TRANSIENT RESPONSE ANALYSIS

Test signals:

- Impulse
- Step
- Ramp
- Sin and/or cos

Transient Response: for t between 0 and T

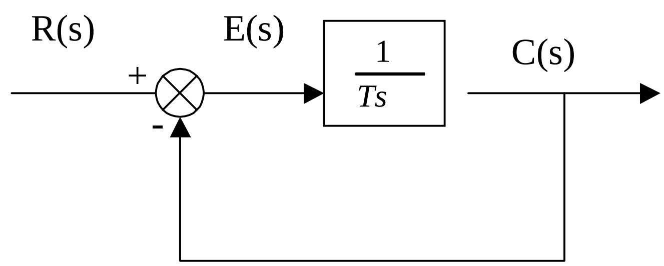
Steady-state Response: for $t \to \infty$

System Characteristics:

- Stability → transient
- Relative stability → transient
- Steady-state error → steady-state

First order systems

$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1}$$



Unit step response:

$$C(s) = \frac{1}{Ts+1} \cdot \frac{1}{s} = \frac{1}{s} - \frac{T}{sT+1}$$

$$c(t) = 1 - e^{-t/T} \qquad t \geq 0$$

$$e(t) = r(t) - c(t) = e^{-t/T} \qquad e(\infty) = 0$$

$$c(T) = 1 - e^{-1} = 0.632$$

$$\frac{dc(t)}{dt}\bigg|_{t=0} = \frac{1}{T} e^{-t/T}\bigg|_{t=0} = \frac{1}{T}$$

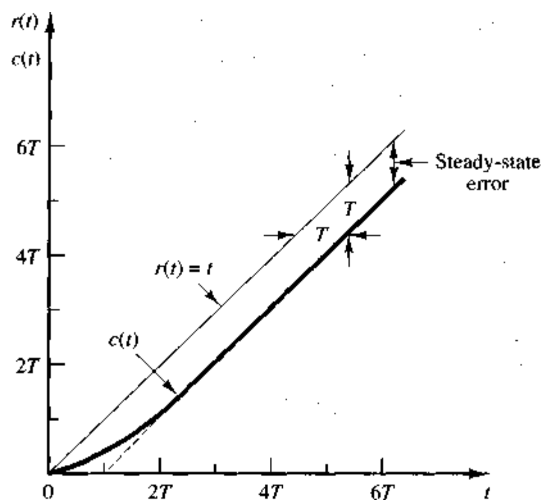
Unit ramp response

$$C(s) = \frac{1}{Ts+1} \cdot \frac{1}{s^2} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

$$c(t) = t - T + Te^{-t/T} \qquad t \geq 0$$

$$e(t) = r(t) - c(t) = T\left(1 - e^{-t/T}\right) \qquad t \geq 0$$

$$e(\infty) = T$$



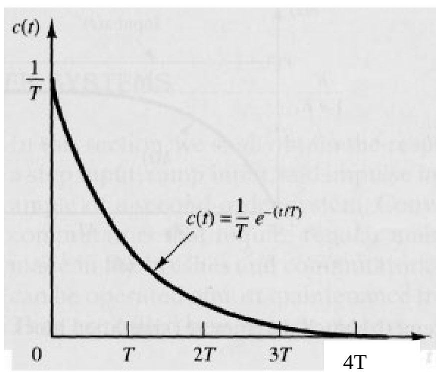
Unit-ramp response of the system

Impulse response:

$R(s) = 1$ $\qquad\qquad$ $r(t) = \delta(t)$

$$C(s) = \frac{1}{sT + 1}$$

$$c(t) = \frac{e^{-t/T}}{T} \qquad t \geq 0$$



Unit-impulse response of the system

Input			Output	
Ramp	r(t) = t	$t \geq 0$	$c(t) = t - T + Te^{-t/T}$	$t \geq 0$
Step	r(t) = 1	$t \geq 0$	$c(t) = 1 - e^{-t/T}$	$t \geq 0$
Impulse	r(t) = δ(t)		$c(t) = \frac{e^{-t/T}}{T}$	$t \geq 0$

Observation:

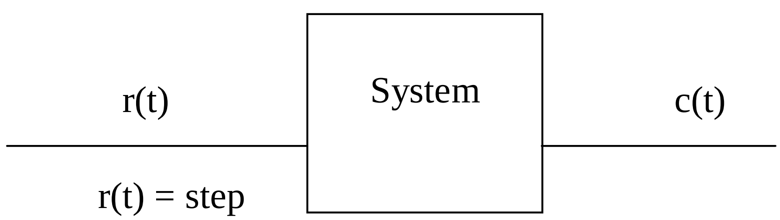
Response to the derivative of an input equals to derivative of the response to the original signal.

$Y(s) = G(s)\,U(s)$ U(s): input

$U_1(s) = s\,U(s)$ $Y_1(s) = s\,Y(s)$ Y(s): output

$G(s)\,U_1(s) = G(s)\,s\,U(s) = s\,Y(s) = Y_1(s)$

How can we recognize if a system is 1st order ?



Plot $\log|c(t) - c(\infty)|$

If the plot is linear, then the system is 1st order

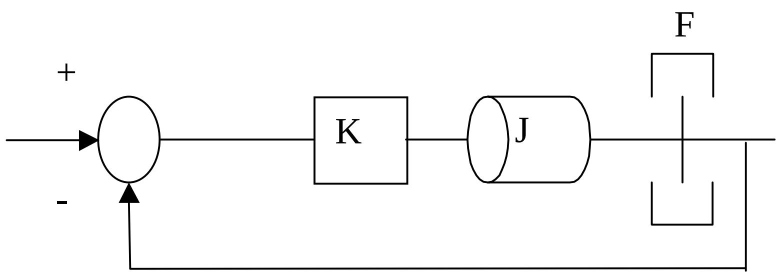
Explanation:

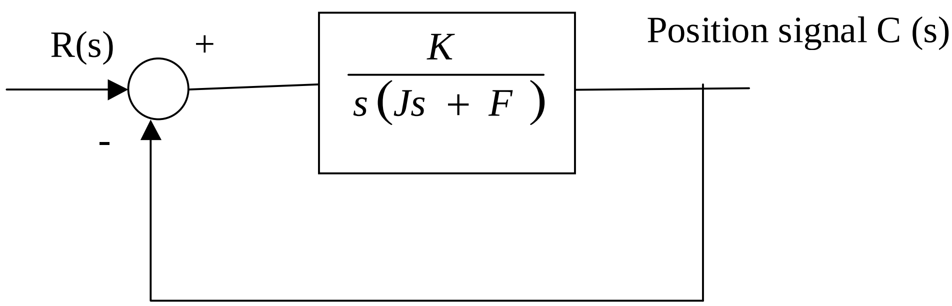
$$c(t) = 1 - e^{-t/T}$$ $c(\infty) = 1$

$$\log|c(t) - c(\infty)| = \log|e^{-t/T}| = \frac{t}{T}$$

Second Order Systems

Block Diagram





Transfer function:

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Fs + K}$$

$$= \frac{\frac{K}{J}}{\left[s + \frac{F}{2J} + \sqrt{\left(\frac{F}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{F}{2J} - \sqrt{\left(\frac{F}{2J}\right)^2 - \frac{K}{J}}\right]}$$

