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TECHNICAL NOTE

A PROCEDURE FOR DETERMINING RIGID BODY TRANSFORMATION PARAMETERS

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Abstract—For many biomechanical applications it is necessary to determine the parameters which describe the transformation of a rigid body from one reference frame to another. These parameters are a scaling factor, an attitude matrix, and a translation vector. The paper presents a new procedure for the determination of these parameters incorporating the work of Arun *et al.* [*IEEE Trans. Pattern Anal. Machine Intell.* **9**, 698–700 (1987)] but expanding their analysis to allow for the determination of a scale factor, the scalar weighting of the least-squares problem, and the problem of obtaining the incorrect determinant when determining the attitude matrix. The procedure, which requires the coordinates of three or more non-collinear points, is based around the singular value decomposition, and provides a least-squares estimate of the rigid body transformation parameters. Examples are presented of the use of this procedure for determining the attitude of a rigid body, and for osteometric scaling. When used for osteometric scaling mirror transformations are possible, therefore a right-hand specimen can be scaled to the left-hand side of another specimen.

INTRODUCTION

In many biomechanical analyses it is necessary to determine the rigid body transformation parameters; parameters which describe the transformation of points from one reference frame to another. These parameters are a scale factor, an attitude matrix, and a position vector. The scale factor allows these parameters to be used to describe the transformation required to map points between reference frames of different scales. One application of scaling of this sort is osteometric scaling, which allows the locations of the origins and insertions of muscles and ligaments that are inaccessible on a live subject, and therefore not easily measurable, to be obtained from the results of cadaver dissection and measurement. Lew and Lewis (1977) demonstrated that the direct application of dry bone data to a live subject is inappropriate and that some form of scaling is appropriate as the physical dimensions of the bones of the cadaver and the experimental subject are likely to be different. Osteometric scaling has been performed for biomechanical analysis by Morrison (1970) who scaled between a dry bone specimen and an experimental subject by scaling along axes defined within the bone of interest. Lew and Lewis (1977) described a technique which incorporates a uniform stretch of differing amounts along three mutually orthogonal axes defined in both rigid bodies. Neither of these approaches takes account of the errors in the measured positions of landmarks on either the specimen or subject. Sommer *et al.* (1982) however, proposed a scaling technique which employed linear least-squares principles and therefore took some account of such errors.

If the scale factor is equal to unity and the attitude matrix is proper orthonormal then the rigid body transformation parameters can be used to describe the position and orientation or movement of a rigid body. The attitude matrix is often parameterised to describe the orientation of a rigid body. For example, helical axis parameters are extracted (e.g. Siegler *et al.*, 1988), or a set of angles defined by a sequence of 'planar' rotations is computed (e.g. Woltring, 1991). A number of different procedures have been proposed for the determination of these rigid body transformation parameters. Spoor and Veldpaus (1980) proposed a least-squares method which required the calculation of the eigenvectors of a three by three matrix. Veldpaus *et al.* (1988) presented a similar technique but it obviated the need to compute eigenvectors. In contrast to these techniques which used least-squares principles Laub and Shiflett (1982) proposed a method based on linear algebra and applying matrix perturbation theory to take account of the lack of precision in the data; a similar procedure was presented by Angeles (1986).

Unfortunately any measurement procedure involves errors, those used in biomechanics are no different. The positions of the points used to determine the rigid body transformation parameters will contain errors, therefore it is important that procedures adopted for the determination of these parameters attempt to take account of these errors. It is the purpose of this paper to present a procedure which allows the rigid body transformation parameters to be determined using a linear least-squares method. It is based on the work of Arun *et al.* (1987) but expands their analysis to allow for: the determination of a scale factor; the scalar weighting of the least-squares problem; and the problem of obtaining the incorrect determinant when determining the attitude matrix. Similar equations to those of Arun *et al.* (1987) were presented by Woltring (1992). Examples of the use of this procedure to determine the orientation of a rigid body (using varying numbers of markers to define the rigid body), and for osteometric scaling are given.

