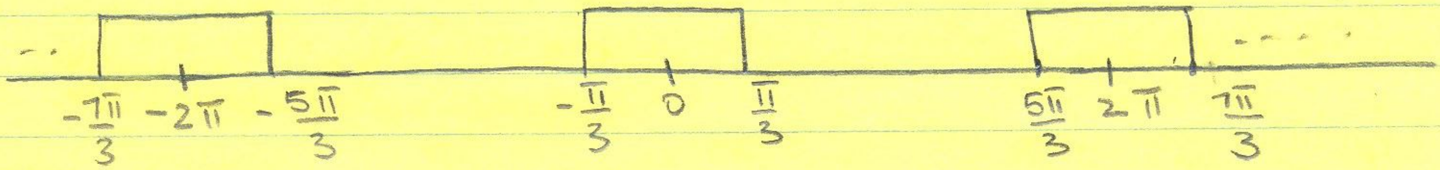
ELEC 310 Midterm Part A Spring 2010.

Question 1. $h[n] = \dfrac{\sin \frac{\pi n}{3}}{\pi n} \xleftrightarrow{\text{DTFT}} H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| < \frac{\pi}{3} \\ 0 & \frac{\pi}{3} < |\omega| < \pi \end{cases}$

ⓐ This system is a Low Pass Filter with cut-off frequency $\omega_c = \frac{\pi}{3}$



ⓑ We can solve this question in more than one way.

① time-domain: we use the properties of δ with regards to convolution.

$$y[n] = x[n] * h[n] = \left(\delta[n+1] - \delta[n-1]\right) * \frac{\sin \frac{\pi n}{3}}{\pi n}$$

$$y[n] = \frac{\sin \frac{\pi}{3}(n+1)}{\pi (n+1)} - \frac{\sin \frac{\pi}{3}(n-1)}{\pi (n-1)}$$

② in frequency domain.

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega}) = e^{j\omega} H(e^{j\omega}) - e^{-j\omega} H(e^{j\omega})$$

time-shifting

$$y[n] = h[n+1] - h[n-1]$$

$$y[n] = \frac{\sin \frac{\pi}{3}(n+1)}{\pi (n+1)} - \frac{\sin \frac{\pi}{3}(n-1)}{\pi (n-1)}$$

Question 1c.

Any signal with a spectrum contained in $\left(\frac{\pi}{3}, \pi\right]$ will be rejected. For instance:

$$x[n] = \sin\left(\frac{\pi}{2} n\right).$$

Question 2.

$$1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega} = \left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 + \frac{1}{4}e^{-j\omega}\right).$$

We perform partial fraction expension.

$$X(e^{j\omega}) = \frac{A}{1 - \frac{1}{2}e^{-j\omega}} + \frac{B}{1 + \frac{1}{4}e^{-j\omega}} = \frac{1 - \frac{1}{3}e^{-j\omega}}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 + \frac{1}{4}e^{-j\omega}\right)}$$

$$A\left(1 + \frac{1}{4}e^{-j\omega}\right) + B\left(1 - \frac{1}{2}e^{-j\omega}\right) = 1 - \frac{1}{3}e^{-j\omega}.$$

$$1 - \frac{1}{2}e^{-j\omega} = 0 \quad \left(e^{-j\omega} = 2\right) \Rightarrow A\left(1 + \frac{2}{4}\right) = 1 - \frac{2}{3}$$

$$A \cdot \frac{3}{2} = \frac{1}{3}. \qquad \boxed{A = \frac{2}{9}}$$

$$1 + \frac{1}{4}e^{-j\omega} = 0 \quad \left(e^{-j\omega} = -4\right) \Rightarrow B\left(1 + 2\right) = 1 + \frac{4}{3}.$$

$$3B = \frac{7}{3} \qquad \boxed{B = \frac{7}{9}}$$

$$X(e^{j\omega}) = \frac{2/9}{1 - \frac{1}{2}e^{-j\omega}} + \frac{7/9}{1 + \frac{1}{4}e^{-j\omega}}$$

Using Fourier Transform pairs: $x[n] = \frac{2}{9}\left(\frac{1}{2}\right)^n u[n] + \frac{7}{9}\left(-\frac{1}{4}\right)^n u[n]$

ELEC 310.

MIDTERM PART B 2010.

Question 1.

$$H(z) = \frac{3}{1+\frac{1}{3}z^{-1}}$$

(a) $H(z)$ has 1 pole $\boxed{z_1 = -\frac{1}{3}}$

For the system to be stable, the ROC needs to contain the unit circle. ROC is thus defined as.

$$\left\{ z \mid |z| > \frac{1}{3} \right\}.$$

(b) $$X(z) = \frac{1}{1-z^{-1}} \qquad |z| > 1.$$

$$Y(z) = H(z) \cdot X(z) = \frac{1}{1-z^{-1}} \cdot \frac{3}{1+\frac{1}{3}z^{-1}}$$

We compute the inverse transform of $Y(z)$ via partial fraction expansion.

First we need to determine the ROC of $Y(z)$.

$$ROC_Y \underset{\uparrow}{=} ROC_X \cap ROC_H.$$

because there is no pole-zero cancellation.

$ROC_Y : |z| > 1$

$$Y(z) = \frac{A}{1-z^{-1}} + \frac{B}{1+\frac{1}{3}z^{-1}} = \frac{3}{1+\frac{1}{3}z^{-1}}$$

$$A\left(1+\frac{1}{3}z^{-1}\right) + B\left(1-z^{-1}\right) = 3 \qquad z^{-1} = -3 \Rightarrow 4B = 3 \quad \boxed{B = \frac{3}{4}}$$

$$z^{-1} = 1 \Rightarrow \frac{4}{3}A = 3 \quad \boxed{A = \frac{9}{4}}$$

$$y[n] = \frac{3}{4} \cdot \left(-\frac{1}{3}\right)^n u[n] + \frac{9}{4} u[n].$$

Question 2

a) We take the Z transform in both members of the identity.

$$Y(z) - 0.25 z^{-2} Y(z) = X(z) - z^{-1} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - 0.25 z^{-2}} = \frac{1 - z^{-1}}{(1 - 0.5 z^{-1})(1 + 0.5 z^{-1})}$$

$H(z)$ has two poles. $z^{-1} = \frac{1}{0.5} \rightarrow z_1 = 0.5$

$z^{-1} = -\frac{1}{0.5} \rightarrow z_2 = -0.5.$

The system is causal $\Rightarrow$ ROC : $|z| > 0.5$.

b). ROC includes the unit circle $\Rightarrow$ the system is stable.

c) $Y(z) = 1 - z^{-1}$.

$$X(z) = \frac{Y(z)}{H(z)} = (1 - 0.5 z^{-1})(1 + 0.5 z^{-1}).$$

$$X(z) = 1 - 0.25 z^{-2} \Rightarrow x[n] = \delta[n] - 0.25\, \delta[n-2].$$