Substitute in the transfer function:

$$\frac{K}{J} = \omega_n^2$$

$$\frac{F}{J} = 2\,\zeta\omega_n = 2\sigma$$

$$\zeta = \frac{F}{2\sqrt{JK}}$$

-

ζ: damping ratio
ω_n: undamped natural frequency
σ: stability ratio

to obtain

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- ***Underdamped*** case: $0 < \zeta < 1$

  $F^2 - 4\,J\,K < 0$ two *complex conjugate* poles

- ***Critically damped*** case: $\zeta = 1$

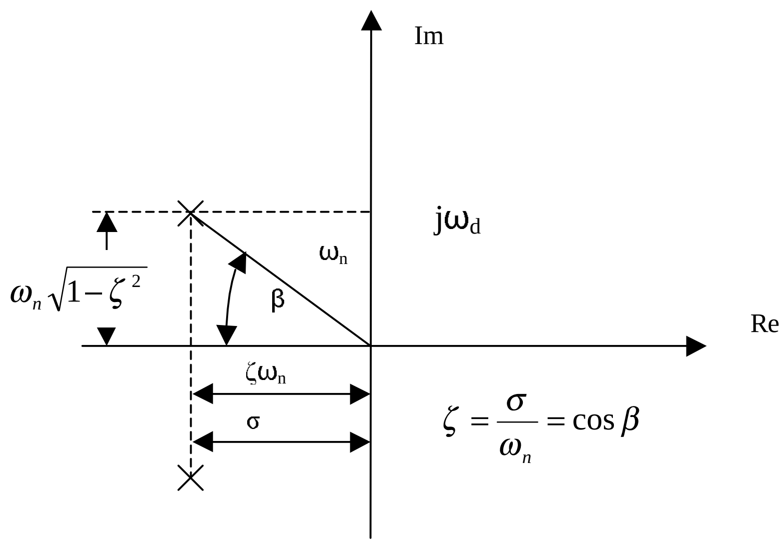
  $F^2 - 4\,J\,K = 0$ two *equal real* poles

- ***Overdamped*** case: $\zeta > 1$

  $F^2 - 4\,J\,K > 0$ two *real* poles

Under damped case $(0 < \zeta < 1)$:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^{\ 2}}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$



$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

ω_n : undamped natural frequency

ω_d : damped natural frequency

ζ : damping ratio

Unit step response:

R(s) = 1/s

$$C(s)=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^{\ 2}}-\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^{\ 2}}$$

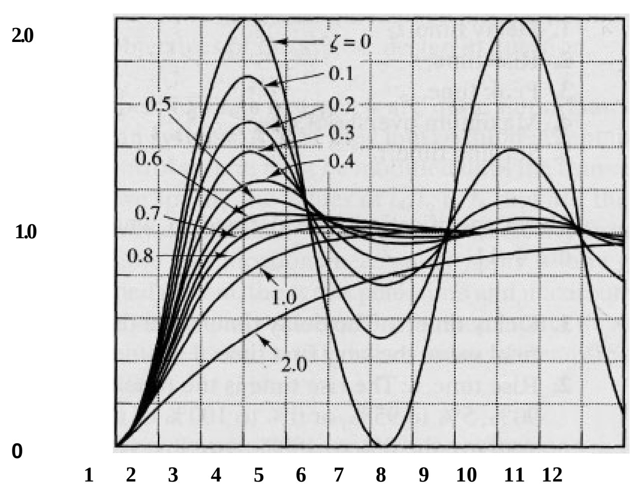
$$c(t)=1-e^{-\zeta\omega_n t}\left(\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\qquad t\geq 0$$

or

$$c(t)=1-\frac{1}{\beta}e^{-\zeta\omega_n t}\sin(\omega_n\beta t+\theta)\qquad t\geq 0$$

$$\beta=\sqrt{1-\zeta^2}\qquad\qquad \theta=\tan^{-1}\frac{\beta}{\zeta}$$

$$\mathrm{e(t)=r(t)-c(t)}=e^{-\zeta\omega_n t}\left(\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\qquad t\geq 0$$



Unit step response curves of a second order system

Undamped case (ζ = 0):

Unit step response:

$$c(t) = 1 - \cos \omega_n t \qquad t \geq 0$$

Critically damped case (ζ = 1):

Unit step Response: R(s) = 1/s

$$\frac{C(s)}{R(s)} = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n + \omega_n^{\ 2}} = \frac{\omega_n^{\ 2}}{(s + \omega_n)^2}$$

$$C(s) = \frac{1}{s(s + \omega_n)^2}$$

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) \qquad t \geq 0$$

Overdamped case (ζ >1):

Unit step Response: R(s) = 1/s

$$C(s) = \frac{\omega_n^2}{\left(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)\left(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)}\cdot\frac{1}{s}$$

$$c(t) = 1+\frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1}-\frac{e^{-s_2 t}}{s_2}\right) \qquad t \geq 0$$

with $s_1 = \left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n$

$$s_2 = \left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n$$

if $|s_2| << |s_1|$, the transfer function can be approximated by

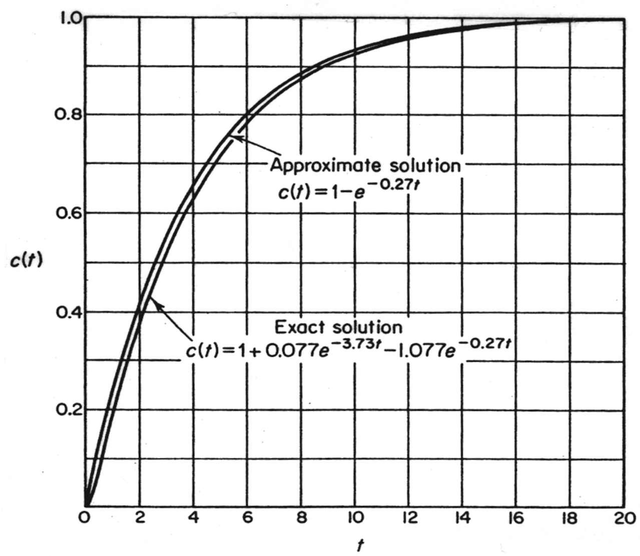
$$\frac{C(s)}{R(s)} = \frac{s_2}{s+s_2}$$

