ELEC 405
Error Control Coding and Sequences

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Dept. of Electrical and Computer Engineering

Topics

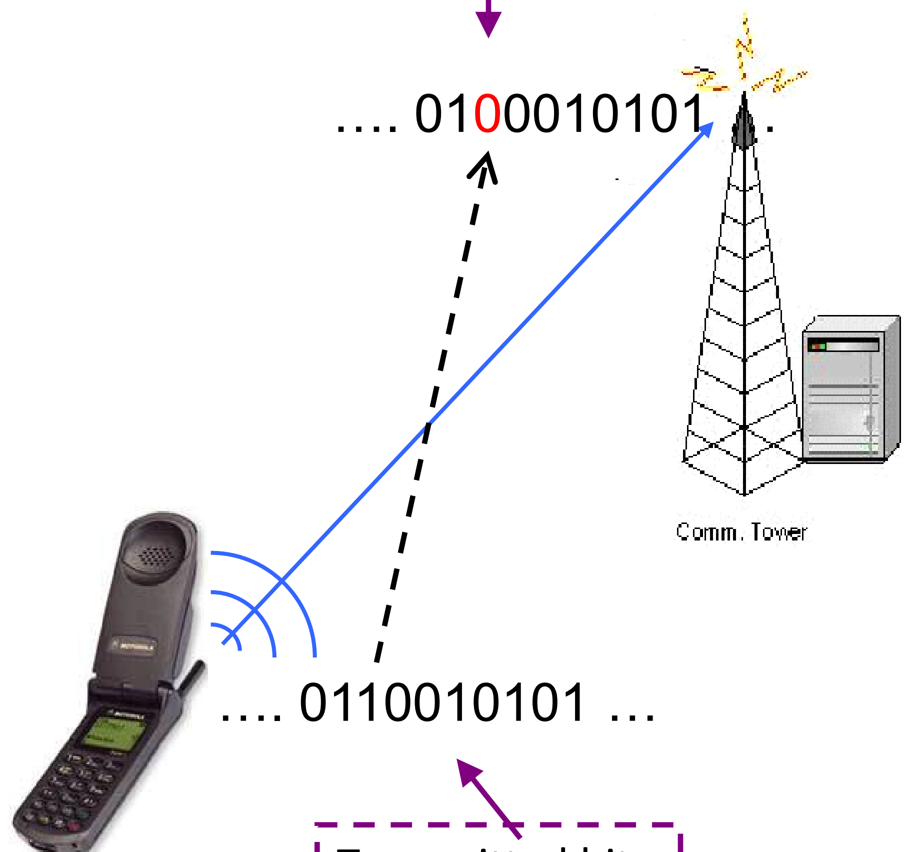
- Introduction – The Channel Coding Problem
- Linear Block Codes
- Cyclic Codes
- BCH and Reed-Solomon Codes
- Convolutional Codes
- Trellis Coded Modulation
- Sequence Design

Errors in Information Transmission

Digital Communications:
Transporting information from one place to another using a sequence of symbols, e.g. bits.

Noise and interference:
The received sequence may differ from the transmitted one.

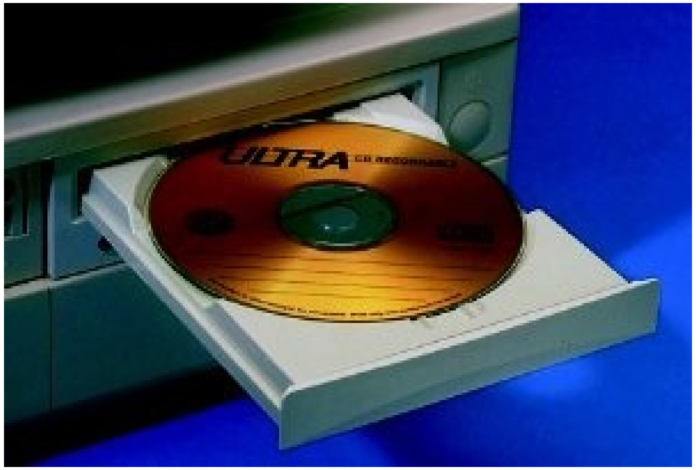
Received bits: a corrupted version of the transmitted bits