DETERMINATION OF RIGID BODY TRANSFORMATION PARAMETERS

The purpose of this section is to present a technique, based around the singular value decomposition (SVD), for computing the parameters associated with rigid body transforma-

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tions. The SVD is a technique for the factorisation of a single matrix into three (Golub and Reinsch, 1971). The technique presented here makes use of the properties of these three matrices to determine the rigid body transformation parameters.

The rigid body transformation parameters can be used to transform points measured in one reference frame to another, this relationship can be described as

$$y_i = s \cdot [R]x_i + v, \tag{1}$$

where y_i is the position of ith point measured in reference frame $\{B\}$; s the scale factor; $[R]$ the attitude matrix (sometimes called the rotation matrix); x_i the position of ith point measured in reference frame $\{A\}$ and v the position vector of the origin of reference frame $\{A\}$ measured in reference frame $\{B\}$.

When the scale factor is equal to unity then the rigid body transformation parameters can be used to describe the orientation of a rigid body relative to a given reference frame, or the movement of a rigid body from one position to another. Analyses for either of these purposes require that matrix $[R]$ is a proper orthonormal matrix, which therefore has the following properties

$$[R]^T[R] = [R][R]^T = [R]^{-1}[R] = [I], \tag{2}$$

$$\det([R]) = +1, \tag{3}$$

where $[I]$ is the identity matrix and $\det([R])$ denotes the determinant of matrix $[R]$.

Given the rigid body transformation described in equation (1) and assuming for the preliminary part of this section that $s = 1$, equation (1) becomes

$$y_i = [R]x_i + v, \tag{4}$$

where x_i, y_i are the points on a rigid body measured in two different reference frames; $[R]$ is the attitude matrix and v is a translation vector.

Using a least-squares method the problem of determining $[R]$ and v is equivalent to minimising

$$\frac{1}{n}\sum_{i=1}^{n}([R]x_i + v - y_i)^T([R]x_i + v - y_i), \tag{5}$$

where n is the number of non-collinear points measured in both reference frames ($n \geqslant 3$).

To simplify the problem v can be eliminated as an unknown. The mean vectors ($\bar{x}$ and $\bar{y}$) are computed

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \tag{6}$$

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i. \tag{7}$$

The vector v can be determined from these mean vectors

$$v = \bar{y} - [R]\bar{x}. \tag{8}$$

Substitution of this relationship in equation (5) gives

$$\frac{1}{n}\sum_{i=1}^{n}([R]x_i - y_i + \bar{y} - [R]\bar{x})^T([R]x_i - y_i + \bar{y} - [R]\bar{x}). \tag{9}$$

Two new sets of vectors can be defined which can be used to simplify equation (9) as

$$x_i' = x_i - \bar{x}, \tag{10}$$

$$y_i' = y_i - \bar{y}. \tag{11}$$

Appropriate substitution of these vectors into equation (9) and basic matrix algebra gives

$$\frac{1}{n}\sum_{i=1}^{n}(y_i' - [R]x_i')^T(y_i' - [R]x_i'). \tag{12}$$

Expansion of this expression gives

$$(\frac{1}{n}\sum_{i=1}^{n}(y_i'^T y_i' - y_i'^T[R]x_i' - \{[R]x_i'\}^T y_i' + \{[R]x_i'\}^T[R]x_i'), \tag{13}$$

which can be reduced to

$$\frac{1}{n}\sum_{i=1}^{n}(y_i'^T y_i' + x_i'^T x_i' - 2y_i'^T[R]x_i') \tag{14}$$

since the following equivalents exist

$$\{[R]x_i'\}^T y_i' = y_i'^T[R]x_i', \tag{15}$$

$$\{[R]x_i'\}^T[R]x_i' = x_i'^T[R]^T[R]x_i' = x_i'^T x_i'. \tag{16}$$

Therefore minimising equation (5) is equivalent to maximising

$$\frac{1}{n}\sum_{i=1}^{n}(y_i'^T[R]x_i'), \tag{17}$$

which can be re-arranged, and summed to give the following to maximise

$$\frac{1}{n}\sum_{i=1}^{n}(y_i'^T[R]x_i') = \operatorname{tr}\left\{[R]^T\frac{1}{n}\sum_{i=1}^{n} y_i' x_i'^T\right\} = \operatorname{tr}([R]^T[C]). \tag{18}$$

where tr( ) refers to the trace of a given matrix and $[C]$ is the cross-dispersion matrix (also known as the correlation matrix).

