INTERNATIONAL JOURNAL OF NUMERICAL MODELLING: ELECTRONIC NETWORKS, DEVICES AND FIELDS
Int. J. Numer. Model. 2001; **14**: 451–456 (DOI: 10.1002/jnm.427)

Huygens' principle in the transmission line matrix method (TLM). Global theory

Peter Enders*,†

Fischerinsel 2, D-10179 Berlin, Germany

SUMMARY

Huygens' principle (HP) is understood as a *universal* principle governing not only the propagation of light, but also of acoustic waves, heat and matter diffusion, Schrödinger's matter waves, random walks, and many more. According to Hadamard's rigorous definition, HP comprehends the principle of action-by-proximity (cf. Faraday's field theory, etc.) and the superposition of secondary wavelets (Huygens' construction). This definition is reformulated for discrete spaces. The global aspect concerns the propagation of fields (e.g. wavefronts). Within TLM, the appropriate field propagator (Green's function) is the Johns matrix. The compatibility with HP explains the success of TLM in computing propagation, transport, and other evolution processes from a different point of view. A possible practical application of these results for computing eigenmodes is mentioned. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: Huygens' principle; transport; propagation; transmission line matrix method

1. INTRODUCTION

Huygens' ideas on how light propagates [1, 2] have become basic ingredients of our physical picture of the world, and their mathematical formulation is related to fundamental methodological problems of the physics of propagation. The notion 'Huygens' principle' (HP), however, is not uniquely used, cf. Reference [3], and there is some confusion in the literature, in particular, on the role of sharp wave fronts and the range of applicability. We have shown earlier [4, 5], how these difficulties can be resolved. Here, we will concentrate on discrete spaces and reformulate Hadamard's definition of HP for them.

The relationship of TLM to HP has been addressed as early as 1974 [6]. Hoefer [7] examined the scattering process in more detail, but a satisfying solution has been found for two-dimensional (2D) TLM networks only (see also References [8, 9]). Recently, it has been demonstrated, how HP is actually realized in the scattering process of isotropic TLM networks [10], i.e. its validity has been shown independent of spatial dimension and coordination number, respectively. Here, we wish to complement this local picture of HP by the global picture provided by field propagators

*Correspondence to: Peter Enders, Fischerinsel 2, D-10179, Berlin, Germany
†E-mail: Peter.Enders@Cheerful.com

Published online 31 July 2001

Received 20 January 2000
Revised 3 August 2000

(Green's functions) and the Chapman–Kolmogorov equation [4], where the results of this paper and of Reference [10] are independent of each other.

Thus, this paper is intended to provide the basics for further explorations of the role of HP in TLM, where we are guided by the modelling philosophy of TLM itself [11]. Section 2 starts with the reformulation of Hadamard's rigorous definition of HP for discrete spaces. In Section 3, the superposition of secondary waves will be represented and illustrated by means of general field propagators in the discrete space-time domain. Section 4 shows, how this role is fulfilled by the Johns matrix. Section 5, finally, concludes this paper.

2. HADAMARD'S SYLLOGISTIC DEFINITION OF HUYGENS' PRINCIPLE

A syllogism is a form of logical conclusion, which originally developed by Aristoteles. The conclusion is derived from two premises, a major and a minor one. For discrete spaces, Hadamard's beautiful formulation of HP [3, Section 33] reads:

(A) *Major premise*: The action of phenomena produced at time step k_0 on the state of matter at the time k_1 takes place by the mediation of every intermediate time step $k = k_0 + 1, k_0 + 2, \ldots, k_1 - 1$.

(B) *Minor premise*: 'The propagation of light pulses proceeds without deformation (spreading, tail building) of the pulse'.

(C) *Conclusion*: In order to calculate at time step $k = k_2$ the effect of an initial luminous phenomenon produced at node ω_0 at $k = k_0$, we may replace the latter by a proper system of disturbances taking place at $k = k_1$, $k_0 < k_1 < k_2$, and being distributed over all cells of distance $(k_1 - k_0)$.

