The Z transform (2)

Today

- Properties of the region of convergence (3.2)
- Read examples 3.7, 3.8

Announcements:

- ELEC 310 FINAL EXAM:

April 14 2010, 14:00 pm ECS 123

- Assignment 2 due tomorrow by 4:00 pm

Observations

- Specification of the Z transform requires both algebraic expression and region of convergence

- Rational Z-transforms are obtained if x[n]=linear combination of exponentials for $n>0$ and $n<0$

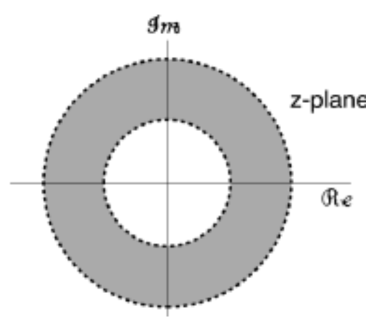
$$X(z) = \frac{N(z)}{D(z)}$$

Polynomials in z

- Rational Z-transforms are completely characterized by their poles and zeros (except for the gain)

Properties of the ROC of the Z-transform

1. The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin
   - Convergence is dependent only on r, not on ω
   - In some cases, the inner boundary can extend inward to the origin (ROC=disc)
   - In other cases, the outer boundary can extend outward to infinity (ROC= the exterior of a circle)



Properties of the ROC of the Z-transform

- 2. The FT of x[n] is convergent if and only if the ROC of the ZT of x[n] contains the unit circle

- Proof:

Properties of the ROC of the Z-transform

- **3. The ROC does not contain any poles**

  Proof: At a pole X(z) is infinite and therefore does not converge.

Properties 4, 5, 6, and 7 are consequences of 1 and 3.

Properties of the ROC of the Z-transform

4. If x[n] is of finite duration, then the ROC is the entire z-plane, except possibly z=0 and/or z=∞.

Proof:

As $|z| \to 0$ terms involving negative powers of z become unbounded.

As $|z| \to \infty$ terms involving positive powers of z become unbounded.

$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}$$

We need to explore three cases:

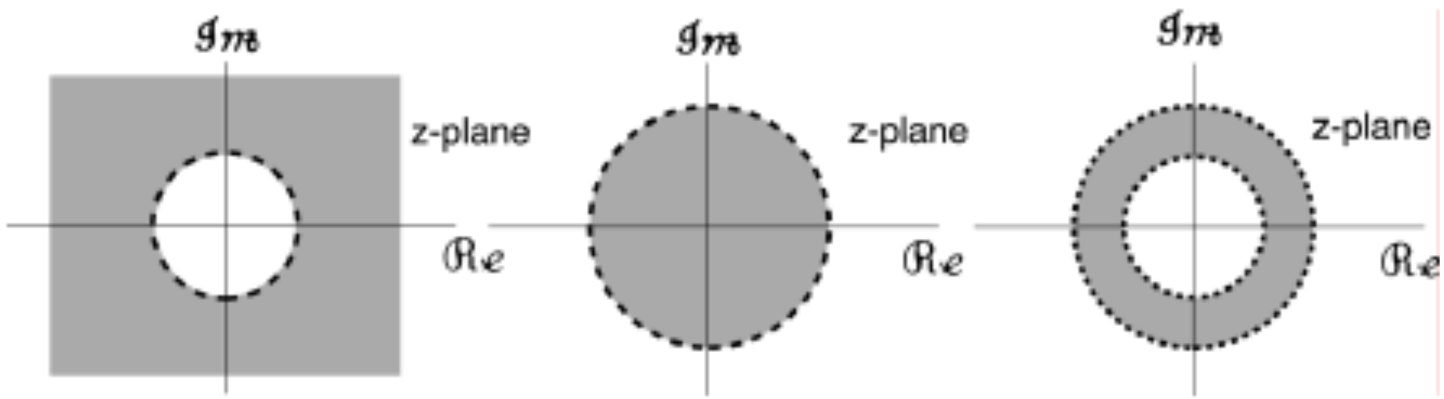
$N_1 \geq 0$ summation includes only terms with negative powers. ROC includes z=∞

$N_2 \leq 0$ summation includes only terms with positive powers. ROC includes z=0

N1<0, N2>0 summation includes terms with both positive and negative powers of z. z=0 and z=∞ are not in the ROC.

Properties of the ROC of the Z-transform

- 5. If x[n] is a right-sided sequence, the ROC extends outward from the outermost finite pole in X(z), possibly including z=∞
- 6. If x[n] is a left-sided sequence, the ROC extends inward from the innermost finite pole in X(z), possibly including z=0
- 7. If x[n] is two-sided, the ROC is a ring in the z-plane bounded on the interior and exterior by poles.



right-sided left-sided two-sided

Properties of the ROC of the Z-transform

- **8. The ROC must be a connected region.**
  - This is a direct consequence of property 1 (which specifies three possible shapes of the ROC).
  - It is useful for evaluating the existence of ZT when x[n] is a sum of two sequences, one left-sided and the other right-sided
  - See example 3.7

Stability, causality, and the ROC

- **We can evaluate the stability and causality of LTI systems in the Z-domain.**
- Suppose our LTI system is given by $h[n]$, by $H(e^{j\omega})$ in the frequency domain, and by $H(z)$ in the z-domain
  - The system is causal if $h[n]=0$ for $n<0$ (right-sided)

    The ROC of a causal system is the exterior of a circle (property 5), and it contains $z=\infty$
  - The system is anti-causal if $h[n]=0$ for $n>0$ (left-sided)

    The ROC of an anti-causal system is the interior of a circle (property 6) and it contains $z=0$.

Stability, causality, and the ROC (cont'd)

- The LTI system is stable if and only if h[n] is absolutely summable (which is equivalent to the fact that $H(e^{j\omega})$ exists)
- Using Property 2 of ROC, we conclude that:

The ROC of a stable system include the unit circle ($|z|=1$)

See example 3.8

Example 1.

- Let $x[n]$ be an absolutely summable signal with rational Z-transform $X(z)$. If $X(z)$ is known to have a pole at $z=1/2$, could $x[n]$ be:
- a) a finite-duration signal?
- b) a left-sided signal?
- c) a right-sided signal?
- d) a two-sided signal?

Example 2

- Find the Z-Transform, the ROC and the Fourier Transform (if it exists) of the following signals:

$$a)\ x[n] = 3\delta[n-1] - \delta[n-2] + 2\delta[n-3]$$

$$b)\ x[n] = \begin{cases} \left(\dfrac{1}{3}\right)^n \cos\left(\dfrac{\pi}{4}n\right) & n \le 0 \\ 0 & n > 0 \end{cases}$$

Summary

- The properties of the ROC depend on the nature of the signal. We assume the the ZT is a rational function.
- There are 8 main properties, stated on p. 111.

- Evaluation of causality and stability in the Z-domain

causal LTI: $H(z)$ has the ROC represented by the exterior of a circle and including $z=\infty$

anti-causal LTI: $H(z)$ has the ROC represented by the interior of a circle and including $z=0$

stable LTI: the ROC of $H(z)$ includes the unit circle.