The Discrete Time Fourier Transform-4. Properties (cont'd)

Today

- Applications of the Fourier Transform in image analysis

- Properties of the Fourier Transform (cont'd)

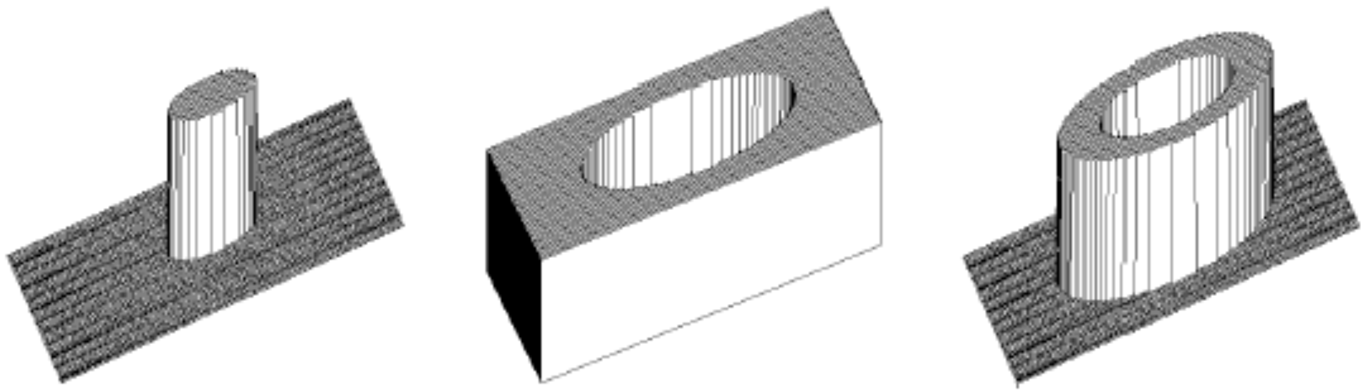


Figure 5.26: Frequency filters displayed in 3D. (a) Low-pass filter. (b) High-pass filter. (c) Band-pass filter.

