**4.1.**

$$
\begin{aligned}
x[n] &= x_c(nT) \\
&= \sin\left(2\pi(100)n\frac{1}{400}\right) \\
&= \sin\left(\frac{\pi}{2}n\right)
\end{aligned}
$$

**4.2.** The discrete-time sequence

$$x[n] = \cos(\frac{\pi n}{4})$$

results by sampling the continuous-time signal

$$x_c(t) = \cos(\Omega_o t).$$

Since $\omega = \Omega T$ and $T = 1/1000$ seconds, the signal frequency could be:

$$\Omega_o = \frac{\pi}{4} \cdot 1000 = 250\pi$$

or possibly:

$$\Omega_o = (2\pi + \frac{\pi}{4}) \cdot 1000 = 2250\pi.$$

**4.3.** (a) Since $x[n] = x_c(nT)$,

$$\begin{aligned} \frac{\pi n}{3} &= 4000\pi n T \\ T &= \frac{1}{12000} \end{aligned}$$

(b) No. For example, since

$$\cos(\frac{\pi}{3}n) = \cos(\frac{7\pi}{3}n),$$

T can be 7/12000.

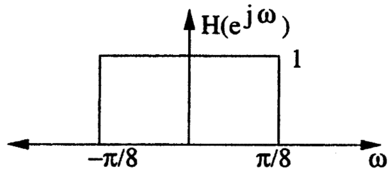
**4.4.** (a) Letting $T = 1/100$ gives

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \sin\left(20\pi n \frac{1}{100}\right) + \cos\left(40\pi n \frac{1}{100}\right) \\ &= \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right) \end{aligned}$$

(b) No, another choice is $T = 11/100$:

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \sin\left(20\pi n \frac{11}{100}\right) + \cos\left(40\pi n \frac{11}{100}\right) \\ &= \sin\left(\frac{11\pi n}{5}\right) + \cos\left(\frac{22\pi n}{5}\right) \\ &= \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right) \end{aligned}$$

**4.5.** A plot of $H(e^{j\omega})$ appears below.




(a)

$$x_c(t) = 0, \quad , |\Omega| \geq 2\pi \cdot 5000$$

The Nyquist rate is 2 times the highest frequency. $\Rightarrow T = \frac{1}{10{,}000}$ sec. This avoids all aliasing in the C/D converter.

(b)

$$\begin{aligned} \frac{1}{T} &= 10kHz \\ \omega &= T\Omega \\ \frac{\pi}{8} &= \frac{1}{10,000}\Omega_c \\ \Omega_c &= 2\pi \cdot 625\text{rad/sec} \\ f_c &= 625Hz \end{aligned}$$

(c)

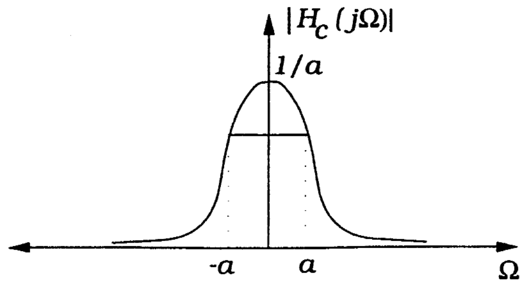
$$\begin{aligned} \frac{1}{T} &= 20kHz \\ \omega &= T\Omega \\ \frac{\pi}{8} &= \frac{1}{20,000}\Omega_c \\ \Omega_c &= 2\pi \cdot 1250\text{rad/sec} \\ f_c &= 1250Hz \end{aligned}$$

**4.6.** (a) The Fourier transform of the filter impulse response

$$\begin{aligned} H_c(j\Omega) &= \int_{-\infty}^{\infty} h_c(t)e^{-j\Omega t}\,dt \\ &= \int_{0}^{\infty} a^{-at}e^{-j\Omega t}\,dt \\ &= \frac{1}{a+j\Omega} \end{aligned}$$

