Assignment 2 solution.

Question 1

(a) $$X(e^{j\omega}) = \frac{1 + 3e^{-j3\omega}}{1 + \frac{1}{4}e^{-j\omega}}.$$

We will use $$a^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - a e^{-j\omega}}.$$

and the time-shifting property.

$$X(e^{j\omega}) = \frac{1}{1 + \frac{1}{4}e^{-j\omega}} + 3 \cdot \frac{e^{-j3\omega}}{1 + \frac{1}{4}e^{-j\omega}}.$$

$$\left(-\frac{1}{4}\right)^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 + \frac{1}{4}e^{-j\omega}}$$

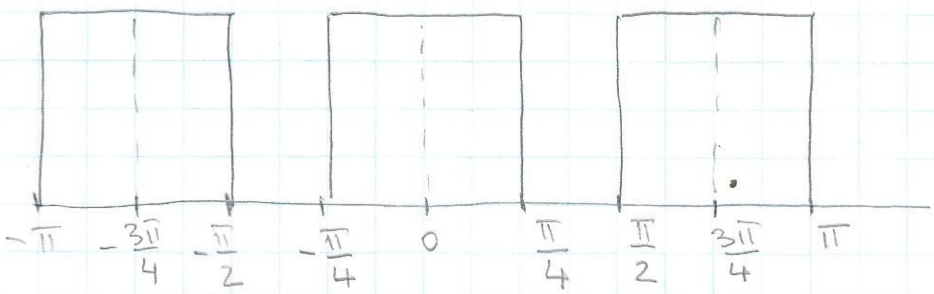
time shift (3 unit delay)

$$\left(-\frac{1}{4}\right)^{n-3} u[n-3] \longleftrightarrow \frac{e^{-j3\omega}}{1 + \frac{1}{4}e^{-j\omega}}$$

$$x[n] = \left(-\frac{1}{4}\right)^n u[n] + 3\left(-\frac{1}{4}\right)^{n-3} u[n-3]$$

Question 1

(b) $X(e^{j\omega})$ for this problem looks like:



$X(e^{j\omega})$ = a sum of 3 ideal low-pass filters of equal width $\left(\frac{\pi}{4}\right)$. Two of these filters are shifted to $\omega_1 = \frac{3\pi}{4}$ and $\omega_2 = -\frac{3\pi}{4}$

$\downarrow$

We will use the frequency-shift property.

$$X\left(e^{j(\omega-\omega_0)}\right) \overset{\mathcal{F}}{\longleftrightarrow} e^{j\omega_0 n} x[n].$$

In the time domain we can write the signal as a sum of three sinc functions.

$$x[n] = e^{-j\frac{3\pi}{4}n} \cdot \frac{\sin\frac{\pi}{4}n}{\pi n} + \frac{\sin\frac{\pi}{4}n}{\pi n} + e^{j\frac{3\pi}{4}n} \cdot \frac{\sin\frac{\pi}{4}n}{\pi n}$$

$$= \left(1 + e^{-j\frac{3\pi}{4}n} + e^{j\frac{3\pi}{4}n}\right) \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}.$$

$$= \left(1 + 2\cos\frac{3\pi}{4}n\right) \frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}.$$

Question 2.

$$H(e^{j\omega}) = e^{-j\omega} \cdot \frac{1 + e^{-j2\omega} + 4e^{-j4\omega}}{1 + \frac{1}{2}e^{-j2\omega}}$$

$$x[n] = \cos\frac{\pi}{2}n = \frac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2}$$

Using the eigenfunction property

$$e^{j\frac{\pi}{2}n} \xrightarrow[H]{} H(e^{j\frac{\pi}{2}}) \cdot e^{j\frac{\pi}{2}n}$$

$$e^{-j\frac{\pi}{2}n} \xrightarrow[H]{} H(e^{-j\frac{\pi}{2}}) \cdot e^{-j\frac{\pi}{2}n}$$

We compute $H(e^{j\frac{\pi}{2}})$ and $H(e^{-j\frac{\pi}{2}})$

$$H(e^{j\frac{\pi}{2}}) = e^{-j\frac{\pi}{2}} \frac{1 + e^{-j\pi} + 4e^{-j2\pi}}{1 + \frac{1}{2}e^{-j\pi}}$$

$$= (-j) \cdot \frac{1 - 1 + 4}{1 - \frac{1}{2}} = -8j$$

$$H(e^{-j\frac{\pi}{2}}) = e^{j\frac{\pi}{2}} \frac{1 + e^{j\pi} + 4e^{+j2\pi}}{1 + \frac{1}{2}e^{+j\pi}} = 8j$$

$$y[n] = \frac{1}{2} H(e^{j\frac{\pi}{2}}) e^{j\frac{\pi}{2}n} + \frac{1}{2} H(e^{-j\frac{\pi}{2}}) e^{-j\frac{\pi}{2}n}$$

$$= -4j\, e^{j\frac{\pi}{2}n} + 4j\, e^{-j\frac{\pi}{2}n} = 8 \sin\frac{\pi}{2}n$$

...estion 3.

