ELEC 310. Lecture 13. Representations of DT signals by Fourier Transforms

Textbook: section 2.7

Eigenfunctions of LTI systems: main formulas

The eigenfunction property stated in the most general way:

$$x[n] = z^n \Rightarrow y[n] = H(z) \cdot z^n$$

where $H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$

For analyzing DT signals and systems in the frequency domain

we need to consider a particular class of eigenfunctions

$$x[n] = e^{j\omega n} \Rightarrow y[n] = H\left(e^{j\omega}\right) e^{j\omega n}$$

$H\left(e^{j\omega}\right)$ is called the frequency response of the LTI system; it offers a complete representation of the system in the frequency domain; this representation is equivalent to the time-domain representation h[n]

$$H\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

Eigenfunctions of LTI systems: main formulas (cont'd)

There is a broad class of signals x[n] that can be expressed as a finite sum of periodic eigenfunctions

For these signals, we can find the input y[n] of an LTI system by using the frequency response $H(e^{j\omega})$

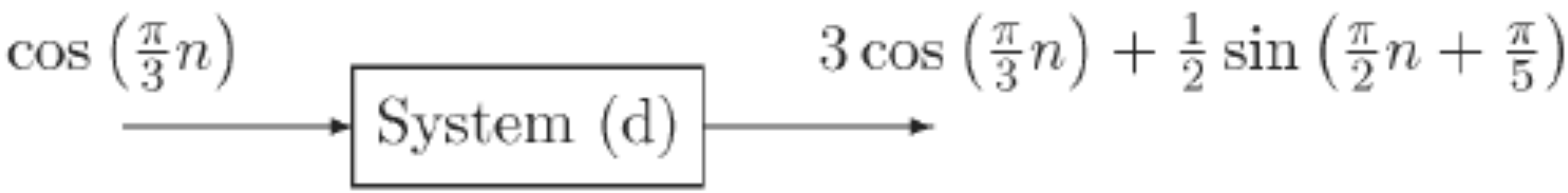
if $x[n] = \sum_k e^{j\omega_k n}$ then

$$y[n] = \sum_k H(e^{j\omega_k}) e^{j\omega_k n}$$

Eigenfunctions – a last example

- All LTI systems need to verify the eigenfunction property
- If this property is not verified, then we can conclude that the system is not LTI.
- Thus, the eigenfunction property can be used for testing whether a system is LTI or not.

2.27 e) Check whether the system can be LTI or not.



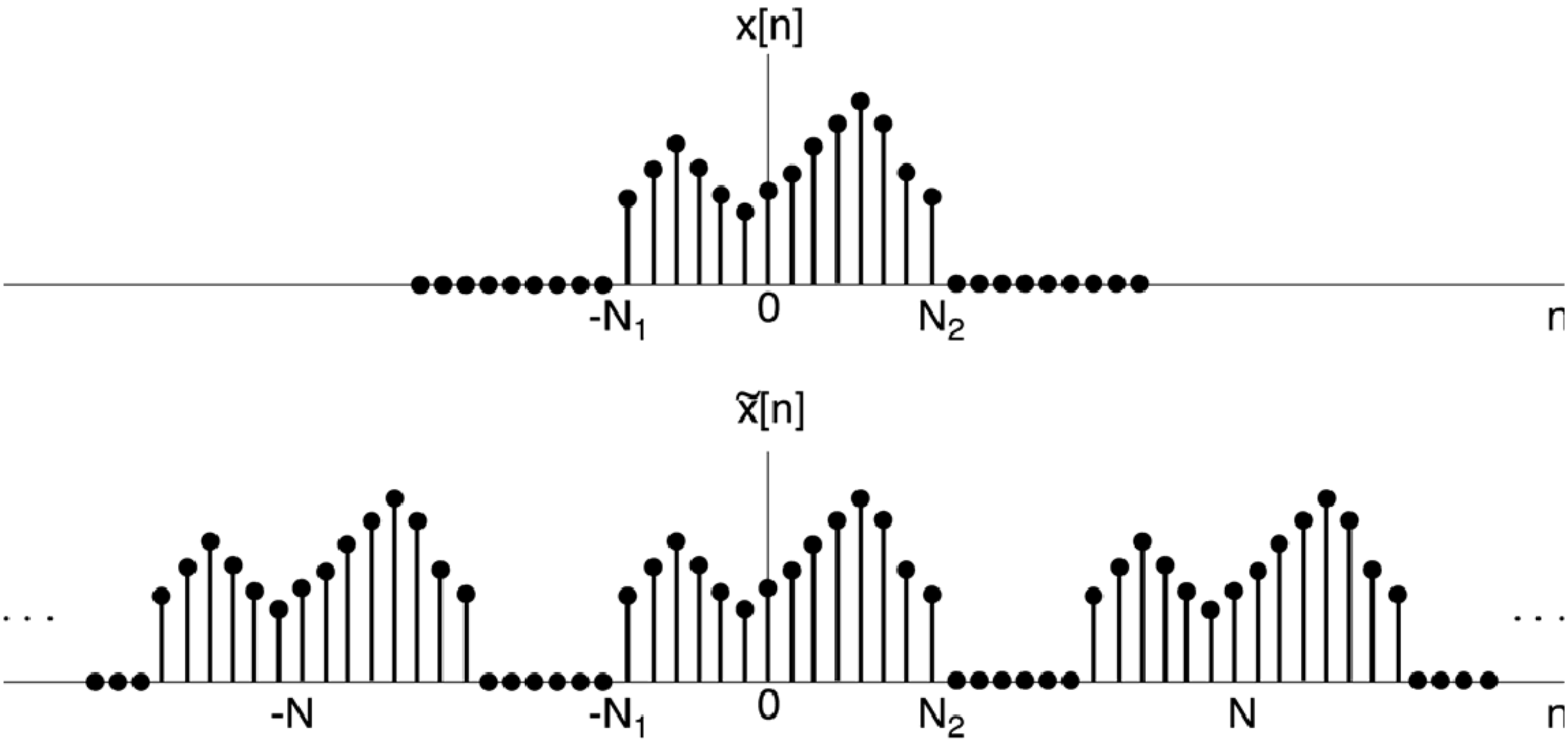
Representations of DT signals by the Fourier Transform

