
SAMPLE SOLUTIONS

DIGITAL SIGNAL PROCESSING: *Signals, Systems, and Filters*

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SA.1 A periodic signal can be represented by the equation

$$\tilde{x}(t) = \sum_{k=1}^{10} A_k \sin(\omega_k t + \phi_k)$$

where the frequencies ω_k , amplitudes A_k , and phase angles ϕ_k are given in Table SA.1.

The signal is to be processed first by an ideal bandpass filter and then by a differentiator, as shown in Fig. SA.1. The bandpass filter will pass frequencies in the range $4 \leq \omega \leq 6$ rad/s and reject all other frequencies and the differentiator will differentiate the signal with respect to time.

- Obtain a time-domain representation for the signal at the outputs of the bandpass filter and differentiator, i.e., at nodes B and C, respectively, in Fig. SA.1.
- Obtain a frequency-domain representation for the signal at the output of the bandpass filter.
- Obtain a frequency-domain representation for the signal at the output of the differentiator.

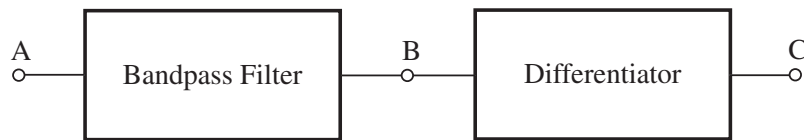


Figure SA.1

Table SA.1 Frequency Spectrum

k	ω_k , rad/s	A_k	ϕ_k
1	1	0.3819	-0.3478
2	2	0.3614	0.8222
3	3	0.8575	2.3502
4	4	0.0629	-0.3292
5	5	0.1342	-0.1693
6	6	0.8648	0.6648
7	7	0.5155	-2.4473
8	8	0.6797	1.7780
9	9	0.7001	-1.5824
10	10	0.3	1.1

Solution

(a) Node B:

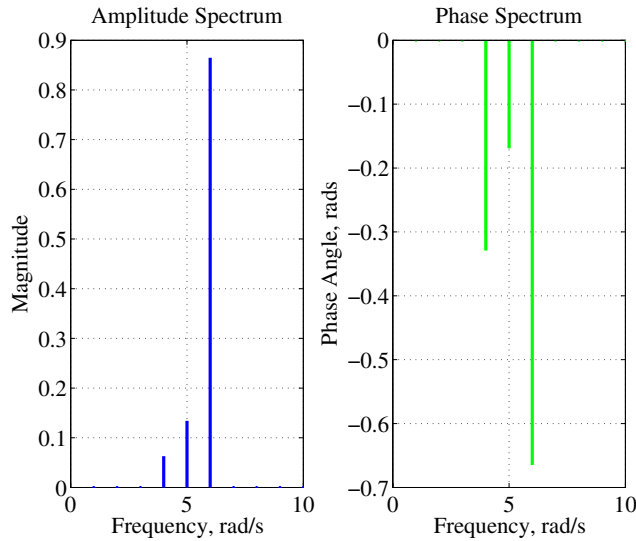
$$x_B(t) = \sum_{k=4}^6 A_k \sin(\omega_k t + \phi_k) \quad \blacksquare$$

Node C:

$$\begin{aligned} x_C(t) &= \frac{d}{dt} \left[\sum_{k=4}^6 A_k \sin(\omega_k t + \phi_k) \right] = \sum_{k=4}^6 \frac{d}{dt} [A_k \sin(\omega_k t + \phi_k)] \\ &= \sum_{k=4}^6 \omega_k A_k \cos(\omega_k t + \phi_k) = \sum_{k=4}^6 \omega_k A_k \sin(\omega_k t + \phi_k + \pi/2) \quad \blacksquare \end{aligned}$$

(b) The amplitude and phase spectrums at Node B are given in Table SA.2 and are plotted in Fig. SA.2. \blacksquare **Table SA.2** Frequency Spectrum at Node B

k	ω_k , rad/s	A_k	ϕ_k
4	4	0.0629	-0.3292
5	5	0.1342	-0.1693
6	6	0.8648	0.6648

**Figure SA.2**(c) Similarly, the amplitude and phase spectrums for Node C are given in Table SA.3 and are plotted in Fig. SA.3. \blacksquare **Table SA.3** Frequency Spectrum at Node B

