Radiation pressure—the history of a mislabeled tensor[a)]

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The acoustic radiation pressure has found practical application in recent years in instruments measuring sound intensity and in experiments on acoustic levitation. The concept of radiation pressure has, however, long fascinated both optical and acoustical physicists. The history of light radiation pressure goes back more than 200 years to Leonhard Euler, while the concept of acoustic radiation pressure dates from the time and work of Rayleigh. It was pointed out by Brillouin that what we call radiation pressure is not a pressure at all, but a diagonal tensor, all the diagonal terms of which are not identical. The size of the effect is small, and the values obtained for the radiation pressure are very sensitive to boundary conditions and to the approximations that must necessarily be employed. In addition, although the phenomenon is primarily one of nonlinear acoustics, it can be observed down to the lowest sound intensities under certain conditions. Thus, the Rayleigh radiation pressure vanishes for the linear case, but the usually measured Langevin pressure does not. It might be said that radiation pressure is a phenomenon that the observer thinks he understands—for short intervals, and only every now and then.

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INTRODUCTION

The subject of radiation pressure has been one of the most widely studied "small subjects" of acoustics, with virtually every big name in the field applying himself to the subject at one time or another, and with many little names doing likewise. The literature is replete with original studies, reviews, and reviews of reviews, of which this is, perhaps, just one more.[1] And hidden in these papers are bright ideas and erroneous assumptions and conclusions, making the writing of another review a dangerous exercise.

In the year of my birth, the novelist Edith Wharton wrote in *The Age of Innocence*

> "...An unalterable and unquestioned law of the music world required that the German text of French operas sung by Swedish artists should be translated into Italian for the clearer understanding of English-speaking audiences."[2]

I am reminded of that quotation as I begin the historical background of the subject of radiation pressure with a quotation from an article written in Latin by a Swiss scientist, but translated into French in order to appear in a German journal, and given its English translation by this American speaker. I can only hope that a clearer understanding will result in this case.

It was in 1746 that Leonhard Euler wrote

> "If it is established that... there is a propagation of streams of light through the ether... in such a way that this light propagation in the ether resembles that of sound in air, then it appears to be more difficult to explain how such streams can carry away the particles that tumble in the atmosphere. While a sound vigorously excites not only a vibratory motion in the air particles, but one also observes a real motion in small, very light dust particles which tumble in the air, it cannot be doubted that the vibratory motion caused by the light produces a similar effect."[3]

It was Euler's idea that the rays of light, hitting loose particles near the limbs of a comet, exerted sufficient force on them to knock them loose but, since the gravitational pull of the comet itself continued to be exerted on the particles, they would fall into a train behind the comet, thus forming its tail. Figure 1 is taken from Euler's original paper.

Euler did not give any reason why the alternation of the wave should give rise to a steady force, but seemed to think that the action was much like that of the pushing of a swing, in which case the force is periodic but the effect is one directional.

The next move forward came 100 years later when Maxwell published his treatise on *Electricity and Magnetism* (1874) in which, with an appeal to Faraday, he wrote that

> "... in a medium in which waves are propagated, there is a pressure in the direction normal to the waves, and numerically equal to the energy in unit of volume."[4]

Time prevents my citing the interesting works of others such as Boltzmann, Lebedev, and Poynting.[5-7] It is curious, however, to reflect that one seems always to

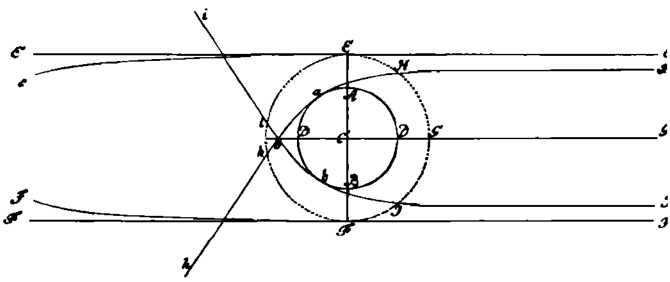

FIG. 1. Displacement of the outer portions of comet by radiation force as envisaged by Euler in 1746.[3] ADBD = solid core of comet; ESF = limit of atmosphere of comet; EHGJF are sun's rays, entering from the right.