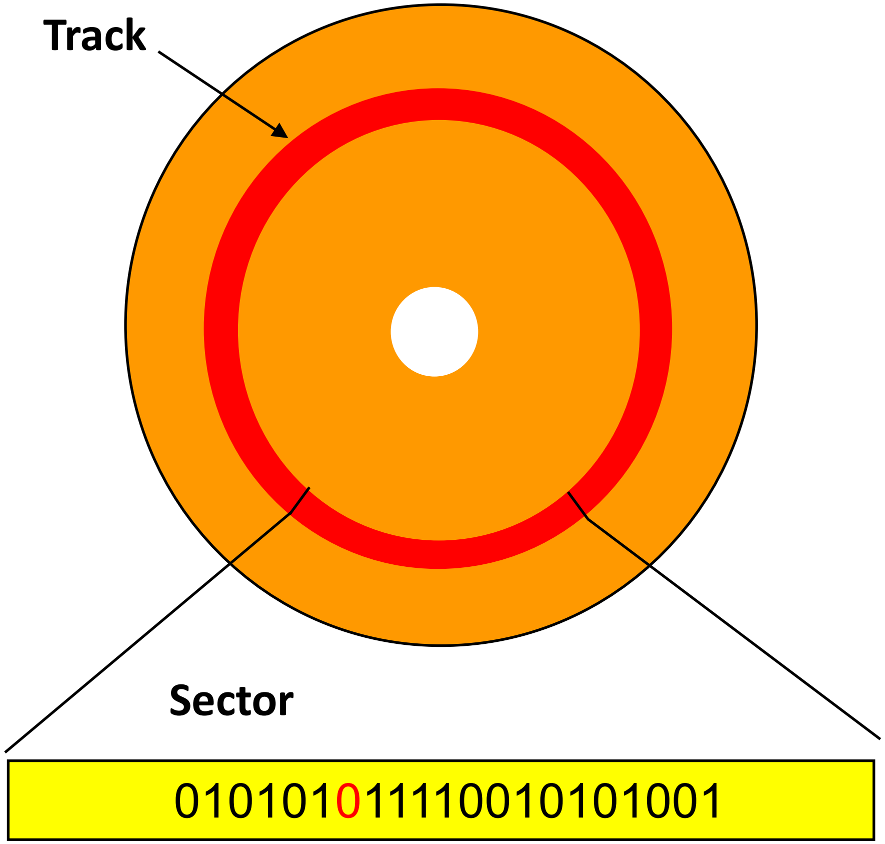
Transmitted bits

Errors in Information Transmission

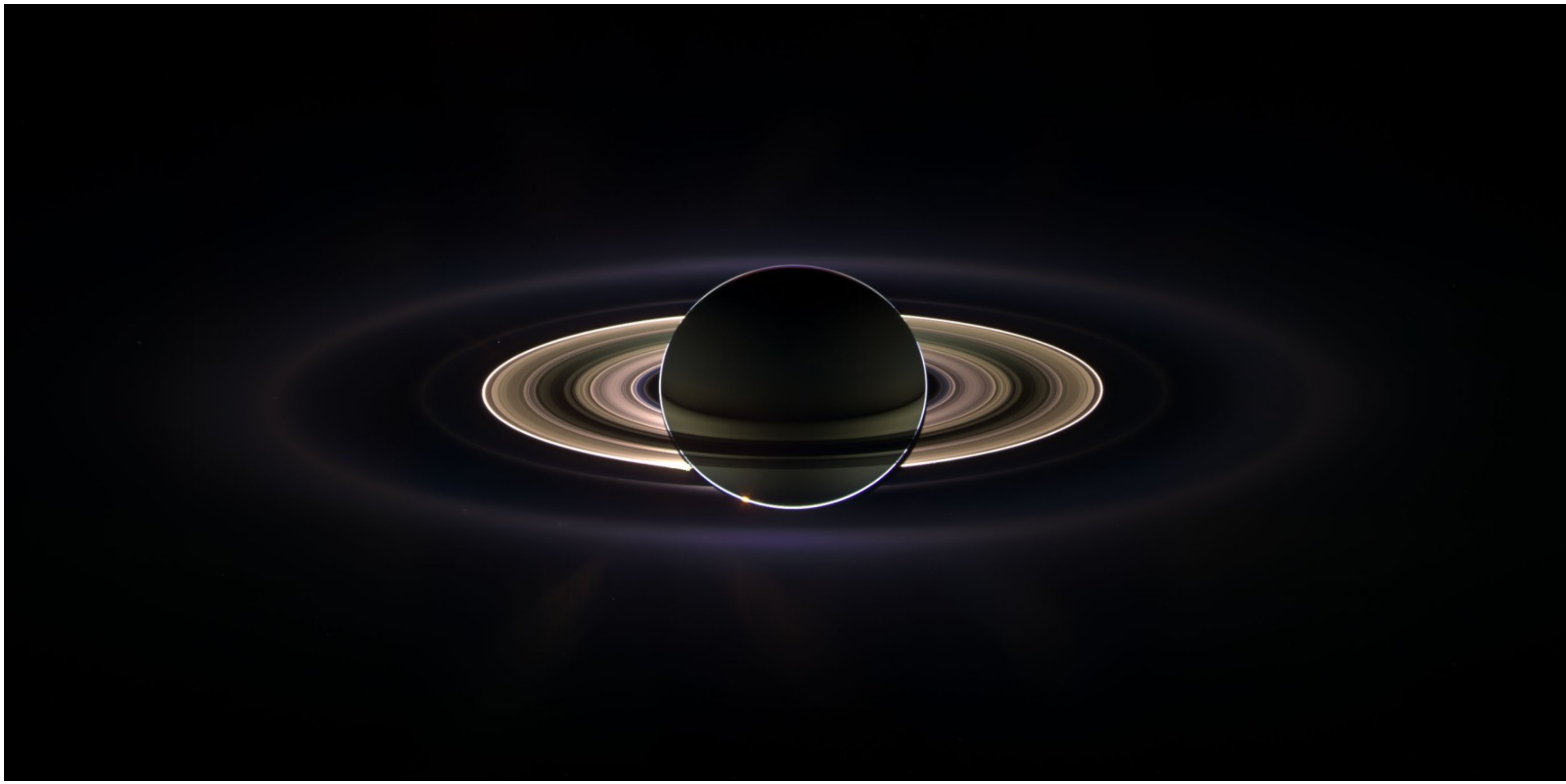
Magnetic recording



Some of the bits may change during storage or reading from the disk



Deep Space Communications



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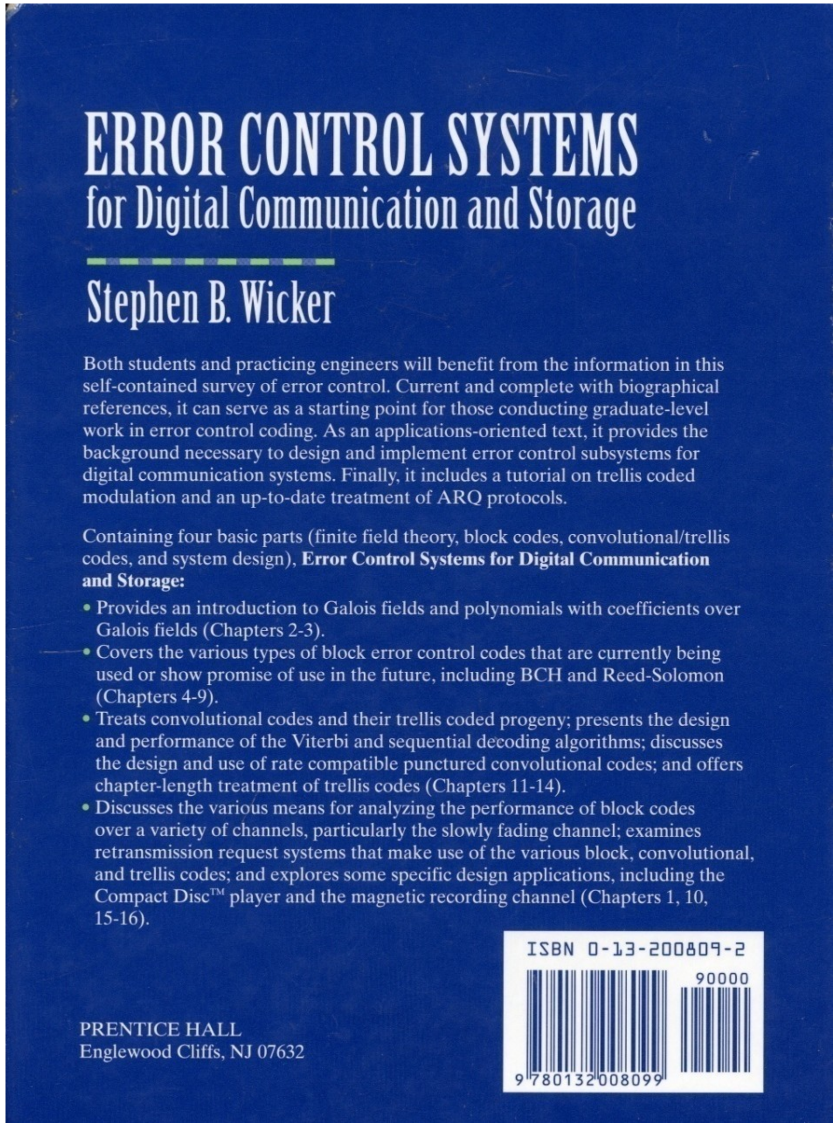

ERROR CONTROL SYSTEMS
for Digital Communication and Storage
Stephen B. Wicker
Both students and practicing engineers will benefit from the information in this self-contained survey of error control. Current and complete with biographical references, it can serve as a starting point for those conducting graduate-level work in error control coding. As an applications-oriented text, it provides the background necessary to design and implement error control subsystems for digital communication systems. Finally, it includes a tutorial on trellis coded modulation and an up-to-date treatment of ARQ protocols.
Containing four basic parts (finite field theory, block codes, convolutional/trellis codes, and system design), Error Control Systems for Digital Communication and Storage:
• Provides an introduction to Galois fields and polynomials with coefficients over Galois fields (Chapters 2-3).
• Covers the various types of block error control codes that are currently being used or show promise of use in the future, including BCH and Reed-Solomon (Chapters 4-9).
• Treats convolutional codes and their trellis coded progeny; presents the design and performance of the Viterbi and sequential decoding algorithms; discusses the design and use of rate compatible punctured convolutional codes; and offers chapter-length treatment of trellis codes (Chapters 11-14).
• Discusses the various means for analyzing the performance of block codes over a variety of channels, particularly the slowly fading channel; examines retransmission request systems that make use of the various block, convolutional, and trellis codes; and explores some specific design applications, including the Compact Disc™ player and the magnetic recording channel (Chapters 1, 10, 15-16).
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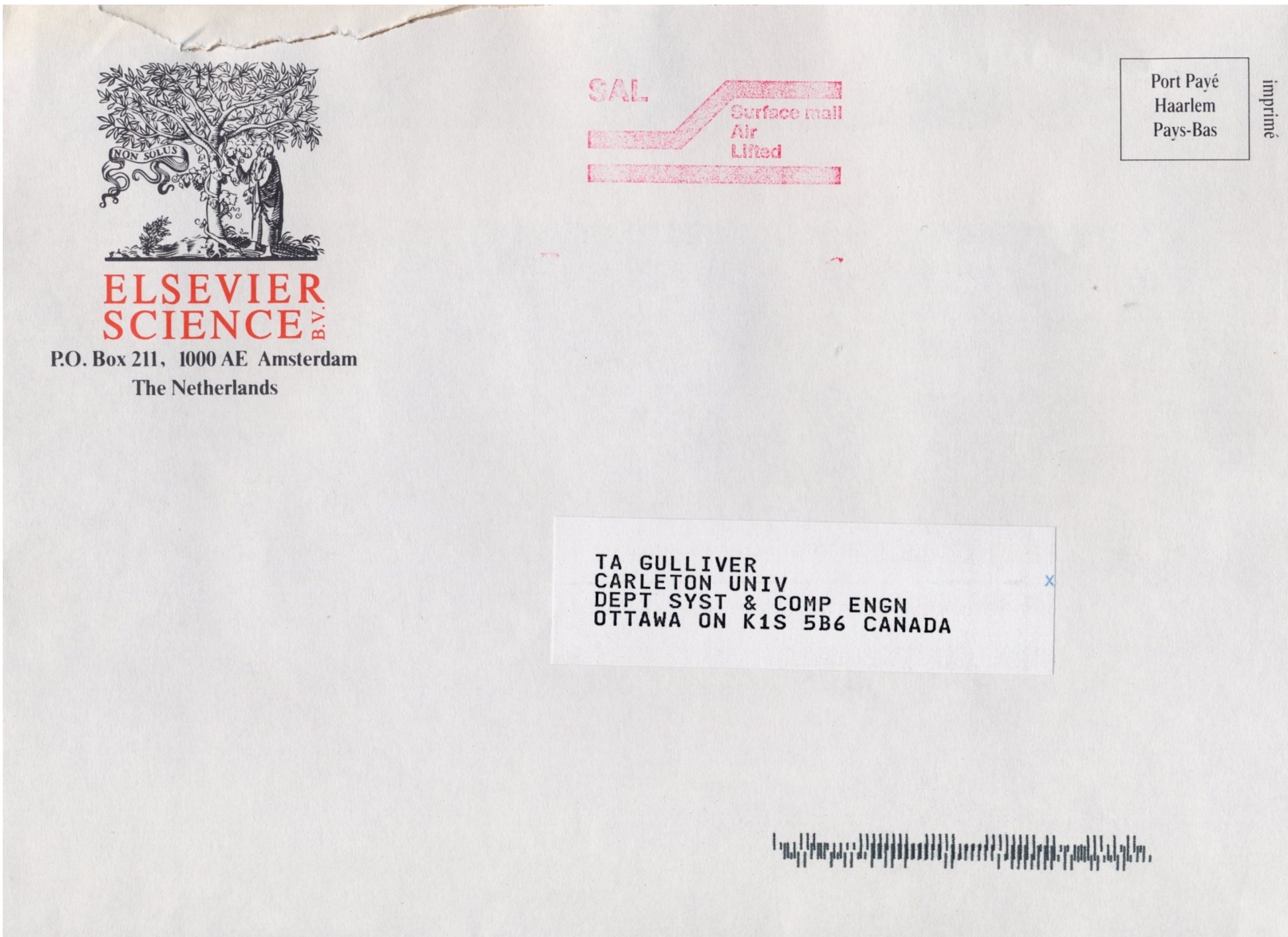
Errors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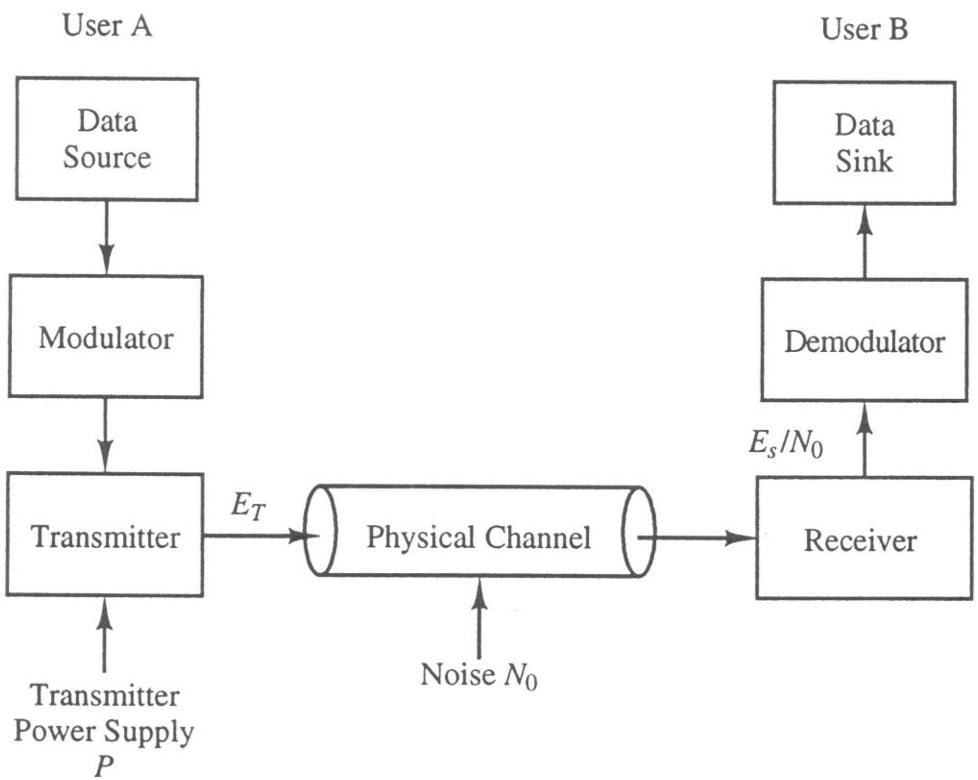
Digital Communication System



Modulator/Transmitter

- Matches the message to the channel
- Modulation encodes the message into the amplitude, phase or frequency of the carrier signal (PSK, FSK, QAM, OFDM, PAM, PPM)
- Advantages:
  - Reduces noise and interference
  - Multiplexing
  - Channel assignment
- Digital modulation is used in television, radio, 802.11, cellphones, bluetooth, GPS, …

Receiver/Demodulator

- The receiver amplifies and filters the signal
- The ideal receiver output is a scaled, delayed version of the message signal
- The demodulator extracts the message from the receiver output
- Converts the receiver output to bits or symbols

Channel

- Physical medium that the signal is transmitted through (or stored on)
- Examples: Air, wires, coaxial cables, fiber optic cables, space (CD, DVD)
- Every channel introduces some amount of distortion, noise and interference
- The channel properties determine
    - Data throughput of the system
    - System bit error rate (BER)
    - Quality of service (QoS) offered by the system

Noise and Interference

- Internal Noise
  - Generated by components within a communication system (thermal noise)
- External Noise and Interference
  - Atmospheric noise (electrical discharges)
  - Man-made noise (ignition noise)
  - Multipath interference (multiple transmission paths)
  - Multiple access interference (signals from other users)

