2.2. For an LTI system, the output is obtained from the convolution of the input with the impulse response of the system:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

(a) Since $h[k] \neq 0$, for $(N_0 \leq n \leq N_1)$,

$$y[n] = \sum_{k=N_0}^{N_1} h[k]x[n-k]$$

The input, $x[n] \neq 0$, for $(N_2 \leq n \leq N_3)$, so

$$x[n-k] \neq 0, \text{ for } N_2 \leq (n-k) \leq N_3$$

Note that the minimum value of $(n-k)$ is N_2. Thus, the lower bound on n, which occurs for $k = N_0$ is

$$N_4 = N_0 + N_2.$$

Using a similar argument,

$$N_5 = N_1 + N_3.$$

Therefore, the output is nonzero for

$$(N_0 + N_2) \leq n \leq (N_1 + N_3).$$

(b) If $x[n] \neq 0$, for some $n_o \leq n \leq (n_o + N - 1)$, and $h[n] \neq 0$, for some $n_1 \leq n \leq (n_1 + M - 1)$, the results of part (a) imply that the output is nonzero for:

$$(n_o + n_1) \leq n \leq (n_o + n_1 + M + N - 2)$$

So the output sequence is $M + N - 1$ samples long. This is an important quality of the convolution for finite length sequences as we shall see in Chapter 8.

2.7. $x[n]$ is periodic with period N if $x[n] = x[n+N]$ for some integer N.

(a) $x[n]$ is periodic with period 12:

$$e^{j(\frac{\pi}{6}n)} = e^{j(\frac{\pi}{6})(n+N)} = e^{j(\frac{\pi}{6}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{\pi}{6}N, \text{for integers } k, N$$

Making $k = 1$ and $N = 12$ shows that $x[n]$ has period 12.

(b) $x[n]$ is periodic with period 8:

$$e^{j(\frac{3\pi}{4}n)} = e^{j(\frac{3\pi}{4})(n+N)} = e^{j(\frac{3\pi}{4}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{3\pi}{4}N, \text{for integers } k, N$$

$$\Longrightarrow N = \frac{8}{3}k, \text{for integers } k, N$$

The smallest k for which both k and N are integers are is 3, resulting in the period N being 8.

(c) $x[n] = [\sin(\pi n/5)]/(\pi n)$ is not periodic because the denominator term is linear in n.

(d) We will show that $x[n]$ is not periodic. Suppose that $x[n]$ is periodic for some period N:

$$e^{j(\frac{\pi}{\sqrt{2}}n)} = e^{j(\frac{\pi}{\sqrt{2}})(n+N)} = e^{j(\frac{\pi}{\sqrt{2}}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{\pi}{\sqrt{2}}N, \text{for integers } k, N$$

$$\Longrightarrow N = 2\sqrt{2}k, \text{for some integers } k, N$$

There is no integer k for which N is an integer. Hence $x[n]$ is not periodic.

2.12. The difference equation:

$$y[n] = ny[n-1] + x[n]$$

Since the system is causal and satisfies initial-rest conditions, we may recursively find the response to any input.

(a) Suppose $x[n] = \delta[n]$:

$$\begin{aligned} y[n] &= 0, \text{ for } n < 0 \\ y[0] &= 1 \\ y[1] &= 1 \\ y[2] &= 2 \\ y[3] &= 6 \\ y[4] &= 24 \end{aligned}$$

$$y[n] = h[n] = n!u[n]$$

(b) To determine if the system is linear, consider the input:

$$x[n] = a\delta[n] + b\delta[n]$$

performing the recursion,

$$\begin{aligned} y[n] &= 0, \text{ for } n < 0 \\ y[0] &= a + b \\ y[1] &= a + b \\ y[2] &= 2(a + b) \\ y[3] &= 6(a + b) \\ y[4] &= 24(a + b) \end{aligned}$$

Because the output of the superposition of two input signals is equivalent to the superposition of the individual outputs, the system is LINEAR.

(c) To determine if the system is time-invariant, consider the input:

$$x[n] = \delta[n-1]$$

the recursion yields

$$\begin{aligned} y[n] &= 0, \text{ for } n < 0 \\ y[0] &= 0 \\ y[1] &= 1 \\ y[2] &= 2 \\ y[3] &= 6 \\ y[4] &= 24 \end{aligned}$$

Using $h[n]$ from part (a),

$$h[n-1] = (n-1)!u[n-1] \neq y[n]|_{x[n]=\delta[n-1]}$$

Conclude: NOT TIME INVARIANT.