[a)]This tutorial paper was given at the 92nd Meeting of the Acoustical Society of America, San Diego, California, 16 November 1976 [J. Acoust. Soc. Am. 60, S21(A) (1976)].

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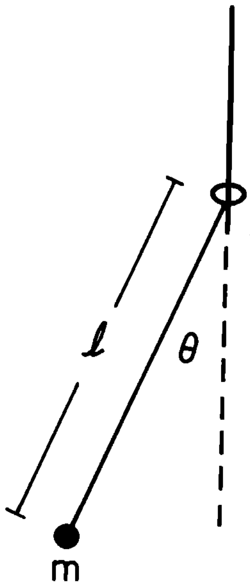

FIG. 2. Pendulum with string constrained by ring.

have accepted the idea that the pressure was exerted in a specific direction, which is, of course, not what we mean by pressure at all. We shall see this again in the acoustic case.

Now the study of one form of wave motion does not long lag behind the other. If they have photons, then we have phonons, if they have radar, we have sonar, and if they have radiation pressure, then we must have it too.

As usual, the story begins with Rayleigh. In 1902, Rayleigh noted the various efforts mentioned above and remarked

"... it would be of interest to inquire whether other kinds of vibration exercise a pressure, and if possible to frame a general theory of the action."[6–8]

Three years and 28 papers later, Rayleigh arrived at the acoustic radiation pressure which bears his name, i.e., the difference between the average pressure at a surface moving with the sound displacements (the Lagrangian pressure) and the pressure that would have existed in the fluid of the same mean density at rest.[9]

Rayleigh first solved the problem of a vibrating pendulum, the length of which is constrained by a ring that can move only vertically (Fig. 2). If the tension in the string is approximately equal to the weight of the bob, mg (for small angles θ), then the net upward force on the ring will be $mg(1-\cos\theta)$. But the potential energy of the bob V is equal to $V=mgl(1-\cos\theta)$. Hence the mean upward force is equal to the mean value of V/l. But the mean value of the potential energy is one-half the mean value of the total energy, so that the mean force is equal to one-half the mean energy per unit length of the pendulum.

Writing a generation later, Brillouin[10] noted that this could be derived more generally from the Boltzmann–Ehrenfest theory of adiabatic invariance.[11] That is, under a very slow and continuous change of a constraint parameter, the product of the period τ and the mean value of the kinetic energy $\overline{T}$ remains a constant, i.e., $\delta(\tau T)=0$. Thus, as we shorten the length of a pendulum, the mean kinetic energy will change, and an amount of work $dW=-dT=-dE$ will be done on the constraint parameter. The force that is exerted is therefore

$$F\,dl=-dE\ .$$

But

$$\frac{d\tau}{\tau}+\frac{dE}{E}=0 \text{ and } \tau=2\pi(l/g)^{1/2}\ ,$$

so that

$$F=\frac{E}{\tau}\frac{d\tau}{dl}=\frac{E}{2l}\ . \tag{1}$$

I. VIBRATING STRING

If we follow the analysis of either Rayleigh or Brillouin for the vibrating string, one finds that again

$$F=\frac{E}{\tau}\frac{d\tau}{dl}\ .$$

If the velocity c of the wave in the string remains constant, then $c\tau=\lambda=2l$ for the fundamental mode, so that

$$\frac{d\tau}{dl}=\frac{2}{c} \quad \text{and} \quad F=\frac{E}{l}\ .$$

However, if c is not a constant, then

$$\frac{d\tau}{\tau}+\frac{dc}{c}=\frac{dl}{l}\ ,$$

and our expression for the force becomes

$$F=\frac{E}{l}\left(1-\frac{l}{c}\frac{dc}{dl}\right)\ . \tag{2}$$

II. FLUID COLUMN

Brillouin then noted that if we are dealing with a fluid column of length l, cross section S, density ρ, volume $V_f=Sl$ the mass remains constant, so that $Sl\rho=\text{const}$; then

$$\frac{dl}{l}=\frac{-d\rho}{\rho}=\frac{dV_f}{V_f}$$

so that

$$F=\frac{E}{l}\left(1-\frac{V_f}{c}\frac{dc}{dV_f}\right)$$

or

$$p_{\text{rad}}=\frac{F}{S}=\langle E\rangle\left(1+\frac{\rho}{c}\frac{dc}{d\rho}\right)=\langle E\rangle\left(1-\frac{V_f}{c}\frac{dc}{dV_f}\right)\ , \tag{3}$$

where $\langle E\rangle$ is the energy per unit volume.