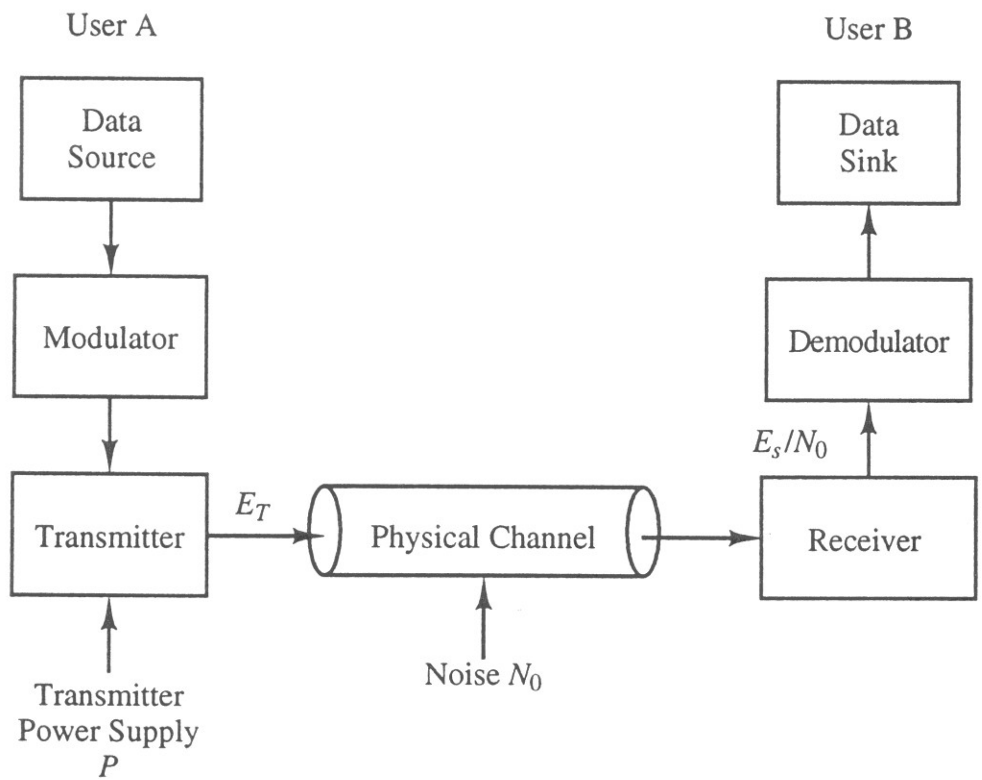
Communications System Design Goals

- Maximize bit rate - R_b
- Minimize the probability of error (BER) - P_b
- Minimize required SNR - E_b/N_0
- Minimize required bandwidth - W
- Maximize system utilization – Capacity
- Minimize system complexity and cost - $

System Constraints

- Minimum bandwidth based on the modulation used
- Channel capacity
- Government regulations
- Technological limitations
- Other requirements (i.e. cost and complexity)

Digital Communication System



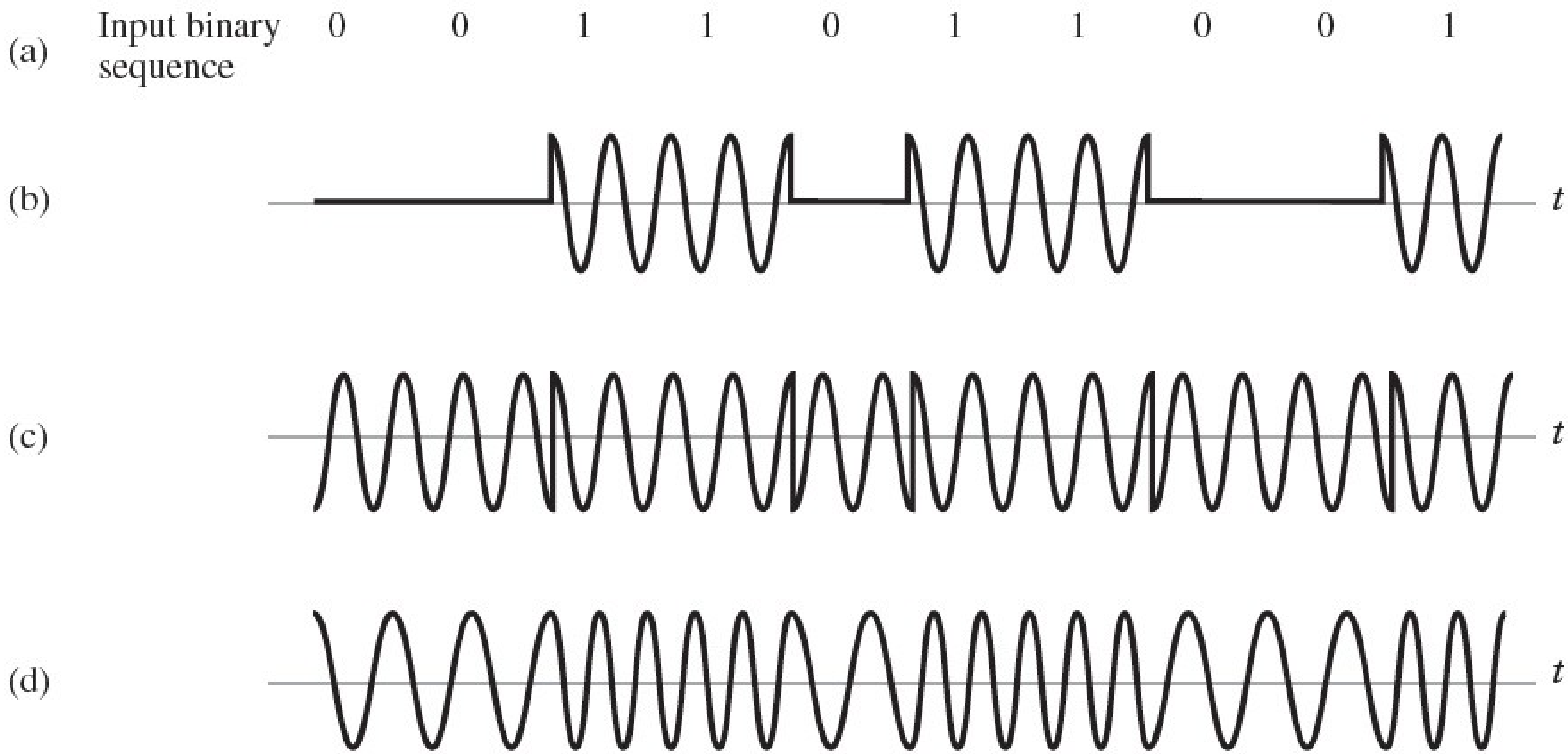


FIGURE 7.1 The three basic forms of signaling binary information. (*a*) Binary data stream. (*b*) Amplitude-shift keying. (*c*) Phase-shift keying. (*d*) Frequency-shift keying with continuous phase.

BPSK

$$\text{data } 0 \quad s_0(t) = \sqrt{2P}\cos(2\pi f_c t) \qquad 0 \le t \le T$$

$$\text{data } 1 \quad s_1(t) = -\sqrt{2P}\cos(2\pi f_c t) \quad 0 \le t \le T$$

$$\text{or} \quad \sqrt{2P}\cos(2\pi f_c t + \pi)$$

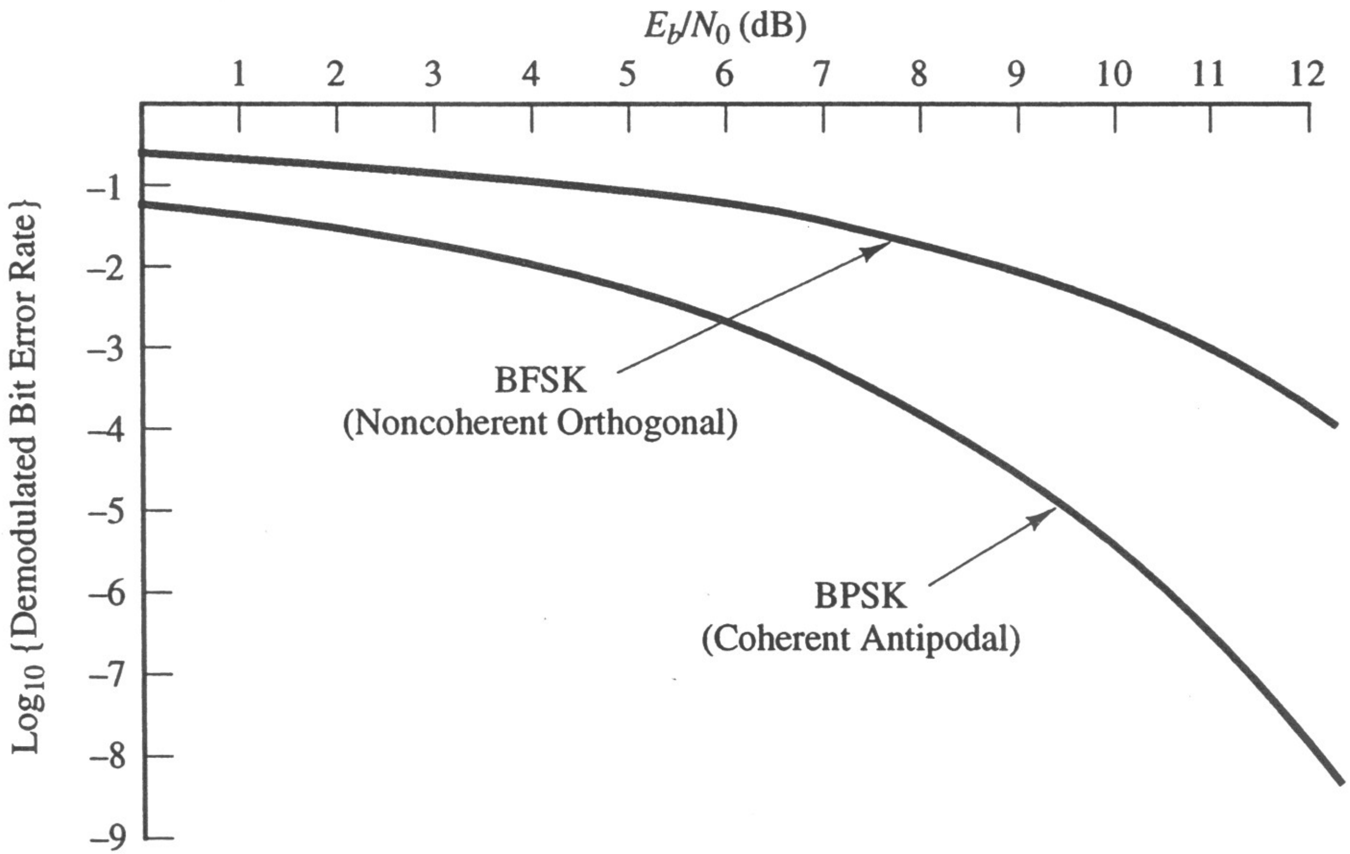
BFSK

$$\text{data } 0 \quad S_0(t) = \sqrt{2P}\cos(2\pi f_0 t) \quad 0 \le t \le T$$

$$\text{data } 1 \quad S_1(t) = \sqrt{2P}\cos(2\pi f_1 t) \quad 0 \le t \le T$$

$$\begin{aligned} E &= PT \rightarrow P = E_b/T_b = E_b R_b \\ R_b &= 1/T_b \text{ bits/sec} \\ P_{dB} &= 10\log_{10} P_{watts} \end{aligned}$$