- A broad class of DT signals can be represented as a linear combination (a finite sum) of complex exponentials
- For these signals, we can:
    - Find the output of an LTI system if we know its frequency response
    - Characterize the signals in the frequency domain (i.e. draw their frequency spectrum). EXAMPLE
- What can we say about this class of signals?
- How about signals that do not belong to this class?

Representations of DT signals by the Fourier Transform (cont'd)

- For x[n] aperiodic, we can represent it as a linear combination of complex exponentials infinitesimally close in frequency and with amplitudes

$$\frac{X(e^{j\omega})}{2\pi}d\omega$$



$$\tilde{x}[n] = x[n] \text{ for } any\ n \text{ as } N \to \infty$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad \text{Synthesis equation}$$

Any 2π interval in ω

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{Analysis equation}$$

- The synthesis equation is a representation of x[n] as a linear combination of complex exponentials located infinitesimally close in frequency and with amplitudes $X(e^{j\omega})(d\omega/2\pi)$
- For this reason $X(e^{j\omega})$ is called **the spectrum** of x[n] since it provides information about the contribution of complex exponentials at different frequencies to the signal x[n].

Discussion

- 1) $X(e^{j\omega})$ is a continuous function of ω
- 2) $X(e^{j\omega})$ is generally complex
  - magnitude and phase shift
- 3) $X(e^{j\omega})$ is a periodic function of ω with period 2π
  - because $X(e^{j\omega})$ is periodic with period 2π, the inverse DTFT requires integration over one period only

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad \text{Synthesis equation}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{Analysis equation}$$

Any 2π interval in ω

Example 1

Find the Fourier Transform of the following signals

- $x_1[n]=\delta[n]$
- $x_2[n]=\delta[n-n_0]$
- $x_3[n]= \delta[n]+ \delta[n-1]$

Example 2

Find the Discrete Fourier Transform of

$x[n]=a^n u[n]$, $|a|<1$

Convergence issues

- Synthesis equation: usually none (we integrate over a finite interval)
- Analysis equation: Conditions that are similar to the CT case

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \quad \text{— Finite energy}$$

or

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad \text{— Absolutely summable}$$

Examples (cont'd)

Find the Discrete Fourier Transform of

$x[n]=a^n u[n], |a|<1$

Provide a graphical representation of its magnitude.

Important formulas in this section

- 2.130, 2.131- the direct and inverse Fourier Transform (also called the analysis and synthesis equation)
- 2.133- computing h[n] from $H(e^{j\omega})$ with the inverse Fourier Transform
- Absolute summability and square summability

$\left|X(e^{j\omega})\right| < \infty$ (the integral is convergent) if

a) the sequence x[n] is absolutely summable

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right| < \infty$$

or

b) the sequence x[n] is square summable (a weaker condition than a)

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 < \infty$$