Proposition (A) is the *principle of action-by-proximity* known, e.g. from cellular automata and random walks: each cell acts within one time step only on its neighbouring cells. Proposition (B) postulates the propagation of *sharp wave fronts* (there is no back-reflection of the incident wavefront). Proposition (C) is essentially *Huygens' construction.*

However, in conclusion (C), isotropy of re-irradiation along the lattice axes can be generalized to re-irradiation according to the actual local propagation conditions. This means, that the secondary sources will represent the propagation properties of the material under consideration (or that of free space). For instance, in anisotropic media, the reaction of the secondary sources is anisotropic, while in non-linear media, their excitation and re-irradiation is not proportional to the amplitude of the exciting field.

Proposition (B) is rather special. It is necessary (and useful) for geometrically constructing the wavefront, but not for, (i), the most fundamental principle of action-by-proximity and, (ii), the cycle of excitation and re-irradiation of secondary wavelets. Although having important practical implications (quality of speech over long transmission lines, e.g.), it is of only minor importance for the general theory of HP which concerns the points (i) and (ii) just mentioned. Moreover, on discrete lattices, Hadamard's conjecture is violated, that there are sharp wavefronts in homogeneous isotropic spaces of odd dimension greater than one. This means, that an initial Kronecker excitation, δ_{i,i_0}, should lead after k time steps a pattern, where only cells at a distance of k mesh sizes exhibit finite field values. For instance, on a 3D Cartesian lattice (coordinates $i_1, \ldots, i_3$), the initial Kronecker excitation $\delta_{i_1,0}\delta_{i_2,0}\delta_{i_3,0}$ should lead after k times steps to a field distribution proportional to $\{\delta_{i_1,k_1}\delta_{i_2,k_2}\delta_{i_3,k_3};\ |k_1| + |k_2| + |k_3| = k\}$. However, on a 3D lossless scalar TLM lattice, there is a finite back-reflection at each node, and the solution to the TLM difference equations is not of this 'sharp' form. For this, we will relax Premise B (cf. also References [6, 12]). Note, however, that Johns' [13] symmetric condensed node for solving Maxwell's equations in free 3D space seems to obey Premise B [14], since there is no back-scattering.

Thus, we will call Huygens's principle (HP) the combination of action-by-proximity ('elastic waves in ether' in Huygens' pictural imagination) and superposition of secondary wavelets (i.e. in case of sharp wavefronts, Huygens' construction). The shape of the wave front may vary from case to case, without influence on these basic ingredients of propagation, while the essentials of Huygens' (and Faraday's) imagination of propagation are conserved. The advantage of this notion of HP consists in that its applicability becomes rather universal; in fact, in this form, HP qualifies to be a clue for unifying the physical and mathematical description of many different transport and propagation processes.

3. DISCRETE PROPAGATORS FOR REPRESENTING HUYGENS' PRINCIPLE

Feynman [15] emphasized, that HP applies to Schrödinger wave mechanics, since the quantum mechanical transition probability amplitudes from state a to state c, P_{ca}, obey the Chapman–Kolmogorov equation in state space,

$$P_{ca} = \Sigma_b P_{cb} P_{ba} \tag{1}$$

where the sum runs over all possible intermediate states b. Obviously, this equation provides a rather general mathematical formulation of HP. The set of intermediate states b represents the intermediate wavefront and the amplitudes P_{cb} the secondary wavelets. Since the states a, b and c may be adjacent to each other, action-by-proximity is also included.

Now the Chapman–Kolmogorov equation is the equation of motion of Markov processes, and each TLM algorithm realizes a discrete Markov process [16]. Thus, the question arises, what is the TLM analogue to P_{ab}.

