

# On the Roots of Digital Signal Processing—Part II

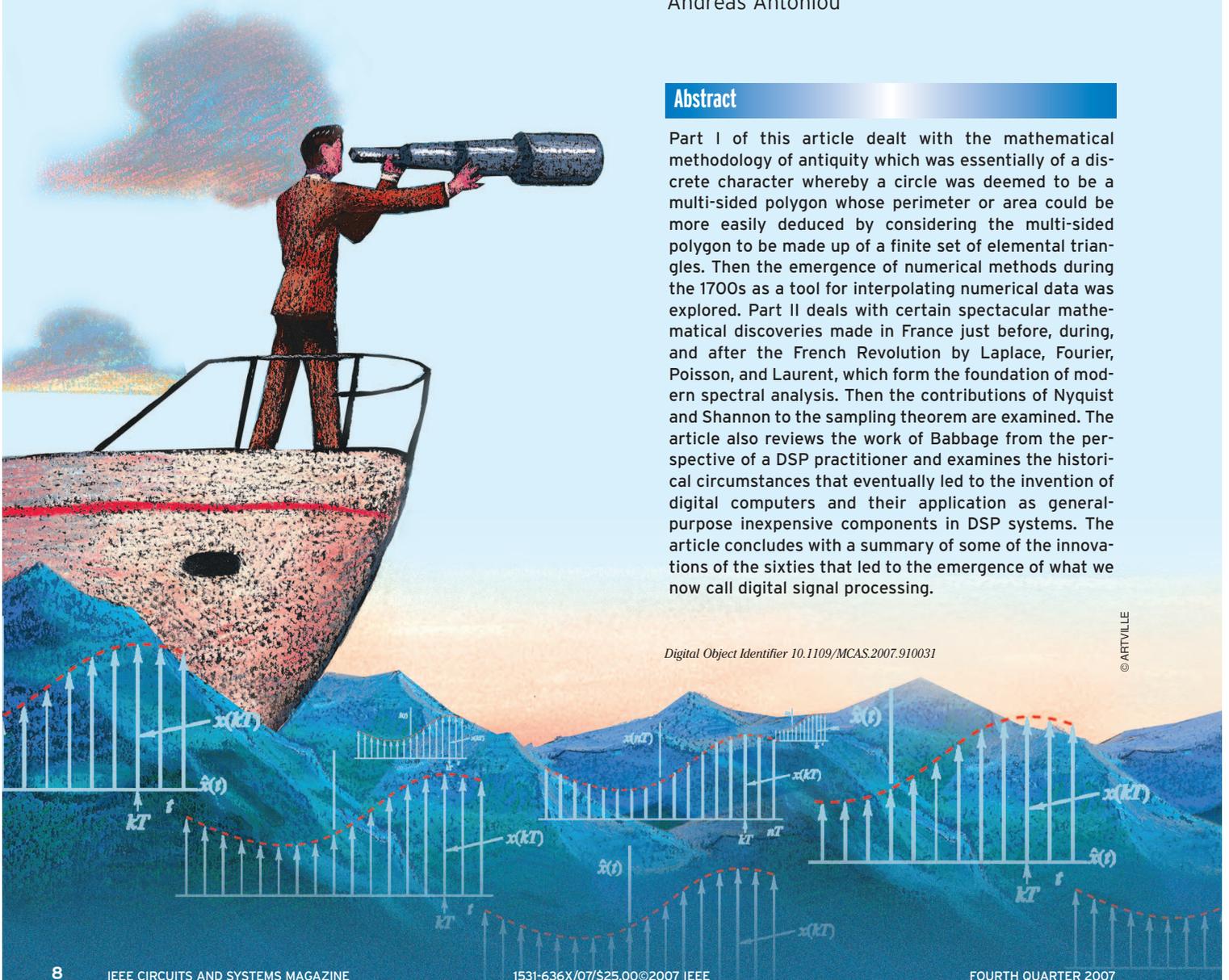
Andreas Antoniou

## Abstract

Part I of this article dealt with the mathematical methodology of antiquity which was essentially of a discrete character whereby a circle was deemed to be a multi-sided polygon whose perimeter or area could be more easily deduced by considering the multi-sided polygon to be made up of a finite set of elemental triangles. Then the emergence of numerical methods during the 1700s as a tool for interpolating numerical data was explored. Part II deals with certain spectacular mathematical discoveries made in France just before, during, and after the French Revolution by Laplace, Fourier, Poisson, and Laurent, which form the foundation of modern spectral analysis. Then the contributions of Nyquist and Shannon to the sampling theorem are examined. The article also reviews the work of Babbage from the perspective of a DSP practitioner and examines the historical circumstances that eventually led to the invention of digital computers and their application as general-purpose inexpensive components in DSP systems. The article concludes with a summary of some of the innovations of the sixties that led to the emergence of what we now call digital signal processing.

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## I. Introduction

Part I of this article [1] has highlighted the landmark events in mathematics from the perspective of the DSP practitioner starting with the great work of Archimedes on the evaluation of  $\pi$ , continuing with the contributions of Wallis, Gregory, and Newton on the evolution of calculus, and concluding with the emergence of numerical methods during the 18th century as a tool for the interpolation of numerical data.

Part II of this review deals with some key breakthroughs made by Laplace, Fourier, Poisson, and Laurent in France just before, during, and soon after the French Revolution. These achievements along with the contributions of Nyquist and Shannon led to frequency-domain characterizations for continuous- and discrete-time signals and the discovery of the sampling theorem. Since a bandlimited continuous-time signal can be reconstructed from a set of samples of the signal, a continuous-time signal can be represented by a sampled version which can be processed by a discrete-time system. By applying interpolation to a processed sampled signal, a continuous-time signal can be constructed, which can be deemed to be a processed version of the original continuous-time signal. In effect, the sampling theorem may be deemed to be the basis of a large segment of today's DSP methodology.

The rapid advancements in mathematics and most other sciences during the Renaissance led to commensurate advancements in engineering, manufacturing, transportation, navigation, trade, banking, etc. Consequently, a great need for numerical calculations emerged be it to estimate the position of a ship using astronomical measurements, to establish the trajectory of a heavenly body, or to design a bridge or steam engine. To expedite such calculations, published numerical tables, such as logarithm and trigonometric tables, had been in use since the 1600s. The calculations necessary to construct numerical tables were carried out by people who spent endless monotonous

hours performing manual calculations. The tables were then set to print by people who had little or no knowledge or interest in the numerical calculations involved. The end result was that published tables contained numerous errata. From the 17th century on, a number of notable scientists and engineers, including Pascal and Leibniz, attempted to construct calculating machines to alleviate the burden of numerical calculations. The most ambitious of these individuals was Babbage who attempted to construct machines he called difference engines that would perform the necessary computations as well as print the numerical tables without human intervention thus resolving the *crisis of the tables* [2]–[4]. In due course, during and soon after World War II, renewed interest in constructing efficient computing machines emerged, and again the principal motivation was to construct numerical tables. In fact, the most renown of the computers of the 1940s, the Electronic Numerical Integrator and Computer (ENIAC), was actually built to construct numerical tables just like Babbage's numerical difference engines. In Sec. VII of this article, the work of Babbage is reviewed in the context of DSP and, as will be shown, what he spent the largest part of his professional life on was to design and build a mechanical discrete system rather than a general-purpose computing machine.