Brillouin was sufficiently pleased with this result that he summed up his results in the form of a table of components of the mean stress tensor in the interior of a fluid traversed by a plane acoustic wave. We would write this today in the form

$$S_{ij}=-\rho v_i v_j-p\delta_{ij}\ , \tag{4}$$

where δ_{ij} is the Kronecker delta. Thus, for a sound wave traveling in the x direction, $v_2=v_3=0$ and we obtain

$$\begin{pmatrix} -\langle p(x_1)\rangle - \langle \rho v_1^2\rangle & 0 & 0 \\ 0 & -\langle p(x_1)\rangle & 0 \\ 0 & 0 & -\langle p(x_1)\rangle \end{pmatrix}, \quad (5)$$

where, of course, $\langle \rho v_1^2\rangle$ is equal to the mean energy density $\langle E\rangle$. The problem is then the evaluation of the mean pressure at a point x_1. This corresponds to the mean pressure in Eulerian coordinates. As can be seen from above, the quantity $p(x_1)$ was identified by Brillouin with the term $-\langle E\rangle(V_f/c)(dc/dV_f)$.

Brillouin was not concerned with the evaluation of the terms involving dc/dV and didn't observe that he had, in fact, made an error in calculating the terms of the tensor. The correct analysis was made by Fubini,[12] who was a student of Brillouin's, but Brillouin did not alter this in his book on tensors in 1938.[13] This correct form was publicized by Westervelt[14] in 1950 and we shall now proceed to its development.

We shall carry out the calculation for an ideal gas, but the case can be extended to liquids by replacing γ—the ratio of specific heats—by $1+(B/A)$, where B/A is called the parameter of nonlinearity, $=\rho_0 c_0(\partial c/\partial p)_S$.

If we deal in Lagrangian or material coordinates, the fluid density ρ^L is related to the space derivative $\partial\xi/\partial x \equiv \xi_x$ by the equation

$$\rho^L/\rho_0 = 1/(1+\xi_x) \ . \quad (6)$$

We now consider the adiabatic relation $p = p_0(\rho/\rho_0)^\gamma$. If we substitute for ρ^L and expand in powers of ξ_x, we obtain for the Lagrangian pressure at the position of a particle that was at rest at the point x:

$$\begin{aligned} p^L &= p_0\{1-\gamma\xi_x + \tfrac{1}{2}[\gamma(\gamma+1)]\xi_x^2\} \\ &= p_0 + \rho_0 c_0^2[-\xi_x + \tfrac{1}{2}(\gamma+1)\xi_x^2] \ , \end{aligned} \quad (7)$$

where $c_0^2 = \gamma p_0/\rho_0$. It will be useful also to write out the Eulerian expression for the pressure at a fixed point. Since an Eulerian quantity $\mathcal{E}$ is related to the corresponding Lagrangian quantity $\mathcal{L}$ through the expression

$$\mathcal{E} = \mathcal{L} - \frac{\mathcal{L}}{\mathcal{E}}\frac{\partial\mathcal{L}}{\partial x} \ , \quad (8)$$

then p^E is given by

$$\begin{aligned} p^E &= p_0 + p_0 c_0^2[-\xi_x + \tfrac{1}{2}(\gamma+1)\xi_x^2 + \xi\xi_{xx}] \\ &= p^L + \rho_0 c_0^2\,\xi\xi_{xx} \ . \end{aligned} \quad (9)$$

III. RAYLEIGH AND LANGEVIN RADIATION PRESSURES

We are now in the position to work out the detailed differences of Rayleigh and Langevin pressures. Rayleigh first. This pressure has been defined as the difference between the average pressure at a surface moving with the particle (the mean p^L) and the pressure that would have existed in the fluid of the same mean density at rest (p_0). Thus,

$$\begin{aligned} p^{\mathrm{Ray}} &= \langle p^L\rangle - p_0 \\ &= \rho_0 c_0^2\langle -\xi_x + \tfrac{1}{2}(\gamma+1)\xi_x^2\rangle \ . \end{aligned} \quad (10)$$

