be/uVCVrZOIl9s` [duration: 00:02:49]

The following are links to particular offsets within the video:
- ◇ 00:00: [lt] Unit: Laplace Transform (LT)
- ◇ 00:32: [lt] Motivation Behind the Laplace Transform
- ◇ 01:35: [lt] Motivation Behind the Laplace Transform (Continued)

G.2.1.36 Laplace Transform — Laplace Transform

The following is a link to the full video:
- ◇ `https://youtu.be/UOeBiWrAfDs` [duration: 00:18:20]

The following are links to particular offsets within the video:
- ◇ 00:00: [lt] Section: Laplace Transform
- ◇ 00:15: [lt] (Bilateral) Laplace Transform
- ◇ 02:07: [lt] Bilateral and Unilateral Laplace Transforms
- ◇ 03:18: [lt] Relationship Between Laplace and Fourier Transforms
- ◇ 06:22: [lt] Derivation: LT FT Relationship (Special Case)
- ◇ 07:40: [lt] Derivation: LT FT Relationship (General Case)
- ◇ 08:54: [lt] Laplace Transform Examples
- ◇ 09:08: [lt] Example 7.3
- ◇ 13:36: [lt] Example 7.4

G.2.1.37 Laplace Transform — Region of Convergence

The following is a link to the full video:

◇ https://youtu.be/DoaZUx55OYw [duration: 00:23:29]

The following are links to particular offsets within the video:

◇ 00:00: [lt] Section: Region of Convergence (ROC)
◇ 00:26: [lt] Left-Half Plane (LHP)
◇ 01:21: [lt] Right-Half Plane (RHP)
◇ 02:08: [lt] Intersection of Sets
◇ 02:49: [lt] Adding a Scalar to a Set
◇ 03:45: [lt] Multiplying a Set by a Scalar
◇ 05:22: [lt] Region of Convergence (ROC)
◇ 06:06: [lt] ROC Property 1: General Form
◇ 07:23: [lt] ROC Property 2: Rational Laplace Transforms
◇ 08:38: [lt] ROC Property 3: Finite-Duration Functions
◇ 09:38: [lt] ROC Property 4: Right-Sided Functions
◇ 10:56: [lt] ROC Property 5: Left-Sided Functions
◇ 12:13: [lt] ROC Property 6: Two-Sided Functions
◇ 13:17: [lt] ROC Property 7: More on Rational Laplace Transforms
◇ 15:20: [lt] General Form of the ROC
◇ 17:11: [lt] Example 7.7

G.2.1.38 Laplace Transform — Properties of the Laplace Transform

The following is a link to the full video:

◇ https://youtu.be/qld0TLDFaxc [duration: 01:00:16]

The following are links to particular offsets within the video:

◇ 00:00: [lt] Section: Properties of the Laplace Transform
◇ 00:21: [lt] Properties of the Laplace Transform
◇ 00:39: [lt] Laplace Transform Pairs
◇ 02:34: [lt] Linearity
◇ 05:00: [lt] Example 7.8
◇ 09:57: [lt] Example 7.9
◇ 14:41: [lt] Time-Domain Shifting
◇ 15:42: [lt] Example 7.10
◇ 17:16: [lt] Laplace-Domain Shifting
◇ 18:56: [lt] Example 7.11
◇ 22:37: [lt] Time-Domain/Laplace-Domain Scaling
◇ 24:31: [lt] Example 7.12
◇ 28:13: [lt] Conjugation
◇ 29:14: [lt] Example 7.13
◇ 33:07: [lt] Time-Domain Convolution
◇ 35:22: [lt] Example 7.14
◇ 37:48: [lt] Time-Domain Differentiation
◇ 40:16: [lt] Example 7.15
◇ 41:43: [lt] Laplace-Domain Differentiation
◇ 42:37: [lt] Example 7.16
◇ 44:29: [lt] Time-Domain Integration
◇ 46:56: [lt] Example 7.17
◇ 49:13: [lt] Initial Value Theorem
◇ 51:23: [lt] Final Value Theorem
◇ 53:36: [lt] Example 7.18
◇ 55:26: [lt] More Laplace Transform Examples