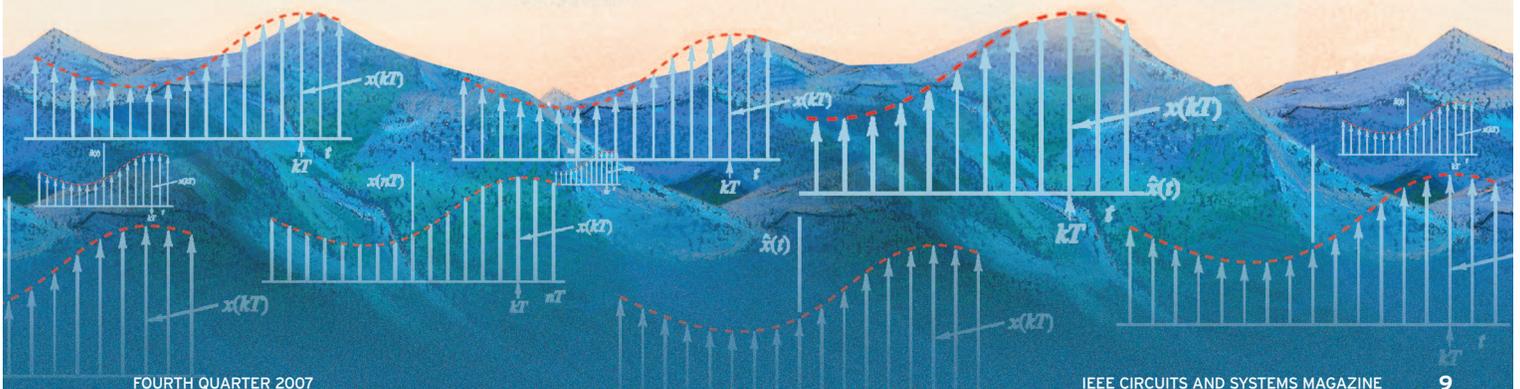
## II. Fourier

One of the most substantial contributions to the mathematics of signal analysis must be attributed to Jean Baptiste Joseph Fourier (1768–1830) who proposed the Fourier series as part of a paper on heat transfer.

Although very interested in mathematics during his teenage years, Fourier decided in 1787 to train for the priesthood.<sup>1</sup> However, he continued to be interested in mathematics and in 1790 he abandoned his religious ambitions to become a teacher at the school where he completed his secondary education. In due course, he

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## The rapid advancements in mathematics and most other sciences during the Renaissance led to commensurate advancements in engineering, manufacturing, transportation, navigation, trade, banking, etc.

got involved in the politics of the French Revolution and was imprisoned as a consequence of a speech he gave in Orleans but was later released. In 1794 he was nominated to study at École Normale in Paris where he was taught by Lagrange (1736–1813) and Laplace (1749–1827). Soon after he was appointed at the École Centrale des Travaux Publics which was soon to be renamed École Polytechnique. His earlier political problems resurfaced and caused him to be imprisoned again but he was later released, some say through the pleas of Lagrange and Laplace, to resume his teaching at École Polytechnique in 1795. In 1797 he succeeded Lagrange as Chair of Analysis and Mechanics. In 1798 Fourier was selected to accompany Napoleon's army in its invasion of Egypt as scientific advisor. His duties included the establishment of educational facilities in Egypt and in this capacity he was appointed Secretary of the Institut d' Egypte.

In 1801, Fourier returned with what remained of Napoleon's expeditionary force to France to take up his post at École Polytechnique. Soon after he was appointed by Napoleon as the Prefect of the Department of Isere in Grenoble. In this capacity, he had to supervise the draining of the swamps of Bourgoin and the construction of a new highway from Grenoble to Turin. During the period 1804–1807, he also found time to carry out research work, presumably in his spare time, on heat transfer. He presented his results in a paper (memoir) entitled *On the Propagation of Heat in Solid Bodies* which was read to the Institut de France in 1807. The paper caused controversy from the start. The committee appointed to report on the work, which included Fourier's past teachers Lagrange and Laplace as members, opposed the work on account of 'analytic difficulty' in the derivation of the heat transfer equations involved and the extensive use of trigonometric series, now known universally as the Fourier series, in the derivations; on the other hand, a certain Biot complained that Fourier did not give due reference to his paper on the derivation of the equations which, incidentally, was found to be in error in recent years! To resolve the issue once and for all, the Institut de France made the propagation of heat the subject of the Grand Prize in mathematics for 1811. There was another candidate in addition to Fourier and the committee set up to select

the winning submission, which included Lagrange and Laplace, awarded the prize to Fourier. Nevertheless, the written report of the committee expressed reservation about the generality and rigor of the work. Formal publication of the work did not take place until 1822 when a treatise *Théorie Analytique de la Chaleur*, authored by Fourier, was finally published by the Academie des Sciences. The controversy continued among mathematicians for some years until Dirichlet, a student of Fourier, published a convergence theorem for the Fourier series in 1829 [7].

The Fourier series together with the Fourier transform, i.e.,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

provide a complete frequency-domain description of periodic and nonperiodic signals. On the other hand, the analytic continuation of the Fourier transform into the  $s$  domain, which amounts to replacing  $j\omega$  by a complex variable  $s$ , gives the Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

The Laplace transform of the impulse response of a continuous-time system gives the transfer function of the system and by converting the continuous- into a discrete-time transfer function, a great variety of digital filters can be obtained from analog filters (see Chaps. 11, 12, and 17 in [8]).

### III. Poisson

Another student of Lagrange and Laplace who made substantial contributions to science and mathematics was Siméon Denis Poisson (1781–1840). His name is associated with an integral in potential theory, a probability distribution, a summation formula, a ratio in elasticity, and a constant in electricity.

His father wanted Siméon to become a surgeon and sent him off to serve as an apprentice surgeon under the tutelage of an uncle. Handicapped by a dreadful lack of dexterity not to mention a lack of motivation for the medical profession, Siméon soon failed in his attempt. With his father's consent, he began to study his favorite subject, mathematics, at École Polytechnique in 1798. In 1800 he became lecturer, advanced to deputy profes-

<sup>1</sup>The historical details found in this and the following two sections are based on information found at the Mac Tutor History of Mathematics Archive [5] and the historical works of Bower and Merzbach [6] and Burton [7].

son in 1802, and was appointed in 1806 to the professorship vacated by Fourier upon his departure for Grenoble.

The one of Poisson's contributions that is of fundamental importance to DSP is his summation formula which is a general mathematical result associated with the Fourier series. In the context of signal analysis, it can be expressed as

$$\sum_{n=-\infty}^{\infty} x(t+nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(jn\omega_s)e^{jn\omega_s t}$$

where  $x(t)$  is a signal,  $X(j\omega)$  is its Fourier transform or frequency spectrum,  $T$  is a period in s, and  $\omega_s = 2\pi/T$  is a frequency in rad/s. By using the Poisson summation formula, one can show that the spectrum of a sampled signal  $\hat{x}(t)$  is given by

$$\hat{X}(j\omega) = X_D(e^{j\omega T}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s) \quad (1)$$

where  $X_D(e^{j\omega T})$  is the z transform of  $x(nT)$  evaluated on the unit circle  $z = e^{j\omega T}$  of the z plane,  $T$  is the sampling period, and  $\omega_s$  is the sampling frequency (see pp. 290–291 [8] for derivation). In effect, *given the frequency spectrum  $X(j\omega)$  of a continuous-time signal  $x(t)$ , the spectrum of the corresponding discrete-time signal  $x(nT)$  can be readily obtained.*

Given a continuous-time signal

$$x(t) = u(t)e^{-at} \sin \omega_0 t \quad (2)$$

with a frequency spectrum  $X(j\omega)$ , the frequency spectrum of discrete-time signal  $x(nT)$  can be obtained from  $X(j\omega)$  by summing an infinite series of shifted copies of  $X(j\omega)$  and then multiplying the sum so obtained by the factor  $1/T$  according to Eq. (1). The usefulness of Eq. (1) can be illustrated by plotting the amplitude spectra of  $x(t)$  and  $x(nT)$  for the sampling frequencies of 15, 25, and 40 rad/s as illustrated in Fig. 1(a) to (c).