The cross-dispersion matrix is computed from

$$[C] = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})^T = \frac{1}{n}\sum_{i=1}^{n} y_i' x_i'^T. \tag{19}$$

The SVD of $[C]$ is computed

$$[C] = [U][W][V]^T, \tag{20}$$

where $[U]$ and $[V]$ are orthogonal matrices, and $[W]$ is a diagonal matrix which contains the singular values of matrix $[C]$, the number of non-zero singular values indicating the rank of $[C]$. If the results of the SVD are substituted into equation (18) the following relationships exist:

$$\begin{aligned}\operatorname{tr}\{[R]^T[C]\} &= \operatorname{tr}\{[R]^T[U][W][V]^T\} \\ &= \operatorname{tr}\{[V]^T[R]^T[U][W]\}.\end{aligned} \tag{21}$$

A new matrix $[Q]$ is defined by

$$[Q] = [V]^T[R]^T[U]. \tag{22}$$

Since $[V]$, $[R]$, and $[U]$ are all orthogonal matrices $[Q]$ must also be orthogonal. The Euclidean vector norm of the main diagonal of $[Q]$ must be equal to or less than unity ($|Q_{ii}| \leqslant 1$) implied by equation (2) as a basic property of orthogonal matrices. As $[W]$ is a diagonal matrix it is only the elements along the diagonal of $[Q]$ which have an influence on the results of the computations described in equation (18). Therefore equation (18) is a maximum when $[Q]$ is equal to the identity matrix, which implies that

$$[R] = [U][V]^T. \tag{23}$$

The description presented thus far parallels that of Arun *et al.* (1987), albeit with a different derivation. For certain cases when determining the rotation matrix for describing rigid body motion or relative orientation, this formulation does not work, and rather than getting a matrix of determinant of $+1$, the matrix $[R]$ has a determinant of -1, in which case it represents a reflection. The following modification accounts for this. If the SVD of $[C]$ has been computed then $\operatorname{tr}([R]^T[C])$ is maximised when

$$[R] = [U]\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \det([U]\cdot[V]^T)\end{bmatrix}[V]^T. \tag{24}$$

The matrix product $([U]\cdot[V]^T)$ does not need to be computed explicitly as the determinants of the two matrices can be multiplied to give the determinant of a matrix which is the product of the matrices. As $[U]$ and $[V]$ are both orthogonal matrices their determinants are ± 1, obviously it is possible that certain combinations of $[U]$ and $[V]^T$ will result in a matrix $[R]$ which has a determinant of -1. If the value of $\det([U].[V]^T)$ is equal to one then the intermediate matrix is not needed as it only represents the identify matrix. When the determinant is equal to -1, without this correction the matrix $[R]$ would represent a reflection matrix (determinant of -1) and would not be suitable for determining rigid body attitudes and positions. A two-dimensional example is given to illustrate how $[R]$ can be incorrectly determined. Figure 1 shows two lines before and after a transformation, the transformation required for this mapping can be described by either a reflection in the Y axis, or by a rotation; therefore a matrix representing either a reflection or rotation would give a maximum to equation (18). It is possible for the example shown that with noise present either the reflection or rotation matrix could provide a maximum to equation (18). For describing rigid body attitudes it would be necessary to constrain the value of the determinant of $[R]$ to $+1$. If the SVD based procedure is used for osteometric scaling then this stage can be omitted as a mirror transformation (determinant of -1) allows a right-hand specimen to be scaled to the left-hand side of another specimen or experimental subject.