◇ 55:42: [lt] Example 7.19

G.2.1.39 Laplace Transform — Determination of Inverse Laplace Transform

The following is a link to the full video:
◇ https://youtu.be/yW0nCwrwaCQ [duration: 00:20:17]
The following are links to particular offsets within the video:
◇ 00:00: [lt] Section: Determination of Inverse Laplace Transform
◇ 00:10: [lt] Finding Inverse Laplace Transform
◇ 01:18: [lt] Example 7.27
◇ 09:55: [lt] Example 7.28

G.2.1.40 Laplace Transform — Laplace Transform and LTI Systems

The following is a link to the full video:
◇ https://youtu.be/MWZV3c6TzJI [duration: 00:33:11]
The following are links to particular offsets within the video:
◇ 00:00: [lt] Section: Laplace Transform and LTI Systems
◇ 00:37: [lt] System Function of LTI Systems
◇ 03:08: [lt] Block Diagram Representations of LTI Systems
◇ 03:58: [lt] Interconnection of LTI Systems
◇ 05:47: [lt] Causality
◇ 09:14: [lt] Example 7.31
◇ 12:31: [lt] BIBO Stability
◇ 15:22: [lt] Example 7.32
◇ 17:24: [lt] Example 7.33
◇ 19:24: [lt] Example 7.34
◇ 23:38: [lt] Invertibility
◇ 26:01: [lt] Example 7.35
◇ 27:53: [lt] LTI Systems and Differential Equations
◇ 29:51: [lt] Example 7.36
◇ 31:39: [lt] Example 7.37

G.2.1.41 Laplace Transform — Application: Circuit Analysis

The following is a link to the full video:
◇ https://youtu.be/cf8JIy83DdQ [duration: 00:15:42]
The following are links to particular offsets within the video:
◇ 00:00: [lt] Section: Application: Circuit Analysis
◇ 00:12: [lt] Electronic Circuits
◇ 01:15: [lt] Resistors
◇ 01:17: [lt] Inductors
◇ 01:19: [lt] Capacitors
◇ 01:24: [lt] Circuit Analysis With the Laplace Transform
◇ 03:25: [lt] Example 7.38

G.2.1.42 Laplace Transform — Application: Design and Analysis of Control Systems

The following is a link to the full video:
◇ https://youtu.be/SgK69mQdgiw [duration: 00:29:03]
The following are links to particular offsets within the video:
◇ 00:00: [lt] Section: Application: Design and Analysis of Control Systems
◇ 00:13: [lt] Control Systems

- ◇ 02:35: [lt] Feedback Control Systems
- ◇ 05:27: [lt] Stability Analysis of Feedback Systems
- ◇ 07:40: [lt] Example 7.40 — Stabilization Example: Unstable Plant
- ◇ 08:52: [lt] Example 7.40 — Stabilization Example: Using Pole-Zero Cancellation
- ◇ 11:04: [lt] Example 7.40 — Stabilization Example: Using Feedback (1)
- ◇ 13:42: [lt] Example 7.40 — Stabilization Example: Using Feedback (2)
- ◇ 17:04: [lt] Example 7.40 — Stabilization Example: Using Feedback (3)
- ◇ 17:55: [lt] Example 7.40 — Remarks on Stabilization Via Pole-Zero Cancellation
- ◇ 20:12: [lt] Exercise 7.30

G.2.1.43 Laplace Transform — Unilateral Laplace Transform

The following is a link to the full video:

- ◇ `https://youtu.be/ac6Nbs6hf7M` [duration: 00:20:52]

The following are links to particular offsets within the video:

- ◇ 00:00: [lt] Section: Unilateral Laplace Transform
- ◇ 00:32: [lt] Unilateral Laplace Transform
- ◇ 03:15: [lt] Inversion of the Unilateral Laplace Transform
- ◇ 05:40: [lt] Unilateral Versus Bilateral Laplace Transform
- ◇ 07:39: [lt] Properties of the Unilateral Laplace Transform
- ◇ 09:14: [lt] Unilateral Laplace Transform Pairs
- ◇ 10:21: [lt] Solving Differential Equations Using the Unilateral Laplace Transform
- ◇ 11:31: [lt] Example 7.42
- ◇ 14:25: [lt] Example 7.43

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