The shifted copies of  $X(j\omega)$  or sidebands, namely,  $\dots$ ,  $X(j\omega - j2\omega_s)$ ,  $X(j\omega - j\omega_s)$ ,  $X(j\omega + j\omega_s)$ ,  $X(j\omega + j2\omega_s)$ ,  $\dots$  overlap with the baseband  $-\omega_s/2 < \omega < \omega_s/2$  and, therefore, the sum in Eq. (1) can be expressed as

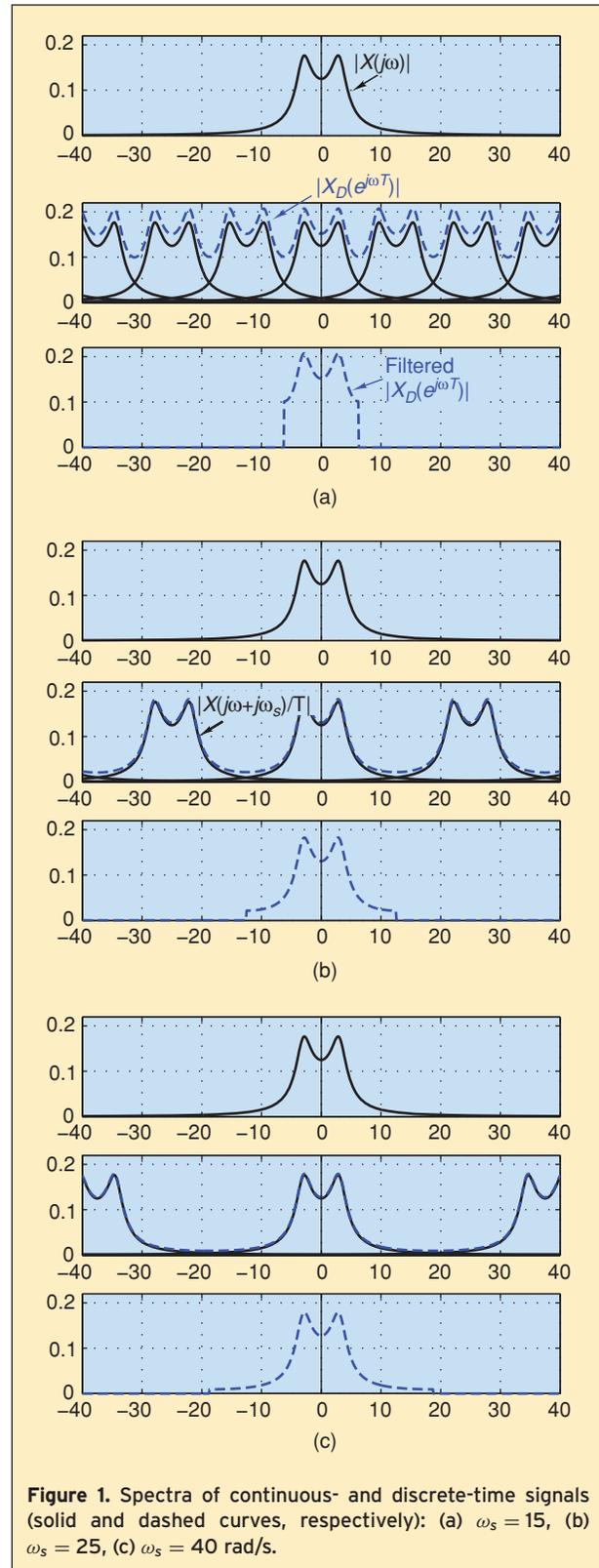
$$\hat{X}(j\omega) = \frac{1}{T} [X(j\omega) + E(j\omega)] \quad (3)$$

where

$$E(j\omega) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} X(j\omega + jk\omega_s)$$

is the contribution of the sidebands to the baseband.

Now if we filter the sampled version of the signal in Eq. (2), namely,  $\hat{x}(t)$ , using an ideal lowpass filter with a frequency response



**Figure 1.** Spectra of continuous- and discrete-time signals (solid and dashed curves, respectively): (a)  $\omega_s = 15$ , (b)  $\omega_s = 25$ , (c)  $\omega_s = 40$  rad/s.

$$H(j\omega) = \begin{cases} T & \text{for } -\omega_s/2 < \omega < \omega_s/2 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

we will get a signal  $y(t)$  whose spectrum is given by

$$\begin{aligned} Y(j\omega) &= H(j\omega)\hat{X}(j\omega) \\ &= H(j\omega) \cdot \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s) \\ &= X(j\omega) + E(j\omega) \end{aligned}$$

according to Eqs. (1)–(4). In other words, the output of the filter will be signal  $x(t)$  plus an error  $e(t)$  given by

$$e(t) = \mathcal{F}^{-1}E(j\omega)$$

which is commonly referred to as the *aliasing error*. With a sampling frequency of 15 rad/s, the magnitude of  $E(j\omega)$ , i.e., the discrepancy between the solid and dashed curves in Fig. 1(a) is large. As the sampling frequency is increased to 25, the sidebands are spread out and  $|E(j\omega)|$  will be decreased quite a bit as shown in Fig. 1(b), and a further increase to 40 rad/s will render  $|E(j\omega)|$  for all practical purposes negligible as shown in Fig. 1(c). Evidently, *a bandlimited signal can be recovered from a sampled version of the signal by using an ideal lowpass filter provided that a sufficiently large sampling frequency is used*. This simple application of Poisson's summation formula demonstrates the essence of the sampling theorem.

Poisson's passion for mathematics is illustrated by a remark he used to make on occasion: *life is good for only two things, discovering mathematics and teaching it*.

#### IV. Laurent

Like Fourier and Poisson, Pierre Alphonse Laurent (1813–1854) studied at École Polytechnique. He started his studies in 1830 and graduated in 1832 ranking among the best students of his class. Upon graduation, he joined the engineering corps of his country as a second lieutenant and before too long he was sent to Algeria where an armed conflict was taking place between the French and Algerians. Laurent returned to France around 1840 to spend the next six years directing operations for the enlargement of the port of Le Havre. It is at that point in time when he began to write some mathematical work for publication.

He submitted a paper entitled “*Mémoire sur le Calcul des Variations*” for the Grand Prize of the Academie des Sciences for 1842, unfortunately, after the deadline for submission. The paper was reviewed by Cauchy but as it was not submitted on time, it was not considered seriously for the Grand Prize. Convinced of its value, Cauchy

recommended later on to the Academie des Sciences that the contribution be published in the academy's *Recueil de Savants Étrangers* but that attempt also failed. Laurent wrote one more mathematical paper later on but, unfortunately, that too received the same fate. Disappointed with his inability to have his mathematical papers accepted by the Academie, Laurent turned his attention to the theory of light waves.

The one mathematical contribution of Laurent that survived is the Laurent series which was part of the paper he had submitted for the Grand Prize but the work was not published until after his early death at the age of 41.