If we now assume a plane harmonic wave that is undergoing the distortion of nonlinear acoustics, we can write for its first terms

$$\begin{aligned} \xi &= \xi_0[1-\cos(\omega t - kx] + \tfrac{1}{8}[M^2(\gamma+1)x] \\ &\quad \times [1-\cos 2(\omega t - kx)] \ , \end{aligned} \quad (11)$$

where $M = \xi_0 k/c_0$ is the acoustic Mach number. Substituting, we get

$$p^{\mathrm{Ray}} = \rho_0 c_0^2[-\tfrac{1}{8}(\gamma+1) + \tfrac{1}{4}(\gamma+1)]M^2 = \rho_0 c_0^2\,\tfrac{1}{8}(\gamma+1)\,M^2 \ . \quad (12)$$

Therefore

$$\begin{aligned} p^{\mathrm{Ray}} &= \tfrac{1}{4}(\gamma+1)\langle E\rangle \qquad \text{(gas)} \ , \\ p^{\mathrm{Ray}} &= \tfrac{1}{2}[1+(B/2A)]\langle E\rangle \quad \text{(liquid)} \ . \end{aligned} \quad (13)$$

Let me now point out a few places where one can and does go wrong here. If we fail to take the second harmonic of the sound wave into account we get

$$p^{\mathrm{Ray}} = \tfrac{1}{2}(\gamma+1)\langle E\rangle \ , \quad (14)$$

which is the value used by Hertz and Mende,[15] and which I used in my 1950 review[16]; it appears again in the 1972 article by Rooney and Nyborg.[17] The same result can be obtained from the Brillouin stress tensor, since we can calculate

$$\frac{\rho}{c}\frac{\partial c}{\partial\rho} = \frac{\gamma-1}{2}$$

so that the radiation pressure for the fluid column becomes

$$p_{\mathrm{rad}} = \langle E\rangle[1+\tfrac{1}{2}(\gamma+1)] = \tfrac{1}{2}(\gamma+1)\langle E\rangle \ . \quad (15)$$

Incidentally, it is clear that the components of the stress tensor in the y and z direction should be negative of the Eulerian pressure, as our first tensor indicates. Thus, the radiation stress tensor should be written

$$\begin{pmatrix} -\tfrac{1}{4}(\gamma+1)\langle E\rangle & 0 & 0 \\ 0 & -\tfrac{1}{4}(\gamma-3)\langle E\rangle & 0 \\ 0 & 0 & -\tfrac{1}{4}(\gamma-3)\langle E\rangle \end{pmatrix}. \quad (16)$$

Another possible error is to consider an elastic linear liquid. This in effect sets $\tfrac{1}{2}(\gamma+1)=0$ and the Rayleigh radiation pressure vanishes. It is thus fair to say that the Rayleigh pressure depends only on nonlinear terms and is wholly a nonlinear phenomenon.

Unfortunately, we rarely if ever measure this quantity, but rather carry out a measurement of the so-called Langevin radiation pressure p^{Lan}, which is defined as the difference between the mean force per unit area at a wall and the pressure in the same acoustic medium at rest behind the wall, where the fluid medium is in complete contact on the two sides of the wall.

It is perhaps worth remarking that Langevin published the derivation of his radiation pressure only on the blackboards of the Collège de France in Paris. However, one of his students, Pierre Biquard, wrote an account of his derivation in Revue d'Acoustique so that Langevin's contribution can be documented.[18]

