Acoustic quality factor and energy losses in cylindrical pipes

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The quality factor Q of a damped oscillator equals 2π times the ratio of stored energy to the energy dissipated per cycle. This makes Q a sensitive probe of energy losses. Using modest equipment, we measured the acoustical Q for a set of cylindrical pipes having the same resonant frequency, but different diameters D. The graph of Q vs D could be well fitted with two parameters, one of which corresponds to energy loss via radiation from the ends of the pipe, and the other to thermal and viscous losses very close to the pipe wall. The wall loss parameter was quite constant no matter where the pipes were located, but the radiative loss parameter varied significantly with location inside a room, suggesting that room reflections affected the sound radiated from the pipe. This study offers valuable insights at no great expense, and could be the basis of an upper-division undergraduate laboratory experiment.
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I. INTRODUCTION

A driven acoustical resonator was constructed by placing a speaker near one end of a PVC pipe, open at both ends, as shown in Fig. 1, in order to determine its acoustical quality factor Q. The response was obtained via a small, inexpensive microphone in the center of the pipe, and the amplified signal was sent to an oscilloscope. Three frequencies were measured to determine Q: the ''central'' frequency f_0 of maximum response, and the frequencies on either side of f_0 where the response is 0.707 of maximum.

Q was found for a set of pipes of different diameters but all having the same resonant frequency. The graph of Q versus diameter for these pipes can be understood in terms of two loss mechanisms sketched in this section and more fully explained later in this paper.

Q is the stored energy E divided by the energy loss per radian. It may also be expressed as $\omega E/P$, where ω is the angular frequency, and P is the dissipated power. In a cylindrical pipe, the stored energy is proportional to the cross-sectional area, or to D^2, where D is the pipe inner diameter. The sound power radiated at low frequencies is proportional to the square of the ''source strength'' (the volume flow rate). Since the source strength is proportional to the pipe cross-sectional area, the rate of energy radiation is proportional to D^4. Thus, for losses due to radiation of sound, Q_{rad} is proportional to $1/D^2$. The power dissipated very close to the pipe walls, however, is proportional to the pipe circumference, which means Q_{wall} is proportional to D. Because the overall loss rate is the sum of loss rates from the two mechanisms, the inverses of the separate Q's add to give the inverse of the total Q: $1/Q=1/Q_{\mathrm{wall}}+1/Q_{\mathrm{rad}}$. Figures 2–4 illustrate the behavior of Q vs D: an almost-linear increase of Q at small D, then a maximum, and a steep decrease at large D. This is due to the competition between the two loss mechanisms.

II. THEORY

A. Quality factor Q

The quality factor Q of an oscillator provides information about energy dissipation and may be defined as:[1]

$$Q=\frac{\text{energy}}{\text{energy lost per radian}}=2\pi\frac{\text{energy}}{\text{energy lost per cycle}}=\frac{\omega\ \text{energy}}{\text{power dissipated}}. \quad (1)$$

High Q means that energy dissipation is low, while low Q implies large damping of amplitude and energy. Two important contexts for the quality factor Q are (1) retention of energy in a decaying oscillator, and (2) the response of an oscillator to a sinusoidal driving force. (We are only interested here in lightly damped oscillators, whose Q values are greater than 10.)

Q of a decaying oscillator is a measure of how long energy is retained. A certain 349-Hz tuning fork rigidly clamped to a bench takes 6 s to have its amplitude reduced by a factor of e, while the same fork mounted in putty on a benchtop requires only 4 s for the same reduction in amplitude. Mounting in putty reduces the energy more quickly and results in a lower Q (about 4400 mounted in putty vs 6600 clamped to the bench).

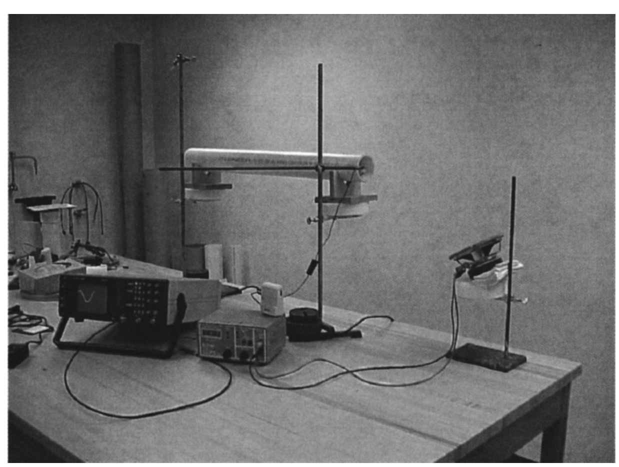

Fig. 1. Experimental setup for measuring Q of an open-ended tube driven by a loudspeaker. Amplitude of response is registered by a tie-clip microphone centered in the tube, amplified, and sent to an oscilloscope.