So, we take the magnitude

$$|H_c(j\Omega)| = \left(\frac{1}{a^2+\Omega^2}\right)^{\frac{1}{2}}$$



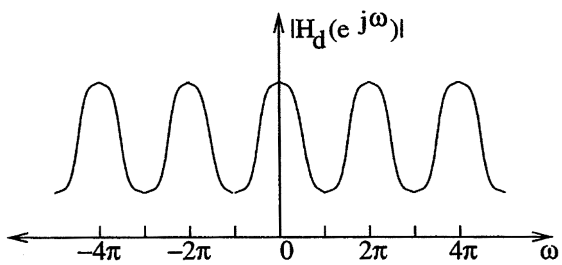

(b) Sampling the filter impulse response in (a), the discrete-time filter is described by

$$\begin{aligned} h_d[n] &= Te^{-anT}u[n] \\ H_d(e^{j\omega}) &= \sum_{n=0}^{\infty} Te^{-anT}e^{-j\omega n} \\ &= \frac{T}{1-e^{-aT}e^{-j\omega}} \end{aligned}$$

Taking the magnitude of this response

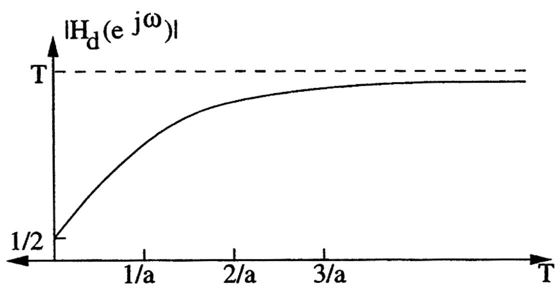
$$|H_d(e^{j\omega})| = \frac{T}{(1-2e^{-aT}\cos(\omega)+e^{-2aT})^{\frac{1}{2}}}.$$

Note that the frequency response of the discrete-time filter is periodic, with period $2\pi$.




(c) The minimum occurs at $\omega = \pi$. The corresponding value of the frequency response magnitude is

$$\begin{aligned} |H_d(e^{j\pi})| &= \frac{T}{(1+2e^{-aT}+e^{-2aT})^{\frac{1}{2}}} \\ &= \frac{T}{1+e^{-aT}}. \end{aligned}$$



**4.7.** The continuous-time signal contains an attenuated replica of the original signal with a delay of $\tau_d$.

$$x_c(t) = s_c(t) + \alpha s_c(t - \tau_d)$$

(a) Taking the Fourier transform of the analog signal:

$$X_c(j\Omega) = S_c(j\Omega) \cdot (1 + \alpha e^{-j\tau_d\Omega})$$

Note that $X_c(j\Omega)$ is zero for $|\Omega| > \pi/T$. Sampling the continuous-time signal yields the discrete-time sequence, $x[n]$. The Fourier transform of the sequence is

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{T}\sum_{r=-\infty}^{\infty} S_c(\frac{j\omega}{T} + j\frac{2\pi r}{T}) \\ &+ \frac{\alpha}{T}\sum_{r=-\infty}^{\infty} S_c(\frac{j\omega}{T} + j\frac{2\pi r}{T})e^{-j\tau_d(\frac{\omega}{T}+\frac{2\pi r}{T})} \end{aligned}$$

(b) The desired response:

$$H(j\Omega) = \begin{cases} 1 + \alpha e^{-j\tau_d\Omega}, & \text{for } |\Omega| \le \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases}$$

Using $\omega = \Omega T$, we obtain a discrete-time system which simulates the above response:

$$H(e^{j\omega}) = 1 + \alpha e^{-j\frac{\tau_d\omega}{T}}$$

(c) We need to take the inverse Fourier transform of the discrete-time impulse response of part (b).

