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Viscous potential flow

By D. D. JOSEPH

Department of Aerospace Engineering and Mechanics, University of Minnesota, MN 55455, USA
joseph@aem.umn.edu

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Potential flows $\boldsymbol{u} = \nabla\phi$ are solutions of the Navier–Stokes equations for viscous incompressible fluids for which the vorticity is identically zero. The viscous term $\mu\nabla^2\boldsymbol{u} = \mu\nabla\nabla^2\phi$ vanishes, but the viscous contribution to the stress in an incompressible fluid (Stokes 1850) does not vanish in general. Here, we show how the viscosity of a viscous fluid in potential flow away from the boundary layers enters Prandtl's boundary layer equations. Potential flow equations for viscous compressible fluids are derived for sound waves which perturb the Navier–Stokes equations linearized on a state of rest. These linearized equations support a potential flow with the novel features that the Bernoulli equation and the potential as well as the stress depend on the viscosity. The effect of viscosity is to produce decay in time of spatially periodic waves or decay and growth in space of time-periodic waves.

In all cases in which potential flows satisfy the Navier–Stokes equations, which includes all potential flows of incompressible fluids as well as potential flows in the acoustic approximation derived here, it is neither necessary nor useful to put the viscosity to zero.

1. Introduction

Potential flows of incompressible fluids admit a pressure (Bernoulli) equation when the divergence of the stress is a gradient as in inviscid fluids, viscous fluids, linear viscoelastic fluids and second-order fluids (for which a term proportional to the square of the velocity gradient called a viscoelastic pressure appears). All of the classical results for inviscid potential flows hold for viscous potential flows with the caveat that the viscous stresses are not generally zero (Stokes 1850). The differences between inviscid and viscous and viscoelastic potential flow together with a review of the literature prior to 1994 are discussed by Joseph & Liao (1994*a*, *b*).

Potential flows will not generally satisfy boundary conditions which require that the tangential component of velocity and the shear stress should be continuous across the interface separating the fluid from a solid or another fluid. The velocity and pressure are the same in inviscid and viscous potential flows when fluid–fluid interfaces or free surfaces are not present. The viscosity enters interface problems explicitly through the viscous term in the normal stress balance across the interface (see Funada & Joseph 2002 and the references therein). The Navier–Stokes equations for compressible fluids linearized on a state of rest, treated here, are yet another case in which the viscosity enters the solution for the potential explicitly.

2. Boundary layer theory

At speeds large enough so that the vorticity in the outer flow is zero, we should like to see how the viscosity in the viscous potential flow enters the boundary layer equation.

To make things simple, consider flow in two dimensions over a body obeying the equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2.2}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial}{\partial y}\left[p - \mu\frac{\partial v}{\partial y}\right] + \mu\frac{\partial^2 v}{\partial x^2}. \tag{2.3}$$

In Prandtl's theory it is argued (Batchelor 1967) that "...the pressure is approximately uniform across the boundary layer; and if it happens that the variation of p with x just outside the boundary layer is known ... perhaps from consideration of the inviscid flow equations in the regions outside the boundary layer ... the pressure term [in our (2.2)] can be regarded as given."

Now consider that the region outside the boundary layer is a potential flow, as is usual, and instead of putting the viscosity to zero let it be whatever it is. Then we have potential flow of a viscous fluid with curl $\boldsymbol{U} = 0$ (where $\boldsymbol{U}, P$ are in the free stream) as in the inviscid case and the same potential Φ with $\nabla^2\Phi = 0$.

Equation (2.3) is written in such a way as to show that the y derivative of

$$p - \mu\frac{\partial v}{\partial y} \tag{2.4}$$

is $O(\delta)$ across the boundary layer of thickness δ; hence it can be taken as

$$P - \mu\frac{\partial V}{\partial y} = P - \mu\frac{\partial^2\Phi}{\partial y^2}, \tag{2.5}$$

which is the value of (2.4) at the edge of that layer. Using now the Bernoulli equation for steady flow, which holds even when the viscosity is not zero, we have as usual

$$P = -\frac{\rho U^2}{2} + \text{const} \tag{2.6}$$

and combining (2.5) and (2.6)

$$-\rho\frac{U^2}{2} + \text{const} - \mu\frac{\partial V}{\partial y} = p - \mu\frac{\partial v}{\partial y} \tag{2.7}$$

with a small error, and at the wall where $\partial v/\partial y = 0$

$$\frac{\partial p}{\partial x} = -\rho U\frac{\partial U}{\partial x} - \mu\frac{\partial^2 V}{\partial x \partial y} \tag{2.8}$$

drives the flow in (2.2). Using (2.1) we can write

$$\frac{\partial V}{\partial y} = -\frac{\partial U}{\partial x}, \qquad (U, V) = \left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}\right).$$