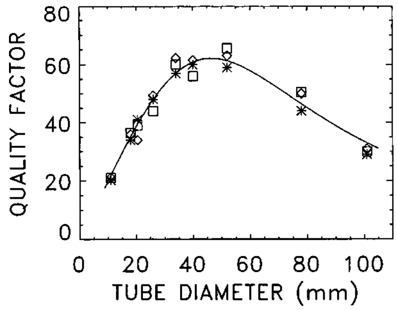

Fig. 2. Indoor data from oscilloscope. The solid curve is Eq. (10) with $c_w = 2.01\ \mathrm{mm}^{-1}$ and $c_R = 4.00 \times 10^5\ \mathrm{mm}^2$.

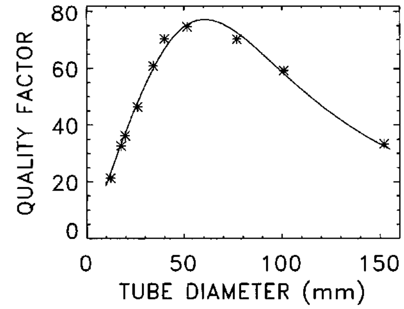

Fig. 4. Outdoor data with signal averaging. The solid curve is Eq. (10) with $c_w = 1.9\ \mathrm{mm}^{-1}$ and $c_R = 8.5 \times 10^5\ \mathrm{mm}^2$.

The theoretical displacement response of three oscillators to a sinusoidal driving force is shown in Fig. 5. Each oscillator has the same resonant frequency but different damping, giving Q values of 10, 20, and 30. The response at resonance of each curve is almost exactly Q times the dc response (unity for each oscillator).

Another feature of a response curve is its ''resonance width'' $\Delta\omega$, the frequency difference between points where the response is $1/\sqrt{2}$ of the maximum. As Q increases, $\Delta\omega$ decreases, so the response at higher Q is both higher and narrower.

The amplitude as a function of time for an underdamped oscillator is[1]

$$y(t) = Ae^{-\beta t}\cos(\omega_1 t + \theta). \tag{2}$$

(The undamped frequency for this oscillator is ω_0. For light damping, the oscillation frequency $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ is very close to ω_0.) Using the definition of Q as in Eq. (1), one may readily show[1] for a lightly damped oscillator that

$$Q = \omega_1 / 2\beta. \tag{3}$$

For the 349-Hz tuning fork mounted with putty, we find a Q of about 4400 from Eq. (3) with $\omega = 2\pi(349)\ \mathrm{s}^{-1}$ and $\beta = (1/4)\ \mathrm{s}^{-1}$. ($\beta$ was found by fitting a straight line to the log of microphone response versus time.)

When a sinusoidal force of amplitude F_0 drives an oscillator, the steady-state displacement amplitude is[1]

$$B = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\omega\beta)^2}}. \tag{4}$$

B is a sharply peaked function of ω when $\beta \ll \omega_0$. It reaches a maximum of $B_R \cong F_0/(2m\omega_0\beta)$ when $\omega = \omega_R = \sqrt{\omega_0^2 - 2\beta^2}$ (very close to ω_0). And when $\omega = \omega_0 \pm \beta$, the amplitude is $B = B_R/\sqrt{2}$, so it is natural to designate $\Delta\omega = 2\beta$ as the ''resonance width.'' Using this in Eq. (3) lets us write

$$Q \cong \frac{\omega}{\Delta\omega} \cong \frac{\omega}{2\beta} \cong \frac{f_0}{f_2 - f_1}, \tag{5}$$

where f_0 is the frequency of maximum displacement amplitude, and f_1 and f_2 are frequencies where the oscillator steady-state amplitude is 0.707 of the maximum.

From Eq. (4), the zero-frequency displacement amplitude is $B_0 = F_0/(m\omega_0^2)$. Using this with B_R and Eq. (5) we can see that the peak response at resonance is Q times the zero-frequency (dc) response of the oscillator. This indicates that a high-Q oscillator will give a large amplitude response, but only very close to its resonant frequency.