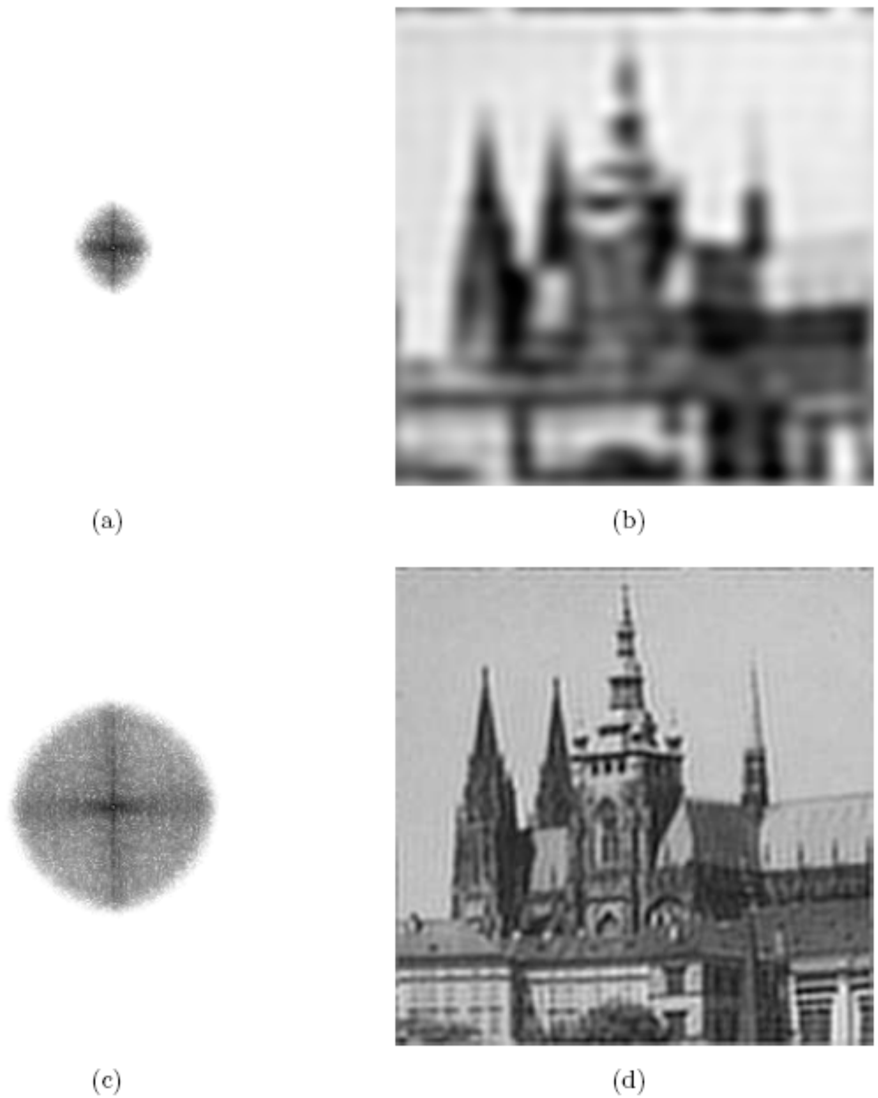


Figure 5.27: Low-pass frequency-domain filtering—for the original image and its spectrum see Figure 3.7. (a) Spectrum of a low-pass filtered image, all higher frequencies filtered out. (b) Image resulting from the inverse Fourier transform applied to spectrum (a). (c) Spectrum of a low-pass filtered image, only very high frequencies filtered out. (d) Inverse Fourier transform applied to spectrum (c).