For simplicity, consider first a link-line TLM chain with $R = Z$. The node voltage, ϕ, obeys the equation of motion of a simple random walk.

$$\phi_{k,i} = \tfrac{1}{2}(\phi_{k-1,i-1} + \phi_{k-1,i+1}), \quad k = 0, 1, 2, \ldots, \quad -\infty < i < +\infty \tag{2}$$

This is the well-known Euler forward scheme for the diffusion equation $\partial T/\partial t = \partial^2 T/\partial x^2$. For the Kronecker initial distribution $\phi_{0,i} = \delta_{i,0}$, the node voltage distribution after k steps is equal to the 'fundamental' solution to Equation (2),

$$\phi^f_{k,i} = \begin{Bmatrix} 2^{-k}\dbinom{k}{n}, & n \equiv \dfrac{k-|i|}{2} \text{ being integer} \\ 0 & n \equiv \dfrac{k-|i|}{2} \text{ being half-integer} \end{Bmatrix} \tag{3}$$

where $\binom{k}{n} \equiv k!/n!(k-n)!$ and $|i|$ being the modulus of i. This 'discrete Gaussian' is the analogue of the fundamental solution of the diffusion equation. The corresponding Green's function (GF;

called 'Green probability' in Reference [17]),

$$G_{k,k';i,i'} = \phi^{f}_{k-k',i-i'} \tag{4}$$

possesses the so-called 'Markov property, (1):

$$G_{k,k';i,i'} = \sum_{i''} G_{k,k'';i,i''} G_{k'',k';i'',i'} \quad \text{for any } k'' \ni [k,k'] \tag{5}$$

The term *discrete Huygens propagator* has been proposed for such GF [4]. In one time step, they connect no states, except next-neighbouring ones, and possess the Markov property (5). Eventually, there are appropriate boundary conditions to be fulfilled, as in the continuum case.

Another example is Pascal's triangle being a simple, but instructive example of Markovean 'number diffusion' obeying Huygens' recipe of construction [18].

4. THE JOHNS MATRIX AS A HUYGENS' PROPAGATOR

For *first*-order processes, such as simple random walk, the Huygens propagator proves to be identical with the GF of the difference equation. This perfectly parallels the continuum case, where the Chapman–Kolmogorov equation is fulfilled by GF of equations of *first* order in the time variable. In contrast, the GF of a *multi*-step equation of motion for the node voltage as obtained in the case $R \neq Z$ [cf. Equation (6) below] is, in general, *not* a (discrete) Huygens propagator. To get such, one has 'to return' to a system of one-step equations of motion for the travelling (incident or reflected) voltage (or current) pulses (the far-reaching consequences of this observation have been treated in Reference [4]).

As an example, consider a 1D lossy TLM chain with $R \neq Z$ as presented by Johns [19]. The node voltage obeys the second-order equation

$$\phi_{k+2,i} = \tau(\phi_{k+1,i-1} + \phi_{k+1,i+1}) + (\rho^2 - \tau^2)\phi_{k,i} \tag{6}$$

The corresponding GF [16] is *not* a Huygens propagator, since Equation (5) is not fulfilled. However, the incident voltage pulses obey a system of two coupled partial difference equations of *first* order,

$$\begin{aligned} {}_{k+1}^{\;\;i}V_i^L &= \rho \cdot {}_k^iV_{i-1}^R + \tau \cdot {}_k^iV_{i-1}^L \\ {}_{k+1}^{\;\;i}V_i^R &= \rho \cdot {}_k^iV_{i+1}^L + \tau \cdot {}_k^iV_{i+1}^R \end{aligned} \tag{7}$$