B. Viscous penetration depth δ

There is an exponential change of fluid velocity near a solid wall. This is elegantly shown[2] by considering a flat plate in the x–z plane that oscillates sinusoidally in the z direction. The fluid lying above the plate moves in parallel strata in the y direction, acted on only by viscous forces. The viscous force per unit area acting across a plane boundary in a fluid is given by

$$\frac{F_{\mathrm{visc}}}{A} = \eta \frac{\partial v}{\partial y} = \rho v \frac{\partial v}{\partial y},$$

where v is the z velocity, η is the viscosity coefficient, ρ is the density, and ν is the kinematic viscosity. A thin sheet of fluid (area A, thickness Δy) has two viscous forces acting on it, one on the top face at $y + \Delta y$ and another in the opposite

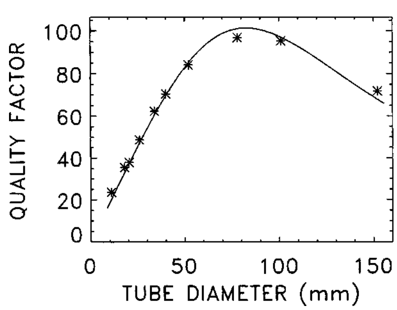

Fig. 3. Indoor data with signal averaging. The solid curve is Eq. (10) with $c_w = 1.84\ \mathrm{mm}^{-1}$ and $c_R = 2.07 \times 10^6\ \mathrm{mm}^2$.

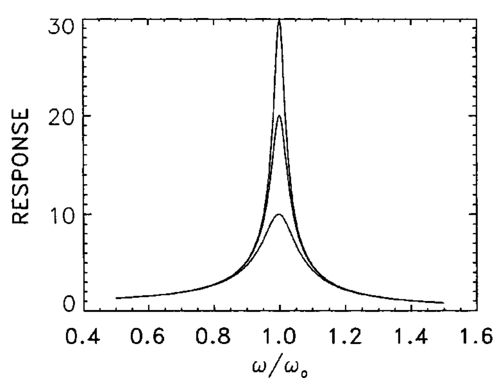

Fig. 5. Relative response of driven, damped oscillators with Q values of 10, 20, and 30. Response is in arbitrary units with unit response at $\omega = 0$.

direction on the bottom face at height y above the oscillating plate. The net viscous force on the sheet, the sum of these forces, causes the sheet to accelerate:

$$F_{\text{visc,net}}=\rho A\Delta y\,\frac{\partial v}{\partial t}=A\rho\nu\,\frac{\partial^2 v}{\partial y^2}\,\Delta y,$$

or

$$\nu\,\frac{\partial^2 v}{\partial y^2}=\frac{\partial v}{\partial t}. \tag{6}$$

The z velocity of the sheet is sinusoidal, varying with y, and is given by $v(y,t)=ae^{i\omega t}f(y)$. Using $v(y,t)$ in Eq. (6) gives $\nu\partial^2 f(y)/\partial y^2=i\omega f(y)$, which has a solution $f(y)=Ae^{-my}+Be^{+my}$, where $m=\sqrt{i\omega/\nu}=(1+i)/\delta$ and $\delta=\sqrt{2\nu/\omega}$.

We may think of δ as a ''penetration depth'' for viscous effects similar to the ''skin depth'' for electromagnetic fields in a conductor. The value of ν for air at 20 °C is $1.5\times10^{-5}\ \text{m}^2/\text{s}$, so δ is quite small (at a frequency of 190 Hz, $\delta=0.16$ mm).

C. Q from viscous and thermal effects

A wave being attenuated as it travels in z may be written as $y(t)=Ae^{-\alpha z}\cos(\omega_1 t+\theta)$, where α is the absorption coefficient as a function of distance. When the absorption is viewed as occurring while the wave travels back and forth in the pipe, z may be replaced by ct (c is the speed of sound). This gives the equivalent of Eq. (2) with $\beta=\alpha c$.

The effect of viscosity and thermal losses close to the walls of a cylindrical pipe of radius R is given as an absorption coefficient[3–6]

$$\alpha=\frac{\delta\omega}{2Rc}\left[1+\sqrt{\frac{\chi}{\nu}}\left(\frac{C_p}{C_v}-1\right)\right], \tag{7}$$

where ν is the kinematic viscosity, C_p and C_v are specific heat capacities at constant pressure and volume, respectively, and χ is the thermometric conductivity, defined by $\chi=(\text{thermal conductivity})/(\text{density }C_p)$.