Hence

$$\frac{\partial p}{\partial x} = -\rho U\frac{\partial U}{\partial x} + \mu\frac{\partial^2 U}{\partial x^2}, \qquad U = \frac{\partial\Phi}{\partial x}. \tag{2.9}$$

Noting next that $\partial^2 u/\partial x^2 = 0$ but $\partial^2 U/\partial x^2 \neq 0$ at the wall, using also the usual boundary layer assumptions we have a modified Prandtl equation

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=\rho U\frac{\partial U}{\partial x}-\mu\frac{\partial^2 U}{\partial x^2}+\mu\frac{\partial^2 u}{\partial y^2}. \tag{2.10}$$

When the outer fluid is inviscid, the viscous term $\mu\partial^2 U/\partial x^2$ vanishes; for high $Re = U_0 L/\nu$ this term is small.

The generally understood principle that the effects of viscosity in the outer flow vanish at surpassingly high Reynolds numbers remains true even when the outer potential flow is viscous. In this limit, our derivation shows that the conventional boundary layer equations are not changed. The imprint of viscosity on the outer flow may possibly be studied at lower, but not low, Reynolds numbers by modified boundary layer equations like (2.10). There are many cases of irrotational flow in which viscous effects are important; for example, in cases of interfacial instability. The generalization of the classical analysis of interfacial instability (Joseph, Belanger & Beavers 1999), Kelvin–Helmholtz instability (Funada & Joseph 2001) and capillary instability (Funada & Joseph 2002) from inviscid to viscous potential flow allows one to compare exact solutions with inviscid and viscous potential flow. All three theories collapse at high Reynolds numbers, as expected, but at lower – but not very low – Reynolds numbers the effects of viscosity are sensible and can be very large even when the differences between viscous potential flow and the exact results are very small.

3. Potential flow solutions of the Navier–Stokes equations for viscous compressible fluids

Potential flows are not in general solutions of the Navier–Stokes equations for viscous compressible fluids. To have such solutions it is necessary to show that curl $\boldsymbol{u} = 0$ is a solution of the vorticity equation. The gradients of density and viscosity which are spoilers for the general vorticity equation do not enter the equations which perturb the state of rest with uniform pressure p_0 and density ρ_0.

The stress for a compressible viscous fluid is given by

$$T_{ij} = -\left(p+\tfrac{2}{3}\mu\,\mathrm{div}\,\boldsymbol{u}\right)\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right). \tag{3.1}$$

Here, the second coefficient of viscosity is selected so that $T_{ii} = -3p$. (The results to follow will apply also to the case when other choices are made for the second coefficient of viscosity.)

The equations of motion are given by

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)=\mathrm{div}\,\boldsymbol{T} \tag{3.2}$$

together with

$$\frac{\partial\rho}{\partial t}+\boldsymbol{u}\cdot\nabla\rho+\rho\,\mathrm{div}\,\boldsymbol{u}=0. \tag{3.3}$$

To study acoustic propagation, these equations are linearized; putting

$$[\boldsymbol{u}, p, \rho] = [\boldsymbol{u}', p_0+p', \rho_0+\rho'], \tag{3.4}$$

where $\boldsymbol{u}'$, p' and ρ' are small quantities, we obtain

$$T_{ij} = -\left(p_0 + p' + \tfrac{2}{3}\mu_0 \operatorname{div} \boldsymbol{u}'\right)\delta_{ij} + \mu_0\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right), \tag{3.5}$$

$$\rho_0 \frac{\partial \boldsymbol{u}'}{\partial t} = -\nabla p' + \mu_0\left(\nabla^2 \boldsymbol{u}' + \tfrac{1}{3}\nabla \operatorname{div} \boldsymbol{u}'\right), \tag{3.6}$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \operatorname{div} \boldsymbol{u}' = 0, \tag{3.7}$$

where p_0, ρ_0 and μ_0 are constants. For acoustic problems, we assume that a small change in ρ induces small changes in p by fast adiabatic processes; hence

$$p' = C_0^2 \rho', \tag{3.8}$$

where C_0 is the speed of sound.

