A New Algorithm for Compressive Sensing Based on Total-Variation Norm

J. K. Pant, W.-S. Lu, and A. Antoniou

University of Victoria
Victoria, British Columbia, Canada

May 21, 2013

Outline

- Compressive Sensing and Signal Recovery

Outline

- Compressive Sensing and Signal Recovery
- Image Recovery Using Total-Variation Minimization

Outline

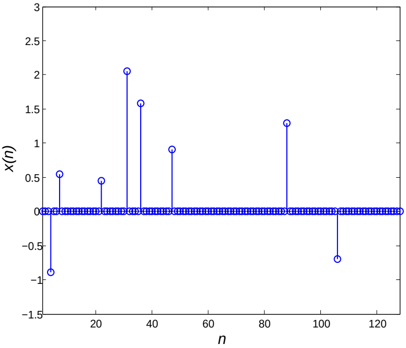
- Compressive Sensing and Signal Recovery
- Image Recovery Using Total-Variation Minimization
- Image Recovery Using Nonconvex Total-Variation Minimization

Outline

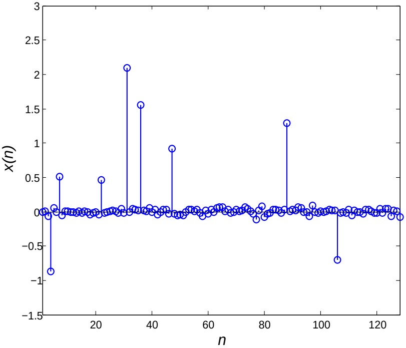
- Compressive Sensing and Signal Recovery
- Image Recovery Using Total-Variation Minimization
- Image Recovery Using Nonconvex Total-Variation Minimization
- Performance Evaluation

- A signal $\mathbf{x}(n)$ of length N is K-sparse if it contains K nonzero components with $K \ll N$.

- A signal $\mathbf{x}(n)$ of length N is K-sparse if it contains K nonzero components with $K \ll N$.
- A signal is near K-sparse if it contains K significant components.



A 8-sparse signal

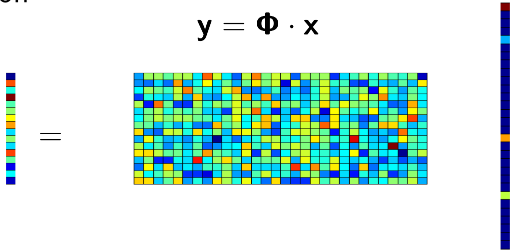


A near 8-sparse signal

- Compressive sensing (CS) is a data acquisition process whereby a sparse signal $\mathbf{x}$ or an image $\mathbf{X}$ represented by a vector $\mathbf{x}$ of length N can be determined using a small number of projections represented by a matrix $\mathbf{\Phi}$ of dimension $M \times N$.

- Compressive sensing (CS) is a data acquisition process whereby a sparse signal $\mathbf{x}$ or an image $\mathbf{X}$ represented by a vector $\mathbf{x}$ of length N can be determined using a small number of projections represented by a matrix $\mathbf{\Phi}$ of dimension $M \times N$.
- In CS, measurement vector $\mathbf{y}$ and signal vector $\mathbf{x}$ are interrelated by the equation

$$\mathbf{y} = \mathbf{\Phi} \cdot \mathbf{x}$$



16 measurements | projection matrix of size 16×30 | 4-sparse signal of length 30

- A sparse signal $\mathbf{x}$ can be recovered by using an ℓ_1-norm minimization that solves the problem

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & ||\mathbf{x}||_1 = \sum_{i=1}^{N} |x_i| \\ \text{subject to} & \mathbf{y} = \mathbf{\Phi x} \end{array}$$

- A sparse signal $\mathbf{x}$ can be recovered by using an ℓ_1-norm minimization that solves the problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & ||\mathbf{x}||_1 = \sum_{i=1}^{N} |x_i| \\ \text{subject to} \quad & \mathbf{y} = \mathbf{\Phi x} \end{aligned}$$

- An ℓ_p-pseudonorm minimization that solves the problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & ||\mathbf{x}||_p^p = \sum_{i=1}^{N} |x_i|^p \\ \text{subject to} \quad & \mathbf{y} = \mathbf{\Phi x} \end{aligned}$$

where a p in the range $0 < p < 1$ can be used to yield a sparser signal.

Image Recovery Using Total-Variation Minimization

- Many synthetic and natural images have a *spatially sparse* gradient.