The Laurent series is nowadays treated as part of a theorem of complex analysis, the Laurent theorem [9], which specifies the conditions under which an analytic function of a complex variable,  $F(z)$ , can be expanded into a Laurent series of the form

$$F(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^{-n} \quad (5)$$

where  $a$  is an arbitrary complex constant. According to the Laurent theorem,  $F(z)$  has as many Laurent series as there are annuli of convergence in the  $z$  plane but each series is unique in its annulus of convergence. The coefficients  $a_n$  for each series are given by a contour integral of the form

$$a_n = \frac{1}{2\pi j} \oint_{\Gamma} F(z)(z-a)^{n-1} dz \quad (6)$$

where  $\Gamma$  is a closed contour in the annulus of convergence that encircles point  $z = a$ . The series is a generalization of the Taylor series and includes the Maclaurin and binomial series as special cases.

If we compare the  $z$  transform of a signal  $x(nT)$ , which is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT)z^{-n}$$

with the Laurent series in Eq. (5), the relevance of the Laurent series to DSP becomes immediately obvious. We note that the  $z$  transform is a Laurent series with  $a_n = x(nT)$  and  $a = 0$ . From the Laurent theorem, this is the unique Laurent series of  $X(z)$  about the origin of the  $z$  plane that converges in the annulus

$$R < |z| < \infty$$

where  $R$  is the radius of a circle that encloses all the singularities<sup>2</sup> of  $X(z)$  (see p. 85 in [8]). Therefore, *a signal*

$x(nT)$  can be uniquely obtained from its  $z$  transform by using Eq. (6) (7), we have

$$B = \frac{N}{2} f_0 = \frac{N}{2} \cdot \frac{1}{N\tau} = \frac{1}{2\tau}$$

and if we let  $1/\tau = f_s$ , we get

$$B = \frac{f_s}{2} \text{ in Hz} \quad \text{or} \quad \frac{\omega_s}{2} \text{ in rad/s}$$

where  $\omega_s = 2\pi f_s$ . In other words, the pulse signal can be uniquely determined from the spectrum of the signal over the frequency range 0 to  $f_s/2$  where  $f_s/2$  is known as the Nyquist frequency.

Fourier and Laurent are related through their association with École Polytechnique in Paris, the first having served as a teacher and the second having studied at that institution some 30 years later. Interestingly, their respective contributions to DSP are also related. The Fourier transform of an impulse-modulated signal such as that in Fig. 2(a) is numerically equal to the  $z$  transform of the corresponding discrete-time signal shown in Fig. 2(b) evaluated on the unit circle  $z = e^{j\omega T}$  of the  $z$  plane (see p. 288 of [8]).<sup>3</sup>

### V. Nyquist

The next chapter on the spectral representation of signals began to be written during the early part of the 20th century, and a notable contributor of that period was Harry Nyquist (1889–1976). Born in Nilsy, Sweden, he emigrated to the USA in 1907. Having received a bachelor's degree and a master's degree in electrical engineering from the University of North Dakota in 1914 and 1915, respectively, he moved to Yale University in 1917 from where he received a PhD. He spent his professional life until his retirement in 1954 at Bell Telephone Laboratories [10].

He made many contributions to circuits and systems and is known for the Nyquist plot which is constructed as part of the Nyquist stability criterion to determine whether an amplifier is stable or not. He also carried out important work on thermal noise which is often referred to as Johnson-Nyquist noise.

The connection of Nyquist to DSP relates to the sampling theorem. What he did was to show that a periodic pulse signal constructed from a sequence of  $N$  equally spaced rectangular pulses of arbitrary amplitudes, such as the one illustrated in Fig. 3, can be uniquely determined from the amplitudes and phase angles of the first  $N/2$  sinusoidal components of the Fourier series of the periodic signal by solving a set of  $N$  simultaneous equations [11]. The fundamental of such a signal in Hz is given by

$$f_0 = \frac{1}{T} = \frac{1}{N\tau} \quad (7)$$

where  $T$  is the period of the pulse signal and  $\tau$  is the duration of each rectangular pulse. If  $B$  is the bandwidth from 0 up to and including harmonic  $N/2$ , then from Eq.

<sup>2</sup>Actually poles in DSP.

<sup>3</sup>In this relationship, it is implicitly assumed that the Fourier transform of the impulse-modulated signal exists or, equivalently, that the  $z$  transform of the corresponding discrete-time signal converges on the unit circle of the  $z$  plane.

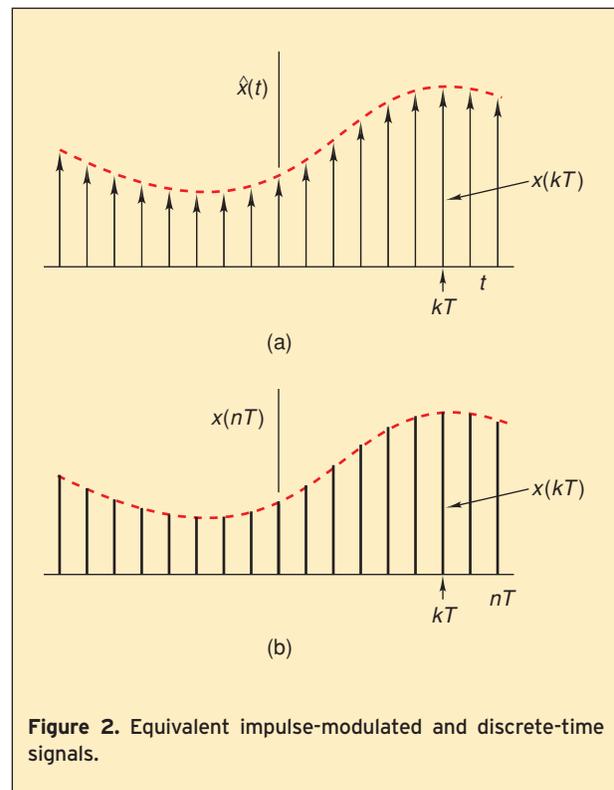


Figure 2. Equivalent impulse-modulated and discrete-time signals.

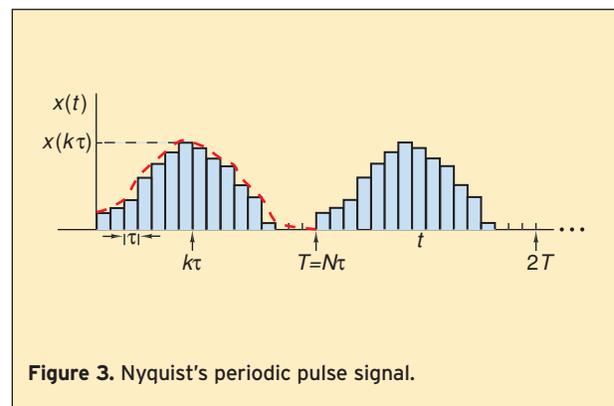


Figure 3. Nyquist's periodic pulse signal.

The above conclusion was deduced in the context of telegraphy whereby ‘intelligence’, in Nyquist’s terminology, was to be transmitted over a telegraph line. The sampling of a continuous-time signal in the modern context or the recovery of such a signal from a sampled version was not involved. Nevertheless, if we make  $\tau$  in Fig. 3 approach a very small value, say,  $\varepsilon$ , the pulse signal would for all practical purposes represent the continuous-time signal  $x(t)$  shown in terms of the dashed curve in Fig. 3. Under these circumstances, the pulse signal can be deemed to be a sampled signal and, in turn,  $f_s$  and  $f_s/2$  would be the *sampling* and *Nyquist* frequencies, respectively, as we know them today. However, deducing the signal by solving a set of  $N$  simultaneous equations would be of little practical value because the number of pulses involved,  $N$ , and, as a consequence, the number of equations required would be very large if  $\tau$  were very small.