To illustrate the distinction between the Rayleigh and

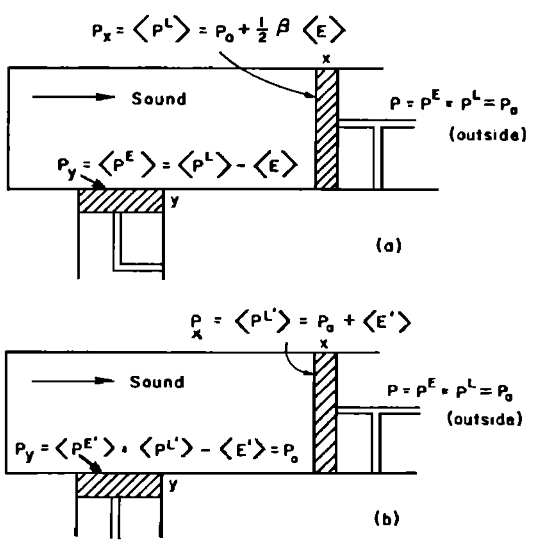

FIG. 3. Models for radiation pressure. (a) Rayleigh pressure and (b) Langevin pressure. Here $\beta = 1 + B/2A$.

Langevin pressures, we look at Fig. 3, which has been modified from a figure in the paper of Hertz and Mende. A somewhat similar drawing appears in Brillouin's book.

In the first picture, both pistons X and Y are fixed in position, and a plane harmonic wave passes from left to right in the medium. Although the piston X is fixed, its inside face must move under the action of the sound wave, in order to satisfy continuity at the boundary. We shall assume that X is perfectly absorbing. The pressure at X is then equal to the Lagrangian pressure $\langle p^L \rangle = p_0 + \frac{1}{2}\beta\langle E \rangle$. The pressure at Y is the Eulerian value of the pressure $p^E = p_0 = \frac{1}{4}(\gamma - 3)\langle E \rangle = \langle p^L \rangle - \langle E \rangle$.

We therefore obtain the Rayleigh pressure from the difference between the pressures on the two sides of X:

$$p^{\mathrm{Ray}} = \langle p^L \rangle - p_0 = \tfrac{1}{2}\beta\langle E \rangle = \tfrac{1}{4}(\gamma + 1)\langle E \rangle \ . \tag{17}$$

We now consider the second drawing, in which Y is free to move. As can be seen, the mean pressure at the upper face of Y is different from that in the outside medium. The piston Y will therefore move until the pressure inside the cylinder is equal to the pressure outside p_0. Hence

$$p_Y = \langle p^{E'} \rangle = \langle p^{L'} \rangle - \langle E' \rangle = p_0 \ .$$

Substituting in the expression for p_X, we obtain

$$p_X = \langle p^{L'} \rangle = p_0 + \langle E' \rangle \ .$$

The Langevin radiation pressure is therefore given by the difference of p_X and p_0, i.e.,

$$p^{\mathrm{Lan}} = \langle E' \rangle \approx \langle E \rangle \ , \tag{18}$$

the differences between $\langle E \rangle$ and $\langle E' \rangle$ being of higher order.

We now remove the cylinder and piston Y, keeping only X. The situation remains unchanged, except that the sound beam will now change gradually from a maximum on its axis to zero at large transverse distances. The hydrostatic pressure makes the adjustment of the second picture gradually instead of the single adjustment at piston Y, but the effect must be the same.

In this simplified presentation, I have just passed over many complicated variations of the radiation problem in order to emphasize the basic principles involved. One of these complications is the existence of reflections. Without going into detail, it can be shown that the Langevin "pressure" becomes equal to the total mean energy density, or $(1+R)\langle E \rangle$, where R is the reflection coefficient.

Another complication is the incidence of the beam at an angle. This was treated by Brillouin, who showed that the term in his radiation stress tensor involving $\langle \rho v_1^2 \rangle$ must be multiplied by $\cos^2\theta$, where θ is the angle of incidence. As you can see, as $\theta \to 90^\circ$, this term disappears and we are left only with the Eulerian pressure $\langle p_1(x) \rangle$.

Of course, when there is imperfect reflection at an interface, some sound penetrates into the next medium. Hence there will be a radiation pressure in the second medium. If little sound is present in this medium (for example, if the second medium is air), the net force of the radiation pressure will be in the direction liquid $\to$ air, and a fountain results (see Fig. 4).[19]

Hertz and Mende studied a number of different liquid combinations. In Fig. 5(a) we have water over CCl_4. The sound source is below and an upward fountain results. In Fig. 5(b) we have water over aniline. Because of density differences here, the energy density in the upper medium (water) is greater than in the lower, and an inverse fountain is produced.