Image Recovery Using Total-Variation Minimization

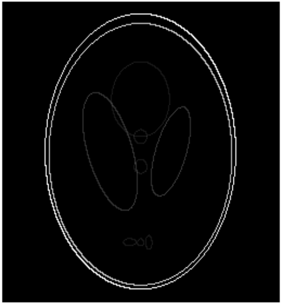
- Many synthetic and natural images have a *spatially sparse* gradient.
- The spatial gradient of an image $\mathbf{X}$ of size $n_1 \times n_2$ can be obtained as a matrix $\mathbf{G}$ of size $n_1 \times n_2$ whose $\{i, j\}$th component is given by

$$g_{i,j} = \begin{cases} \sqrt{(x_{i,j} - x_{i+1,j})^2 + (x_{i,j} - x_{i,j+1})^2} & \text{for } \begin{cases} 1 \le i < n_1, \\ 1 \le j < n_2 \end{cases} \\ |x_{i,j} - x_{i+1,j}| & \text{for } \begin{cases} j = n_2, \\ 1 \le i < n_1 \end{cases} \\ |x_{i,j} - x_{i,j+1}| & \text{for } \begin{cases} i = n_1, \\ 1 \le j < n_2 \end{cases} \\ 0 & \text{for } \; i = n_1, j = n_2 \end{cases}$$

where $x_{i,j}$ is the $\{i, j\}$th component of $\mathbf{X}$.

- The Shepp-Logan Phantom image has a sparse spatial gradient:

Phantom image



Sparse spatial gradient of Phantom image

- The Cameraman image has near-sparse spatial gradient:

Cameraman image

Near-sparse spatial gradient of Cameraman image

- The sparsity of the spatial gradient of an image **X** can be measured in terms of the total-variation norm given by

$$TV(\mathbf{X}) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} g_{i,j}$$

where $g_{i,j}$ is the $\{i, j\}$th element of matrix **G**.

Image Recovery Using Total-Variation Minimization, cont'd

- The sparsity of the spatial gradient of an image **X** can be measured in terms of the total-variation norm given by

$$TV(\mathbf{X}) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} g_{i,j}$$

  where $g_{i,j}$ is the $\{i, j\}$th element of matrix **G**.

- The smaller the $TV(\mathbf{X})$, the sparser the gradient of **X**.

Image Recovery Using Total-Variation Minimization, cont'd

- The sparsity of the spatial gradient of an image $\mathbf{X}$ can be measured in terms of the total-variation norm given by

$$TV(\mathbf{X}) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} g_{i,j}$$

where $g_{i,j}$ is the $\{i,j\}$th element of matrix $\mathbf{G}$.

- The smaller the $TV(\mathbf{X})$, the sparser the gradient of $\mathbf{X}$.

- An image $\mathbf{X}$ with sparse spatial gradient represented by a vector $\mathbf{x}$ can be recovered from measurements $\mathbf{y}$ by solving the optimization problem

$$\underset{\mathbf{x}}{\text{minimize}} \quad \frac{1}{2}\,||\mathbf{\Phi x} - \mathbf{y}||_2^2 + \lambda TV(\mathbf{X})$$

where λ a regularization parameter.

Image Recovery Using Nonconvex Total-Variation Minimization

- Inspired by the success of ℓ_p over ℓ_1 minimization in CS, we consider the nonconvex version of the TV norm, called the TV_p pseudonorm, given by

$$
\begin{aligned}
TV_p(\mathbf{X}) \;=\; & \left[\sum_{i=1}^{n_1-1} \sum_{j=1}^{n_2-1} \left({x'^{i}_{i,j}}^2 + {x'^{j}_{i,j}}^2 \right)^{p/2} \right. \\
& \left. + \sum_{i=1}^{n_1-1} \left({x'^{i}_{i,n_2}}^2 \right)^{p/2} + \sum_{j=1}^{n_2-1} \left({x'^{j}_{n_1,j}}^2 \right)^{p/2} \right]^{1/p}
\end{aligned}
$$

where $x'^{i}_{i,j} = x_{i,j} - x_{i+1,j}$, $x'^{j}_{i,j} = x_{i,j} - x_{i,j+1}$, and $0 < p < 1$.