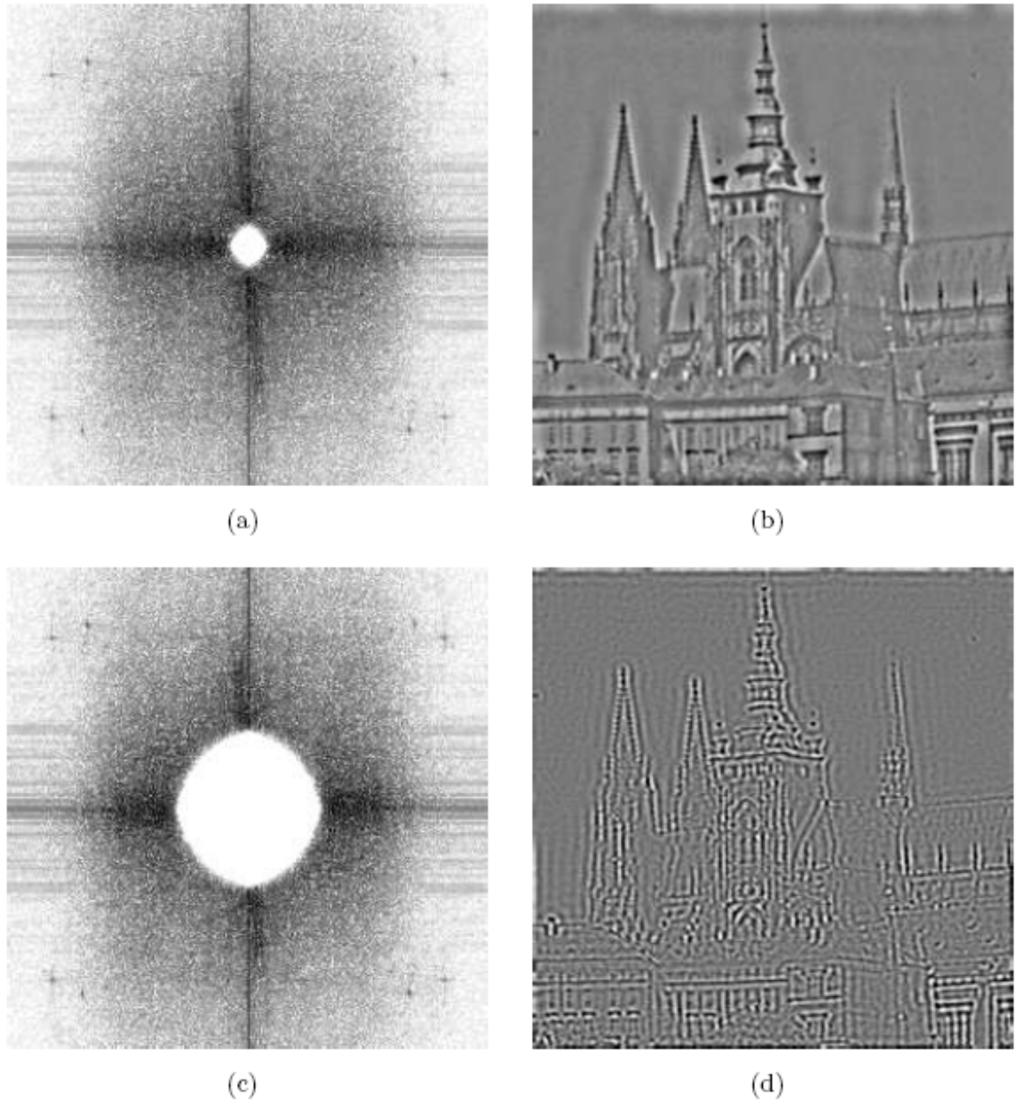
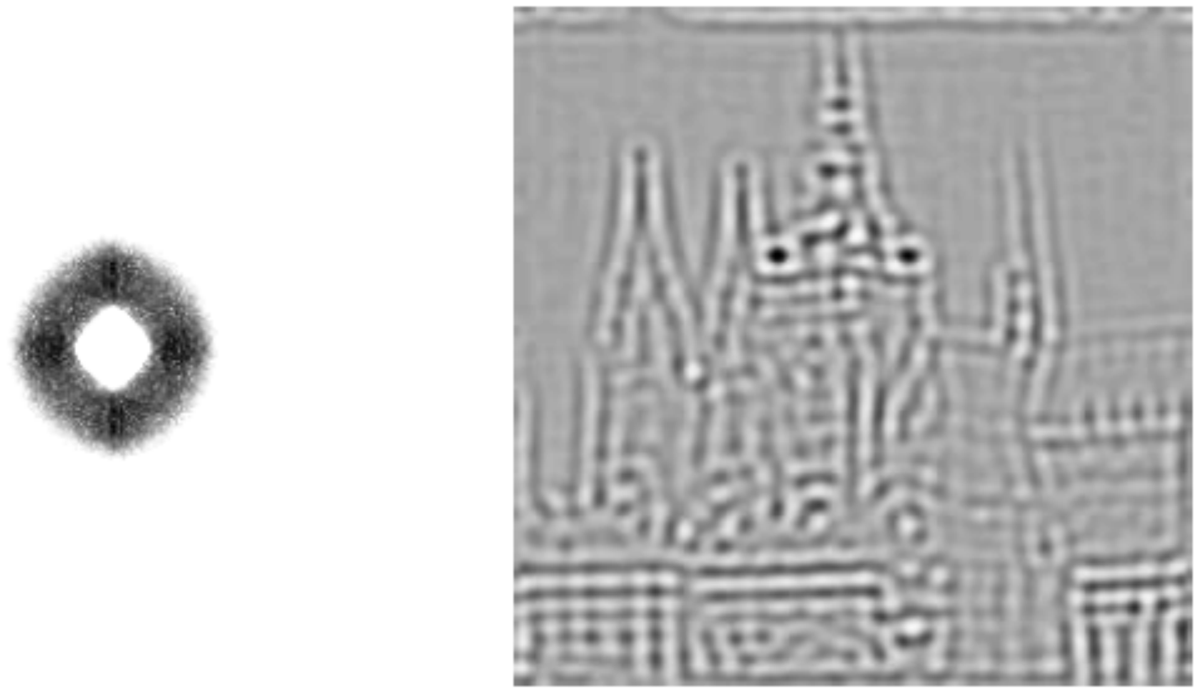


Figure 5.28: High-pass frequency domain filtering. (a) Spectrum of a high-pass filtered image, only very low frequencies filtered out. (b) Image resulting from the inverse Fourier transform applied to spectrum (a). (c) Spectrum of a high-pass filtered image, all lower frequencies filtered out. (d) Inverse Fourier transform applied to spectrum (c).



(a) (b)

Figure 5.29: Band-pass frequency domain filtering. (a) Spectrum of a band-pass-filtered image, low and high frequencies filtered out. (b) Image resulting from the inverse Fourier transform applied to spectrum (a).