If scaling is employed then equation (5) becomes

$$\frac{1}{n}\sum_{i=1}^{n}(s\cdot[R]x_i + v - y_i)^T(s\cdot[R]x_i + v - y_i). \qquad (25)$$

This equation can be expanded to give the following expression to be minimised:

$$\sigma_y^2 + s^2\sigma_x^2 - 2s\cdot \mathrm{tr}([R]^T[C]), \qquad (26)$$

where

$$\sigma_y^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

and

$$\sigma_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2.$$

If equation (26) is differentiated with respect to the scale factor, and the value of the resulting function set to zero, then the scale factor can be determined from

$$s = \frac{1}{\sigma_x^2}\mathrm{tr}([R]^T[C]). \qquad (27)$$

When determining the orientation of a rigid body relative to a given reference frame, or the movement of a rigid body from one position to another, any deviations of the scale factor (s) from unity will indicate rigid body deformations and/or errors in the measurements of the data used to define the body. For other applications the scale factor may be of more direct use for example in osteometric scaling. Given the scale factor and mean vectors of the points, the translation vector can be computed

$$v = \bar{y} - s\cdot[R]\bar{x}. \qquad (28)$$

It has been shown that the SVD can be used to compute rigid body transformation parameters. This procedure can be extended to allow for scalar weighting of points, in which case the objective function to be minimised becomes

$$\frac{1}{n}\sum_{i=1}^{n}\omega_i^2[([R]x_i + v - y_i)^T([R]x_i + v - y_i)], \qquad (29)$$

where ω_i is a weighting factor reflecting the accuracy of the ith point.

These weighting factors are normally set so that they are inversely proportional to the expected variance of the coordinates of each of the landmarks. It is common practise to normalise the weighting factors so that they lie between zero and unity. The weighting can allow for differences in the accuracy with which marker locations have been determined in the photogrammetric process, with missing markers being given a weighting of zero. The inclusion of a weighting factor only effects equations (6), (7), and (19), therefore these modified equations become

$$\bar{x} = \sum_{i=1}^{n}\omega_i^2\cdot x_i \Big/ \sum_{i=1}^{n}\omega_i^2, \qquad (30)$$

$$\bar{y} = \sum_{i=1}^{n}\omega_i^2\cdot y_i \Big/ \sum_{i=1}^{n}\omega_i^2, \qquad (31)$$

$$[C] = \left(1 \Big/ \sum_{i=1}^{n}\omega_i^2\right)\sum_{i=1}^{n}\omega_i^2(y_i' x_i'^T). \qquad (32)$$

The algorithm described in this section uses basic linear algebra so it can be easily implemented using matrix algebra software libraries such as LINPACK (Dongarra *et al.*, 1978).

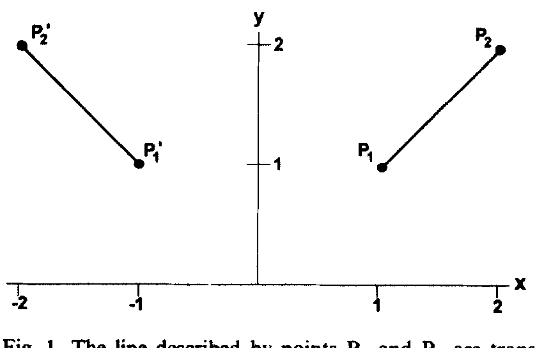

Fig. 1. The line described by points P_1 and P_2 are transformed to new positions P_1' and P_2', respectively. This transformation could be achieved with either a rotation or a reflection in the y axis.

APPLICATION 1—RIGID BODY ORIENTATION

In this section the procedure for determining the rigid body transformation parameters is utilised to examine the influence of the number of markers on the accuracy of angles which specify the relative orientation of one rigid body to another.

The problem was to define the attitude of a cube (length of sides 0.50 m) relative to an inertial reference frame. For the cases examined here it was assumed that the origin of both the inertial and cube reference frames were coincident. A reference frame was defined for the cube and the locations of 36 points on the cube determined in the cube's reference frame. The relative attitude of the cube was defined using helical angles (Woltring, 1991) with the criterion angles in the range 180° to $-180°$ being generated using a random number generator. These angles were than used to specify the orientation of the cube in the inertial reference frame. White noise with an isotropic distribution was then added to the coordinates of the points, as measured in both reference frames. The standard deviation of the noise was 0.0015 m in the cases examined here. The problem was to determine the rigid body transformation parameters from this noisy data, using the SVD based procedure, and then from the attitude matrix to extract the helical angles. These estimated angles could then be compared with their criterion values. A set of 100 helical angles were determined. The procedure presented in this paper requires at least three non-collinear points measured in both reference frames so that the transformation parameters can be determined, but the option is available to use more points. The task was to estimate each of the angles from the noisy coordinate data when the number of markers used to define the cube's reference frame was sequentially varied in increments of one from three to 36. The accuracy of the estimations of these angles was assessed by