and for R(s) = 1/s

$$c(t) = 1-e^{-s_2 t} \qquad t \geq 0$$

with

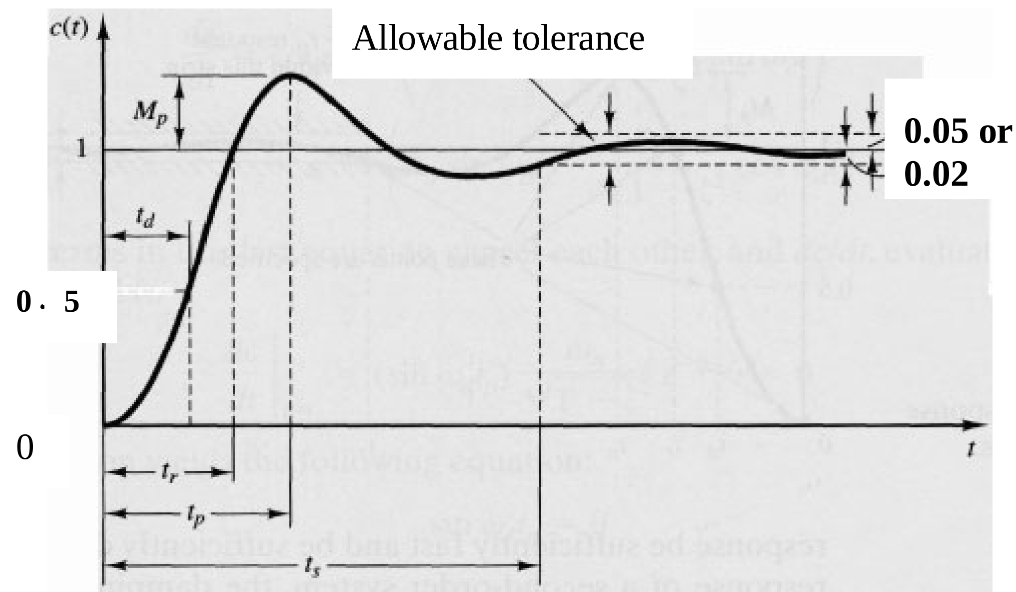
$$s_2 = \left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n$$



Unit step response curves of a critically damped system.

Transient Response Specifications

Unit step response of a 2^{nd} order underdamped system:



t_d *delay time*: time to reach 50% of $c(\infty)$ for the first time.

t_r *rise time* : time to rise from 0 to 100% of $c(\infty)$.

t_p *peak time* : time required to reach the first peak.

M_p *maximum overshoot* : $\frac{c(t_p) - c(\infty)}{c(\infty)} \cdot 100\%$

t_s *settling time* : time to reach and stay within a 2% (or 5%) tolerance of the final value $c(\infty)$.

$$0.4 < \zeta < 0.8$$

Gives a good step response for an underdamped system

Rise time t_r time from 0 to 100% of c(∞)

$$c(t_r)=1 \;\Rightarrow\; 1-e^{-\zeta\omega_n t_r}(\cos\omega_d t_r+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r)=1$$

$$\cos\omega_d t_r+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r=0$$

$$\tan\omega_d t_r=-\frac{\sqrt{1-\zeta^2}}{\zeta}=-\frac{\omega_d}{\sigma}$$

$$t_r=\frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{\sigma}\right)$$

Peak time t_p: time to reach the first peak of c(t)

$$\left.\frac{dc(t)}{dt}\right|_{t=t_p}=0 \quad\Rightarrow\quad (\sin\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p}=0$$

$$\sin\omega_d t_p=0$$

$$t_p=\frac{\pi}{\omega_d}$$

Maximum overshoot M_p:

$$t = t_p = \frac{\pi}{\omega_d}$$

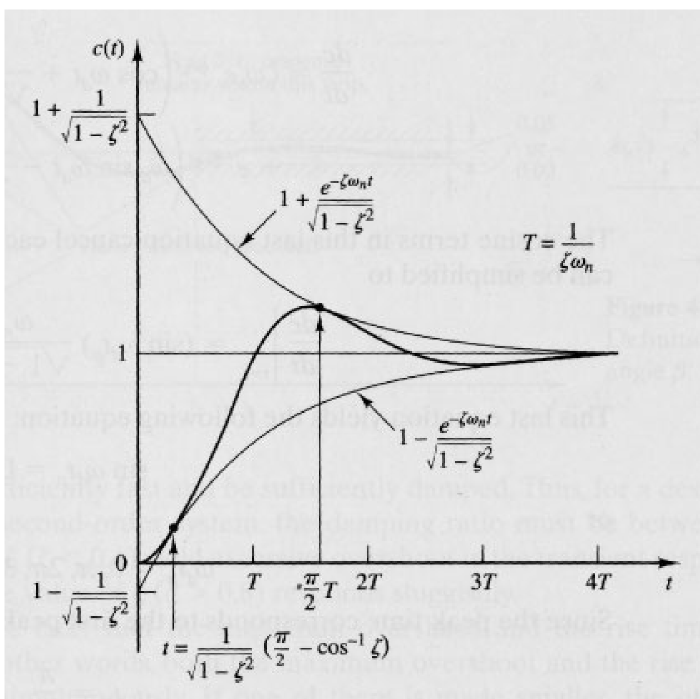
$$M_p = c(t_p) = 1 - e^{-\zeta\omega_n(\pi/\omega_d)}(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi)$$

$$= e^{-\frac{\zeta\omega_n\pi}{\omega_d}} = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{\frac{-\sigma\pi}{\omega_d}}$$

Settling time t_s:

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

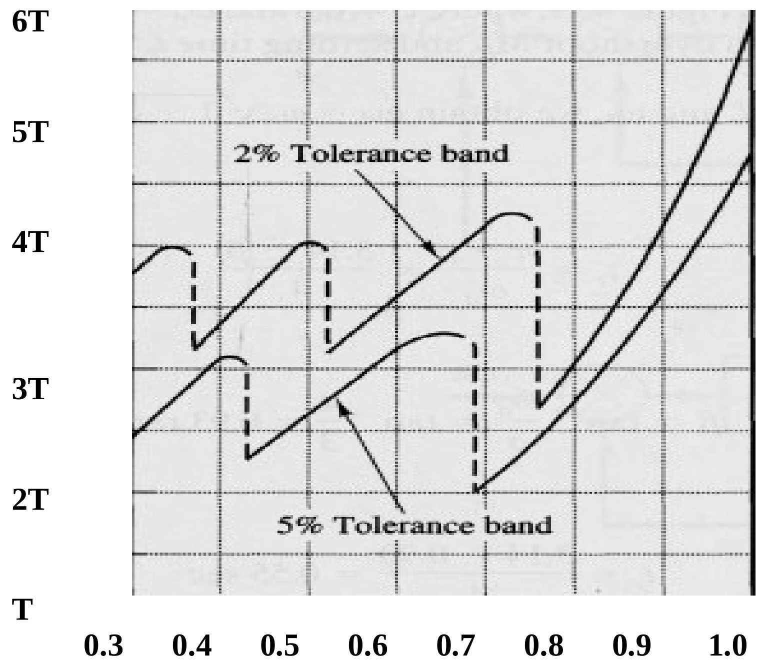
approximate t_s using envelope curves: $env(t) = 1 \pm \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$