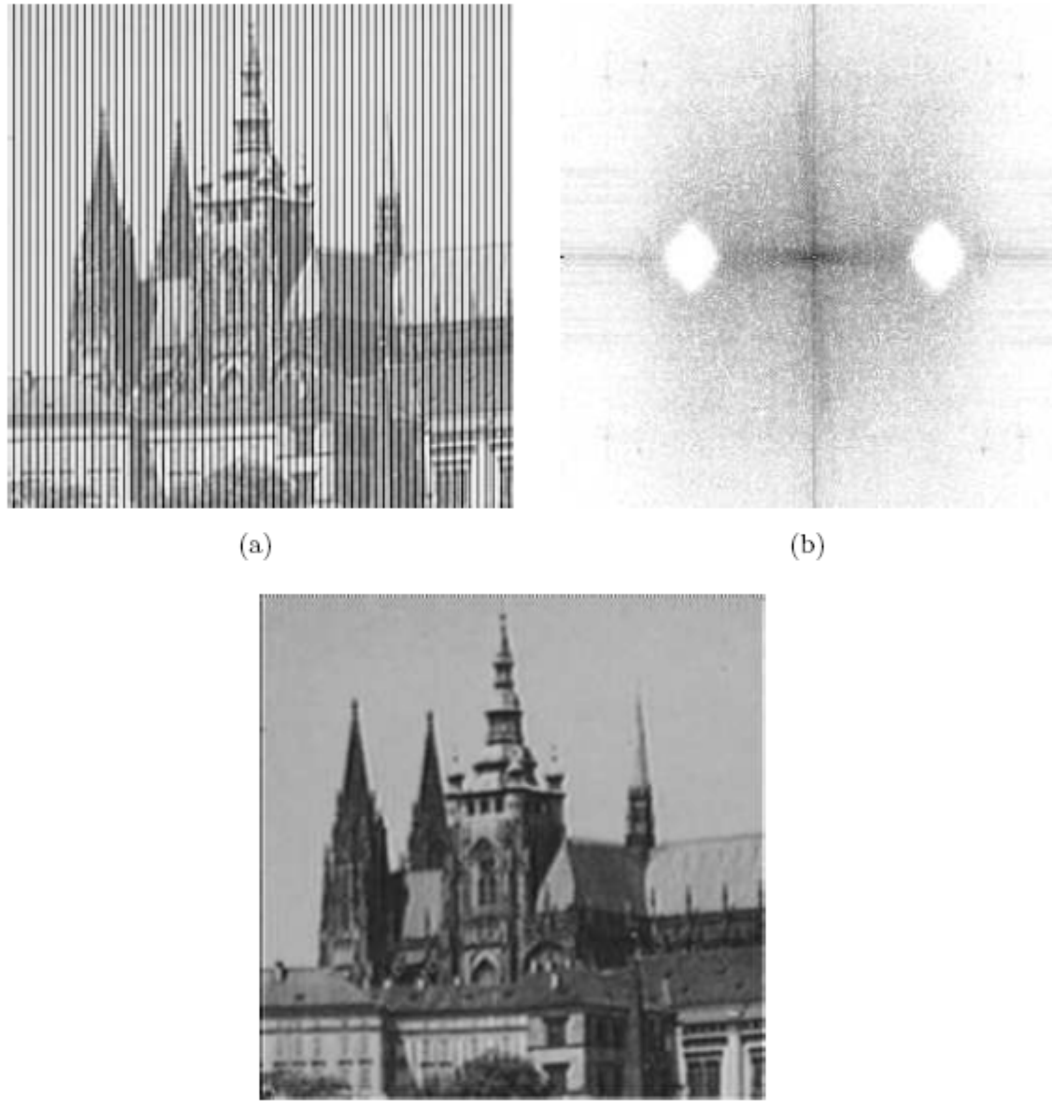


Figure 5.30: Periodic noise removal. (a) Noisy image. (b) Image spectrum used for image reconstruction—note that the areas of frequencies corresponding with periodic vertical lines are filtered out. (c) Filtered image.