In matrix form, Equations (7) read

$$\begin{pmatrix} {}_{k+1}^{\;\;i}\mathbf{V}_i^L \\ {}_{k+1}^{\;\;i}\mathbf{V}_i^R \end{pmatrix} = \begin{pmatrix} \tau\Delta_- & \rho\Delta_- \\ \rho\Delta_+ & \tau\Delta_+ \end{pmatrix} \begin{pmatrix} {}_k^i\mathbf{V}_i^L \\ {}_k^i\mathbf{V}_i^R \end{pmatrix} \equiv \mathbf{D} \begin{pmatrix} {}_k^i\mathbf{V}_i^L \\ {}_k^i\mathbf{V}_i^R \end{pmatrix}, \quad \Delta_{\pm}\,{}_k^i\mathbf{V}_i^{L,R} \equiv {}_k^i\mathbf{V}_{i\pm 1}^{L,R} \tag{8}$$

It is an easy excercise to prove, that $\mathbf{G}_{k,i;k',i'} = (\mathbf{D}^{k-k'})_{i,i'}$ is the corresponding Huygens propagator.

It is interesting to note, that, by virtue of the Caley–Hamilton theorem, the eigenvalue equation of the transition matrix $\mathbf{D}$ reads

$$\mathbf{D}^2 = \tau(\Delta_- + \Delta_+)\mathbf{D} + (\rho^2 - \tau^2)\mathbf{1} \tag{9}$$

Equation (9) diagonalizes systems (7) and (8), respectively, into the form of Equation (6) for both ${}_k^iV_i^L$ and ${}_k^iV_i^R$, and, consequently, for $\phi_{k,i} = {}_k^iV_i^L + {}_k^iV_i^R$. The corresponding Huygens propagator, $\mathbf{G}_{k,i;k',i'} = (\mathbf{D}^{k-k'})_{i,i'}$, may be called *proper* or *irreducible*, since its elements obey the two-step equation of motion (9), too. This is an quite important property, because in this case, the eigenvalue equation of the transition matrix diagonalizes the first-order equations of motion to a physically relevant equation. A counter-example is given by the difference equations relating (ϕ_{k+1}, ϕ_k) to (ϕ_{k-1}, ϕ_{k-2}).

Hoefer [20] has proposed to call the GF of TLM difference equations the Johns matrix. Thus, the Johns matrix is the proper Huygens propagator within TLM. Its properties realize HP in the sense of action-by-proximity and re-irradiation of secondary wavelets on TLM meshes.

Going over to the Fourier space, $(t, \mathbf{r}) \to (\omega, \mathbf{q})$, Equation (9) suggests the following hypothesis.

The eigenspectrum of the Johns matrix approximates the eigenspectrum of the object under investigation.

This would be another example for the observation, that the discrete formulation of HP yields construction principles for numerical algorithms for a wide variety of problems (cf. References [7, 8, 21]). In particular, it would largely simplify the computation of eigenmodes by means of the Caley–Hamilton theorem.

5. CONCLUSIONS

Huygens' principle (HP) exhibit a local and a global aspect, both being intimately related to each other, of course. The former is the excitation of secondary sources and their irradiation of secondary wavelets, i.e. the scattering process. Its correct description within TLM has been presented in Reference [10]. The global aspect concerns the propagation of wavefronts and other excitations. It is most naturally represented in terms of appropriated Green's functions, the so-called Huygens propagators. Within TLM, the Johns matrix being a proper Huygens propagator plays this role.

The mathematical representation of HP by means of propagators (Green's functions) and the Chapman–Kolmogorov equation provides, perhaps, the most general way to connect HP with concrete differential and difference equations. In this form, HP applies to all explicit finite differencing schemes, including TLM. At once, the difficulties discussed in References [6, 12] are lifted.

Difference equations representing a discrete HP are directly suited for computing *all* propagation processes that can be modelled through explicit differential equations. This should enable the simultaneous and self-consistent computation of interacting fields of different type, e.g. heat diffusion and electromagnetic waves in lasers, in microwave ovens or in lenses and mirrors for high-power beams. Within explicit schemes, self-consistency can be achieved at *every* (time) step, whereby convergency is considerably accelerated.

The Johns matrix as the Green's function of the coupled one-step TLM equations for the incident (or for the reflected) travelling voltage pulses exhibits several computational advantages; its application to computing eigenmodes deserves further investigations (cf. also References [8, 21]).