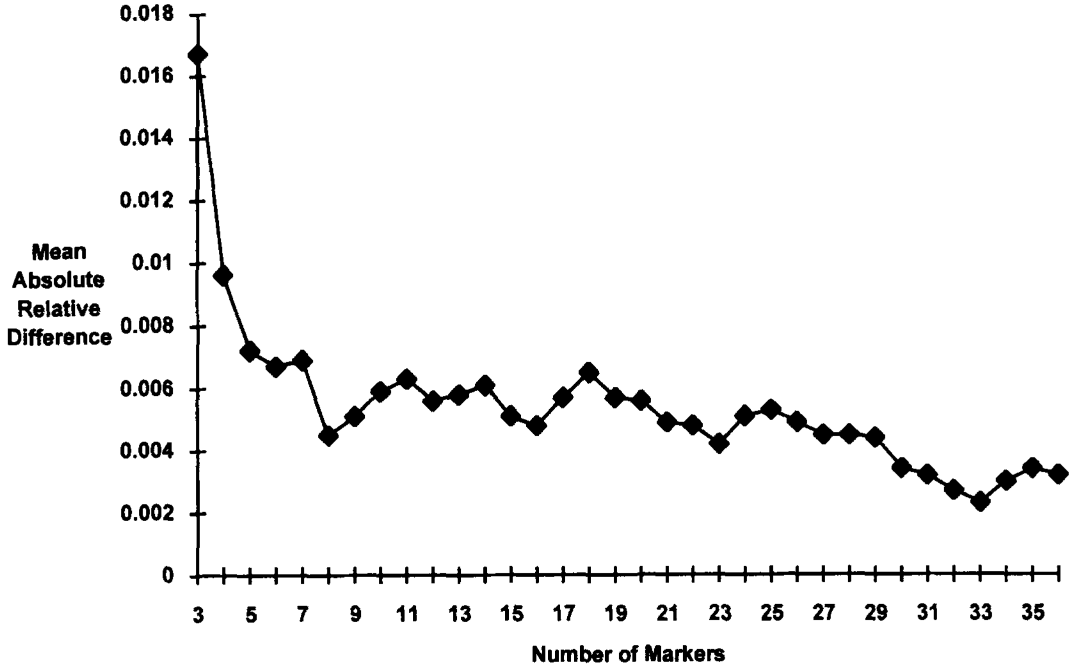

Fig. 2. The variation in the mean absolute relative difference in estimating the orientation of a rigid body with an increasing number of markers on the rigid body. Data showing trend of increased accuracy of orientation estimation with increased number of markers.

computing the mean relative absolute difference between the criterion values of the angles and their estimated values. The mean absolute relative difference was evaluated using the formula

$$\overline{\mathrm{ARDIFF}} = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{\mathrm{CA}_i - \mathrm{EA}_i}{\mathrm{CA}_i}\right|, \tag{33}$$

where $\overline{\mathrm{ARDIFF}}$ is the mean absolute relative difference; n the number of angle sets (100); CA_i the criterion angle value and EA_i the estimated angle value.

The mean absolute relative accuracy of the estimated helical angles was determined as the number of points defining the cube's reference frame was increased from three to 36.

Figure 2 shows that the general trend was that as the number of markers increased, the accuracy of the estimation of the helical angles increased. The increase was most rapid when the number of markers was increased from three to four. In the analysis of human movement it is often not possible to attach more than three markers to a segment in positions where the influence of skin movement will not be too great. The graph illustrates how attitude determination accuracy was influenced by the distribution of the markers. An increase in the number of markers did not always increase the accuracy of the helical angle estimations; thus reflecting that the distribution of the markers is also an important factor in determining the accuracy of the rigid body transformation parameters. During the simulations the adjustment in the determinant of the attitude matrix as described in equation (24) was required, although the number of times a reflection matrix (determinant of -1) occurred was not quantified.