2.18. $h[n]$ is causal if $h[n] = 0$ for $n < 0$. Hence, (a) and (b) are causal, while (c), (d), and (e) are not.

2.19. $h[n]$ is stable if it is absolutely summable.

(a) Not stable because $h[n]$ goes to ∞ as n goes to ∞.

(b) Stable, because $h[n]$ is non-zero only for $0 \leq n \leq 9$.

(c) Stable.

$$\sum_n |h[n]| = \sum_{n=-\infty}^{-1} 3^n = \sum_{n=1}^{\infty} (1/3)^n = 1/2 < \infty$$

(d) Not stable. Notice that

$$\sum_{n=0}^{5} |\sin(\pi n/3)| = 2\sqrt{3}$$

and summing $|h[n]|$ over all positive n therefore grows to ∞.

(e) Stable. Notice that $|h[n]|$ is upperbounded by $(3/4)^{|n|}$, which is absolutely summable.

(f) Stable.

$$h[n] = \begin{cases} 2, & , -5 \leq n \leq -1 \\ 1, & , 0 \leq n \leq 4 \\ 0, & , \text{otherwise} \end{cases}$$

So $\sum |h[n]| = 15$.

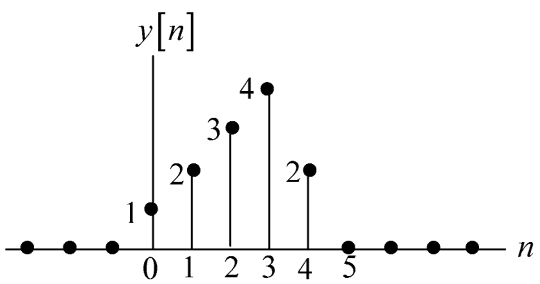
2.29.

A.

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} u[k]h[n-k] \\ &= \sum_{k=0}^{\infty} h[n-k]. \end{aligned}$$

Therefore

$$\begin{aligned} y[0] &= h[0] = 1 \\ y[1] &= h[1] + h[0] = 2 \\ y[2] &= h[2] + h[1] + h[0] = 3 \\ y[3] &= \sum_{k=0}^{\infty} h[3-k] = 4 \\ y[4] &= 2 \\ y[5] &= 0 \\ y[6] &= 0. \end{aligned}$$



B. Since the system is LTI, shifting the input in time shifts the output in time. Thus if

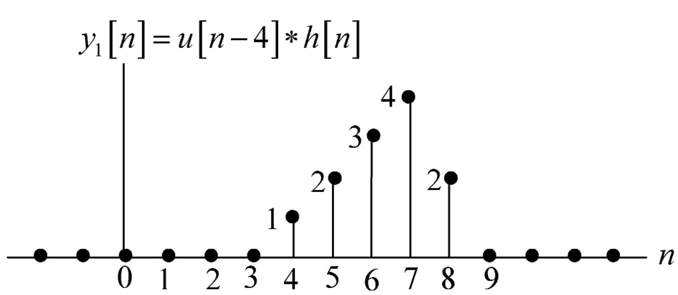
$$y[n] = x[n] * h[n]$$

and

$$y_1[n] = x[n-4] * h[n],$$

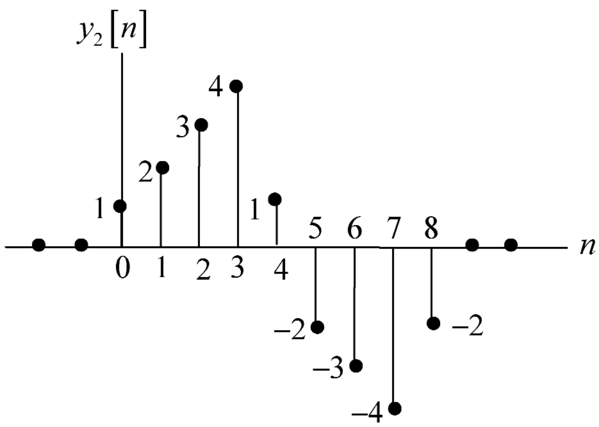
then

$$y_1[n] = y[n-4].$$



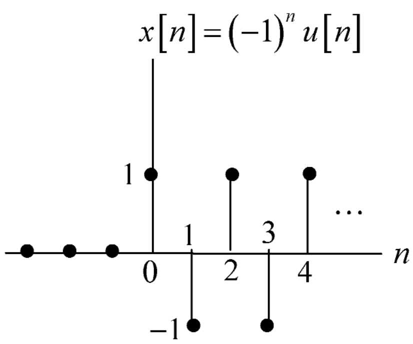
C. Using the principle of linearity, convolution is an associative operation. Then

