ASSIGNMENT 4

Question 1

Using the convolution property of the Fourier Transform we obtain

$$Y(j\omega) = X_1(j\omega) \cdot X_2(j\omega) \cdot X_3(j\omega)$$

Therefore $Y(j\omega) = 0$ for $|\omega| > 500\pi$

This implies that the Nyquist rate for $y(t)$ is $2 \times 500\pi = 1000\pi$

The sampling period T must be less than

$$\frac{2\pi}{1000\pi} = 10^{-3} \text{ seconds.}$$

ASSIGNMENT 4 - Question 2

We need to first find the frequency response of the DT filter.

$$y[n] = \frac{3}{4} y[n-2] + x[n] + \frac{1}{4} x[n-1]$$

$$Y(e^{j\omega}) = \frac{3}{4} e^{-j2\omega} Y(e^{j\omega}) + X(e^{j\omega}) + \frac{1}{4} e^{-j\omega} X(e^{j\omega})$$

$$Y(e^{j\omega})\left(1 - \frac{3}{4} e^{-j2\omega}\right) = X(e^{j\omega})\left(1 + \frac{1}{4} e^{-j\omega}\right)$$

$$H_d(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + \frac{1}{4} e^{-j\omega}}{1 - \frac{3}{4} e^{-j2\omega}}$$

It is given that $X(j\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$ and we have a sampling frequency $\Omega_s = \frac{2\pi}{T}$. Thus there will be no aliasing.

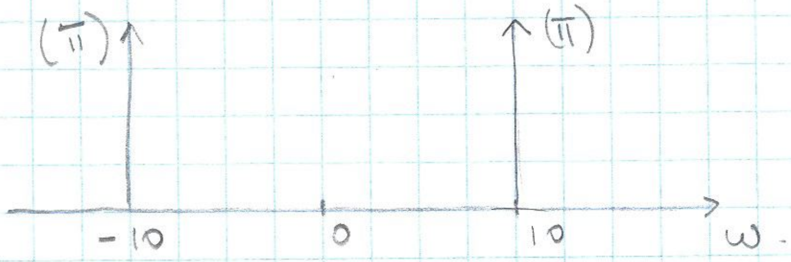
Since we are in a "no aliasing" case, the frequency response of the entire CT system, $H_c(j\Omega)$ is related to the frequency of the DT system $H_d(e^{j\omega})$ $(\omega = \Omega T)$ by:

$$H_c(j\Omega) = \begin{cases} H_d(e^{j\Omega T}), & |\Omega| \leq \frac{\pi}{T} = \frac{\Omega_s}{2} \\ 0 & |\Omega| > \frac{\pi}{T} \end{cases}$$

$$H_c(j\omega) = \begin{cases} \dfrac{1 + \frac{1}{4} e^{-j\Omega T}}{1 - \frac{3}{4} e^{-j2\Omega T}} & \text{if } |\Omega| \leq \frac{\pi}{T} = \frac{\Omega_s}{2} \\ 0 & \text{if } |\Omega| > \frac{\pi}{T} \end{cases}$$

Assignment 4 Question 3.

$x(t) = \cos(10t)$, the Fourier Transform of $x(t)$ is



The sampling function can be expressed as:

$$s(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT) - \sum_{k=-\infty}^{+\infty} \delta\left(t - kT - \frac{T}{2}\right)$$

Taking the Fourier Transform

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right) - \frac{2\pi}{T} e^{-j\omega\frac{T}{2}} \sum_{k=-\infty}^{+\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right)$$

$$= \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right) - \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} e^{-jk\frac{2\pi}{T}\cdot\frac{T}{2}} \delta\left(\Omega - k\frac{2\pi}{T}\right)$$

$$= \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right) - \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \left(e^{-j\pi}\right)^k \delta\left(\Omega - k\frac{2\pi}{T}\right)$$

$$= \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right) - \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} (-1)^k \delta\left(\Omega - k\frac{2\pi}{T}\right)$$

Separating the odd and even terms of k

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=\text{even}} \delta\left(\Omega - k\frac{2\pi}{T}\right) - \frac{2\pi}{T} \sum_{k=\text{even}} \delta\left(\Omega - k\frac{2\pi}{T}\right) +$$