Performance of BPSK and BFSK



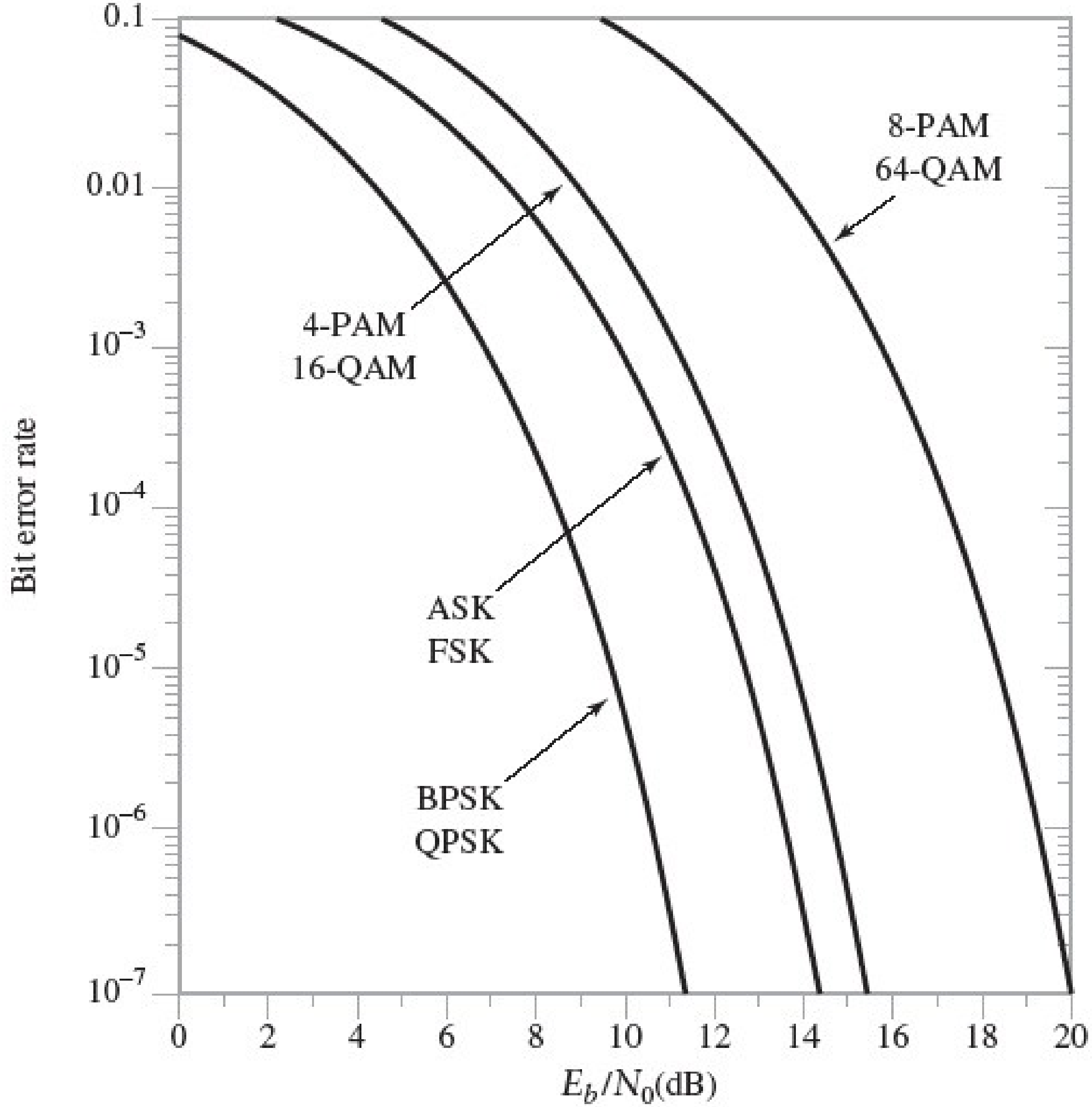
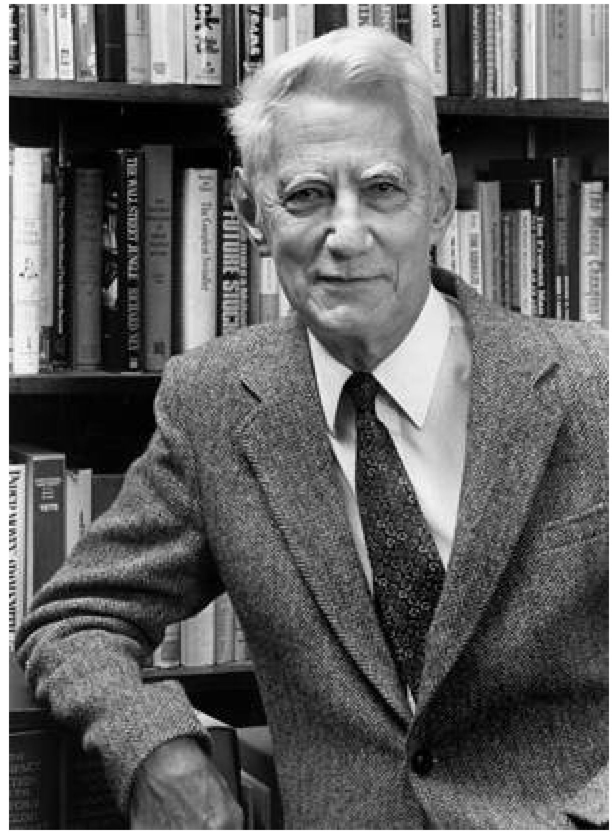


FIGURE 10.16 Comparison of BER versus E_b/N_0 for several digital transmission strategies.

Claude Shannon (1916-2001)



A Mathematical Theory of Communications, BSTJ, July 1948

``The fundamental problem of communication is that of reproducing at one point exactly or approximately a message selected at another point. …

If the channel is noisy it is not in general possible to reconstruct the original message or the transmitted signal with certainty by any operation on the received signal.’’

Channel Capacity

- An important question for a communication channel is the **maximum rate** at which it can transfer information.
- The **channel capacity C** is a theoretical maximum rate below which information can be transmitted over the channel with an arbitrarily low probability of error.

Shannon's Noisy Channel Coding Theorem

- How to achieve capacity?
  - With every channel we can associate a ``channel capacity'' C. There exist **error control codes** such that information can be transmitted across the channel at rates less than C with arbitrarily low bit error rate.
- There are only two factors that determine the capacity of a channel:
  - Bandwidth (W)
  - Signal-to-Noise Ratio (SNR) E_b/N_0

Channel Capacity

$$C = W \log_2 \left(1 + \frac{P}{N_0 W} \right)$$

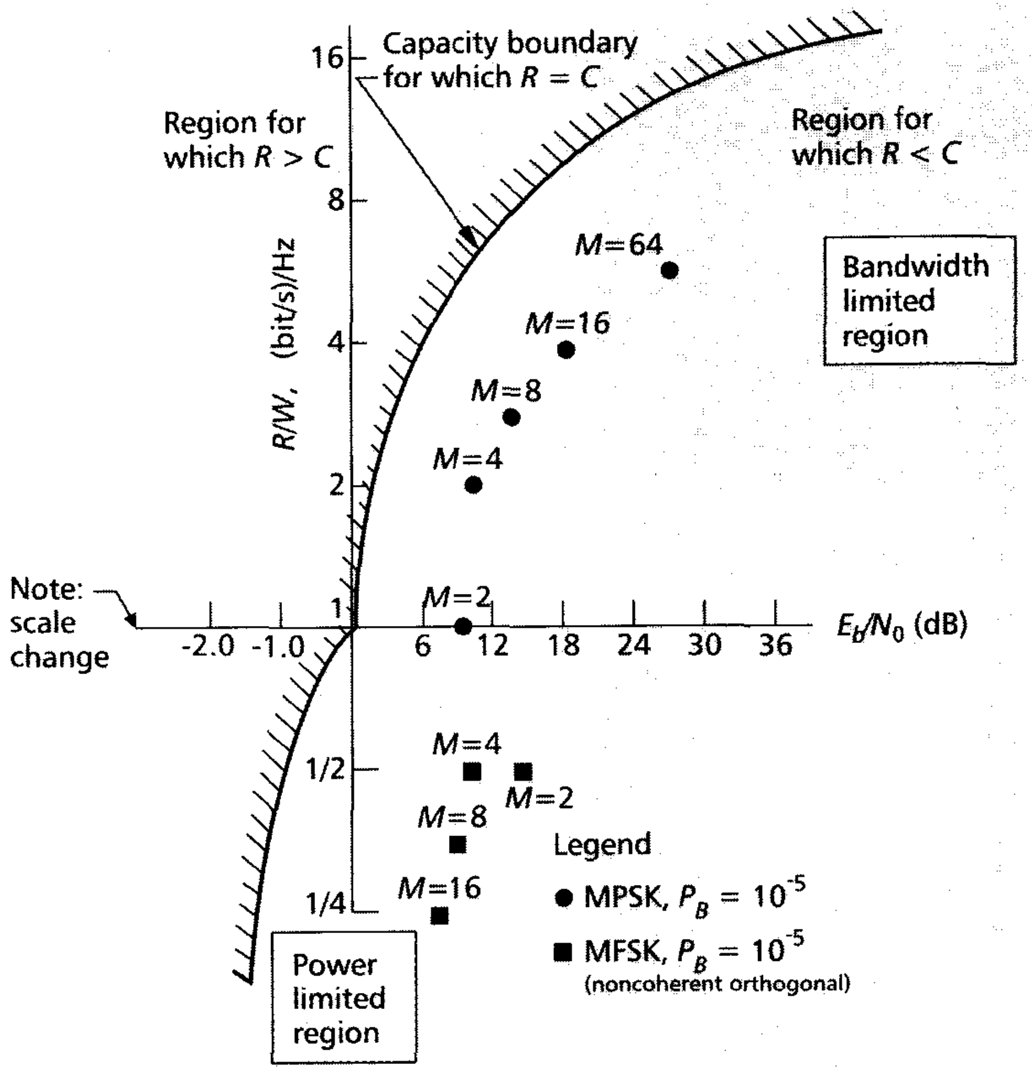
$$E = PT \rightarrow P = E_b R_b$$

$$C = W \log_2 \left(1 + \frac{E_b R_b}{N_0 W} \right)$$

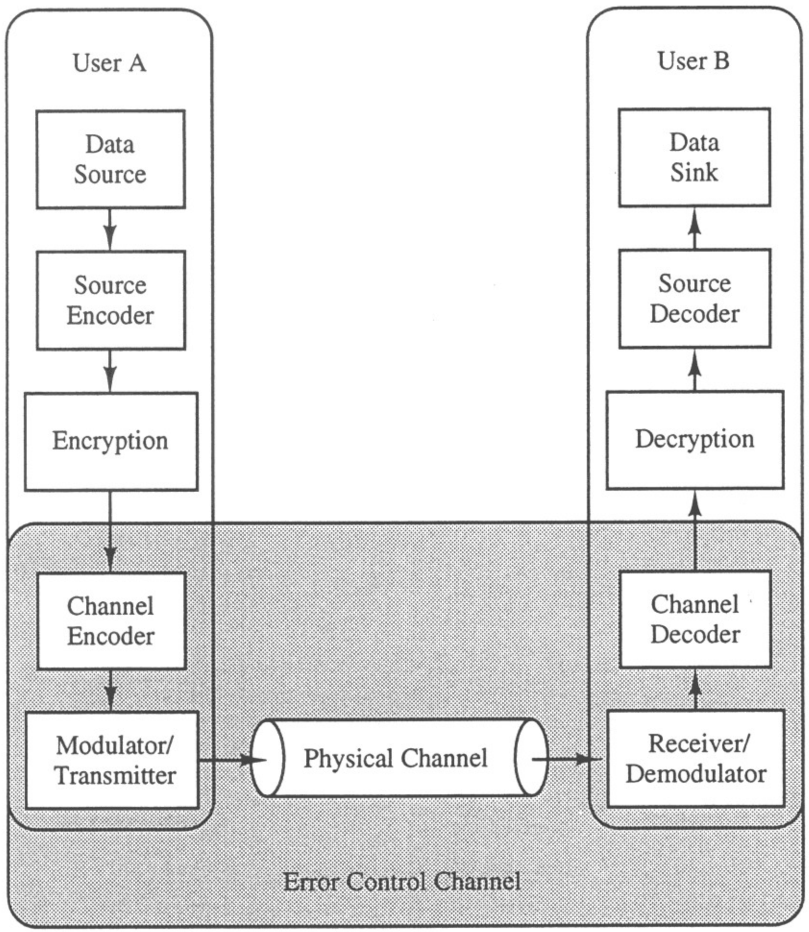
Let $R_b = C$ $$\frac{C}{W} = \log_2 \left(1 + \frac{E_b}{N_0} \frac{C}{W} \right)$$