Using $\beta=\alpha c$, and Eq. (7) in Eq. (3), we find Q due to wall effects

$$Q_{\text{wall}}=\frac{R}{\delta\left[1+\sqrt{\frac{\chi}{\nu}}\left(\frac{C_p}{C_v}-1\right)\right]}. \tag{8}$$

Numerical values from Refs. 4 and 5 in Eq. (8) give for 300 K $Q_{\text{wall}}=(447.2R\sqrt{f})/(1+0.476)$, where f is frequency in hertz and R is in meters. This shows that thermal effects represent an additional 48% loss at the wall beyond viscous losses. Writing $R=D/2$ and using a frequency of 190 Hz, we find

$$Q_{\text{wall}}=2.09D,$$

where D is in millimeters.

D. Q from radiation of sound

A sphere of radius R that oscillates radially with maximum velocity u_0 is said to have a source strength S given by $S=4\pi R^2 u_0$, which is just the maximum flow rate of fluid at the source radius. This sphere is regarded as a ''simple source'' whose radiated power when $ka\ll 1$ ($k=2\pi/\lambda$) is given by[7]

$$\text{radiated power}=\frac{\rho c k^2 S^2}{8\pi}, \tag{9}$$

where ρ is the density and c is the velocity of sound. In simple harmonic motion, the total energy may be written as the maximum kinetic energy. Then the total energy in a cylindrical pipe containing a standing sinusoidal wave with maximum velocity u_0 at one end is readily found to be $E=\frac{1}{4}\rho u_0^2\pi R^2 L_{\text{pipe}}$. The energy radiated per cycle is power divided by frequency, so we may use Eq. (1) to determine Q_{rad}:

$$Q_{\text{rad}}=\frac{2\pi f[\frac{1}{4}\rho\pi R^2 u_0^2 L_{\text{pipe}}]}{\rho c k^2 S^2/(8\pi)}.$$

A pipe open at one end and closed at the other may be regarded as a simple source whose strength is $S=\pi R^2 u_0$, so the expression for Q_{rad} (Ref. 7) becomes $Q_{\text{rad}}=L_{\text{pipe}}c/\pi R^2 f$. Using $c=3.44\times10^5$ mm/s, $L_{\text{pipe}}=430$ mm, and $f=190$ Hz, we find

$$Q_{\text{rad}}=9.91\times10^5/D^2$$

with D in millimeters.

These expressions for Q_{rad} and Q_{wall} also apply to a pipe open at both ends that has twice the length of the pipe closed at one end and open at the other. (The Q for a closed–open pipe is very slightly lower than for the equivalent open–open pipe due to losses at the end wall.)

III. EXPERIMENTAL ARRANGEMENT

A PVC pipe open at both ends with a speaker near one end was used as a driven acoustical resonator, measured in its lowest mode only. A tie-clip microphone[8] on a long cord was positioned at the center of the pipe by inserting it from one end. This eliminated the need for drilling a hole through the pipe wall at the middle of the pipe to allow microphone placement. Leaks around such a hole were found to seriously affect the Q value. The microphone output went to an audio amplifier (an inexpensive microphone and audio amplifier[8] worked very well in this experiment), and then to an oscilloscope. (One might also use a true rms voltmeter to measure the amplified output, although this does not let one see if the amplified signal is clipped or nonsinusoidal.) All pipes resonated close to 190 Hz, with a typical length around 85 cm. Because of the end correction[9] at each open end, lengths varied somewhat, with larger diameter pipes being shorter.

Frequencies f_0, f_1, and f_2 were measured, as mentioned earlier. Q was then calculated using Eq. (5). We estimate an uncertainty of up to 10% in the Q values, due to the difficulty of determining the 0.707 response frequencies.

A PASCO PI-9587C digital function generator[10] was used to drive the speaker. It has excellent low-impedance drive, can be dialed to the desired frequency, and also offers square wave output (used when we did signal averaging on the computer). The ease of dialing to 0.01 Hz was a great help in this experiment. (A conventional function generator and frequency meter, found in most physics labs, would do the same job as the PI-9587C, but would not be as convenient.)

Table I. A comparison of theoretical and measured loss parameters.

	Theory	Figure 2 (indoors)	Figure 3 (indoors)	Figure 4 (outdoors)
C_R (mm^2)	9.9×10^5	4.0×10^5	2.1×10^6	8.5×10^5
C_W (mm^{-1})	2.09	2.01	1.84	1.90

PVC pipe of various diameters ranging from 0.5 to 4 in., as well as a 6 in. cardboard mailing tube were used for our measurements. Each pipe was placed at some moderate height (10–50 cm) off a tabletop, with the speaker at roughly a 45° angle from the axis of the pipe, 25–30 cm or so from one end of the pipe.