Since Nyquist deduced his result by using the Fourier series, what he demonstrated was that the bandwidth condition that carries his name is applicable to periodic signals. In an effort to extend the validity of the condition to nonperiodic signals, he suggested that  $T$  could be made very large, a day or year, in his words, presumably by adding pulses of zero amplitude as shown in Fig. 3. Unfortunately, if one were to attempt to make  $T$  infinite, the analysis would break down because all the Fourier-series coefficients would become zero (see Eqs. (2.10a)–(2.10c) in [8]).

## VI. Shannon

A more general proof that a signal which satisfies the Nyquist condition can be recovered from a set of uniformly spaced values of the signal was described by Shannon in 1949 [12]. He demonstrated the validity of the Nyquist condition by showing that if a bandlimited signal  $x(t)$  that satisfies the condition is passed through an ideal channel with a frequency response

$$H(j\omega) = \begin{cases} 1 & \text{for } -\omega_s/2 < \omega < \omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

then a signal of the form

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{\sin[\omega_s(t - nT)/2]}{\omega_s(t - nT)/2} \quad (8)$$

would be obtained at the receiving end. Since the channel would not disturb the spectrum of the signal, he concluded that the received signal must be an alternative representation of the original signal. The formula in Eq. (8) is essentially an interpolation formula that reconstructs the original signal from its values  $x(nT)$ , for

$n = \dots, -2, -1, 0, 1, 2, \dots$ . Shannon used the Fourier transform in his proof and, in effect, he has shown that the Nyquist condition applies to periodic as well as nonperiodic signals that have a Fourier transform.

Evidently, like Nyquist, Shannon was *not* concerned with sampled signals in today’s context. However, given a continuous-time signal  $x(t)$  with a spectrum  $X(j\omega)$  that satisfies the Nyquist condition, the shifted copies of  $X(j\omega)$  in Eq. (1), i.e.,  $X(j\omega + jn\omega_s)$  for  $-\infty < n < \infty$ , would not overlap with  $X(j\omega)$  and, therefore, the spectrum of the sampled signal with respect to the baseband would simply be  $X(j\omega)/T$  according to Eq. (1). Therefore, Shannon’s proof applies equally well to the situation where a sampled signal is passed through an ideal lowpass filter with a frequency response such as that in Eq. (4) whereby the passband gain of the filter,  $T$ , would cancel the scaling factor  $1/T$  introduced by the sampling process. In effect, unlike Nyquist’s deduction, Shannon’s proof essentially incorporates a practical technique that can be used to recover continuous-time signals from their sampled versions. In practice, we just use a lowpass filter whose frequency response is sufficiently close to that in Eq. (4) to meet the requirements of the application under consideration.

It should be mentioned that Shannon pointed out in his paper that the theorem was common knowledge in the art of communications and that it had been given previously in other forms by mathematicians; in fact, he cites a mathematical treatise by Whittaker [13]. In recent years it has been found out that the sampling theorem was ‘discovered’ independently by several others, e.g., Kotelnikov in 1933, Raabe in 1939, and Someya in 1949, according to a recent article by Lüke [14]. Furthermore, the underlying mathematical principles of the theorem are strongly related to the work of Lagrange on interpolation, who, as may be recalled from Secs. II–III, was one of the teachers of Fourier and Poisson.

Claude Elwood Shannon (1916–2001) studied at the University of Michigan graduating with two Bachelor of Science degrees, one in electrical engineering and the other in mathematics in 1936. He pursued graduate studies at the Massachusetts Institute of Technology earning a master’s degree in electrical engineering and a PhD in mathematics in 1940. He joined the mathematics department at Bell Labs in 1941 and remained affiliated with Bell Labs until 1972. He was offered a visiting position at MIT in 1956, became a permanent member of the faculty as Donner Professor of Science in 1958, and continued from 1978 on as professor emeritus [15].

It should be mentioned that the proof of the sampling theorem was no more than one of Shannon’s minor contributions relative to his monumental achievements in other fields of electrical engineering. As part of his mas-

ter's thesis, he proposed the application of Boolean algebra for the description of switching circuits, which became in due course the standard design methodology for digital circuits and computers, and from 1940 on he began to be involved with the emerging field of communication theory. Over the years, he laid the foundation of what is now known as *information theory* and, in fact, he is regarded by many as the 'father' of that branch of communication theory. More information on Shannon's accomplishments can be found in [15].

**VII. Babbage**

Charles Babbage (1791–1871) studied at Trinity and Peterhouse Colleges, Cambridge University, during 1810–1814 earning a BA degree. He got married in 1814 and a year later the Babbages moved from Cambridge to London where they would spend the rest of their lives. During the period 1815–1819, Babbage kept himself busy writing a number of papers on a variety of mathematical topics. Some of these papers were meritorious enough to get him elected member of the Royal Society of London in 1816 at the early age of 24 and of the Royal Society of Edinburgh in 1820. In 1819 he embarked on the quest of his life to design and build a machine he called a *difference engine*, that would, as mentioned in the introduction, calculate the entries of numerical tables and also print the tables without human intervention. He completed a small model by 1822 which helped him get a grant from the British government in 1823 in the amount of £1500 [5].

He hired an engineer and immediately embarked on the design of *Difference Engine No. 1*. The machine would require 25,000 high-precision mechanical parts. The initial grant soon ran out and Babbage kept returning to the British government for additional funding. In the absence of tangible progress towards a working model by 1834, the British government stopped its support having spent an estimated amount of £17,470, a huge amount in those days that would buy some 22 British-made steam locomotives. The project was formally abandoned by the British government in 1842 after several years of indecision [3].

Convinced about the ultimate success of his endeavor, Babbage soldiered on with his personal resources and between 1847 and 1849, he completed detailed drawings of a simpler and more economical difference engine, *Difference Engine No. 2*. Later on, in 1852, he tried to sell the drawings of his new design to the British government but they were not interested for obvious reasons [3].

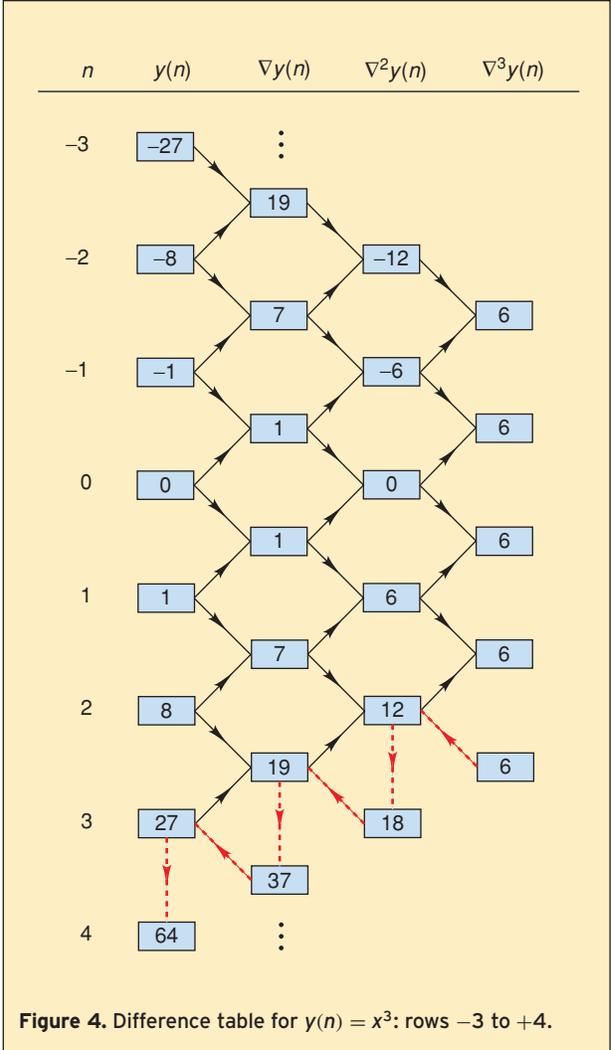
Babbage's difference engine was based on a simple numerical extrapolation technique illustrated in terms of the difference table given in Fig. 4 for the function

$$y(n) = n^3$$

The first and second columns of the difference table give the values of independent variable  $n$  and the corresponding values of the function for  $-3 \leq n \leq 3$  whereas the third, fourth, and fifth columns give the first, second, and third backward differences which are defined as

$$\begin{aligned} \nabla y(n) &= y(n) - y(n - 1) \\ \nabla^2 y(n) &= \nabla[\nabla y(n)] \\ \nabla^3 y(n) &= \nabla[\nabla^2 y(n)] \end{aligned}$$