Forming now the curl of (3.6), we find that

$$\rho_0 \frac{\partial \zeta}{\partial t} = \mu_0 \nabla^2 \zeta, \quad \zeta = \operatorname{curl} \boldsymbol{u}'. \tag{3.9}$$

Hence $\zeta = 0$, is a solution of the vorticity equation and we may introduce a potential

$$\boldsymbol{u}' = \nabla \phi. \tag{3.10}$$

Next, combining (3.10) and (3.6), we obtain

$$\nabla\left[\rho_0 \frac{\partial \phi}{\partial t} + p' - \tfrac{4}{3}\mu_0 \nabla^2 \phi\right] = 0. \tag{3.11}$$

The quantity in the brackets is equal to an arbitrary function of the time, which may be absorbed in ϕ.

A viscosity-dependent Bernoulli equation

$$\rho_0 \frac{\partial \phi}{\partial t} + p' - \tfrac{4}{3}\mu_0 \nabla^2 \phi = 0, \tag{3.12}$$

is implied by (3.11). The stress (3.5) is given in terms of the potential ϕ by

$$T_{ij} = -\left(p_0 - \rho_0 \frac{\partial \phi}{\partial t} + 2\mu_0 \nabla^2 \phi\right)\delta_{ij} + 2\mu_0 \frac{\partial^2 \phi}{\partial x_i \partial x_j}. \tag{3.13}$$

To obtain the equation satisfied by the potential ϕ, we eliminate ρ' in (3.7) with p' using (3.8), then eliminate $\boldsymbol{u}' = \nabla\phi$ and p' in terms of ϕ using (3.12) to find

$$\frac{\partial^2 \phi}{\partial t^2} = \left(C_0^2 + \tfrac{4}{3}\nu_0 \frac{\partial}{\partial t}\right)\nabla^2 \phi, \tag{3.14}$$

where the potential ϕ depends on the speed of sound and the kinematic viscosity $\nu_0 = \mu_0/\rho_0$.

The damped wave equation (3.14) may be derived directly without introducing a potential from the Navier–Stokes equation for compressible fluids in the acoustic approximation; obviously, the viscosity-dependent Bernoulli equation (3.12) requires one to introduce a potential. Lamb (1932) derived (3.14) for the velocity in one space dimension directly from the linearized Navier–Stokes equation (3.6) for plane waves in a laterally unbounded compressible fluid (his equation (4), p. 647). Lighthill (1978) derived the same one-dimensional damped wave equation for the density rather than the velocity without introducing a velocity potential. In Lighthill's equation (205), $\frac{4}{3}\nu_0$ is replaced by δ, a relaxation time for a relaxing gas given by his equation (200), which may be written as

$$p' = C_0^2\rho' + \delta\frac{\partial\rho'}{\partial t}. \tag{3.15}$$

Inserting (3.15) into (3.12) we obtain

$$\rho_0\frac{\partial\phi}{\partial t} + C_0^2\rho' + \delta\frac{\partial\rho'}{\partial t} - \tfrac{4}{3}\mu_0\nabla^2\phi = 0. \tag{3.16}$$

Combining now (3.16) with

$$\frac{\partial\rho'}{\partial t} + \rho_0\nabla^2\phi = 0,$$

we find a generalized damped wave equation

$$\frac{\partial^2\phi}{\partial t^2} = \left(C_0^2 + \left[\delta + \tfrac{4}{3}\nu_0\right]\frac{\partial}{\partial t}\right)\nabla^2\phi. \tag{3.17}$$

A dimensionless form for the potential equation (3.17)

$$\frac{\partial^2\phi}{\partial T^2} = \left(1 + \frac{\partial}{\partial T}\right)\nabla^2\phi, \qquad \nabla^2\phi = \frac{\partial^2\phi}{\partial X^2} + \frac{\partial^2\phi}{\partial Y^2} + \frac{\partial^2\phi}{\partial Z^2}, \tag{3.18}$$

arises from a change of variables

$$t = \frac{\left[\delta + \frac{4}{3}\nu_0\right]}{C_0^2}T, \qquad \boldsymbol{x} = \frac{\left[\delta + \frac{4}{3}\nu_0\right]}{C_0}\boldsymbol{X}. \tag{3.19}$$

The classical theory of sound (see Landau & Lifshitz 1987, chap. VIII) is governed by a wave equation, which may be written in dimensionless form as

$$\frac{\partial^2\phi}{\partial T^2} = \nabla^2\phi. \tag{3.20}$$

The time derivative on the right of (3.18) leads to a decay of the waves not present in the classical theory.