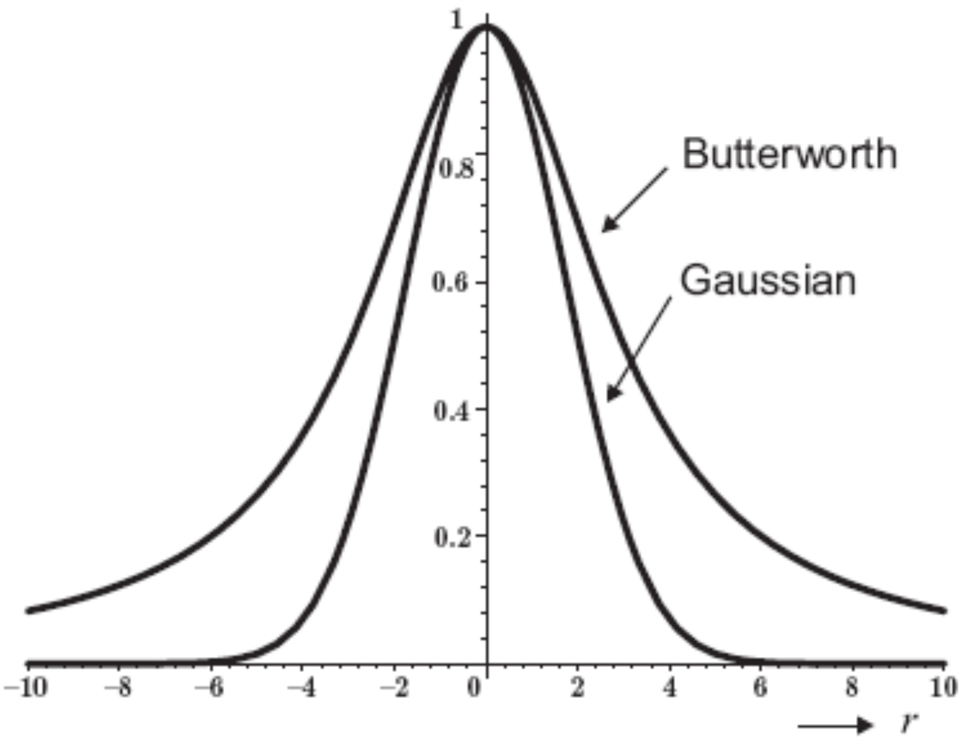


Figure 5.31: Gaussian and Butterworth low-pass filters.

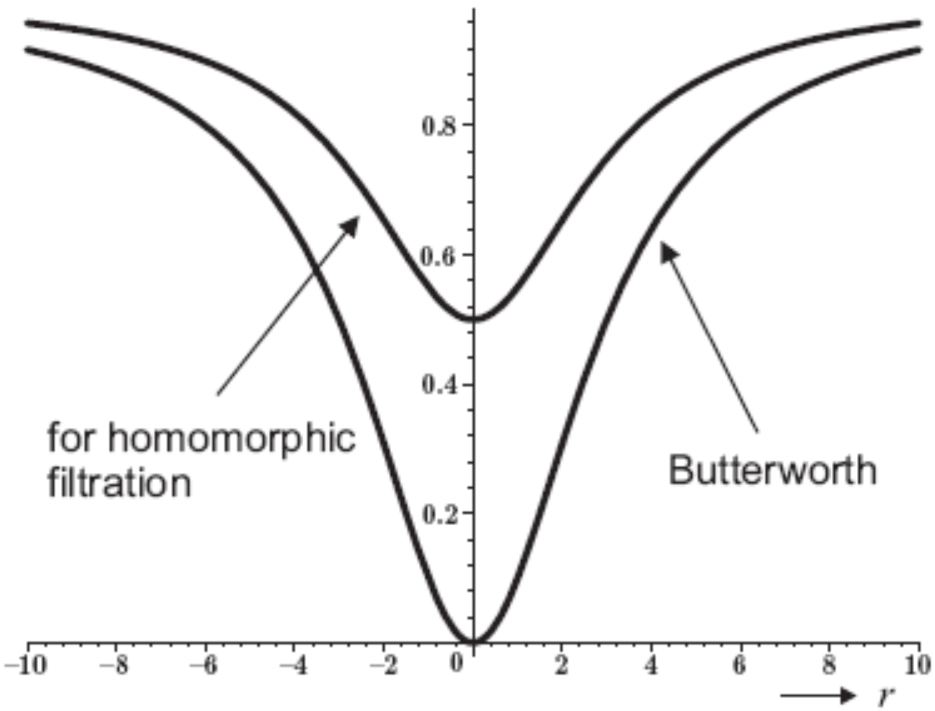


Figure 5.32: High-pass filter used in homomorphic filtering. It is the Butterworth filter damped by a 0.5 coefficients to keep also some low frequencies.