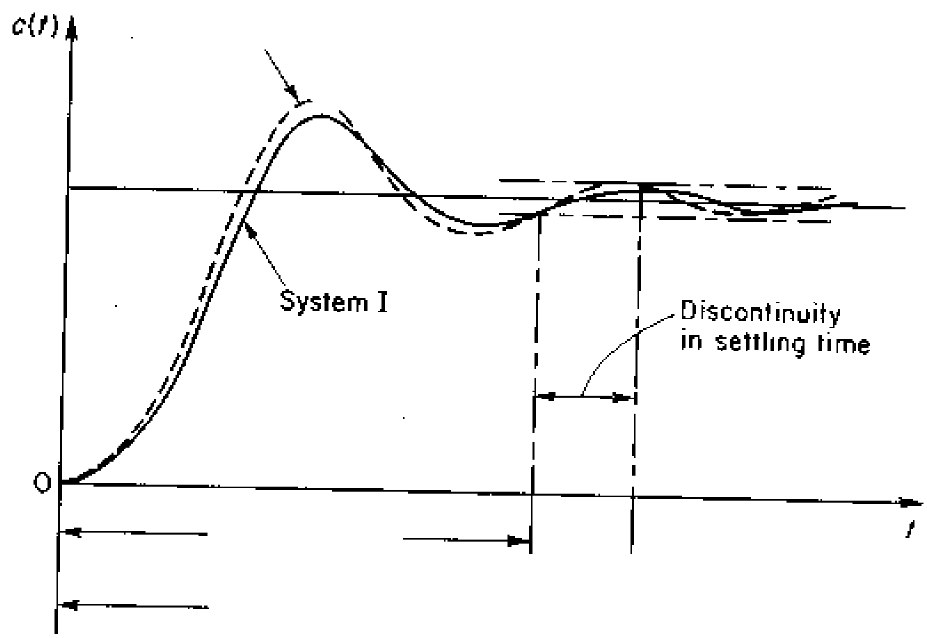


Pair of envelope curves for the unit-step response curve

2% band: $t_s = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n}$ 5% band $t_s = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n}$

Settling time t_s versus ζ curves $\{T = 1/(\zeta\omega_n)\}$





Impulse response of second-order systems

$$C(s) = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2} \qquad\qquad R(s) = 1$$

underdamped case (0< ζ < 1):

$$c(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\omega_n\sqrt{1-\zeta^2}\,t \qquad\qquad t \geq 0$$

the first peak occurs at $t = t_0$

$$t_0 = \frac{\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}}$$

and the maximum peak is

$$c(t_0) = \omega_n \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

critically damped case (ζ = 1):

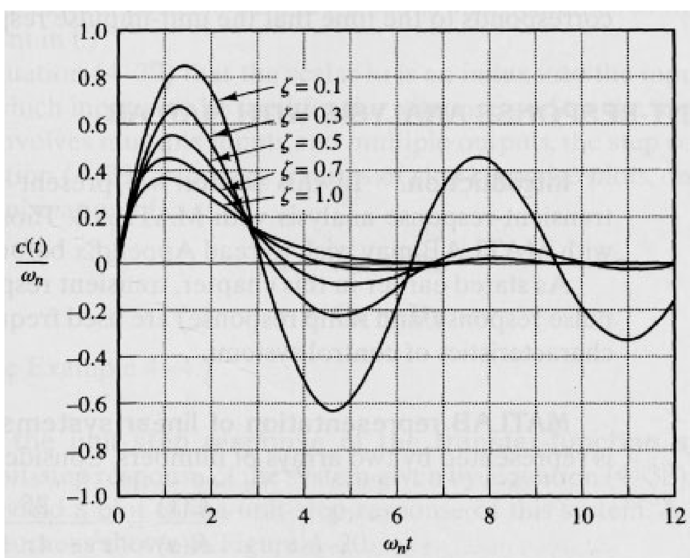
$$c(t) = \omega_n{}^2 t e^{-\omega_n t} \qquad t \geq 0$$

overdamped case (ζ >1):

$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-s_1 t} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-s_2 t} \qquad t \geq 0$$

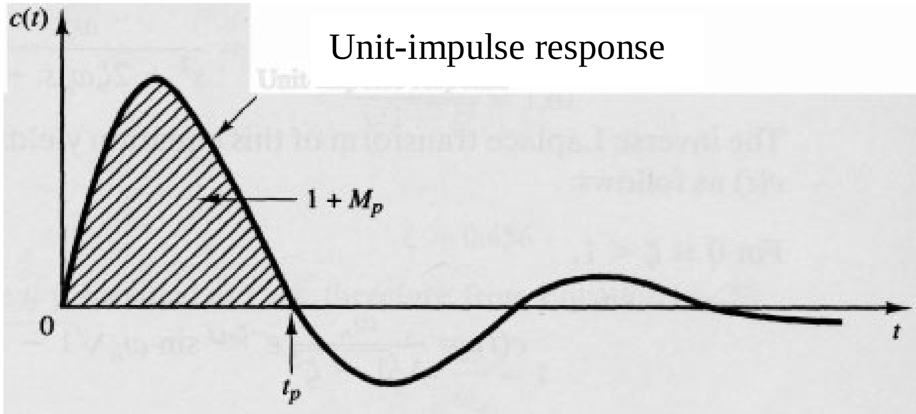
where

$$s_1 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$$
$$s_2 = \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$$



Unit-impulse response for 2nd order systems

Remark: Impulse Response = d/dt (Step Response)



Relationship between t_p, M_p and the unit-impulse response curve of a system

Unit ramp response of a second order system

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} \cdot \frac{1}{s^2}$$

R(s) = 1/ s^2

for an underdamped system $(0 < \zeta < 1)$

$$c(t) = t - \frac{2\zeta}{\omega_n} + e^{-\zeta\omega_n t}\left(\frac{2\zeta}{\omega_n}\cos\omega_d t + \frac{2\zeta^2 - 1}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t\right) \qquad t \geq 0$$

and the error:

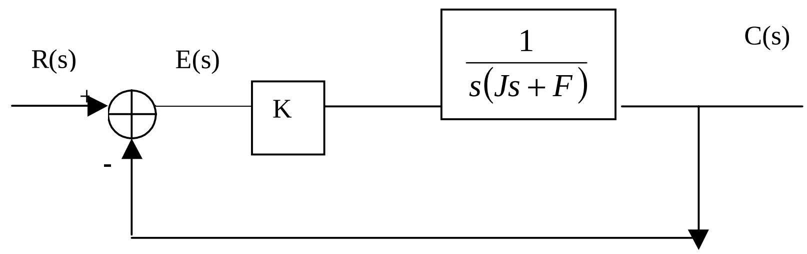
$$e(t) = r(t) - c(t) = t - c(t)$$

at steady-state:

$$e(\infty) = \lim_{t\to\infty} e(t) = \frac{2\zeta}{\omega_n}$$

Examples:

a. Proportional Control



$$\frac{C(s)}{R(s)} = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n + \omega_n^{\ 2}}$$

with

$$\frac{K}{J} = \omega_n^2$$

$$\frac{F}{J} = 2\,\zeta\omega_n = 2\sigma$$

$$\zeta = \frac{F}{2\sqrt{JK}}$$

Choose K to obtain 'good' performance for the closed-loop system

For good *transient response*:

$0.4 < \zeta < 0.8$ → acceptable overshoot

ω_n sufficiently large → good settling time

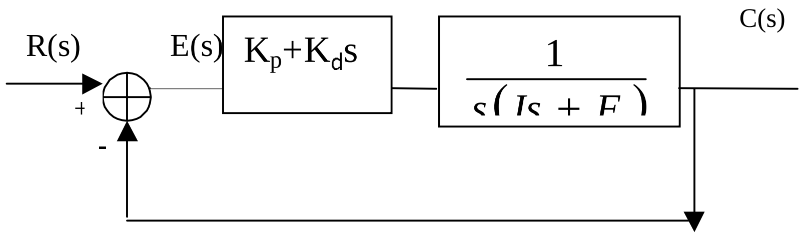
For small *stead- state error in ramp response*:

$$e(\infty) = \lim_{t\to\infty} e(t) = \frac{2\zeta}{\omega_n} = \frac{2F}{2\sqrt{K\zeta}} \cdot \sqrt{\frac{\zeta}{K}} = \frac{F}{K} \quad \rightarrow \quad \text{large K}$$