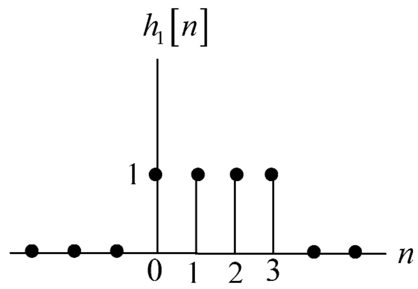
$$\begin{aligned} y_2[n] &= (u[n]-u[n-4]) * h[n] \\ &= u[n]*h[n]-u[n-4]*h[n] \\ &= y[n]-y_1[n]. \end{aligned}$$



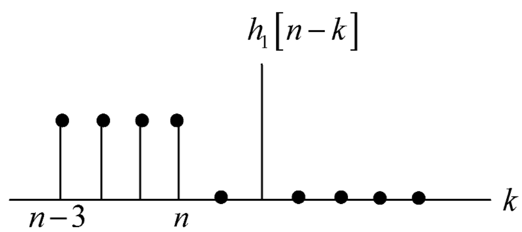
2.30. A. $w[n] = x[n] * h_1[n] = \sum_{k=-\infty}^{\infty} x[k] h_1[n-k]$

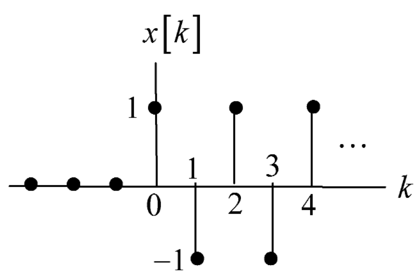
Graphical approach:





Flip and slide $h_1[n]$:





Case 1: When $n < 0$, no overlap so $w[n] = 0$. (Note: causal sequence.)

Case 2: Partial overlap, $n \geq 0$ and $n - 3 < 0$ or $n < 3$

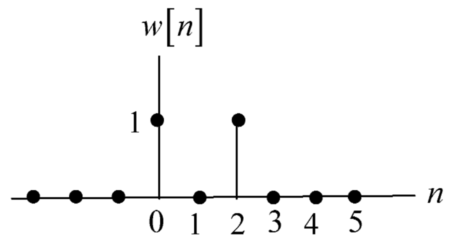
when $n = 0$, $w[0] = 1$

when $n = 1$, $w[1] = 1 - 1 = 0$

when $n = 2$, $w[2] = 1 - 1 + 1 = 1$

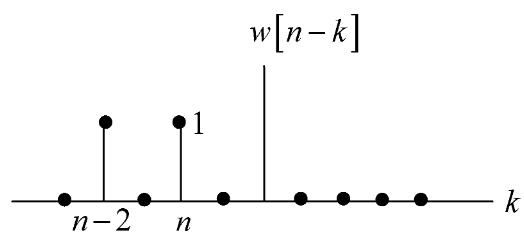
Case 3: Complete overlap, $n \geq 3$. $w[n] = 1 - 1 + 1 - 1 = 0$

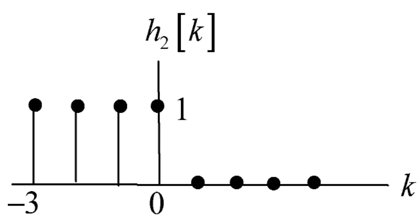
Plotting $w[n]$,



The overall output $y[n] = w[n] * h_2[n] = \sum_{k=-\infty}^{\infty} h_2[k]w[n-k]$

Flip and slide $w[n]$:





Case 1: No overlap, $n < -3$, so $y[n] = 0$

Case 2: Partial overlap, $n \geq -3$ and $n-2 < -3$ or $n < -1$

$y[-3] = 1$

$y[-2] = 0 + 1 = 1$

Case 3: Complete overlap, $n \leq 0$ and $n-2 \geq -3$ or $n \geq -1$

$y[-1] = 1 + 0 + 1 = 2$

$y[0] = 2$

Case 4: Partial overlap, $n > 0$ and $n-2 \leq 0$ or $n \leq 2$

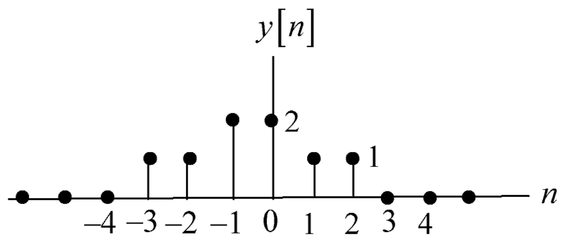
$y[1] = 1$

$y[2] = 1$

Case 5: No overlap, $n-2 > 0$ or $n > 2$

$y[n] = 0$

Plotting $y[n]$,

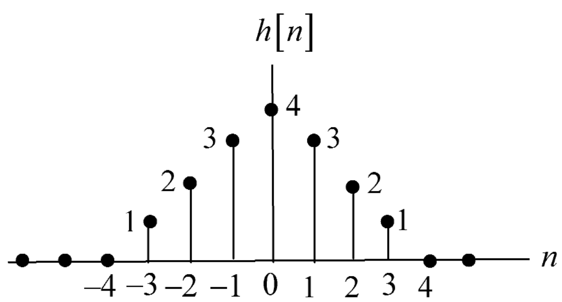


Notice, $\text{length}(y) = \text{length}(w) + \text{length}(h_2) - 1$.

B. The overall response of the cascaded system is given by

$$h[n] = h_1[n] * h_2[n]$$
$$= \sum_{k=-\infty}^{\infty} h_2[k] h_1[n-k]$$

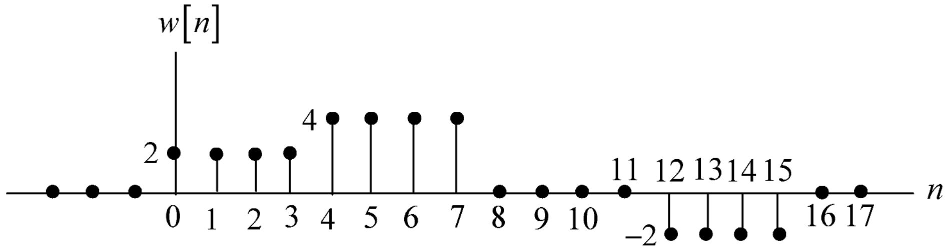
Following identical steps as in A, the convolution result is given by the following:



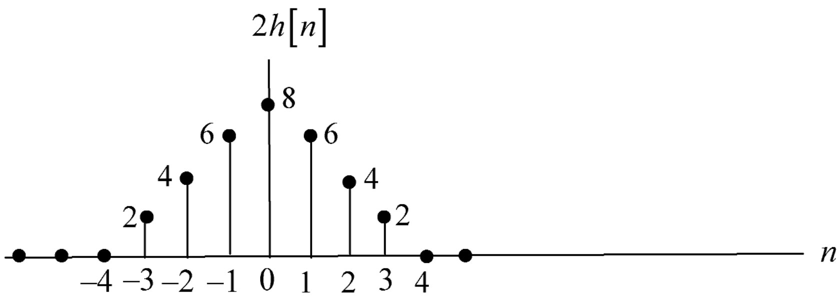
Note the following:

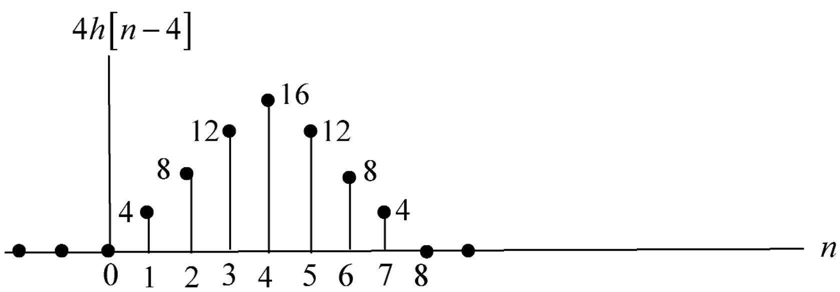
1. $\text{length}(h) = \text{length}(h_1) + \text{length}(h_2) - 1$
2. $h_1[n]$ is zero except $0 \le n \le 3$
   $h_2[n]$ is zero except $-3 \le n \le 0$
   $h[n]$ is zero except $-3 \le n \le 3$ (that is, $0 + (-3) \le n \le 3 + 0$
3. The convolution of two boxes is a triangle.