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (1 + \alpha e^{-j\frac{\tau_d\omega}{T}})e^{j\omega n}\,d\omega \end{aligned}$$

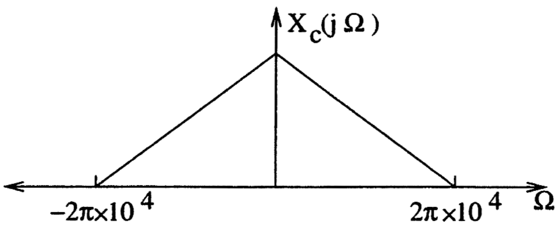
(i) Consider the case when $\tau_d = T$:

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (e^{j\omega n} + \alpha e^{j\omega(n-1)})\,d\omega \\ &= \frac{\sin(\pi n)}{\pi n} + \frac{\alpha\sin[\pi(n-1)]}{\pi(n-1)} \\ &= \delta[n] + \alpha\delta[n-1] \end{aligned}$$

(ii) For $\tau_d = T/2$:

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (e^{j\omega n} + \alpha e^{j\omega(n-\frac{1}{2})})\,d\omega \\ &= \frac{\sin(\pi n)}{\pi n} + \frac{\alpha\sin[\pi(n-\frac{1}{2})]}{\pi(n-\frac{1}{2})} \\ &= \delta[n] + \frac{\alpha\sin[\pi(n-\frac{1}{2})]}{\pi(n-\frac{1}{2})} \end{aligned}$$

**4.8.** A plot of $X_c(j\Omega)$ appears below.




(a) For $x_c(t)$ to be recoverable from $x[n]$, the transform of the discrete signal must have no aliasing. When sampling, the radian frequency is related to the analog frequency by

$$\omega = \Omega T.$$

No aliasing will occur if the sampling interval satisfies the Nyquist Criterion. Thus, for the band-limited signal, $x_c(t)$, we should select $T$ as:

$$T \leq \frac{1}{2 \times 10^4}.$$

(b) Assuming that the system is linear and time-invariant, the convolution sum describes the input-output relationship.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

We are given

$$\begin{aligned} y[n] &= T \sum_{k=-\infty}^{n} x[k] \\ &= T \sum_{k=-\infty}^{\infty} x[k]u[n-k] \end{aligned}$$

Hence, we may infer that the impulse response of the system

$$h[n] = T \cdot u[n].$$

(c) We use the expression for $y[n]$ as given and examine the limit

$$\begin{aligned} \lim_{n\to\infty} y[n] &= \lim_{n\to\infty} T \cdot \sum_{k=-\infty}^{n} x[k] \\ &= T \cdot \sum_{k=-\infty}^{\infty} x[k] \end{aligned}$$

Recall the analysis equation for the Fourier transform:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Hence,

$$\lim_{n\to\infty} y[n] = T \cdot X(e^{j\omega})|_{\omega=0}$$

**(d)** We use the result from part (c). Noting that

$$X(e^{j\omega}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c(\frac{j\omega}{T} + \frac{j2\pi r}{T}).$$

Thus, we have

$$T \cdot X(e^{j\omega})|_{\omega=0} = \sum_{r=-\infty}^{\infty} X_c(\frac{j2\pi r}{T})$$

From the given information, we seek a value of $T$ such that:

$$\begin{aligned}\sum_{r=-\infty}^{\infty} X_c(\frac{j2\pi r}{T}) &= \int_{-\infty}^{\infty} x_c(t)\,dt \\ &= X_c(j\Omega)|_{\Omega=0}\end{aligned}$$

For the final equality to be true, there must be no contribution from the terms for which $r \neq 0$. That is, we require no aliasing at $\Omega = 0$. Since we are only interested in preserving the spectral component at $\Omega = 0$, we may sample at a rate which is lower than the Nyquist rate. The maximum value of $T$ to satisfy these conditions is

$$T \leq \frac{1}{1 \times 10^4}.$$

**4.9.** (a) Since $X(e^{j\omega}) = X(e^{j(\omega-\pi)})$, $X(e^{j\omega})$ is periodic with period $\pi$.

(b) Using the inverse DTFT,

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(e^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(e^{j(\omega-\pi)})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(e^{j\omega})e^{j(\omega+\pi)n}d\omega \\
&= \frac{1}{2\pi}e^{j\pi n}\int_{\langle 2\pi\rangle} X(e^{j\omega})e^{j\omega n}d\omega \\
&= (-1)^n x[n].
\end{aligned}$$

All odd samples of $x[n] = 0$, because $x[n] = -x[n]$. Hence $x[3] = 0$.