Large K reduces e(∞) but also leads to small ζ and large M_p

→compromise necessary

b. Proportional plus derivative control:



$$\frac{C(s)}{R(s)}=\frac{K_p+K_d s}{Js^2+(F+K_d)s+K_p}$$

with

$$\varsigma = \frac{F + K_d}{2\sqrt{K_p J}} \qquad \omega_n = \sqrt{\frac{K_p}{J}}$$

The error for a ramp response is:

$$E(s)=\frac{s^2 J + sF}{s^2 J + s(F+K_d)+K_p}\cdot R(s)$$

and at steady-state:

$$e(\infty)=\lim_{s\to 0} sE(s)=\frac{F}{K_p}$$

using $z = \frac{K_p}{K_d}$

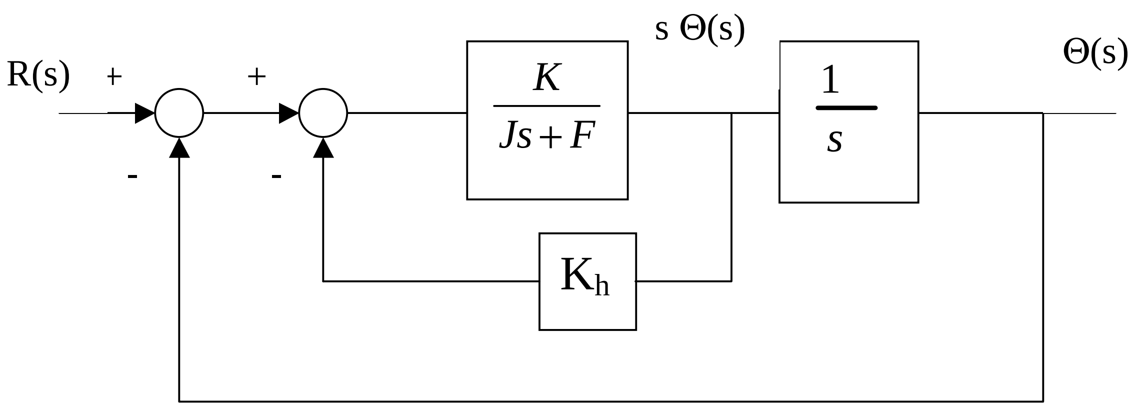
$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{z}\cdot\frac{s+z}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Choose K_p, K_d to obtain 'good' performance of the closed-loop system

For small *steady-state error in ramp response* → K_p large

For good *transient response* → K_d so that $0.4 < \zeta < 0.8$

c. Servo mechanism with velocity feedback



Transfer function

$$\frac{\Theta(s)}{R(s)} = \frac{K}{Js^2 + (F + KK_h)s + K}$$

where

$$\varsigma = \frac{F + KK_h}{2\sqrt{KJ}}$$

$$\omega_n = \sqrt{\frac{K}{J}}$$ (not affected by velocity feedback)

$$e(\infty) = \frac{F}{K}$$ for a ramp

Choose K, K_h to obtain 'good' performance for the closed-loop system

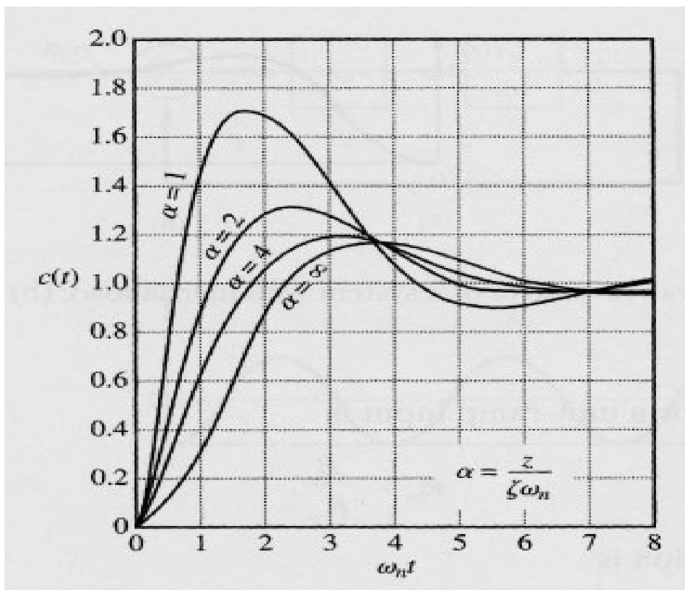
For small *steady-state error in ramp response* → K large

For good *transient response* → K_h so that $0.4 < \zeta < 0.8$

<u>*Remark:*</u> The damping ratio ζ can be increased without affecting the natural frequency ω_n in this case.

Effect of a zero in the step response of a 2nd order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n{}^2}{z} \cdot \frac{s + z}{s^2 + 2\zeta\omega_n s + \omega_n{}^2} \qquad \zeta = 0.5$$



Unit-step response curves of 2nd order systems

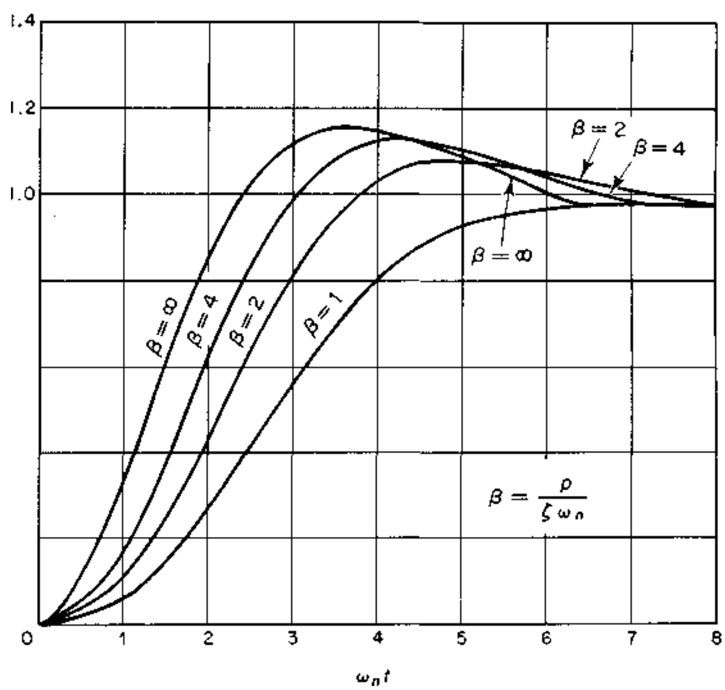
Unit step Response of 3rd order systems

$$\frac{C(s)}{R(s)}=\frac{\omega_n{}^2 p}{\left(s^2+2\zeta\omega_n s+\omega_n{}^2\right)(s+p)} \qquad 0<\zeta<1 \qquad R(s)=1/s$$

$$c(t)=1-\frac{e^{-pt}}{\beta\zeta^2(\beta-2)+1}-\frac{e^{-\xi\omega_n t}}{\beta\zeta^2(\beta-2)+1}\bullet$$

$$\left\{\beta\zeta^2(\beta-2)\cos\sqrt{1-\zeta^2}\,\omega_n t+\frac{\beta\zeta\left[\zeta^2(\beta-2)+1\right]}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

where $\beta=\dfrac{p}{\zeta\omega_n}$



Unit-step response curves of the third-order system, $\zeta = 0.5$

The effect of the pole at $s = -p$ is:

- Reducing the maximum overshoot
- Increasing settling time

Transient response of higher-order systems

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + ... + b_{m-1}s + b_m}{s^n + ... + d_{n-1}s + a_n} = \frac{K(s+z_1)...(s+z_m)}{(s+p_1)...(s+p_n)} \qquad n > m$$

Unit step response

$$C(s) = \frac{K\sum_{i=1}^{m}(s+z_i)}{\sum_{j=1}^{q}(s+p_j)\sum_{k=1}^{r}\left(s^2+2\zeta_k\omega_k s+\omega_k{}^2\right)} \cdot \frac{1}{s}$$

$0 < \zeta_k < 1 \quad k=1,\dots,r \quad$ and $\quad q + 2r = n$

$$C(s) = \frac{a}{s} + \sum_{j=1}^{q}\frac{a_j}{s+p_j} + \sum_{k=1}^{r}\frac{b_k(s+\zeta_k\omega_k)+c_k\omega_k\sqrt{1-\zeta_k{}^2}}{s^2+2\zeta_k\omega_k+\omega_k{}^2}$$

$$c(t) = a + \sum_{j=1}^{q}a_j e^{-p_j t} + \sum_{k=1}^{r}b_k e^{-\zeta_n\omega_k t}\cos\left(\omega_k\sqrt{1-\zeta_k{}^2}\,t\right)$$

$$+\sum_{k=1}^{r}c_k e^{-\zeta_k\omega_k t}\sin\left(\omega_k\sqrt{1-\zeta_k{}^2}\,t\right) \qquad t \geq 0$$

Dominant poles: the poles closest to the imaginary axis.

STABILITY ANALYSIS

$$G(s) = \frac{B(s)}{A(s)} = \frac{\sum_{i=0}^{m} b_i s^{m-i}}{\sum_{i=0}^{n} a_i s^{n-i}}$$

Conditions for Stability:

A. ***Necessary*** condition for stability:

All coefficients of A(s) have the same sign.

B. ***Necessary and sufficient*** condition for stability:

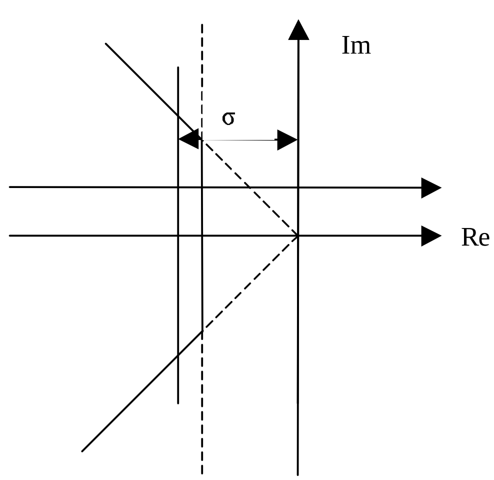
$A(s) \neq 0$ for $\mathrm{Re}[s] \geq 0$

or, equivalently

All poles of G(s) in the left-half-plane (LHP)

Relative stability:

The system is stable and further, all the poles of the system are located in a sub-area of the left-half-plane (LHP).



Necessary condition for stability:

$$
\begin{aligned}
A(s) &= a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n \\
&= a_0 (s + p_1)(s + p_2)\dots(s + p_n) \\
&= a_0 s^n + a_0 (p_1 + p_2 + \dots p_n) s^{n-1} \\
&\quad + a_0 (p_1 p_2 + \dots + p_{n-1} p_n) s^{n-2} \\
&\quad \vdots \\
&\quad + a_0 (p_1 p_2 \dots p_n)
\end{aligned}
$$

$-p_1$ to $-p_n$ are the poles of the system.

If the system is stable → all poles have negative real parts

→ the coefficients of a stable polynomial have the same sign.

Examples:

$A(s) = s^3 + s^2 + s + 1$ can be stable or unstable

$A(s) = s^3 - s^2 + s + 1$ is unstable

Stability testing

Test whether all poles of G(s) (roots of A(s)) have *negative real parts*.

Find all roots of A(s) → too many computations

Easier Stability test?

Routh-Hurwitz Stability Test

$A(s) = \alpha_0 s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n$

s^n	α_0	α_2	α_4	…
s^{n-1}	α_1	α_3	α_5	…
s^{n-1}	b_1	b_2	b_3	
	c_1	c_2		
……				
s^2	e_1	e_2		
s^1	f_1			
s^0	g_1			

$$b_1 = \frac{1}{-a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{1}{-a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$c_1 = \frac{1}{-b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} = \frac{a_3 b_1 - a_1 b_2}{b_1}$$

etc

.

Properties of the Ruth-Hurwitz table:

1. Polynomial A(s) is stable (i.e. all roots of A(s) have negative real parts) if there is *no sign change in the first column*.

2. The *number of sign changes in the first column* is equal to the number of roots of A(s) with positive real parts.

Examples:

A(s) = $a_0 s^2 + \alpha_1 s + \alpha_2$

$$\begin{array}{lll} s^2 & a_0 & a_2 \\ s^1 & a_1 & \\ s^0 & a_2 & \end{array}$$

$\alpha_0 > 0, \quad \alpha_1 > 0, \quad \alpha_2 > 0$ or
$\alpha_0 < 0, \quad \alpha_1 < 0, \quad \alpha_2 < 0$

For 2nd order systems, the condition that all coefficients of A(s) have the same sign is *necessary and sufficient for stability*.

A(s) = $\alpha_0 s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$

$$\begin{array}{lll} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \dfrac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

$\alpha_0 > 0, \quad \alpha_1 > 0, \quad \alpha_3 > 0, \qquad \alpha_1\alpha_2 - \alpha_0\alpha_3 > 0$

(or all first column entries are negative)

Special cases:

1. The properties of the table do not change when all the coefficients of a row are multiplied by the same positive number.

2. If the first-column term becomes zero, replace 0 by ε and continue.
   - If the signs above and below ε are the same, then there is a pair of (complex) imaginary roots.
   - If there is a sign change, then there are roots with positive real parts.

Examples:

$A(s) = s^3 + 2s^2 + s + 2$

s^3	1	1
s^2	2	2
s^1	$0 \to \varepsilon$	
s^0	2	

→ pair of imaginary roots (s= ±j)

$A(s) = s^3 - 3s + 2 = (s-1)^2(s+2)$

s^3	1	-3
s^2	$0 \approx \varepsilon$	2
s^1	$-3 - \frac{2}{\varepsilon}$	
s^0	2	

→ two roots with positive real parts

3. If all coefficients in a line become 0, then A(s) has roots of equal magnitude radially opposed on the real or imaginary axis. Such roots can be obtained from the roots of the auxiliary polynomial.

Example:

$A(s)= s^5 + 2s^4 +24 s^3 + 48s^2 -25s -50$

s^5	1	24	-25
s^4	2	48	-50
s^3	0	0	

→ auxiliary polynomial p(s)

$$p(s) = 2s^4 + 48s^2 - 50$$

$$\frac{dp(s)}{ds} = 8s^3 + 96s$$

s^3	8	96
s^2	24	-50
s^1	112.7	0
s^0	-50	

- A(s) has two radially opposed root pairs (+1,-1) and (+5j,-5j) which can be obtained from the roots of p(s).
- One sign change indicates A(s) has one root with positive real part.