Image Recovery Using Nonconvex Total-Variation Minimization

- Inspired by the success of ℓ_p over ℓ_1 minimization in CS, we consider the nonconvex version of the TV norm, called the TV_p pseudonorm, given by

$$
\begin{aligned}
TV_p(\mathbf{X}) \;=\; & \left[\sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2-1}\left({x'^{i}_{i,j}}^2 + {x'^{j}_{i,j}}^2\right)^{p/2}\right. \\
& \left. + \sum_{i=1}^{n_1-1}\left({x'^{i}_{i,n_2}}^2\right)^{p/2} + \sum_{j=1}^{n_2-1}\left({x'^{j}_{n_1,j}}^2\right)^{p/2}\right]^{1/p}
\end{aligned}
$$

  where $x'^{i}_{i,j} = x_{i,j} - x_{i+1,j}$, $x'^{j}_{i,j} = x_{i,j} - x_{i,j+1}$, and $0 < p < 1$.

- From the nonconvexity and nondifferentiability of the ℓ_p pseudonorm, it follows that function $TV_p(\mathbf{X})$ remains nonconvex and nondifferentiable for $p < 1$.

Image Recovery Using Nonconvex Total-Variation Minimization, cont'd

- To render the TV_p pseudonorm differentiable and to facilitate its optimization, we consider the approximate TV_p pseudonorm given by

$$\begin{aligned} TV_{p,\epsilon}^{p}(\mathbf{X}) &= \sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2-1}\left({x'^{i}_{i,j}}^2 + {x'^{j}_{i,j}}^2 + \epsilon^2\right)^{p/2} \\ &\quad + \sum_{i=1}^{n_1-1}\left({x'^{i}_{i,n_2}}^2 + \epsilon^2\right)^{p/2} + \sum_{j=1}^{n_2-1}\left({x'^{j}_{n_1,j}}^2 + \epsilon^2\right)^{p/2} \end{aligned}$$

where ϵ is a nonzero parameter used to render it differentiable.

Image Recovery Using Nonconvex Total-Variation Minimization, cont'd

- To render the TV_p pseudonorm differentiable and to facilitate its optimization, we consider the approximate TV_p pseudonorm given by

$$
\begin{aligned}
TV_{p,\epsilon}^{p}(\mathbf{X}) \;=\; & \sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2-1}\left({x'^{i}_{i,j}}^{2} + {x'^{j}_{i,j}}^{2} + \epsilon^2 \right)^{p/2} \\
& + \sum_{i=1}^{n_1-1}\left({x'^{i}_{i,n_2}}^{2} + \epsilon^2 \right)^{p/2} + \sum_{j=1}^{n_2-1}\left({x'^{j}_{n_1,j}}^{2} + \epsilon^2 \right)^{p/2}
\end{aligned}
$$

where ϵ is a nonzero parameter used to render it differentiable.

- Note that $TV_{p,\epsilon}^{p}(\mathbf{X}) \rightarrow TV(\mathbf{X})$ as $\epsilon \rightarrow 0, p \rightarrow 1$.

Image Recovery Using Nonconvex Total-Variation Minimization, cont'd

- The reconstruction involves solving the optimization problem

$$(\mathbf{P}\text{-}\mathbf{TV}_p) \qquad \underset{\mathbf{x}}{\text{minimize}} \quad F_{\lambda,p,\epsilon}(\mathbf{X}) = \frac{1}{2}\,||\mathbf{\Phi x} - \mathbf{y}||_2^2 + \lambda\, TV_{p,\epsilon}^{p}(\mathbf{X})$$

for a small values ϵ_T and λ_T of ϵ and λ, respectively, and $p < 1$.

Image Recovery Using Nonconvex Total-Variation Minimization, cont'd

- The reconstruction involves solving the optimization problem

$$(\mathbf{P}\text{-}\mathbf{TV}_p) \qquad \underset{\mathbf{x}}{\text{minimize}} \quad F_{\lambda,p,\epsilon}(\mathbf{X}) = \frac{1}{2}\,||\mathbf{\Phi x} - \mathbf{y}||_2^2 + \lambda\, TV_{p,\epsilon}^p(\mathbf{X})$$

for a small values ϵ_T and λ_T of ϵ and λ, respectively, and $p < 1$.

- The gradient of the objective function $F_{\lambda,p,\epsilon}(\mathbf{X})$ can be evaluated as

$$\mathbf{g} = \mathbf{\Phi}^T\,(\mathbf{\Phi x} - \mathbf{y}) + \lambda p \mathbf{u}$$

where $\mathbf{u}$ is a vector representing the gradient of $TV_{p,\epsilon}^p(\mathbf{X})/p$.