In summary, the TLM description of travelling voltage pulses obeys Huygens' principle in the same sense as many physical transport and propagation processes. This is an additional explanation of its success in many areas of application and strengthens its theoretical foundations.

ACKNOWLEDGEMENTS

I would like to thank Christian Vanneste and Donard de Cogan for critically reading the manuscript and the referees for their comments which helped to clarify some issues.

REFERENCES

1. Huygens C. *Traité de la lumiere*. Pierre van der Aa: Leiden, 1690.
2. Block H, Ferwerda HA, Kuiken HK. 'Huygens' Principle 1690–1990: Theory and Applications. Elsevier Science: Amsterdam, 1992.
3. Hadamard J. *The Cauchy Problem and the Linear Hyperbolic Partial Differential Equations*. Dover: New York, 1953.
4. Enders P. Huygens' principle and the modelling of propagation. *European Journal of Physics*, 1996; **17**:226–235.
5. Enders P. A new view on Huygens' principle. *Presented at OSA Annual Meeting* Rochester, 1996.
6. Johns PB. A new mathematical model to describe the physics of propagation. *The Radio and Electronic Engineer* 1974; **44**:657–666.
7. Hoefer WJR. Huygens and the computer—a powerful allianz in numerical electromagnetics. *Proceedings of the IEEE* 1991; **79**:1459–1471.
8. Hoefer WJR, So PPM, Pompei D, Papiernik A. Numerical analysis of guiding and radiating microwave structures based on Huygens', Principle. In *Huygens' Principle 1690–1990: Theory and Applications*, Block H, Ferwerda HA, Kuiken HK (eds). Elsevier Science: Amsterdam, 1992; 461–472.
9. Christopoulos C. *The Transmission-Line Modeling Method*: *TLM*. IEEE Press: New York, 1995.
10. Enders P, Vanneste C. Huygens' principle in the transmission line matrix method (TLM). *International Journal of Numerical Modelling*: *Electrical Networks*, *Devices & Fields*, submitted.
11. Johns PB. The art of modelling. *Electronics and Power* 1979; **25**:565–569.
12. Miller DAB. Huygens's wave propagation principle corrected. *Optics Letters* 1991; **16**:1370–1372.
13. Johns PB. A symmetrical condensed node for the TLM method. *IEEE Transactions on Microwave Theory Technology* 1987; **MTT-35**:370–377.
14. Johns DP, Enders P. Wave optics computing by a network-based vector wave automaton. *Presented at the 4th International Conference on Physics Computing '92*. Prague, 1992.
15. Feynman RP. Space-time approach to non-relativistic quantum mechanics. *Reviews of Modern Physics* 1948; **20**:367–387.
16. Enders P, de Cogan D. The efficiency of transmission-line matrix modelling—a rigorous viewpoint. *International Journal of Numerical Modelling*: *Electrical Networks*, *Devices & Fields* 1993; **6**:109–26.
17. Keilson J. *Green's Function Methods in Probability Theory*. Griffin & Co.: London, 1965; 80.
18. de Cogan D, Enders P. Microscopic effects in TLM heat flow modelling. *IEE Digest* 1991; **157**:8/1–8/11.
19. Johns PB. A simple explicit and unconditionally stable numerical routine for the solution of the diffusion equation. *International Journal of Numerical Methods in Engineering* 1977; **11**:1307–1328.
20. Hoefer WJR. The discrete time domain Green's function or Johns matrix—a new powerful concept in TLM. *International Journal of Numerical Modelling*: *Electrical Networks*, *Devices & Fields* 1989; **2**:215–225.
21. Enders P, Wlodarczyk AJ. Towards exactly modelling open/absorbing boundaries. *Presented at the 2nd International Workshop on Discrete Time Domain Modelling of Electromagnetic Fields and Networks*, Berlin, 1993.