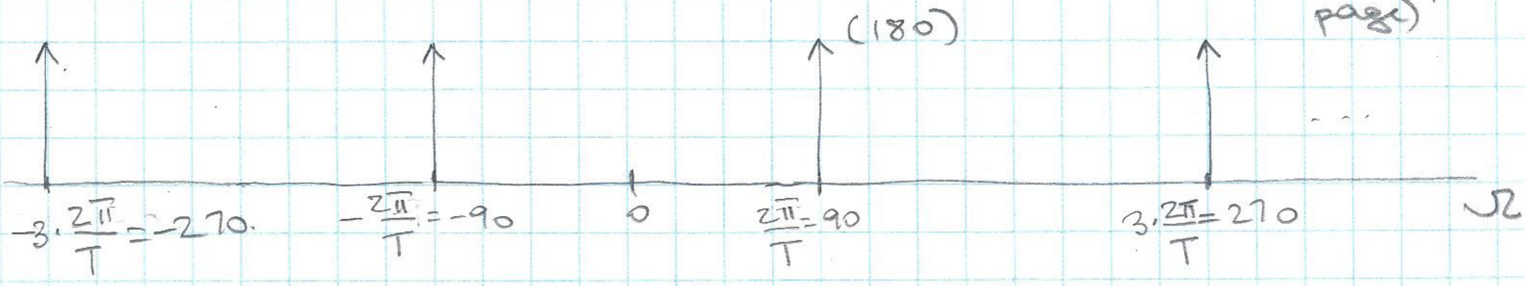
$$+ \frac{2\pi}{T} \sum_{k=\text{odd}} \delta\left(\Omega - k\frac{2\pi}{T}\right) + \frac{2\pi}{T} \sum_{k=\text{odd}} \delta\left(\Omega - k\frac{2\pi}{T}\right) =$$

$$= \frac{4\pi}{T} \sum_{k=\text{odd}} \delta\left(\Omega - k\frac{2\pi}{T}\right)$$

Assignment 4 Question 3 cont'd.

To find $Z(j\Omega)$ we need to convolve $X(j\Omega)$ with the impulse train in $S(j\Omega)$ and scale the result by $\frac{1}{2\pi}$

$S(j\Omega)$ is as sketched below. (according to the formula at the bottom of the previous page)



The convolution will place two scaled impulses (from $X(j\Omega)$) centered at each impulse in the impulse train of $S(j\Omega)$. Finally, the filter $H(j\Omega)$ will only pass impulses that exist between 90 to 180 and −180 to −90 radians. We plot $|Y(j\Omega)|$ (output from $H(j\Omega)$) as follows. (see next page)

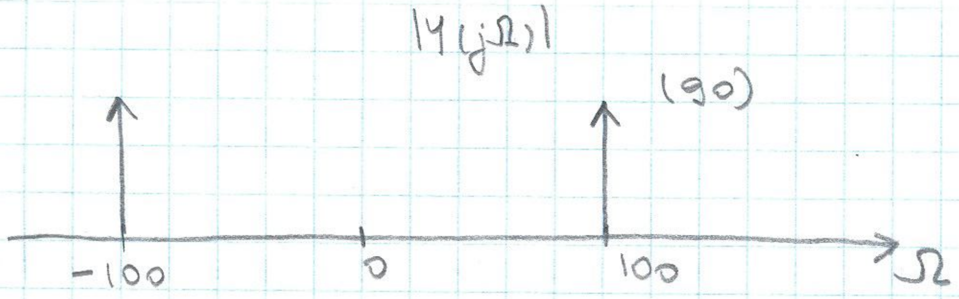
We will compute the magnitude and phase of $Y(j\Omega)$ separately.

$$|Y(j\Omega)| = |X(j\Omega)|\cdot|H(j\Omega)| = X(j\Omega)\cdot|H(j\Omega)|$$

$$\angle Y(j\Omega) = \angle X(j\Omega) + \angle H(j\Omega) = \angle H(j\Omega) = -\frac{\pi\Omega}{200}$$

We will first determine the signal $y_1(t)$ whose Fourier Transform is $|Y(j\Omega)|$.

Assignment 4 Question 3 cont'd



$$y_1(t) = \frac{90}{\pi} \cos(100t)$$

The signal $y(t)$ that corresponds to

$$|Y(j\Omega)| e^{\angle Y(j\Omega)} = |Y(j\Omega)| e^{-j\frac{\pi}{200}}$$

will be a time-shifted version of $y_1(t)$

$$y(t) = y_1\left(t - \frac{\pi}{200}\right) = \frac{90}{\pi} \cos\left(100\left(t - \frac{\pi}{200}\right)\right)$$

$$y(t) = \frac{90}{\pi} \cos\left(100t - \frac{\pi}{2}\right)$$

$$y(t) = \frac{90}{\pi} \sin(100t)$$

Assignment 4 Question 4.