$$\frac{E_b}{N_0} = \frac{2^{C/W} - 1}{C / W}$$

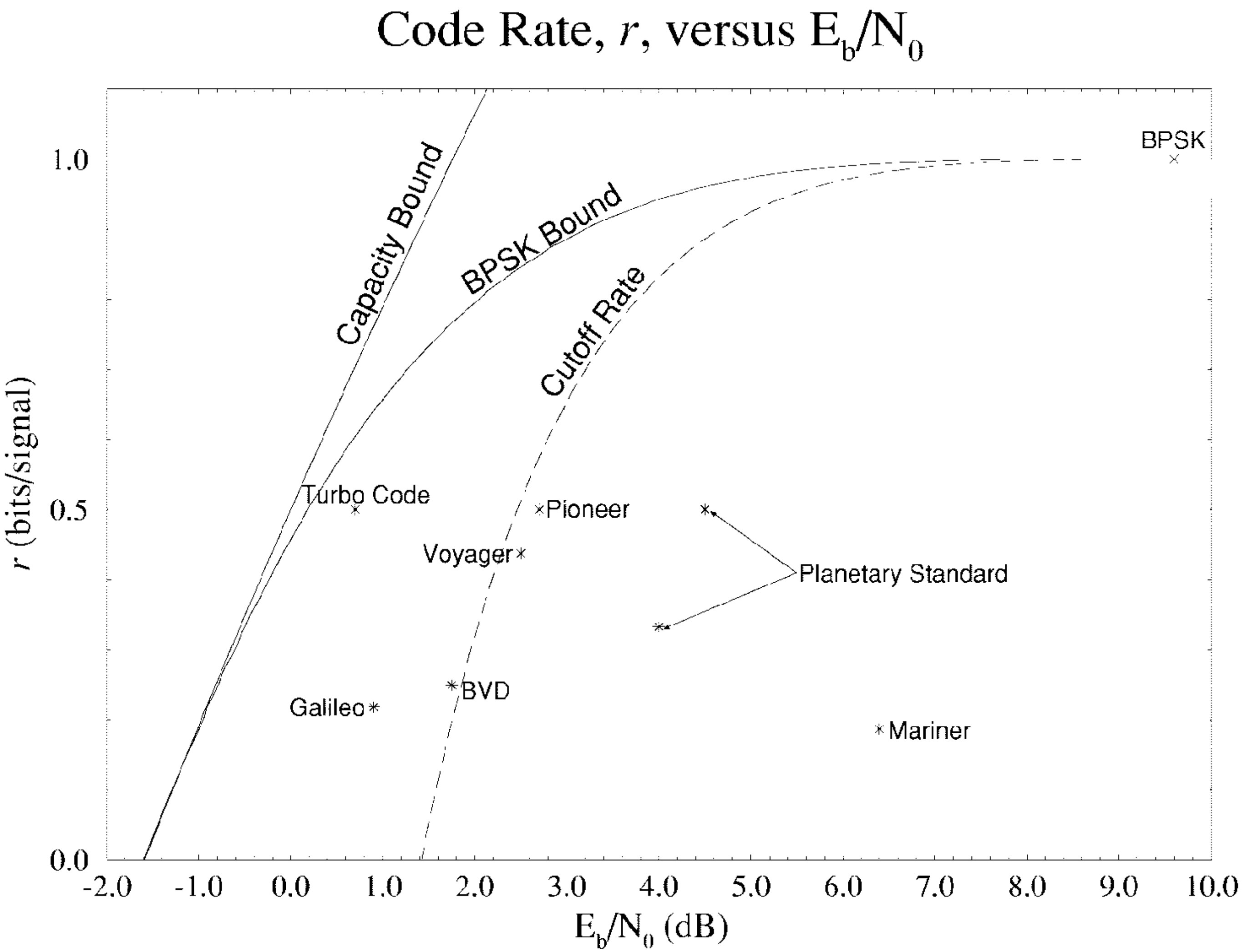
Bandwidth Efficiency versus SNR



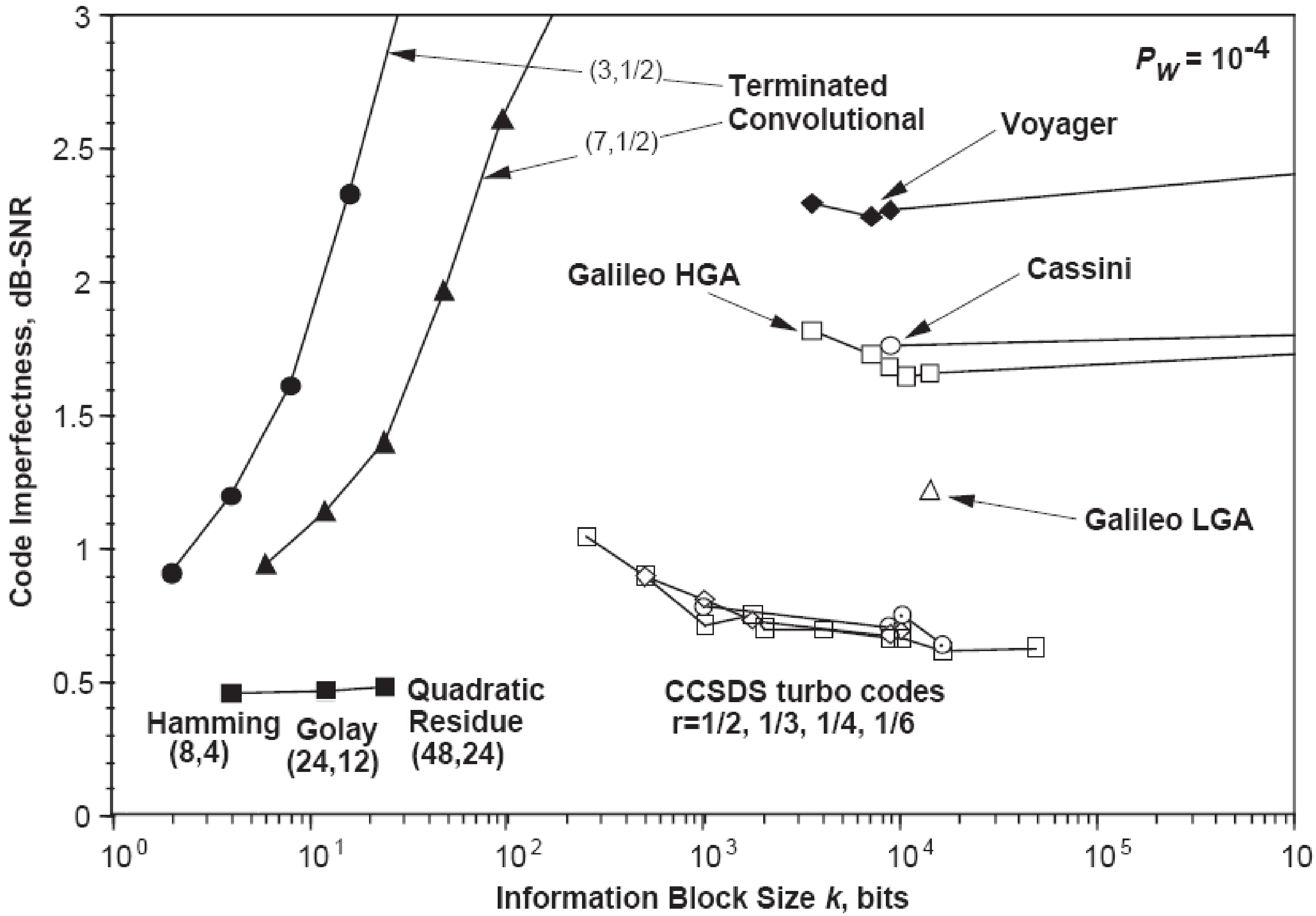
Digital Communication Model with Coding



Power Efficiency



Code Imperfectness Relative to the Sphere-Packing Bound



USA Standard Code for Information Exchange (USASCII)											
				b_7 b_6 b_5	b_7 b_6 b_5	b_7 b_6 b_5	b_7 b_6 b_5	b_7 b_6 b_5	b_7 b_6 b_5	b_7 b_6 b_5	b_7 b_6 b_5
b_4	b_3	b_2	b_1	0 0 0	0 0 1	0 1 0	0 1 1	1 0 0	1 0 1	1 1 0	1 1 1
0	0	0	0	NUL	DLE	SP	0	@	P	`	p
0	0	0	1	SOH	DC1	!	1	A	Q	a	q
0	0	1	0	STX	DC2	″	2	B	R	b	r
0	0	1	1	ETX	DC3	#	3	C	S	c	s
0	1	0	0	EOT	DC4	$	4	D	T	d	t
0	1	0	1	ENQ	NAK	%	5	E	U	e	u
0	1	1	0	ACK	SYN	&	6	F	V	f	v
0	1	1	1	BEL	ETB	´	7	G	W	g	w
1	0	0	0	BS	CAN	(	8	H	X	h	x
1	0	0	1	HT	EM	)	9	I	Y	i	y
1	0	1	0	LF	SUB	*	:	J	Z	j	z
1	0	1	1	VT	ESC	+	;	K	[	k	{
1	1	0	0	FF	FS	,	$<$	L	\	l	$\vert$
1	1	0	1	CR	GS	-	=	M	]	m	}
1	1	1	0	SO	RS	.	$>$	N	$\wedge$	n	$\sim$
1	1	1	1	SI	US	/	?	O	_	o	DEL