Image Recovery Using Nonconvex Total-Variation Minimization, cont'd

- The problem $\mathbf{P\text{-}TV}_p$ can be solved by using the following sequential procedure:
  - Select $\{\epsilon = \epsilon_1, \lambda = \lambda_1\}$ so that $\{\epsilon_1 > \epsilon_T, \lambda_1 > \lambda_T\}$, set the zero vector as initializer, and solve problem $\mathbf{P\text{-}TV}_p$. Denote the resulting solution as $\mathbf{x}^*$.
  - Using $\mathbf{x}^*$ as the initializer, solve problem $\mathbf{P\text{-}TV}_p$ again for smaller values of ϵ and λ.
  - Repeat this procedure until problem $\mathbf{P\text{-}TV}_p$ is solved for the pair $\{\epsilon = \epsilon_T, \lambda = \lambda_T\}$. Denote the final solution as $\mathbf{x}_T^*$.
  - Construct image $\mathbf{X}^*$ from the final solution $\mathbf{x}_T^*$.
  - Output $\mathbf{X}^*$ and stop.

Image Recovery Using Nonconvex Total-Variation Minimization, cont'd

- The problem $\mathbf{P\text{-}TV}_p$ can be solved by using the following sequential procedure:
  - Select $\{\epsilon = \epsilon_1, \lambda = \lambda_1\}$ so that $\{\epsilon_1 > \epsilon_T, \lambda_1 > \lambda_T\}$, set the zero vector as initializer, and solve problem $\mathbf{P\text{-}TV}_p$. Denote the resulting solution as $\mathbf{x}^*$.
  - Using $\mathbf{x}^*$ as the initializer, solve problem $\mathbf{P\text{-}TV}_p$ again for smaller values of ϵ and λ.
  - Repeat this procedure until problem $\mathbf{P\text{-}TV}_p$ is solved for the pair $\{\epsilon = \epsilon_T, \lambda = \lambda_T\}$. Denote the final solution as $\mathbf{x}_T^*$.
  - Construct image $\mathbf{X}^*$ from the final solution $\mathbf{x}_T^*$.
  - Output $\mathbf{X}^*$ and stop.
- The Fletcher-Reeves' conjugate-gradient (FR-CG) technique can be applied to solve problem $\mathbf{P\text{-}TV}_p$ for a given pair of values of $\{\epsilon, \lambda\}$.

- In the FR-CG technique, iterate $\mathbf{x}_k$ is updated to $\mathbf{x}_{k+1}$ as

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$$

where

$$\mathbf{d}_k = -\mathbf{g}_k + \beta_{k-1}\mathbf{d}_{k-1},$$

$$\beta_{k-1} = \frac{||\mathbf{g}_k||_2^2}{||\mathbf{g}_{k-1}||_2^2},$$

and $\mathbf{g}_k$ is the gradient at $\mathbf{x} = \mathbf{x}_k$.

- In the FR-CG technique, iterate $\mathbf{x}_k$ is updated to $\mathbf{x}_{k+1}$ as

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$$

where

$$\mathbf{d}_k = -\mathbf{g}_k + \beta_{k-1}\mathbf{d}_{k-1},$$

$$\beta_{k-1} = \frac{||\mathbf{g}_k||_2^2}{||\mathbf{g}_{k-1}||_2^2},$$

and $\mathbf{g}_k$ is the gradient at $\mathbf{x} = \mathbf{x}_k$.

- Step size α_k is obtained by using the recursion

$$\alpha_{l+1} = G(\alpha_l) \quad \text{for} \quad l = 2, 3, \ldots$$

with $\alpha_0 \geq 0$ where function $G(\alpha)$ depends on $\mathbf{x}_k$, $\mathbf{\Phi}$, $\mathbf{d}_k$, $\mathbf{y}$, ϵ, and p.

Performance Evaluation

- The performance of the proposed TV_p-RLS and conventional TV-RLS algorithms was tested using six images, namely,
  - "Circles", "Resolution Chart", and "Shepp-Logan Phantom" having sparse spatial gradient and
  - "Cameraman", "Aeroplane", and "Clock" having near-sparse spatial gradient.

Performance Evaluation

- The performance of the proposed TV_p-RLS and conventional TV-RLS algorithms was tested using six images, namely,
  - "Circles", "Resolution Chart", and "Shepp-Logan Phantom" having sparse spatial gradient and
  - "Cameraman", "Aeroplane", and "Clock" having near-sparse spatial gradient.
- The image reconstruction performance was measured in terms of the peak signal-to-noise ratio (PSNR) which is defined as

$$PSNR = 20 \log \left(\frac{I_{MAX}}{\sqrt{MSE}} \right) dB$$

where $I_{MAX} = 2^b - 1$ and $b = 8$ is the number of bits used to encode the components of image $\mathbf{X}$.