We note that the entries in the fifth column, namely, the third backward differences, are all equal to 6. In fact, if we were to continue the table for larger values of  $n$  we would find out that all the entries in the fifth column would be equal to 6. The reason behind this phenomenon is connected to the fact that the third derivative of  $n^3$  is a constant. On the basis of this fact, we can generate a new set of differences for the table by starting with the next entry in the fifth column, which is known to be 6, and progressing towards the left ending with the next value of the func-



**Figure 4.** Difference table for  $y(n) = n^3$ : rows  $-3$  to  $+4$ .

tion in the second column. Thus the next entry in the fourth column can be obtained as 18 by adding 6 to 12; the next entry in the third column can be obtained as 37 by adding 18 to 19; and, finally, the next value of the function in the second column can be obtained as 64 by adding 37 to 27. Repeating this simple algorithm, an arbitrarily long table for the cubes of integers can be readily constructed.

Extending the above principles, one can show that the  $n$ th backward differences of an  $n$ th-order polynomial are also all equal to a constant. Therefore, the extrapolation technique just described can be used to evaluate arbitrary polynomials, and this was what Babbage's difference engines were designed to do.

Let us now apply the above procedure for the evaluation of the cube of  $n$ . From the extension of the difference table in Fig. 4 to entries  $n - 3, n - 2, n - 1, n$ , as shown in Fig. 5, we obtain

$$\begin{aligned} \nabla^2 y(n) &= 6 + \nabla^2 y(n - 1) \\ \nabla y(n) &= \nabla^2 y(n) + \nabla y(n - 1) \\ y(n) &= \nabla y(n) + y(n - 1) \end{aligned}$$

and by solving for  $y(n)$ , we get the recursive equation

$$y(n) = 6x(n) + 3y(n - 1) - 3y(n - 2) + y(n - 3) \quad (9)$$

where  $x(n) = u(nT)$  and  $u(nT)$  is the unit-step func-

tion. Equation 9 represents the system in Fig. 6, as can be readily verified, and, in effect, what Babbage was trying to build was actually a discrete system in today's terminology. It should be mentioned that the system in Fig. 6 will give the correct values for the cubes of  $n$  only if the initial values of  $y(-3), y(-2), y(-1), y(-0)$ , i.e.,  $-27, -8, -1$ , and  $0$ , are supplied, and, presumably, the operator of a Babbage difference engine would have to preset certain dials before the engine is put into operation.

As an aside, the response of the system in Fig. 6 to a unit-step would become unbounded as  $n \rightarrow \infty$  and hence the system would be regarded as bounded-input bounded-output unstable nowadays! Nevertheless, it would serve its intended purpose of producing the cubes of any range of integers.

Babbage was not able to build a working difference engine, and friend and foe alike, not to mention the British government of the time, considered the project an abysmal failure.<sup>4</sup> All that was left to posterity is an almost complete set of drawings of Difference Engine No. 2 and certain parts that escaped recycling. However, he was fully vindicated when a team led by Donald D. Swade, sponsored by the Science Museum in London, actually built a working model of Difference Engine No. 2, minus the printing mechanism, based on Babbage's drawings using as far as possible technology

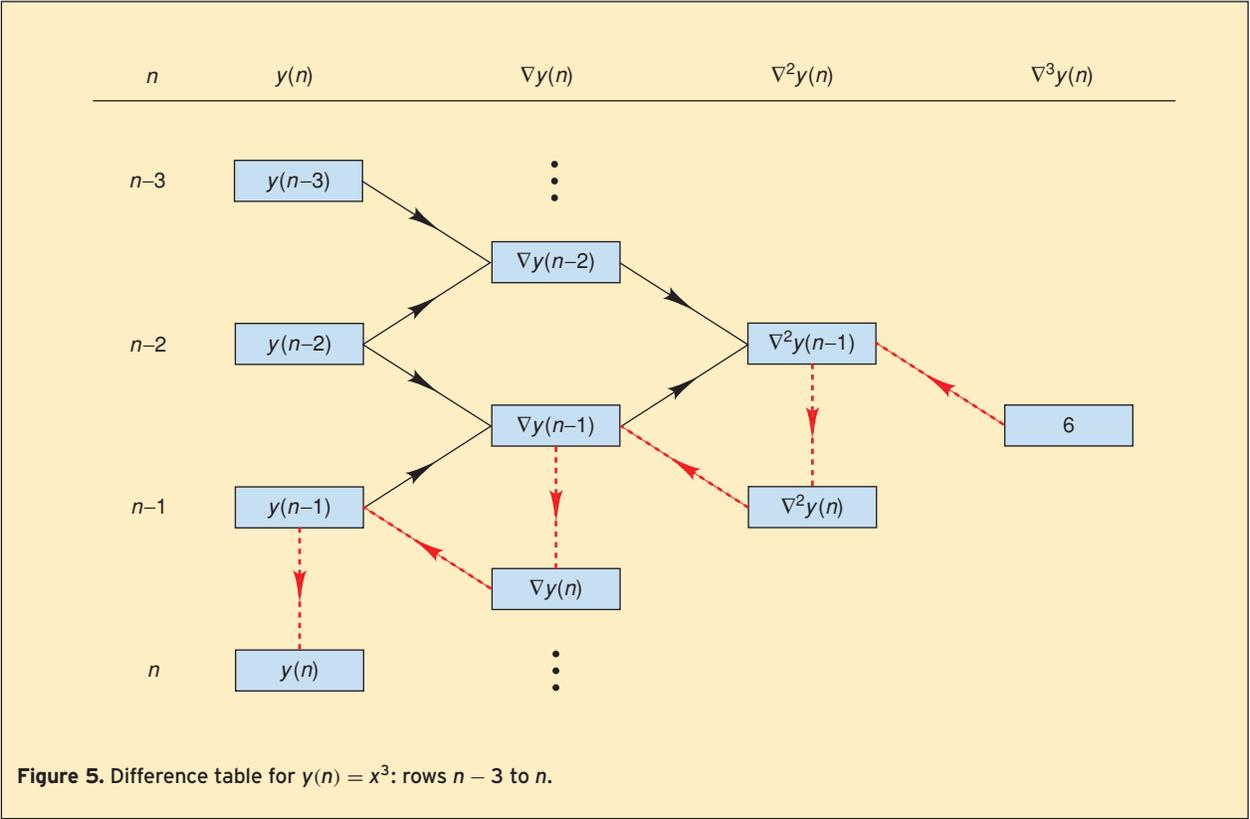


Figure 5. Difference table for  $y(n) = x^3$ : rows  $n - 3$  to  $n$ .