The decay of the amplitude of separable solutions of (3.18) is governed by a telegraph equation. To see this consider the propagation of plane monochromatic travelling waves (see Landau & Lifshitz 1987, p. 253). We can solve the one-dimensional version of (3.18),

$$\frac{\partial^2\phi}{\partial T^2} = \left(1 + \frac{\partial}{\partial T}\right)\frac{\partial^2\phi}{\partial X^2}, \tag{3.21}$$

by separation of variables, $\phi = F(T)G(X)$, where

$$\frac{F''}{F + F'} = \frac{G''}{G} = -k^2. \tag{3.22}$$

The function $F(T)$ satisfies a telegraph equation. If $k^2 > 4$, the solution is

$$\phi = \left(A\mathrm{e}^{-\omega_1 T} + B\mathrm{e}^{-\omega_2 T}\right)\cos(-kX + \alpha) \tag{3.23}$$

where A, B and α are undetermined constants and

$$\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} = \frac{k^2}{2}\begin{bmatrix}1\\ 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}\sqrt{k^4-4k^2}\\ -\sqrt{k^4-4k^2}\end{bmatrix}. \tag{3.24}$$

The solution is a standing periodic wave with a decaying amplitude.

If $k^2 < 4$, the solution is

$$\begin{aligned}\phi = \exp\left(-\tfrac{1}{2}k^2T\right)\big[A\cos\left(-kX - \tfrac{1}{2}(4k^2-k^4)^{1/2}T + \alpha\right)\\ + B\cos\left(-kX + \tfrac{1}{2}(4k^2-k^4)^{1/2}T + \alpha\right)\big],\end{aligned} \tag{3.25}$$

which represents decaying waves propagating to the left and right.

Travelling plane wave solutions which are periodic in T and grow or decay in X are also easily derived by separating variables. The travelling wave

$$\phi = A\mathrm{e}^{-k_1X}\cos(k_2X - \omega T + \alpha) + B\mathrm{e}^{k_1X}\cos(-k_2X - \omega T + \alpha) \tag{3.26}$$

is such a solution provided that

$$k_1 = \frac{1}{\sqrt{2}}\frac{\omega^2}{\left[p + (p^2+\omega^2p^2)^{1/2}\right]^{1/2}}, \qquad k_2 = \frac{1}{\sqrt{2}}\frac{\left[p + (p^2+\omega^2p^2)^{1/2}\right]^{1/2}}{p}\omega,$$

where $p = 1 + \omega^2$.

The separation of variables for plane waves may be greatly generalized by considering solutions of (3.18) of the form $\phi = F(T)G(X, Y, Z)$ leading to a separation of variables like (3.22) in the form

$$\frac{F''}{F + F'} = \frac{\nabla^2 G}{G} = -k^2, \tag{3.27}$$

where $F(T)$ satisfies the same telegraph equation as for plane waves.

4. Concluding remarks

It is not necessary, nor useful, to put the viscosity to zero when studying potential flows. In all cases, the viscous stresses do not vanish even when the vorticity is zero. In some cases, like interface problems in which the viscous normal stress enters, and in the acoustic approximation for compressible fluids, the viscosity enters the solution for the potential explicitly. The derivation of the boundary layer equations based on potential flow of a viscous fluid in the outer flow leads to a modified boundary layer in which the viscous stress in the outer flow is felt as a pressure contribution on the solid boundary. This extra contribution vanishes at very high Reynolds numbers but may play a role at lower Reynolds numbers.

All of the potential flow solutions which perturb the state of rest of an inviscid compressible fluid can be considered for the effects of viscosity using the potential flow equations for flows of viscous compressible fluids derived here. Under ordinary circumstances viscous and relaxation effects will be negligible. In problems of high-frequency ultrasound in liquids however, the effects of viscosity can be important, even dominant. The viscous effects which would enter the study of stress-induced cavitation (Joseph 1998) due to high frequency are two-fold: dissipative effects which are more or less described by a telegraph equation, and 'anistropic' pressures associated with the viscous part of the stress tensor (3.13).

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