SPC Code – Example 1

- ASCII symbols

  E = 1000101 c = 10001011

  G = 1000111 c = 10001110

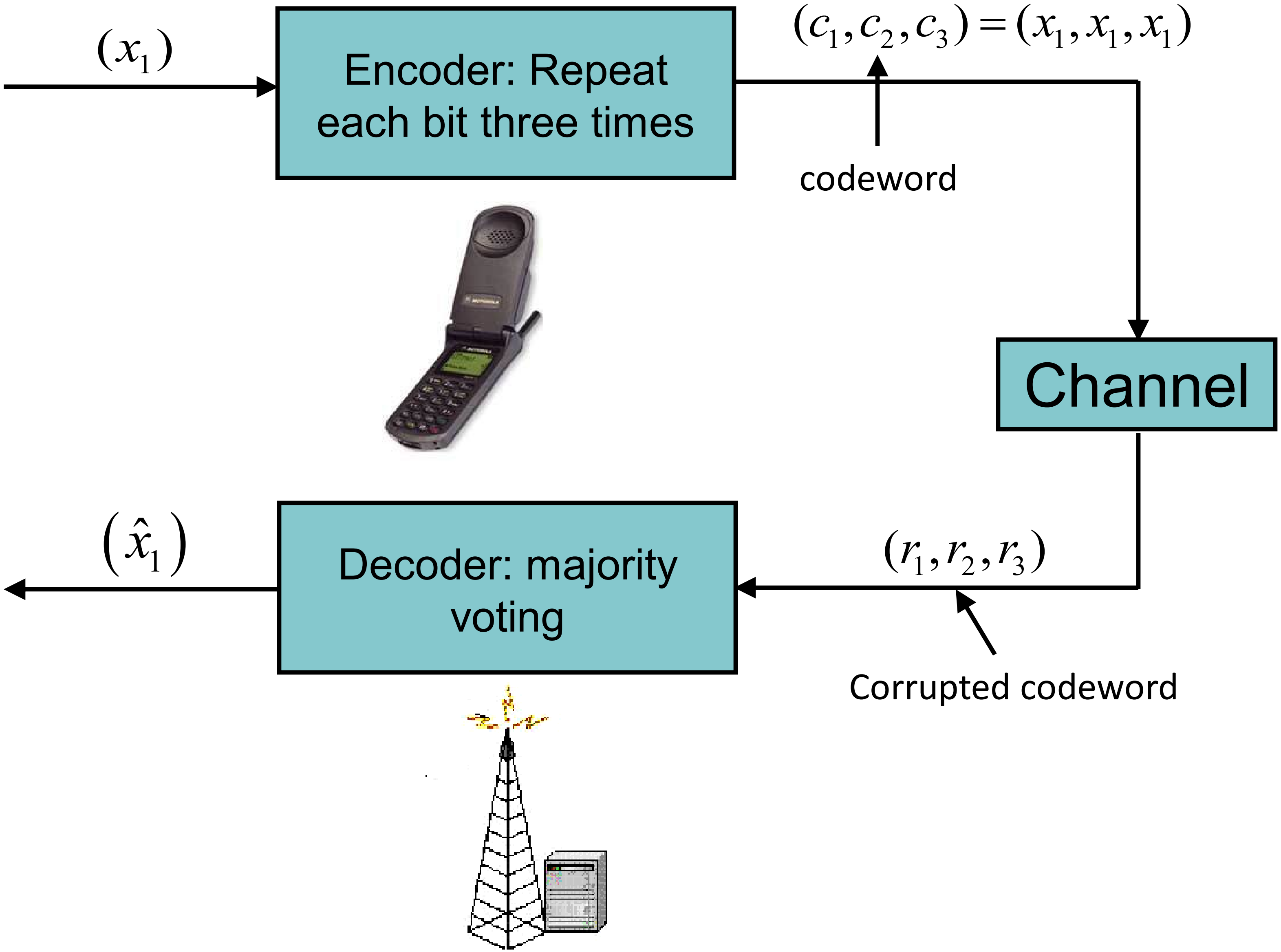
- Received word

  r = 10001010

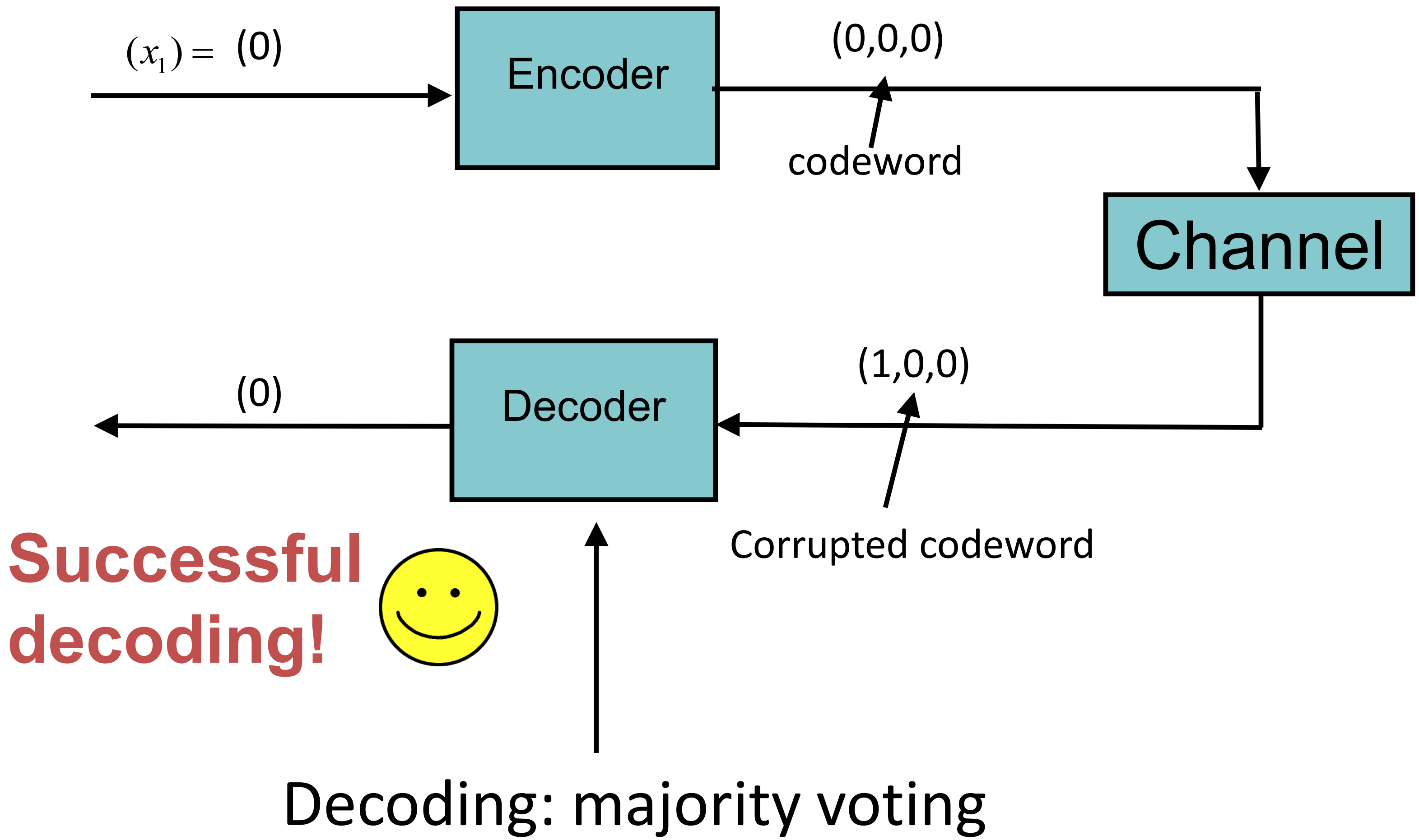
Triple Repetition Code – Decoding

Received Word			Codeword			Error Pattern		
0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	1
0	1	0	0	0	0	0	1	0
1	0	0	0	0	0	1	0	0
1	1	1	1	1	1	0	0	0
1	1	0	1	1	1	0	0	1
1	0	1	1	1	1	0	1	0
0	1	1	1	1	1	1	0	0

Triple Repetition Code

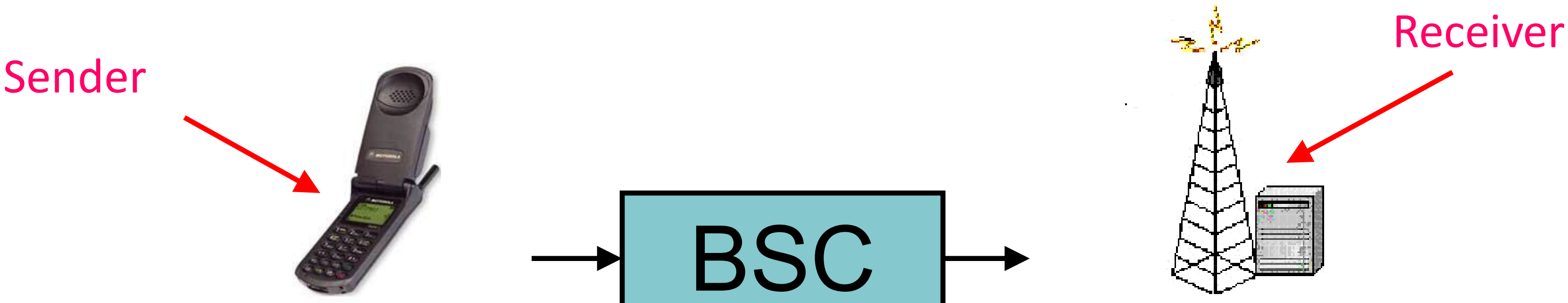


Triple Repetition Code (Cont.)



Transmission Errors

A communication channel can be modeled as a
Binary Symmetric Channel (BSC)



Information bits

10010...10101...

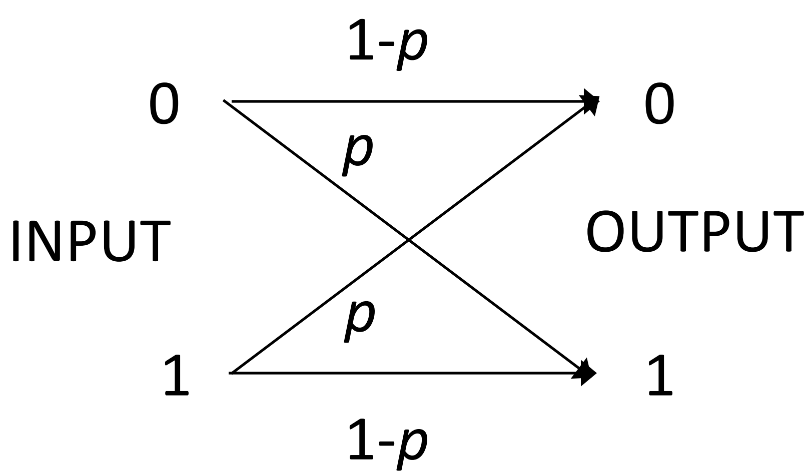
Received bits

10110...00101...

Each bit is flipped with probability *p*

Binary Symmetric Channel

- Transmitted symbols are binary
- Errors affect 0s and 1s with equal probability (symmetric)
- Errors occur randomly and are independent from bit to bit (memoryless)



p is the probability of bit error - Crossover probability

Binary Symmetric Channel

- If *n* symbols are transmitted, the probability of an *m* error pattern is

$$p^m (1-p)^{n-m}$$

- The probability of exactly *m* errors is

$$p^m (1-p)^{n-m} \binom{n}{m}$$

- The probability of *m* or more errors is

$$\sum_{i=m}^{n} p^i (1-p)^{n-i} \binom{n}{i}$$

Example

- The BSC bit error probability is $p < \frac{1}{2}$
- majority vote or nearest neighbor decoding

$$000, 001, 010, 100 \rightarrow 000$$

$$111, 110, 101, 011 \rightarrow 111$$

- the probability of a decoding error is

$$P(E) = 3p^2(1-p) + p^3 = 3p^2 - 2p^3 < p$$

- **Example:** If p = 0.01, then $P(E)$ = 0.000298 and only one word in 3555 will be in error after decoding.