Figure 6 shows a different experimental arrangement. We are viewing the same two liquid combinations as before, but this time, the source of sound is in the upper medium, and is tilted, so that the incident beam penetrates into the second liquid, is reflected from the plate therein, and returns to the original liquid, with a separ-

FIG. 4. Example of acoustic fountain. Sound source is located underwater and sound is directed upward (from L. Bergmann[19]).

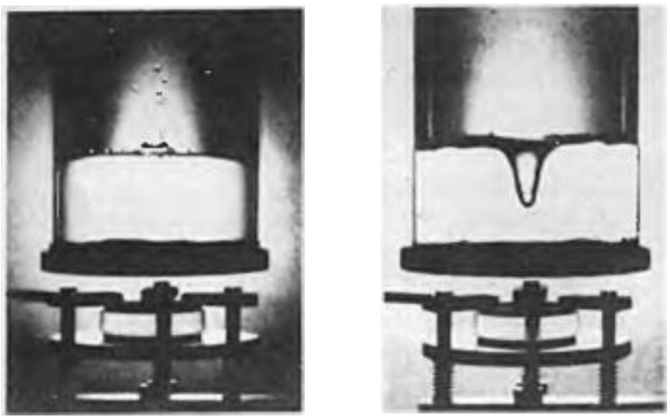

FIG. 5. Effect of radiation pressure at liquid–liquid interface. Source of sound is located at bottom of cylinder and sound is directed upward. (a) water over CCl_4; (b) water over aniline (from G. Hertz and H. Mende[15]).

ation of the beams at the interface. Figure 7 shows conclusively that the direction of the peak or fountain is independent of the direction of the beam, depending as it does only on the relative sound energy densities in the two media.

Still another complication is the existence of a different shape for the detector. The Canadian physicist Louis King worked out a solution for the case of a solid sphere in 1934, and arrived at a value of the radiation pressure that approached the value of E as ka became larger than 3 (k is the wave number of the sound in the fluid, a is the radius of sphere).[20]

More recently, Yosioka and co-workers at Osaka University have studied the effect of the elasticity of the sphere in more detail, and found some resonance departures from the King curve, but substantial agreement, overall, between theory and experiment. An example is shown in Fig. 8.

The ideas of Yosioka have been picked up by Eller,[22] Gould,[23] Crum,[24] and, finally, Apfel.[25] If we are dealing with a standing wave, then the time-averaged force exerted on small bubbles in a host liquid by a beam in which the local pressure amplitude is $p(z)$ is given by

$$\langle F\rangle = bp(dp/dz) \ , \tag{19}$$

where b is a constant characteristic of the medium. In an ideal standing wave, p is given by

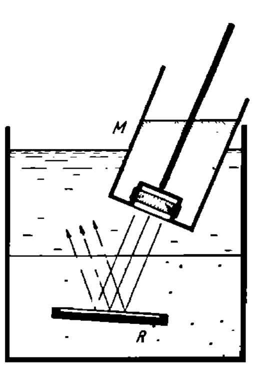

FIG. 6. Experimental arrangement for viewing two separated sound beams at liquid–liquid interface. The sound source is located in cylinder M and the sound beam is reflected from the plate R (from G. Hertz and H. Mende[15]).

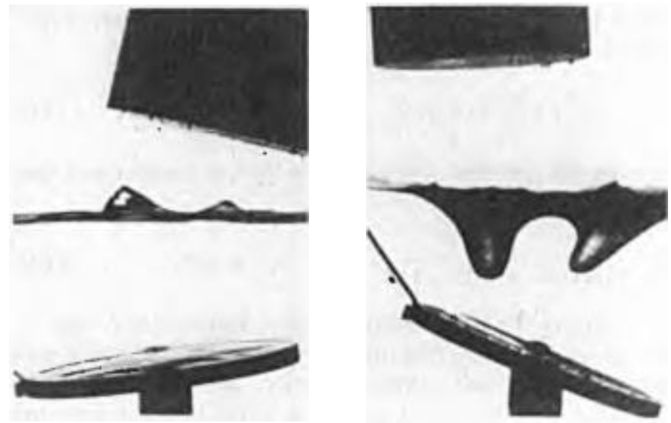

FIG. 7. Results with apparatus of Fig. 6. (a) water over aniline; (b) water over CCl_4 (from G. Hertz and H. Mende[15]).

$$p = p_a \sin kz$$

and hence

$$\langle F\rangle = \tfrac{1}{2} p_a^2 kb \sin 2kz \ . \tag{20}$$

This force can be positive or negative, depending on where we are in the standing wave. If the standing wave is in the vertical direction, then the radiation can force the bubble either up or down, depending on position.