**Numerical methods found their perfect niche in the modern digital computer and considerable progress was achieved through the fifties and sixties in the development of algorithms that can be used to process signals represented in terms of numerical data.**

that was accessible during Victorian times [2]–[4]. The project was completed, despite numerous adversities, just in time to celebrate the bicentennial of Babbage’s birth in 1991. The finished difference engine, shown in Fig. 7, continues to attract thousands of visitors as one of the ground-floor exhibits at the Science Museum. It is made up of 4000 parts, measures  $2.1 \times 3.4 \times 0.5$  m, weighs 3 tons, and calculates differences up to the seventh order. Each turn of the handle at the right produces a 30-digit value in the table of differences and prepares the machine for the next calculation.

After his failed attempt to build Difference Engine No. 1, Babbage conceived a more advanced machine in 1834, the Analytical Engine, which would be programmable to perform additions, subtractions, multiplications, and divisions on command in any order. The machine would comprise a ‘mill’ and a ‘store’, the counterparts of the CPU and memory in a modern digital computer, and its programming would be done by means of punched cards. He worked on this project on and off during the rest of his life but did not attempt to build such a machine.

Sophisticated calculating machines that could perform the full range of arithmetic operations from addition to division made their appearance in the marketplace at the end of the 19th century and successive models served the heavy demand for automated calculations.

The pressures of World War II during the 1940s rekindled strong interest in constructing machines that would perform calculations accurately and efficiently and several machines were built during that period based on the new emerging electronics technology. The most notable of these machines, as is well known, is the Electronic Numerical Integrator and Computer, or ENIAC for short, which was conceived and designed by John Mauchly and J. Presper Eckert of the University of Pennsylvania [16], [17]. ENIAC bears no ancestral relationship to Babbage’s difference and analytical engines, but it is of interest to note that just like the difference engines, ENIAC was designed to construct numerical tables, actually artillery firing tables for the U.S. Army’s Ballistics Research Laboratory.

**VIII. Emergence of the Modern Era**

Numerical methods found their perfect niche in the modern digital computer and considerable progress was achieved through the fifties and sixties in the develop-

ment of algorithms that can be used to process signals represented in terms of numerical data. By the late fifties, a cohesive collection of techniques referred to as ‘*data smoothing and prediction*’ began to emerge through the efforts of pioneers such as Blackman, Bode, Shannon, Tukey [18], [19], and others. During the early sixties, an entity referred to as the ‘*digital filter*’ began to appear in the literature to describe a collection of algorithms that could be used for spectral analysis and data processing [20]–[25].<sup>5</sup> In 1965, Blackman described the state-of-the-art of this new technology in his seminal book on data smoothing and prediction [26], and included in this work certain techniques which he referred to

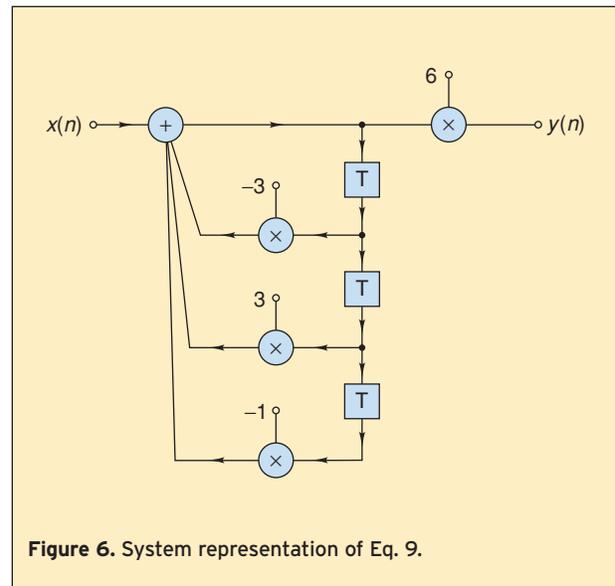


Figure 6. System representation of Eq. 9.

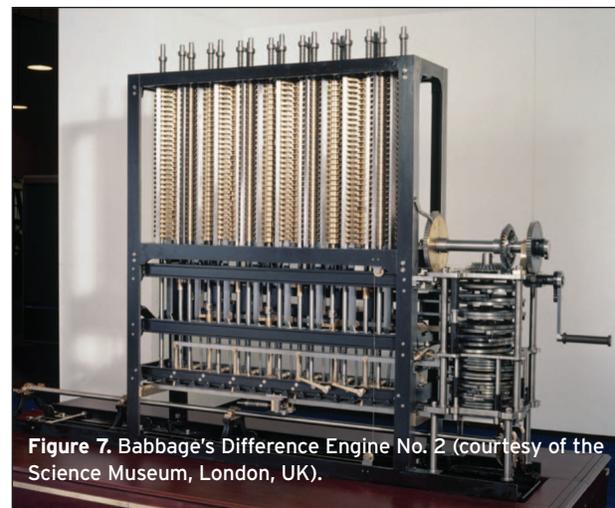


Figure 7. Babbage’s Difference Engine No. 2 (courtesy of the Science Museum, London, UK).

<sup>4</sup>See [4] for the probable reasons for Babbage’s failure.

as ‘numerical filtering’. Within a year, in 1966, Kaiser authored a landmark chapter, entitled *Digital Filters* [27] in which he presented a collection of signal processing techniques that could be applied for the simulation of dynamic systems and analog filters. Digital filters in hardware form began to appear during the late sixties and an early design was reported by Jackson, Kaiser, and McDonald in 1968 [28]. During this very productive period, the discrete Fourier transform was formalized and efficient algorithms for its computation, usually referred to as *Fast Fourier Transforms*, were proposed by Cooley, Tukey, and others [29]–[33].

From the middle sixties on, the analysis and processing of signals in the form of numerical data began to be referred to as *digital signal processing*, and algorithms, computer programs, or systems that could be used for the processing of these signals became firmly established as *digital filters*. The state-of-the-art of the emerging technologies of signal analysis, system analysis and design, digital filters, digital signal processing, and related mathematics was described in the classic textbooks of Papoulis [34], Jury [35], Schwarz and Friedland [36], Gold and Rader [37], and others. In due course, digital computers in the form of inexpensive VLSI chips began to be used as components in DSP systems of considerable complexity.

More recently, DSP has mushroomed into a multifaceted collection of related areas with applications in most disciplines of science and technology. A great variety of digital filters has emerged including

- (FIR) nonrecursive filters
- (IIR) recursive filters
- two- and multi-dimensional filters
- fan filters
- adaptive filters
- multirate systems

On the other hand, DSP methodologies have found applications in numerous areas such as

- digital communication systems such as long-distance and cellular telephone systems
- high-definition TVs
- audio systems such as stereo systems, CD players, and iPods
- speech synthesis
- image processing and enhancement
- the Internet
- instrumentation
- photography
- processing of biological signals such as ECGs

- processing of seismic and other geophysical signals
- artificial cochleas
- remote sensing
- astronomy
- economics
- genetic and proteomic signal processing
- movie making

to name just a few, and new breakthroughs are announced almost daily.

## IX. Conclusions

Part II of this article has shown that a small number of mathematicians who taught or studied at École Polytechnique in Paris laid the mathematical foundations of modern spectral analysis. The contributions of Nyquist and Shannon to the sampling theorem have been examined from a modern perspective. Nyquist has shown that a bandlimited periodic pulse signal comprising  $N$  pulses can be recovered from the amplitudes and angles of the first  $N/2$  sinusoidal components of the Fourier series of the signal by solving a set of  $N$  simultaneous equations. On the other hand, Shannon has shown that an arbitrary bandlimited continuous-time signal that has a Fourier transform can be reconstructed from a set of values of the signal. Although this important theorem has been referred to as the Nyquist, Shannon, Nyquist-Shannon, or Shannon-Nyquist sampling theorem in the past, it is now known that it was actually proposed independently by several engineers or scientists and it seems that mathematicians were well aware of the underlying principle going back to the great Lagrange himself, the teacher of Fourier and Poisson at École Polytechnique.