Example with BFSK

$$p = \frac{1}{2} e^{-E_b/2N_0}$$

$$\mathrm{SNR} \ = E_b/N_0 = 8.93 \text{ dB} \Rightarrow p = 10^{-2}$$

code rate $R = \frac{1}{3}$

$$\hat{E}_b = E_b R = E_b/3 = 4.16 \text{ dB}$$

$$\hat{p} = .136$$

$$P(E) = 3\hat{p}^2(1 - \hat{p}) + \hat{p}^3 = 0.050$$

Example (cont.)

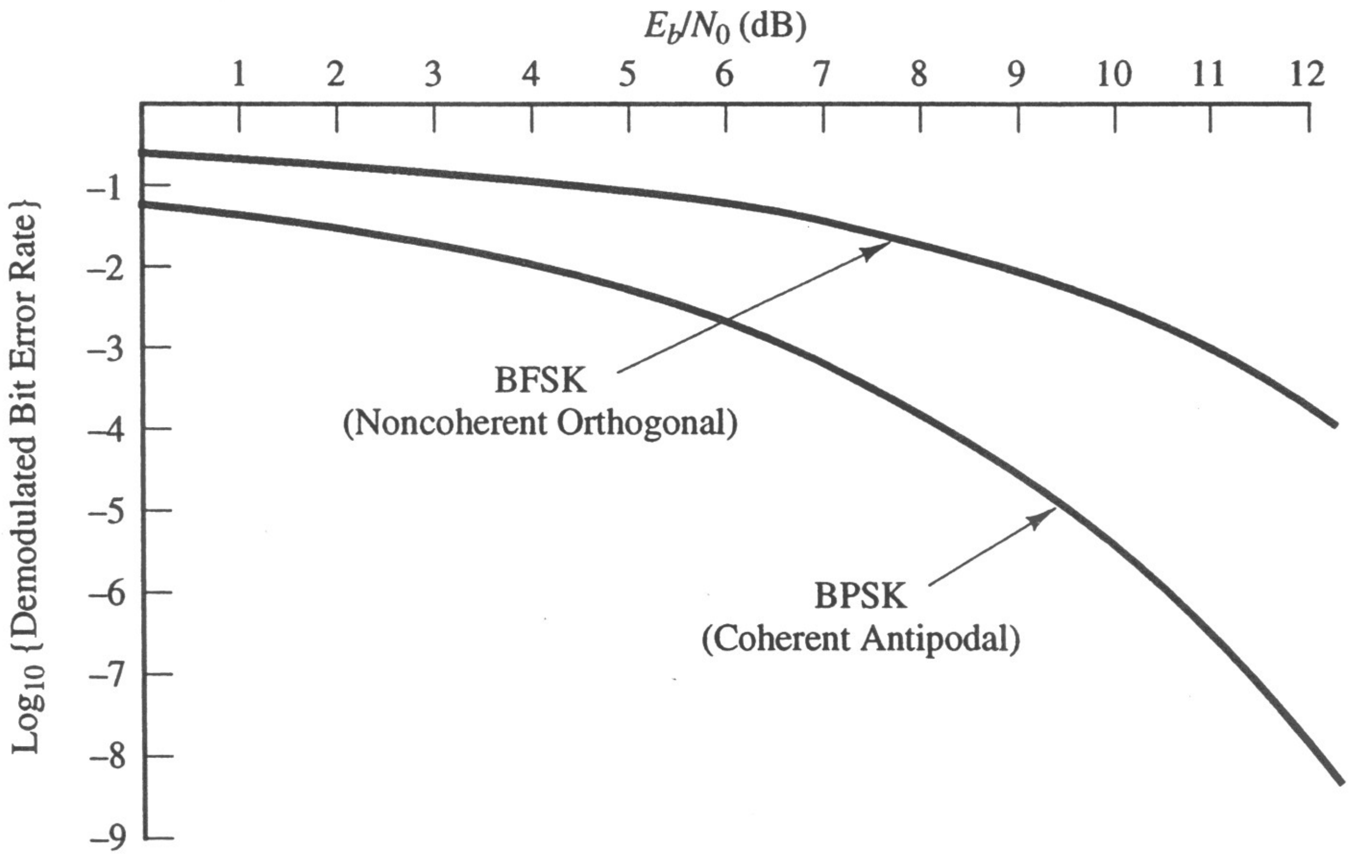
$$p = 10^{-5} \Rightarrow E_b/N_0 = 21.6 \text{ W or } 13.35 \text{ dB}$$

$$\hat{E}_b/N_0 = 13.35 - 4.77 = 8.58 \text{ dB or } 7.21 \text{ W}$$

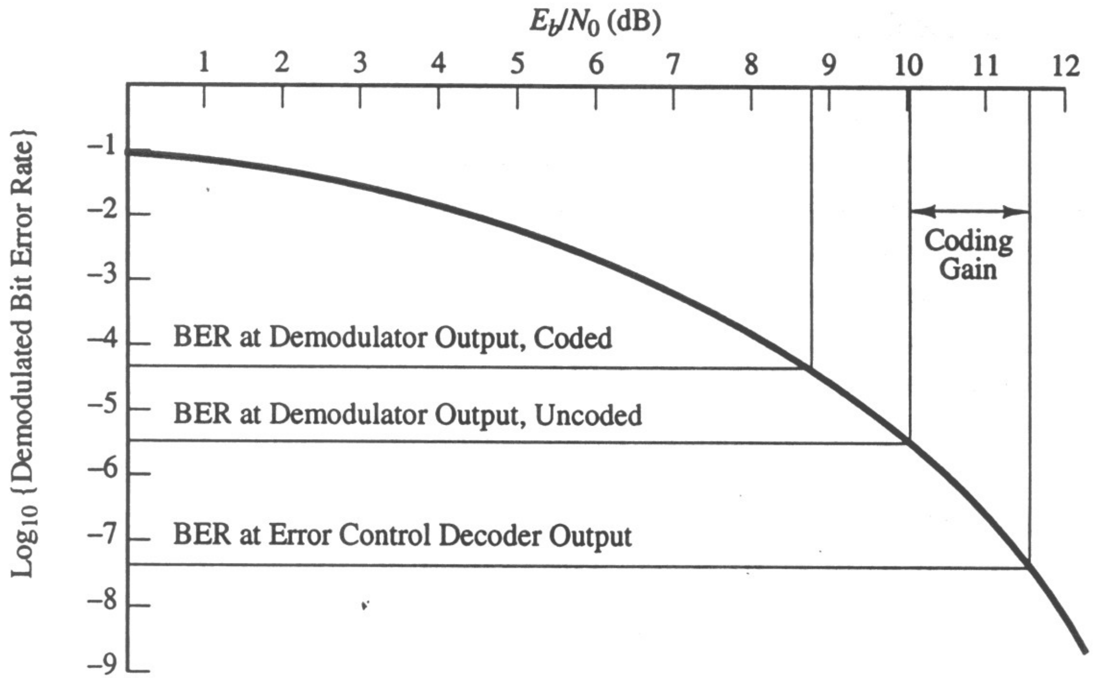
$$\hat{p} = .0136$$

$$P(E) = 5.5 \times 10^{-4}$$

Performance of BPSK and BFSK



Coding Gain with the (15,11) Hamming Code



Single Parity Check Code Example

- Consider the 2^{11} binary words of length 11 as datawords
- Let the probability of error be 10^{-8}
- Bits are transmitted at a rate of 10^{7} bits per second
- The probability that a word is transmitted incorrectly is approximately

$$P(E) = 11p(1-p)^{10} \approx \frac{11}{10^{8}}$$

- Therefore $\frac{11}{10^{8}} \cdot \frac{10^{7}}{11} = 0.1$ words per second are transmitted incorrectly.
- One wrong word is transmitted every 10 seconds, 360 erroneous words every hour and 8640 words every day without being detected!

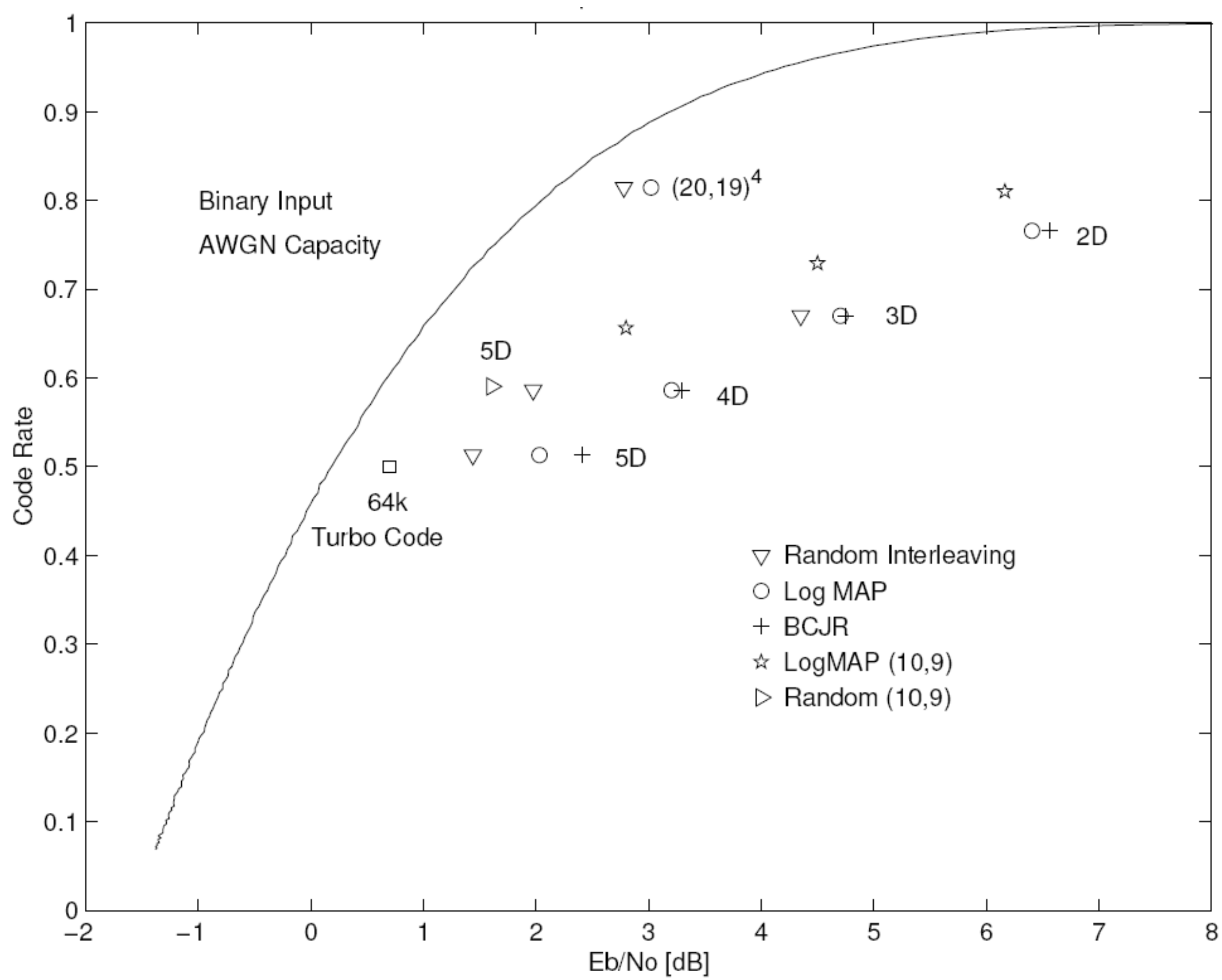
SPC Code Example (Cont.)