(a) (b)

Figure 5.33: Illustration of homomorphic filtration. (a) Original image. (b) Result of homomorphic filtration. *Courtesy of Tomas Svoboda, Czech Technical University, Prague.*

Overview of DTFT properties

We have already discussed (and made use of)

- Periodicity $X(e^{j\omega}) = X(e^{j\omega+2\pi})$
- Linearity

We can group the other properties into meaningful categories:

- 1. properties related to signal symmetry
- 2. properties related to transformations of the independent variable (time domain and frequency domain)
- 3. properties related to time and frequency differentiation
- 4. properties related to convolution
- 5. property related to the energy of the signal (Parseval)

- Our focus will be on understanding the properties and on knowing how to use them, rather than on their mathematical proof

Table 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

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Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

Table 2.2 FOURIER TRANSFORM THEOREMS

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Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

Table 2.3 FOURIER TRANSFORM PAIRS

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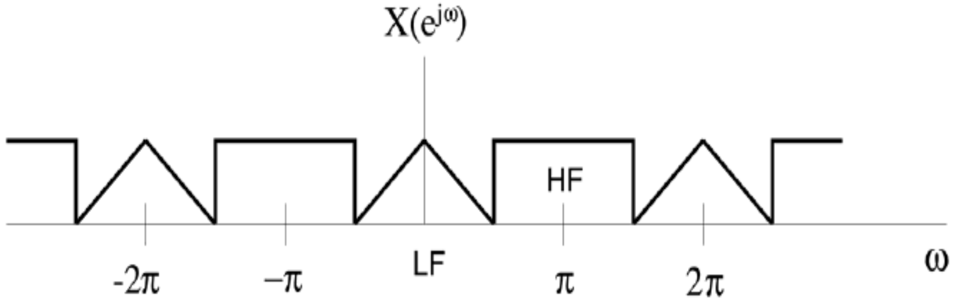
Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\dfrac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{(1-ae^{-j\omega})^2}$
7. $\dfrac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\dfrac{1}{1-2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\dfrac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

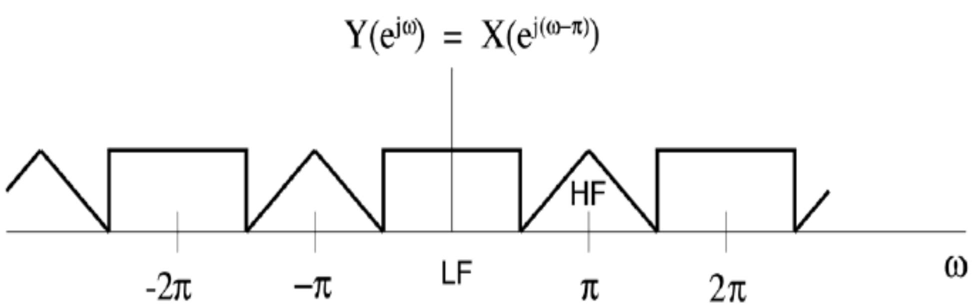
Frequency shifting: discussion

Frequency shifting has important implications because of DTFT periodicity

$$e^{j\omega_0 n}x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$$



$$\omega_0 = \pi\,,\; y[n] = e^{j\pi n}x[n] = (-1)^n x[n]$$



Example 1 (using symmetry properties)

- The following facts are known about a signal x[n] :

$$\text{x[n] is real}$$

$$\text{x[n]} = 0 \text{ for } \text{n} > 0$$

$$\text{x[0]} > 0$$

$$\text{Im}\left\{X\left(e^{j\omega}\right)\right\} = \sin\omega - \sin 2\omega$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X(e^{j\omega})\right|^2 = 3$$

Determine x[n]

Example 2 (using the frequency differentiation property)

- Consider the DFT of x[n]=a^{n}u[n], where 0<a<1
- Show that

$$(n+1)a^n \xleftrightarrow{DTFT} X(e^{j\omega}) = \frac{1}{\left(1 - ae^{-j\omega}\right)^2}$$

Example 3 (using the convolution property)

Consider a discrete LTI system with

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

Determine its response to

$$x[n] = \left(\frac{3}{4}\right)^n u[n]$$