(c) Yes, $y[n]$ contains all even samples of $x[n]$, and all odd samples of $x[n]$ are 0.

$$x[n] = \begin{cases} y[n/2], & n \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

**4.10.** Use $x[n] = x_c(nT)$, and simplify:

(a) $x[n] = \cos(2\pi n/3)$.

(b) $x[n] = \sin(4\pi n/3) = -\sin(2\pi n/3)$

(c) $x[n] = \frac{\sin(2\pi n/5)}{\pi n/5000}$

**4.11.** (a) Pick $T$ such that

$$x[n] = x_c(nT) = \sin(10\pi nT) = \sin(\pi n/4) \qquad \Longrightarrow T = 1/40$$

There are other choices. For example, by realizing that $\sin(\pi n/4) = \sin(9\pi n/4)$, we find $T = 9/40$.

(b) Choose $T = 1/20$ to make $x[n] = x_c(nT)$. This is unique.

**4.12.** (a) Notice first that $H(e^{j\omega}) = 10j\omega, -\pi \leq \omega < \pi$.

(i) After sampling,

$$\begin{aligned}
x[n] &= \cos(\frac{3\pi}{5}n), \\
y[n] &= |H(e^{j\frac{3\pi}{5}})| \cos(\frac{3\pi}{5}n + \angle H(e^{j\frac{3\pi}{5}})) \\
&= 6\pi \cos(\frac{3\pi}{5}n + \frac{\pi}{2}) \\
&= -6\pi \sin(\frac{3\pi}{5}n) \\
y_c(t) &= -6\pi \sin(6\pi t).
\end{aligned}$$

(ii) After sampling, $x[n] = \cos(\frac{7\pi}{5}n) = \cos(\frac{3\pi}{5}n)$, so again, $y_c(t) = -6\pi \sin(6\pi t)$.

(b) $y_c(t)$ is what you would expect from a differentiator in the first case but not in the second case. This is because aliasing has occurred in the second case.

**4.13.** (a)

$$\begin{aligned}
x_c(t) &= \sin(\frac{\pi}{20}t) \\
y_c(t) &= \sin(\frac{\pi}{20}(t-5)) \\
&= \sin(\frac{\pi}{20}t - \frac{\pi}{4}) \\
y[n] &= \sin(\frac{\pi n}{2} - \frac{\pi}{4})
\end{aligned}$$

(b) We get the same result as before:

$$\begin{aligned}
x_c(t) &= \sin(\frac{\pi}{10}t) \\
y_c(t) &= \sin(\frac{\pi}{10}(t-2.5)) \\
&= \sin(\frac{\pi}{10}t - \frac{\pi}{4}) \\
y[n] &= \sin(\frac{\pi n}{2} - \frac{\pi}{4})
\end{aligned}$$

(c) The sampling period $T$ is not limited by the continuous time system $h_c(t)$.

**4.14.** There is no loss of information if $X(e^{j\omega/2})$ and $X(e^{j(\omega/2-\pi)})$ do not overlap. This is true for (b), (d), (e).

**4.15.** The output $x_r[n] = x[n]$ if no aliasing occurs as result of downsampling. That is, $X(e^{j\omega}) = 0$ for $\pi/3 \leq |\omega| \leq \pi$.

(a) $x[n] = \cos(\pi n/4)$. $X(e^{j\omega})$ has impulses at $\omega = \pm\pi/4$, so there is no aliasing. $x_r[n] = x[n]$.

(b) $x[n] = \cos(\pi n/2)$. $X(e^{j\omega})$ has impulses at $\omega = \pm\pi/2$, so there is aliasing. $x_r[n] \neq x[n]$.

(c) A sketch of $X(e^{j\omega})$ is shown below. Clearly there will be no aliasing and $x_r[n] = x[n]$.