The work of Babbage has then been examined in the context of DSP. Although Babbage’s programmable Analytical Engine comprised a mill and a store which are analogous to the CPU and memory of a modern digital computer, that machine remained an incomplete conceptual design. What Babbage spent the greatest part of his professional life on was to design and build a mechanical discrete system that would perform the necessary computations and print numerical tables without human intervention, which would, presumably, be free of numerical errors. With the emergence of electronics as the technology of choice during World War II and soon after, interest in building fast and efficient computing machines was reignited and ENIAC was the outcome. Curiously, though, the principal purpose of ENIAC was to construct numerical tables. As it turned out, digital computers can do many more things than just construct numerical tables. In fact, life would not be the same without them.

<sup>5</sup>It is not known who introduced the terminology ‘digital filters’ but in a recent email exchange, Jim F. Kaiser mentioned to the author that as far as he recalls it arose spontaneously among the early DSP practitioners presumably as the digital counterpart of analog filters.

## References

- [1] A. Antoniou, "On the roots of digital signal processing—part I," *IEEE Circuits and Systems Magazine*, vol. 7, no. 1, pp. 8–18, 2007.
- [2] D. Swade, *Charles Babbage and his Calculating Engines*, Science Museum, London, 1991.
- [3] D. Swade, "Redeeming Charles Babbage's mechanical computer," *Scientific American*, vol. 268, no. 2, pp. 86–91, Feb. 1993.
- [4] D. Swade, *The Difference Engine*, Viking, 2000.
- [5] *Indexes of Biographies*, The MacTutor History of Mathematics Archive, School of Mathematics and Statistics, University of St. Andrews, Scotland. Available: <http://turnbull.mcs.st-and.ac.uk/~history/BiogIndex.html>
- [6] C.B. Bower and U.C. Merzbach, *A History of Mathematics*, Second Edition, Wiley, New York, 1991.
- [7] D.M. Burton, *The History of Mathematics, An Introduction*, Fifth Edition, McGraw-Hill, New York, 2003.
- [8] A. Antoniou, *Digital Signal Processing: Signals, Systems, and Filters*, McGraw-Hill, New York, 2005.
- [9] E. Kreyszig, *Advanced Engineering Mathematics*, Third Edition, Wiley, New York, 1972.
- [10] H. Nyquist, Available: <http://www.geocities.com/bioelectrochemistry/nyquist.htm>
- [11] H. Nyquist, "Certain topics in telegraph transmission theory," *Trans. A.I.E.E.*, pp. 617–644, Feb. 1928. (See also *Proc. IEEE*, vol. 90, no. 2, pp. 280–305, Feb. 2002.)
- [12] C.E. Shannon, "Communication in the presence of noise," *Proc. IRE*, vol. 37, no. 1, pp. 10–21, Jan. 1949. (See also *Proc. IEEE*, vol. 86, no. 2, pp. 447–457, Feb. 1998.)
- [13] J.M. Whittaker, *Interpolatory Function Theory*, *Cambridge Tracts in Mathematics and Physics*, no. 33, ch. IV, Cambridge University Press, Cambridge, UK, 1935.
- [14] H.D. Lüke, "The origins of the sampling theorem," *IEEE Communications Magazine*, vol. 37, no. 4, pp. 106–108, Apr. 1999.
- [15] N.J.A. Sloane and A.D. Wyner, *Biography of Claude Elwood Shannon*, Available: <http://www.research.att.com/~njas/doc/shan-nonbio.html>
- [16] M.H. Weik, "The ENIAC story," *Journal of the American Ordnance Association*, Jan.–Feb. 1961. (See also <http://ftp.arl.mil/~mike/comphist/eniac-story.html>)
- [17] ENIAC, Wikipedia. Available: <http://en.wikipedia.org/wiki/ENIAC>
- [18] R.B. Blackman, H.W. Bode, C.E. Shannon, "Data smoothing and prediction in fire-control systems," Summary Technical Report of Division 7, NDRC, vol. 1, pp. 71–160. Reprinted as Report Series, MGC 12/1 (August 15, 1948), National Military Establishment, Research and Development Board.
- [19] R.B. Blackman and J.W. Tukey, *The Measurement of Power Spectra from the Point of View of Communications Engineering*, Dover, 1959.
- [20] M.A. Martin, *Digital Filters for Data Processing*, General Electric Co., Missile and Space Division, Tech. Inf. Series Report No. 62-SD484, 1962.
- [21] K. Steiglitz, *The General Theory of Digital Filters with Applications to Spectral Analysis*, New York University, New York, AFOSR Report no. 64-1664, May 1963.
- [22] E.B. Anders et al., *Digital Filters*, NASA Contractor Report CR-136, Dec. 1964.
- [23] H.H. Robertson, "Approximate design of digital filters," *Technometrics*, vol. 7, pp. 387–403, Aug. 1965.
- [24] J.F. Kaiser, "Some practical considerations in the realization of digital filters," *Proc. Third Allerton Conf. on Circuits and Systems*, pp. 621–633, Oct. 1965.
- [25] K. Steiglitz, "The equivalence of digital and analog signal processing," *Information and Control*, vol. 8, pp. 455–467, Oct. 1965.
- [26] R.B. Blackman, *Data Smoothing and Prediction*, Addison-Wesley, Reading, MA, 1965.
- [27] F.F. Kuo and J.F. Kaiser, *System Analysis by Digital Computer*, Wiley, New York, 1966.
- [28] L.B. Jackson, J.F. Kaiser, and H.S. McDonald, "An approach to the implementation of digital filters," *IEEE Trans. Audio and Electroacoust.*, vol. 16, pp. 413–421, Sept. 1968.
- [29] J.W. Cooley and J.W. Tukey, "An algorithm for the machine calculation of complex Fourier series," *Math Comp.*, vol. 19, pp. 297–301, Apr. 1965.
- [30] W.T. Cochran, J.W. Cooley, D.L. Favon, H.D. Helms, R.A. Kaenel, W.W. Lang, G.C. Maling, D.E. Nelson, C.M. Rader, and P.D. Welch, "What is the fast fourier transform?," *IEEE Trans. Audio Electroacoust.*, vol. 15, pp. 45–55, June 1967.
- [31] G.D. Bergland, "A guided tour of the Fast Fourier Transform," *IEEE Spectrum*, vol. 6, pp. 41–52, July 1969.
- [32] J.W. Cooley, P.A.W. Lewis, and P.D. Welch, "Historical notes on the Fast Fourier Transform," *IEEE Trans. Audio Electroacoust.*, vol. 15, pp. 76–79, June 1967.
- [33] J.W. Cooley, P.A.W. Lewis, and P.D. Welch, "Application of the fast fourier transform to computation of fourier integrals, Fourier series and convolution integrals," *IEEE Trans. Audio Electroacoust.*, vol. 15, pp. 79–84, June 1967.
- [34] A. Papoulis, *The Fourier Integral and Its Applications*, McGraw-Hill, New York, 1962.
- [35] E.I. Jury, *Theory and Application of the z-Transform Method*, Wiley, New York, 1964.
- [36] R.J. Schwarz and B. Friedland, *Linear Systems*, McGraw-Hill, New York, 1965.
- [37] B. Gold and C.M. Rader, *Digital Signal Processing*, McGraw-Hill, New York, 1969.



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