- Let one parity bit be added
- Any single error can be detected
- The probability of at least 2 errors is

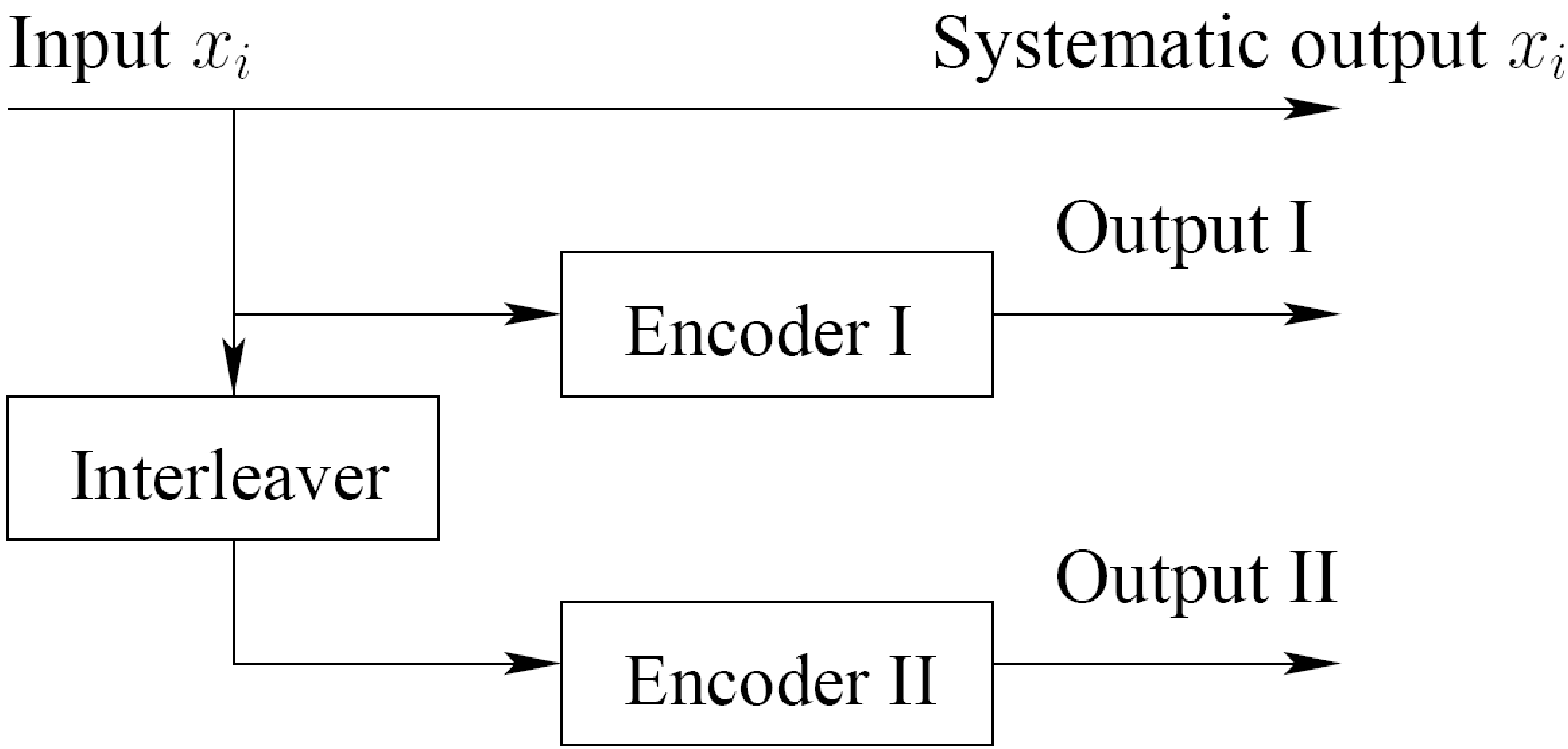
$$1-(1-p)^{12}-12(1-p)^{11}p \approx \binom{12}{2}(1-p)^{10}p^2 \approx \frac{66}{10^{16}}$$

- Therefore approximately $\frac{66}{10^{16}} \cdot \frac{10^7}{12} \approx 5.5 \cdot 10^{-9}$ words per second are transmitted with an undetectable error
- An undetected error occurs only every 2000 days ($2000 \approx 10^9/(5.5 \times 86400)$)

Randomly Interleaved SPC Product Codes with BER=10^{-5}

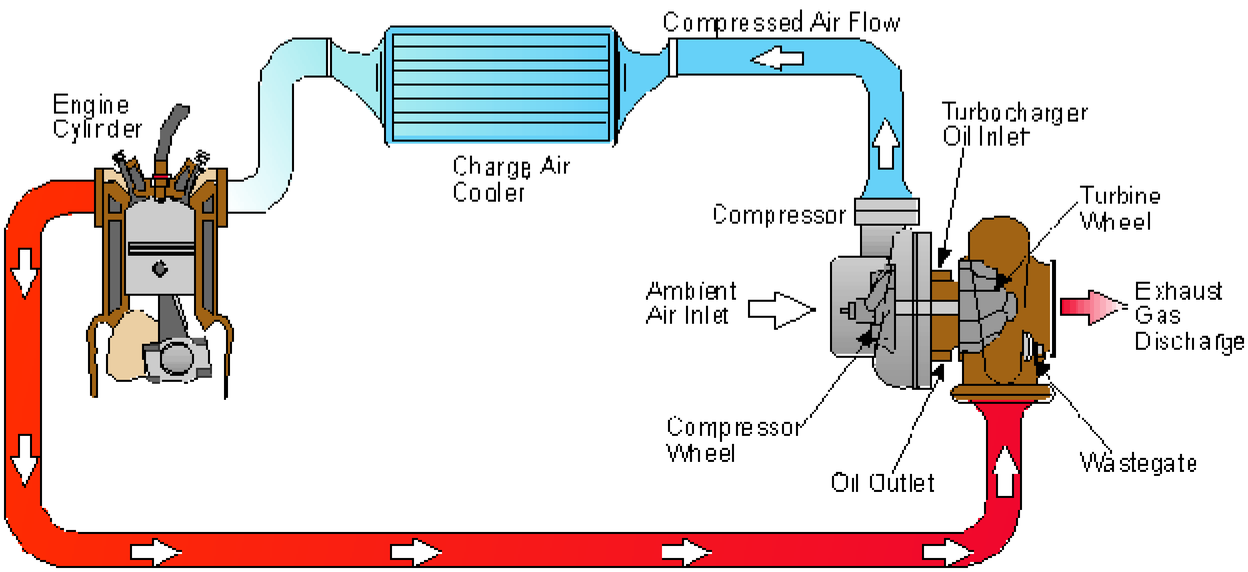


Generic Turbo Encoder

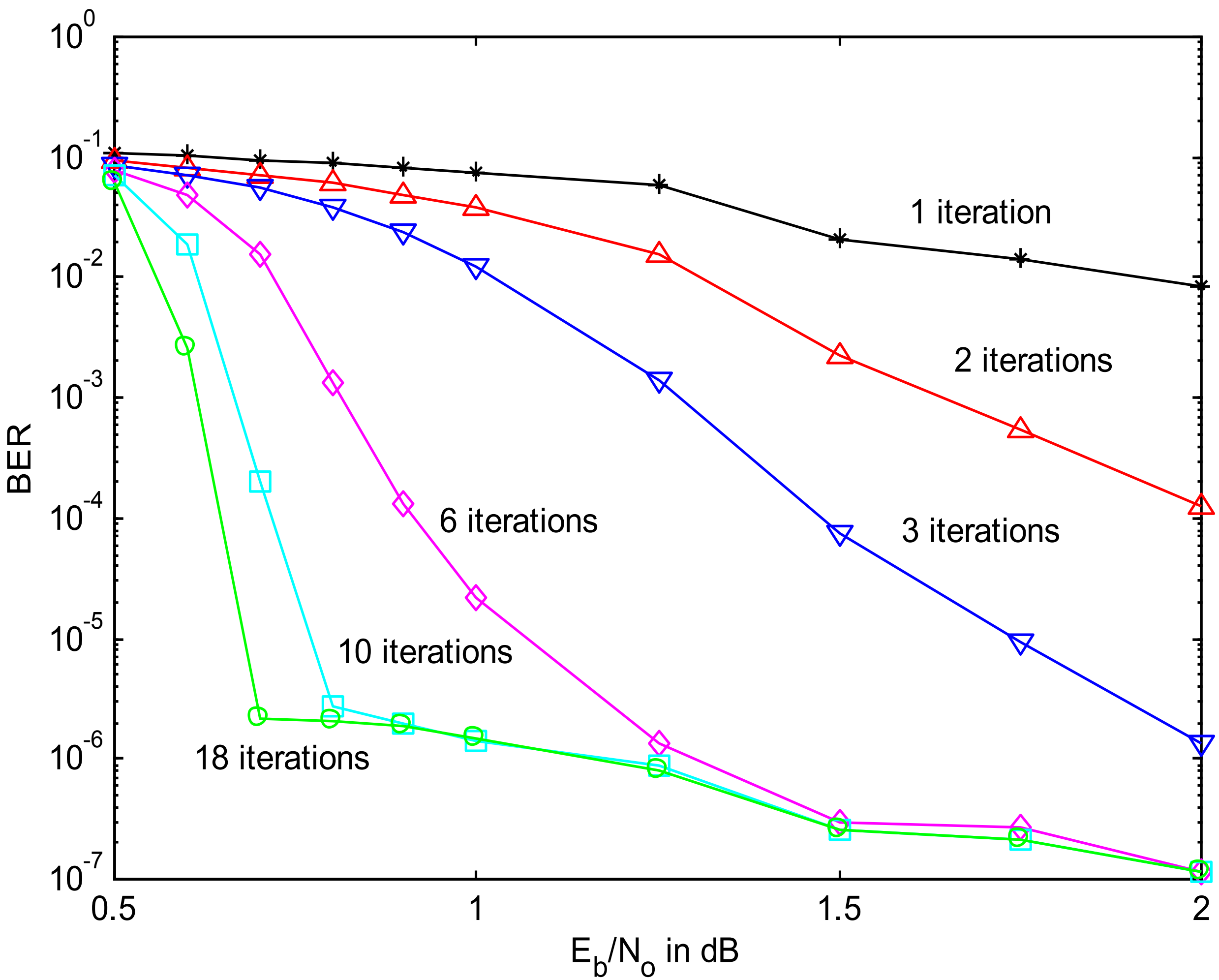


The Turbo Principle

- Turbo codes are so named because the decoder uses feedback, like a turbo-charged engine.



Performance as a Function of Number of Iterations



K=5

R=1/2

n=65,536

The UMTS Turbo Encoder