The signal $y[n] = x[n] \cdot \sum_{k=-\infty}^{+\infty} \delta[n-3k]$

is the sampled version of $x[n]$. The sampling period is $N=3$.

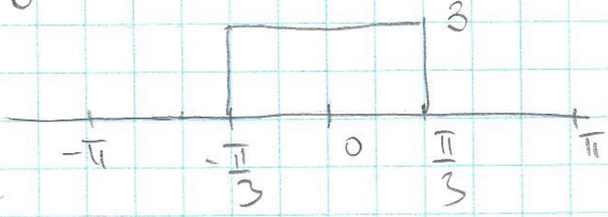
The Fourier Transform of $y[n]$ is

$$Y(e^{j\omega}) = \frac{1}{3} \sum_{k=0}^{2} X\left(e^{j(\omega - \frac{2\pi}{3}k)}\right)$$

The convolution of $y[n]$ with $h[n] = \dfrac{\sin \frac{\pi}{3} n}{\frac{\pi}{3} n}$

represents a filtering operation.

$$\frac{\sin \frac{\pi}{3} n}{\frac{\pi}{3} n} \overset{FT}{\longleftrightarrow}$$



Ideal low-pass filter with cut-off frequency $\frac{\pi}{3}$ and passband gain of 3

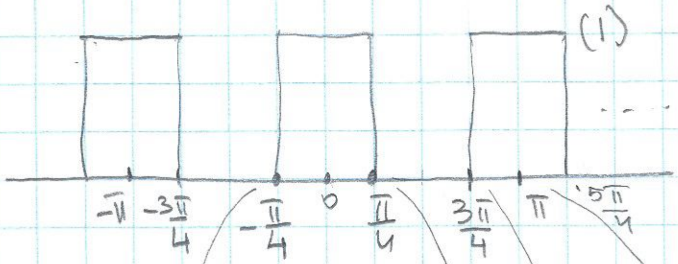
In order to be able to recover $x[n]$ using this lowpass filter, $H(e^{j\omega})$ must completely contain the spectrum of $X(e^{j\omega})$ in $[-\pi, \pi]$

Therefore $X(e^{j\omega}) = 0$ for $\frac{\pi}{3} < |\omega| < \pi$

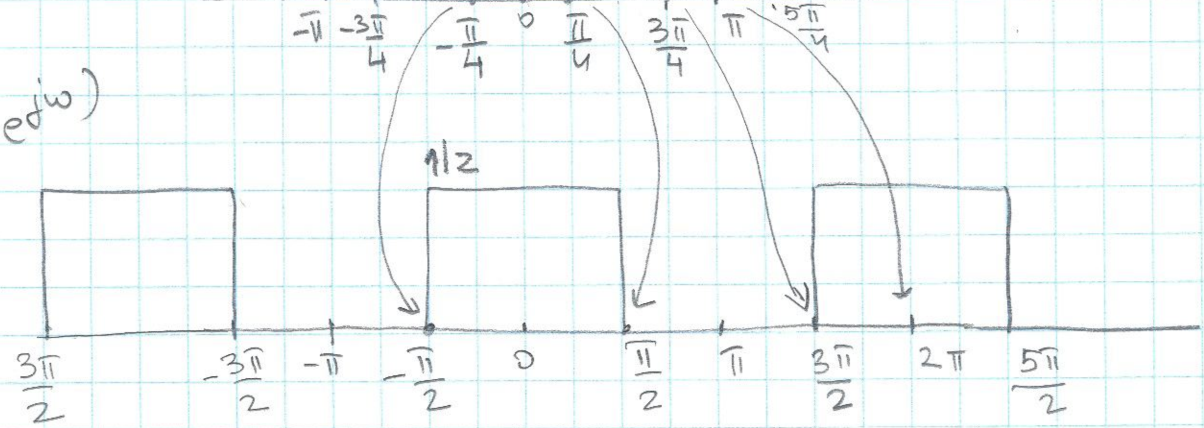
Assignment 4 question 5.

Decimation in time by a factor of 2 is equivalent to expansion in frequency by the same factor.

$H(e^{j\omega})$



$H_1(e^{j\omega})$



$$H_1(e^{j\omega}) = \begin{cases} 1/2 & \text{for } |\omega| < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

Through decimation in time, we have transformed a band-pass filter into a low-pass filter.