APPLICATION 2—OSTEOMETRIC SCALING

Lew and Lewis (1977) presented data of the locations of six bony landmarks on two human tibias; these data were used to examine the utility of the SVD based procedure for determining the rigid body transformation parameters including the scaling factor. The six points measured on the two tibia were the following:

(1) tubercle of Gerdy,
(2) lateral malleolus,
(3) posterior cruciate attachment,
(4) middle of tibial spines,
(5) centre of lateral condylar surface,
(6) tubercle on soleal line.

The bones used by Lew and Lewis (1977, p. 174) were selected so that the 'geometry differences between the bones were as large as possible'.

In this study one bone was used as a reference and the rigid body transformation parameters were determined for mapping from that bone to another bone. Given that there were not many points with which to assess the accuracy of the procedure the cross-validation procedure of Allen (1974) was used. An unbiased estimate of error was obtained by calculating the rigid body transformation parameters for sub-sets of the original data set. The sub-sets were achieved by removing one of the body landmarks from both original data sets. The removed value was then estimated using the rigid body transformation parameters; by doing this sequentially for each of the six points it was possible to get six estimates of error from which the mean absolute relative error was computed. The resulting cross-validation errors indicate the predictive capabilities of the model.

From the results presented in Table 1 it can be seen that the scaling from one bone to another using the procedure described here produces small errors in the estimation of points known on one bone but not on another. For point two, the lateral malleolus, the results are the least accurate. Although no reason for this is obvious this difference could have arisen due to experimental errors in the procedures for measuring and locating the positions of landmarks, or due to the assumption that the geometry differences between bones can be accounted for using homogeneous scaling.

Table 1. The absolute relative differences between the true value and estimated value for each of the points identified on the tibia. The mean absolute relative differences indicate the small errors arising when using the SVD based procedure for osteometric scaling

Point excluded/ predicted	Absolute relative differences		
	x	y	z
1	0.016	0.004	0.008
2	0.072	0.022	0.052
3	0.028	0.010	0.010
4	0.009	0.008	0.009
5	0.025	0.003	0.000
6	0.015	0.007	0.009
Mean	0.027	0.009	0.015

DISCUSSION

A procedure has been presented which allows the estimation of the rigid body transformation parameters in a least-squares sense. When using this procedure for the estimation of the relative attitude or movement of a rigid body the scale factor is assumed to have a value of one. The algorithm will of course estimate the scale factor, any deviations from unity for the scale factor will be due to the combined effects of marker movement, and errors in locating the positions of these markers. Marker movements arise because the segments are not truly rigid, which means that markers may have movement relative to the reference frame they are intended to define. Evidence of the advantage of using a least-squares technique for determining the attitude and position of a rigid body is provided by Challis (1994) who showed that compared with a number of other commonly used procedures a least-squares based procedure gave the most accurate estimates of rigid body position and attitude.

The use of the SVD based procedure for osteometric scaling means the scaling along the three coordinate directions is homogeneous. Therefore this procedure can make use of data such as that of Brand *et al.* (1982) who provided the averaged scaled locations of the origins and insertions of 47 muscles from the dissection of three cadavers (six lower limbs), and White *et al.* (1989) who also provided data suitable for osteometric scaling for 40 muscles of the lower limb. A non-homogeneous technique has been presented by Lewis *et al.* (1980) which is based on finite element principles. Non-homogeneous scaling is a non-linear least-squares problem so is not solvable using the least-squares procedure presented here. The procedure of Lewis *et al.* (1980) requires eight landmarks on both the specimen and experimental subject/specimen. In an analysis of two human femurs the non-homogeneous scaling was more accurate than homogeneous scaling. Unfortunately in the analysis of human subjects *in vivo* without the use of X-ray photogrammetry it is difficult to locate eight bony landmarks so that the non-homogeneous transformation procedure of Lewis *et al.* (1980) can be used. Future developments should permit non-homogeneous scaling with fewer than eight markers, until such time the homogeneous scaling technique presented here is valuable, particularly for *in vivo* studies.

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