Note:

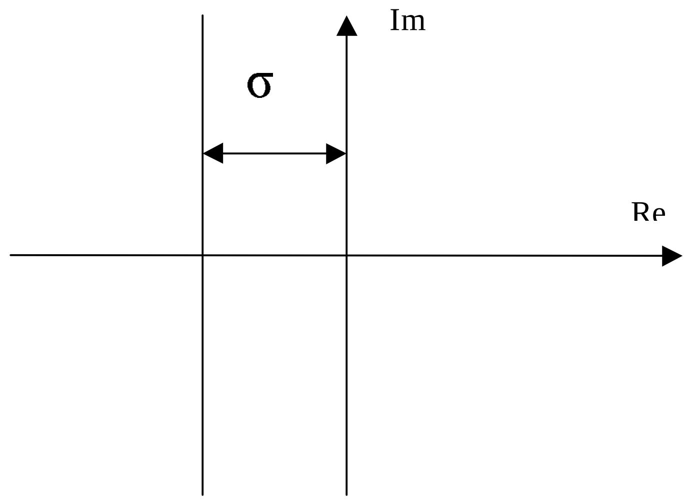
$A(s) = (s+1)(s-1)(s+5j)(s-5j)(s+2)$

$p(s) = 2(s^2-1)(s^2 +25)$

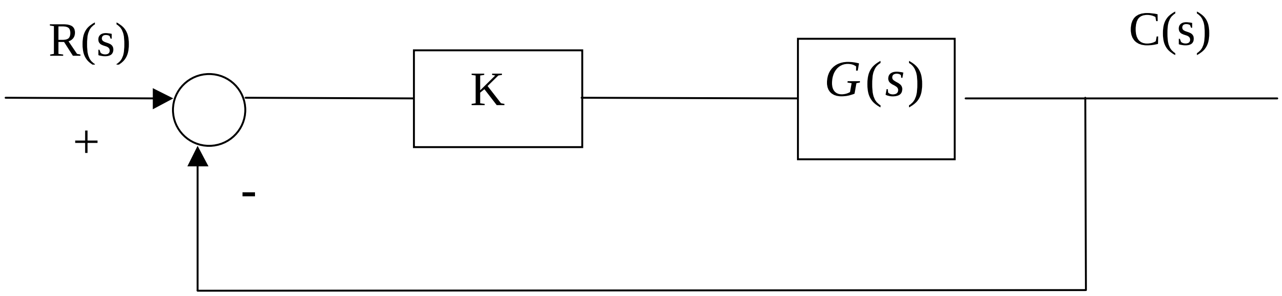
Relative stability

Question: Have all the roots of A(s) a distance of at least σ from the imaginary axis?

Substitute s with $s = z - \sigma$ in A(s) and apply the Routh-Hurwitz test to A(z)



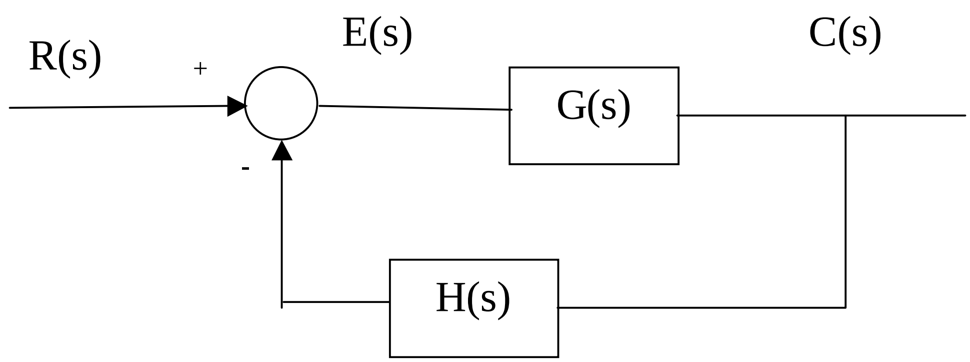
Closed-loop System Stability Analysis



Question: For what value of K is the closed-loop system stable?

Apply the Routh-Hurwitz test to the denominator polynomial of the closed-loop transfer function $\dfrac{KG(s)}{1 + KG(s)}$.

Steady-State Error Analysis



Evaluate the steady-state performance of the closed-loop system using the steady-state error e_{ss}

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

$$E(s) = \frac{1}{1 + G(s)H(s)} \cdot R(s)$$

for the following input signals:
- Unit step input
- Unit ramp input
- Unit parabolic input

Assumption: the closed-loop system is stable

Question: How can we obtain the steady-state error e_{ss} of the closed-loop system from the open-loop transfer function G(s)H(s) ?

Classification of systems:

For an open-loop transfer function

$$G(s)H(s) = \frac{K(T_a s+1)(T_b s+1)\cdots}{s^N(T_1 s+1)(T_2 s+1)\cdots}$$

Type of system: Number of poles at the origin, i.e., N

Static Error Constants: K_p, K_v, K_a

Open-loop transfer function: G(s)H(s)

Closed-loop transfer function: $G_{tot}(s) = \dfrac{G(s)}{1+G(s)H(s)}$

Static Position Error Constant: K_p

Unit step input to the closed-loop system shown in fig, p. B33.

R(s) = 1/s $\quad e_{ss} = \lim_{s\to 0} sE(s) = \dfrac{1}{1+G(0)H(0)}$

Define: $\quad K_p = \lim_{s\to 0} G(s)H(s) = G(0)H(0)$

Type 0 system $\quad K_p = K \quad e_{ss} = \dfrac{1}{1+K_p}$

Type 1 and higher $\quad K_p = \infty \quad e_{ss} = 0$

Static Velocity Error Constant: **K_v**

Unit ramp input to the closed-loop system shown if fig, p. B33.

R(s) = $1/s^2$ $$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \cdot \frac{1}{s^2} = \lim_{s\to 0} \frac{1}{sG(s)H(s)}$$

Define: $$K_v = \lim_{s\to 0} sG(s)H(s)$$

Type 0 system	$K_v = 0$	$e_{ss} = \infty$
Type 1 system	$K_v = K$	$e_{ss} = 1/K_v$
Type 2 and higher	$K_v = \infty$	$e_{ss} = 0$

Static Acceleration Error Constant: K_a

Unit parabolic input to the closed-loop system shown in fig, p. B33

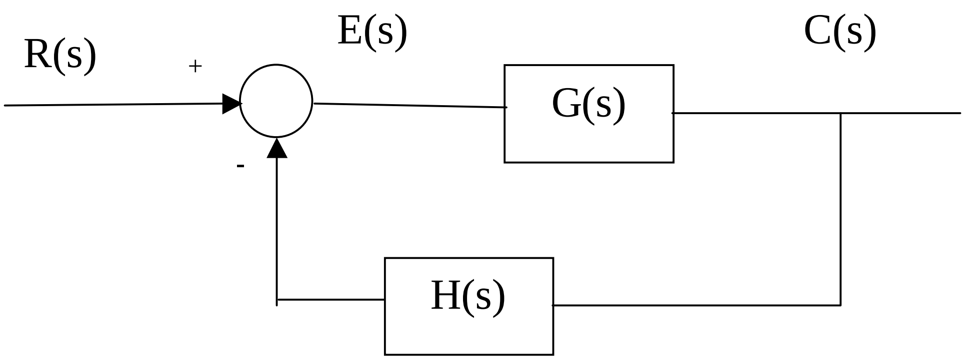
R(s) = $1/s^3$ $$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \cdot \frac{1}{s^3} = \lim_{s\to 0} \frac{1}{s^2 G(s)H(s)}$$