The initial data were taken using an oscilloscope display to measure the response amplitude of the microphone. Later we used a computer to average response amplitude data (typically, 200 samples would be averaged) from the microphone, using the square wave output of the PASCO signal generator as a trigger.

Indoor reflections had an effect on radiated sound from the pipes, which we attempted to remove by making several sets of measurements outdoors. Outdoor data were taken at least 40 ft from the nearest building, and 6 ft or more above a grassy surface. Although working outdoors effectively removed the wall and floor effects, it had its own drawbacks. It was necessary to cope with transient environmental noises, so quite a bit of repetition was required to make sure these sounds had not adversely affected the measurements.

IV. INTERPRETATION OF RESULTS

Q values for the various pipe sizes are displayed in Figs. 2–4. Figure 2 shows data taken indoors on the oscilloscope. Figure 3 also shows data taken indoors but with computer signal averaging, while Fig. 4 shows data taken outdoors with computer signal averaging.

We have already noted that $1/Q=1/Q_{\mathrm{wall}}+1/Q_{\mathrm{rad}}$. At constant frequency, $Q_{\mathrm{wall}}=c_w D$ and $Q_{\mathrm{rad}}=c_R/D^2$, where c_R and c_w are constants associated with radiation and wall losses, respectively. This leads to Q being expressed as

$$Q=\frac{(c_w D)(c_R/D^2)}{c_w D+c_R/D^2}. \tag{10}$$

At small D, $c_R/D^2 \gg c_w D$, so $Q \cong c_w D$, while at large D, $c_w D \gg c_R/D^2$ and $Q \cong c_R/D^2$. Figures 2–4 accordingly show a nearly linear increase in Q at small D, then a maximum, then roughly a $1/D^2$ decrease of Q at large D.

Table I indicates the theoretical values for the c_R and c_w parameters as well as those obtained from the fits to our experimental data for Figs. 2–4. Experimental and theoretical values of c_w agree well for losses due to viscous and thermal effects, but values for the radiative loss parameter, c_R, range from less than half the theoretical value to more than twice this value, measured indoors at different locations. Only the data series taken outdoors where room effects are removed yields a value of c_R close to the theoretical value.

We believe reflections from nearby surfaces indoors are responsible for the variation in indoor c_R values. Such reflections create pressure waves near each end of the resonating tube, the phase of which will influence the amount of energy radiated, either enhancing or reducing it. Items such as tables, desks, supporting columns, etc., in the room make these effects complicated. Our studies so far have not clarified this effect.

V. CONCLUSION

Acoustic Q values were measured for sets of eight to ten pipes of various diameters, all at the same frequency. Only two parameters were needed to fit these data, one for losses near pipe walls and the other for losses from the pipes due to sound radiation. The graphs of Q versus pipe diameter nicely illustrate the competition between energy losses close to the wall and energy losses due to sound radiated from the ends of the pipes.

For data taken outdoors, both parameters are in good agreement with theory. Indoors there is disagreement between the theoretical and experimental sound radiation parameter. This is apparently due to the phase of reflections from nearby surfaces, either enhancing or reducing the radiation of sound.

This experiment offers insight into several important physical effects, and requires only moderate equipment. It would be a valuable addition to an upper-level undergraduate laboratory.

[1]T. L. Chow, *Classical Mechanics* (Wiley, New York, 1995), Secs. 7.4 and 7.7.

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[3]L. D. Landau and E. M. Lifschitz, *Fluid Mechanics* (Academic, New York, 1959), p. 301.

[4]A. H. Benade, ‘‘On the propagation of sound waves in a cylindrical conduit,’’ J. Acoust. Soc. Am. **44**, 616–623 (1968).

[5]T. D. Rossing and N. H. Fletcher, *Principles of Vibration and Sound* (Springer-Verlag, New York, 1994), Sec. 8.2.

[6]L. E. Kinsler, A. R. Frey, A. B. Coppens, and J. V. Sanders, *Fundamentals of Acoustics* (Wiley, New York, 1999), 4th ed., Sec. 8.9.

[7]H. Lamb, *The Dynamical Theory of Sound* (Dover, New York, 1960), 2nd ed., Chaps. VII and IX.

[8]Radio Shack Tie-Clip Microphone Catalog No. 33-3013 (under $30), Radio Shack Audio Speaker-Amplifier Catalog No. 277-1008C (around $12).

[9]See Ref. 6, Sec. 10.2.

[10]PASCO Scientific, 10101 Foothills Blvd, Roseville, CA 95678-9011, or www.pasco.com.