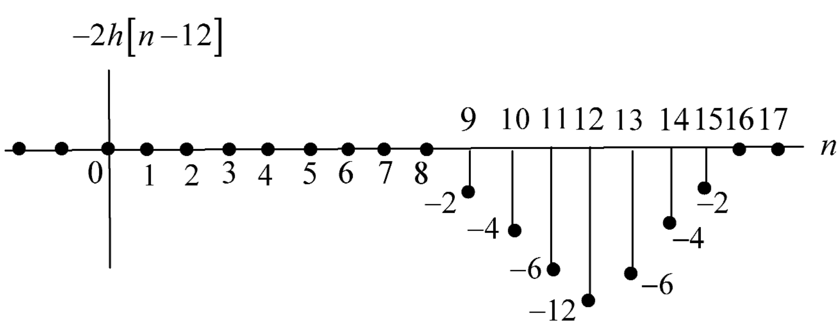
C.
$$w[n] = x[n] * h_1[n]$$
$$= 2\delta[n] * h_1[n] + 4\delta[n-4] * h_1[n] - 2\delta[n-12] * h_1[n]$$
$$= 2h_1[n] + 4h_1[n-4] - 2h_1[n-12]$$

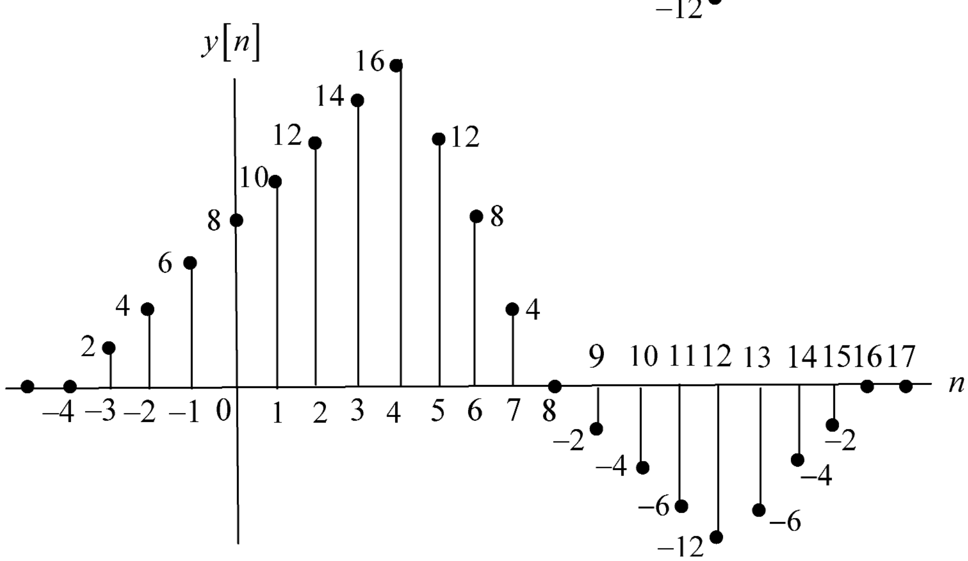


D. Similarly, $y[n] = 2h[n] + 4h[n-4] - 2h[n-12]$









2.31. A. LTI systems are stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ (the summation should converge).

Then

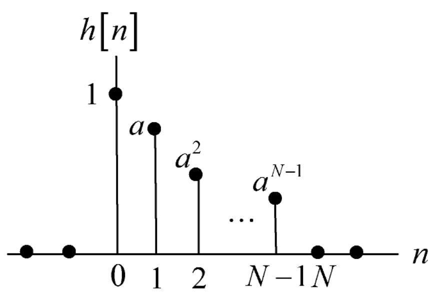
$$S = \sum_{n=-\infty}^{\infty} |a|^n u[n]$$

$$= \sum_{n=0}^{\infty} |a|^n$$

S will converge only when $|a| < 1$ and $S = \frac{1}{1-|a|} < \infty$.

Therefore the system is stable for $|a| < 1$.

B. $y[n] = ay[n-1] + x[n] - a^N x[n-N]$. Therefore,
$h[n] = ah[n-1] + \delta[n] - a^N \delta[n-N]$.
Since the system is causal, $h[-1] = 0$. Then
$h[0] = 0 + 1 - 0 = 1$
$h[1] = a$, $h[2] = a^2$, $h[N] = a^N - a^N = 0$
$h[N+1] = a \times 0 + 0 - 0 = 0$.



C. We see that even though it is a recursive system (with feedback), its impulse response is finite in length. The length of $h[n]$ is N terms. Hence this system is FIR.

D. FIR systems are <u>always</u> stable as the sum $\sum_{n=-\infty}^{\infty} |h[n]|$ has at most a finite number of nonzero terms.