Now, if the density of the bubble is greater than that of the host medium, the bubble will fall until it reaches a point in the standing wave where the upward force counterbalances the net gravitational-buoyancy force. The bubble can therefore be held at this position or raised by the simple expedient of increasing p_a. Apfel calls this acoustic levitation. However, most of his work has been on bubbles of density less than that of the host medium. Since these bubbles would normally rise, he is therefore working in a region where the radiation force pushes the bubble down. I suppose one might call this acoustic gravitation or acoustic depression. But levitation sounds much more cheerful, and the name has stuck.

At equilbrium, Apfel showed that the relation of pres-

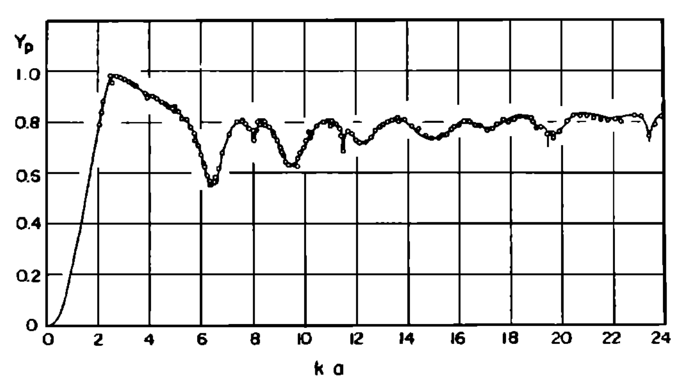

FIG. 8. Plot of $Y_p = \langle F\rangle/\pi a^2\langle E\rangle$ vs ka, where $\langle F\rangle$ is the average force on a sphere of radius a from a sound beam with wave number k and average energy density $\langle E\rangle$. Circles denote experimental values (from T. Hasegawa and K. Yosioka (1975)[21]).

sure to the various compressibilities and densities is given by

$$p\left(\frac{dp}{dz}\right)=\frac{2\rho g}{\beta_s}\ G(\beta_s^*/\beta_s,\ \rho^*/\rho)\ , \tag{21}$$

where the asterisk refers to the bubble parameters and G is given by

$$G=\frac{|1-\lambda|}{\mu-(5\lambda-2)/(2\lambda+1)}\ ,\quad \lambda=\rho^*/\rho,\quad \mu=\beta_s^*/\beta_s\ . \tag{22}$$

With his flair for name coining, Apfel calls G the *compressadensity* function. Working with a bubble with known density and compressibility, and using two different host liquids, it is possible in principle to determine both the density and the compressibility on the unknown bubble.

In this historical–tutorial review of the development of the concept of acoustic radiation pressure, it is, perhaps, not inappropriate to conclude with a quotation from Brillouin's 1936 paper:

> "Lord Rayleigh devoted several memoirs to research on the radiation pressure of sound waves but his calculations are inaccurate because of various errors of detail; the detailed analysis of various mechanisms which take part in the creation of radiation pressure is very delicate; one must take into account a whole series of second order effects which simultaneously deform the wave during its propagation; a direct reasoning, based on the formula of Boltzmann–Ehrenfest, is much more sure."

It takes nerve as well as ability to criticize Lord Rayleigh successfully, and Brillouin had both. But as we have seen, even a Brillouin can make a *faux pas*. On the other hand, so can I, and so can any of you. I've already made plenty on my part; now it's someone else's turn.

IV. CONCLUSION

The work on this paper was supported in part by the United States Navy Ocean Research and Development Activity. Recently, Professor J. Bosquet of the University of Brussels read a paper on the same subject and with very similar conclusions at the International Congress on Acoustics in Madrid, Spain, 4–9 July 1977.

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