Define: $$K_a = \lim_{s\to 0} s^2 H(s)G(s)$$

Type 0 system	$K_a = 0$	$e_{ss} = \infty$
Type 1 system	$K_a = 0$	$e_{ss} = \infty$
Type 2 system	$K_a = K$	$e_{ss} = 1/K_a$
Type 3 and higher	$K_a = \infty$	$e_{ss} = 0$

Summary:

Consider a closed-loop system:



with an open-loop transfer function:

$$G(s)H(s) = \frac{K(T_a s + 1)\cdot(T_b s + 1)\ldots}{s^N (T_1 s + 1)\cdot(T_2 s + 1)\ldots}$$

and static error constants defined as:

$$K_p = \lim_{s \to 0} G(s)H(s) = G(0)H(0)$$

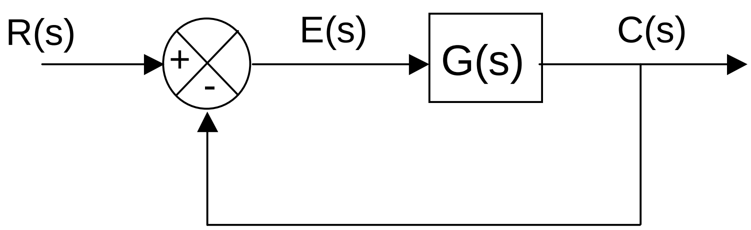
$$K_v = \lim_{s \to 0} sG(s)H(s)$$

$$K_a = \lim_{s \to 0} s^2 H(s)G(s)$$

The steady-state error e_{ss} is given by:

	Unit step $r(t) = 1$	Unit ramp $r(t) = t$	Unit parabolic $r(t) = t^2/2$
Type 0	$e_{ss} = \frac{1}{1+K_p} (= \frac{1}{1+K})$	$e_{ss} = \infty$	$e_{ss} = \infty$
Type 1	$e_{ss} = 0$	$e_{ss} = \frac{1}{K_v} (= \frac{1}{K})$	$e_{ss} = \infty$
Type 2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = \frac{1}{K_a} (= \frac{1}{K})$

Correlation between the Integral of error in step response and Steady-state error in ramp response



$$E(s) = L[e(t)] = \int_0^\infty e^{-st} e(t)dt$$

$$\lim_{s \to 0} E(s) = \lim_{s \to 0} \int_0^\infty e^{-st} e(t)\,dt = \int_0^\infty e(t)\,dt$$

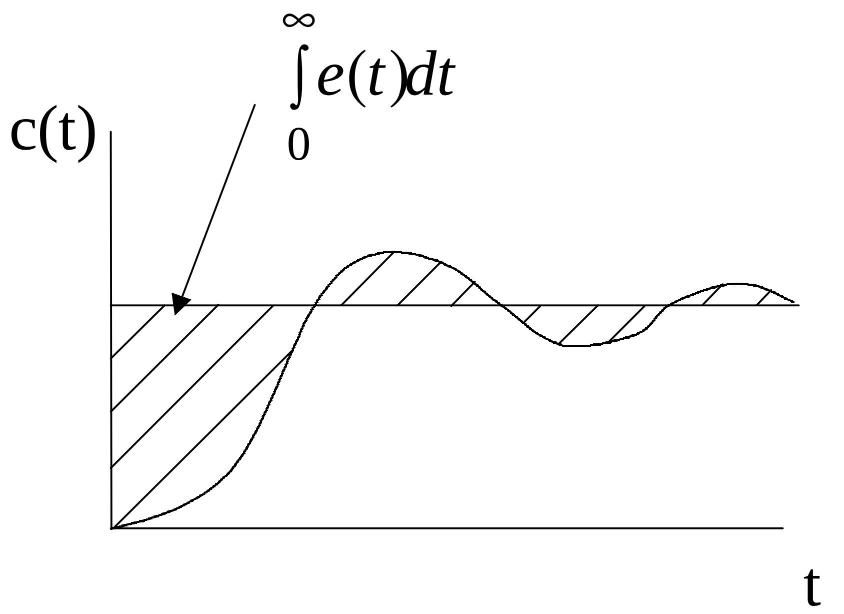
substitute $E(s) = \dfrac{R(s)}{1+G(s)}$ in the above eq.

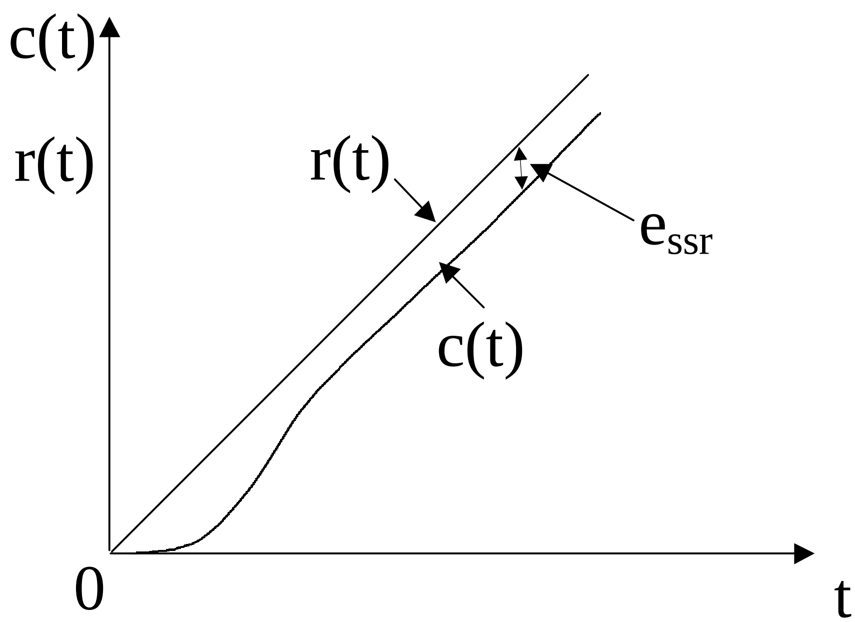
$$\lim_{s\to 0} \frac{R(s)}{1+G(s)} = \int_0^\infty e(t)\,dt \qquad step: \quad R(s) = \frac{1}{s}$$

$$\int_0^\infty e(t)dt = \lim_{s\to 0}\left[\frac{1}{1+G(s)} \cdot \frac{1}{s}\right] = \lim_{s\to 0} \frac{1}{s \cdot G(s)} = \frac{1}{K_v}$$

$\dfrac{1}{K_V}$ = Steady-state error in unit-ramp input = e_{ssr}

$$\boxed{e_{ssr} = \int_0^\infty e(t)dt}$$


∞
∫e(t)dt
0
c(t)
t


c(t)
r(t)
r(t)
e_ssr
c(t)
0
t