Performance Evaluation

- The performance of the proposed TV_p-RLS and conventional TV-RLS algorithms was tested using six images, namely,
  - "Circles", "Resolution Chart", and "Shepp-Logan Phantom" having sparse spatial gradient and
  - "Cameraman", "Aeroplane", and "Clock" having near-sparse spatial gradient.
- The image reconstruction performance was measured in terms of the peak signal-to-noise ratio (PSNR) which is defined as

$$PSNR = 20 \log \left(\frac{I_{MAX}}{\sqrt{MSE}} \right) dB$$

where $I_{MAX} = 2^b - 1$ and $b = 8$ is the number of bits used to encode the components of image $\mathbf{X}$.

- The mean-square error is defined as

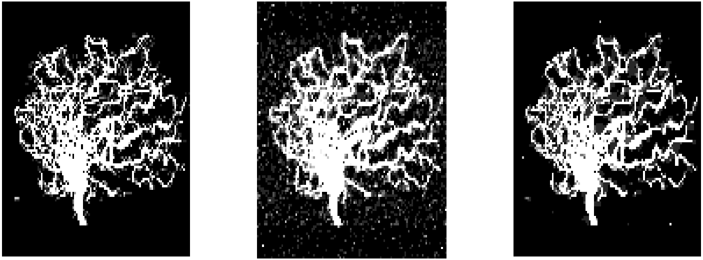
$$MSE = \frac{1}{n_1 n_2} \left\| \mathbf{X} - \hat{\mathbf{X}} \right\|_F^2$$

- Experimental results:

PSNR and CPU Time for TV_p-RLS and TV-RLS Algorithms

Images	TV_p-RLS ($p = 0.5$)		TV-RLS	
	PSNR (dB)	CPU time (seconds)	PSNR (dB)	CPU time (seconds)
Cameraman	32.8	47.1	32.2	952.8
Aeroplane	41.7	49.1	41.5	767.0
Circles	90.1	43.6	58.4	483.0
Clock	38.4	48.1	37.3	911.4
Resolution Chart	74.6	45.0	49.7	1201.7
Shepp-Logan	86.5	44.1	76.2	121.2

- Reconstruction of an angiogram of size 256×256:

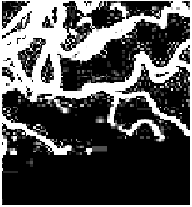


(a) (b) (c)

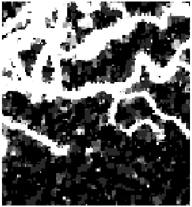
(a) Original angiogram
(b) Angiogram reconstructed using TV_p-RLS algorithm with $p = 0.5$
(c) Angiogram reconstructed using TV-RLS algorithm

Performance Evaluation, cont'd

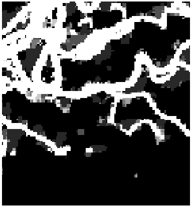
- Segments of the angiograms shown in Slide 17 for the range $120 \leq n_y \leq 220, 120 \leq n_x \leq 220$ where n_y and n_x are pixel indices for vertical and horizontal directions, respectively:



(a)



(b)



(c)

(a) Original angiogram
(b) Angiogram reconstructed using TV_p-RLS algorithm with $p = 0.5$
(c) Angiogram reconstructed using TV-RLS algorithm

Conclusions

- Compressive sensing is an effective technique for sampling sparse signals.

Conclusions

- Compressive sensing is an effective technique for sampling sparse signals.
- ℓ_1 and ℓ_p minimizations work in general for the reconstruction of sparse signals.

Conclusions

- Compressive sensing is an effective technique for sampling sparse signals.
- ℓ_1 and ℓ_p minimizations work in general for the reconstruction of sparse signals.
- Total variation minimization is effective for the reconstruction of images.

Conclusions

- Compressive sensing is an effective technique for sampling sparse signals.
- ℓ_1 and ℓ_p minimizations work in general for the reconstruction of sparse signals.
- Total variation minimization is effective for the reconstruction of images.
- Nonconvex total-variation minimization offers improved reconstruction performance relative to the total-variation minimization for images with sparse spatial gradient.

Thank you for your attention.