2.26. textbook (systems A and B only).

$$\left(\frac{1}{2}\right)^n \longrightarrow \boxed{\text{System A}} \longrightarrow \left(\frac{1}{4}\right)^n.$$

The input $x[n] = \left(\frac{1}{2}\right)^n$ is an eigenfunction of an LTi system.

The response to this input needs to be of the form $A\left(\frac{1}{2}\right)^n$, where A must be a constant.

$$\left(\frac{1}{4}\right)^n = A\left(\frac{1}{2}\right)^n \longrightarrow A = \left(\frac{1}{2}\right)^n \text{ not constant.}$$

The system is not LTi since it violates the eigenfunction property

$$\cos\left(\frac{\pi}{3}n\right) \longrightarrow \boxed{\text{System B}} \longrightarrow 3j \sin \frac{\pi}{3} n.$$

$$\cos\left(\frac{\pi}{3}n\right) = \frac{e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}}{2}$$

$$3j \sin\left(\frac{\pi}{3}n\right) = \frac{3}{2} e^{j\frac{\pi}{3}n} - \frac{3}{2} e^{-j\frac{\pi}{3}n}.$$

$$\frac{e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}}{2} \longrightarrow \boxed{\text{System B}} \longrightarrow \underbrace{H(e^{j\frac{\pi}{3}})}_{3} \cdot \frac{1}{2} e^{j\frac{\pi}{3}n} + \underbrace{H(e^{-j\frac{\pi}{3}})}_{-3} \cdot \frac{1}{2} e^{-j\frac{\pi}{3}n}$$

The system <u>could be</u> LTi since it <u>does not</u> contradict the eigenfunction property.

Question 4

2.33

a) $h[n] = -2\delta[n] + 4\delta[n-1] - 2\delta[n-2]$

b) $H(e^{j\omega}) = -2 + 4e^{-j\omega} - 2e^{-j2\omega}$

$= -2 + 4(\cos\omega - j\sin\omega) - 2(\cos 2\omega - j\sin 2\omega)$

$= (-2 + 4\cos\omega - 2\cos 2\omega) - 4j\left(\sin\omega - \frac{1}{2}\sin 2\omega\right)$

Using $1 + \cos 2\omega = \cos^2\omega$; $\sin 2\omega = 2\sin\omega\cos\omega$; $2\sin^2\omega = 1 - \cos 2\omega$

we find $H(e^{j\omega}) = (4\cos\omega - 4\cos^2\omega) - 4j\sin\omega(1 - \cos\omega)$

$= 4(1-\cos\omega)[\cos\omega - j\sin\omega] = 8\sin^2\frac{\omega}{2}\, e^{-j\omega}$

d) $x_1[n] = e^{j\cdot 0\cdot n} + e^{j\frac{\pi}{2}n}$

$y_1[n] = H(e^{j\cdot 0})\cdot e^{j\cdot 0\cdot n} + H(e^{j\frac{\pi}{2}})\, e^{j\frac{\pi}{2}n}$

$H(e^{j0}) = -2 + 4 - 2 = 0$

$H(e^{j\frac{\pi}{2}}) = -2 + 4e^{-j\frac{\pi}{2}} - 2e^{-j\pi} = -2 + 4j + 2 = -4j$

$y_1[n] = -4j\, e^{j\frac{\pi}{2}n} = 4e^{-j\frac{\pi}{2}} \cdot e^{j\frac{\pi}{2}n} = 4e^{j\frac{\pi}{2}(n-1)}$

2.45.

$$x[n] = \cos(0.4\pi n) + \sin(0.6\pi n) + 5\delta[n-2] + u[n].$$

After passing through the first system.

$$w[n] = x[n] - x[n-1]$$

$$= \cos(0.4\pi n) - \cos(0.4\pi(n-1))$$

$$+ \sin(0.6\pi n) - \sin(0.6\pi(n-1))$$

$$+ 5\delta[n-2] - 5\delta[n-3]$$

$$+ \underbrace{2u[n] - 2u[n-1]}_{= 2\delta[n]}.$$

After passing through the second system (ideal LPF)

1. the cos terms pass through $(0.4\pi < \omega_c = 0.5\pi)$
2. the sin terms are rejected $(0.6\pi > \omega_c = 0.5\pi)$
3. the δ terms give a delayed and scaled impulse response back.

$$y[n] = \cos[0.4\pi n] - \cos[0.4\pi(n-1)]$$

$$+ 5\,\frac{\sin(0.5\pi(n-2))}{\pi(n-2)}$$

$$- 5\,\frac{\sin(0.5\pi(n-3))}{\pi(n-3)}$$

$$+ 2\,\frac{\sin(0.5\pi n)}{\pi n}$$