k	ω_k , rad/s	A_k	ϕ_k
4	4	0.2516	1.2416
5	5	0.6710	1.4015
6	6	5.1888	2.2356

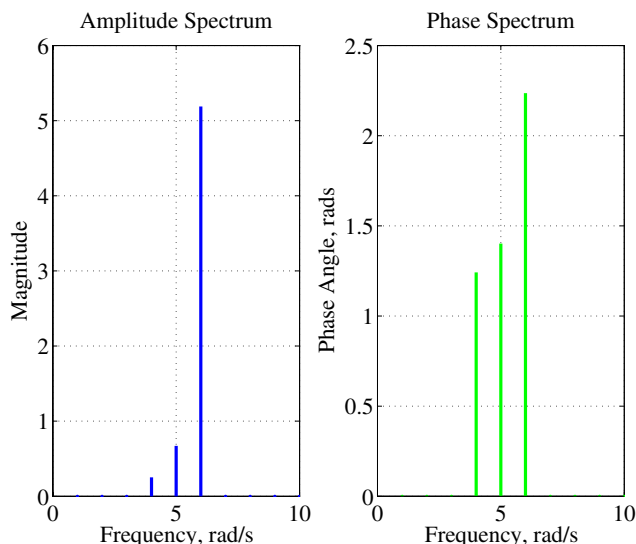


Figure SA.3

SA.2 A periodic signal defined by

$$\tilde{x}(t) = \sum_{r=-\infty}^{\infty} x(t + r\tau_0)$$

where $x(t)$ is zero outside the range $-\tau_0/2 \leq t \leq \tau_0/2$ has a Fourier series of the form

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \quad \text{for } -\tau_0/2 \leq t \leq \tau_0/2$$

where

$$X_k = \frac{1}{\tau_0} \int_{-\tau_0/2}^{\tau_0/2} x(t) e^{-jk\omega_0 t} dt$$

(a) Assuming that $x(t)$ is an even function, show that

$$X_k = \frac{2}{\tau_0} \int_0^{\tau_0/2} x(t) \cos k\omega_0 t dt$$

Justify your steps.

(b) The periodic signal of Fig. SA.4 is described by the equation

$$x(t) = \begin{cases} \cos \omega_0 t / 2 & \text{for } -\tau_0/4 \leq t \leq \tau_0/4 \\ 0 & \text{otherwise} \end{cases}$$

where $\omega_0 = 2\pi/\tau_0$. Using the formula in part (a), obtain an expression for the Fourier series coefficients X_k .

(c) Give expressions for the amplitude and phase spectrums of $\tilde{x}(t)$.

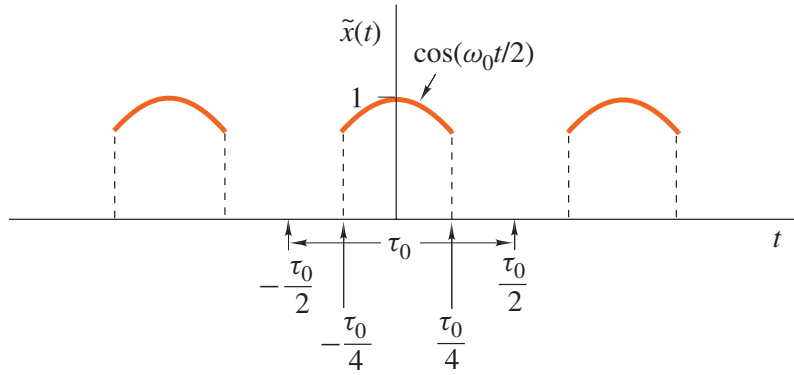


Figure SA.4

Solution

(a) From the definition of the Fourier series

$$\begin{aligned}
 X_k &= \frac{1}{\tau_0} \int_{-\tau_0/2}^{\tau_0/2} x(t) e^{-jk\omega_0 t} dt \\
 &= \frac{1}{\tau_0} \int_{-\tau_0/2}^{\tau_0/2} x(t) (\cos k\omega_0 t - j \sin k\omega_0 t) dt \\
 &= \frac{1}{\tau_0} \int_{-\tau_0/2}^{\tau_0/2} x(t) \cos k\omega_0 t dt - j \frac{1}{\tau_0} \int_{-\tau_0/2}^{\tau_0/2} x(t) \sin k\omega_0 t dt \\
 &= \frac{1}{\tau_0} \left[\int_{-\tau_0/2}^0 x(t) \cos k\omega_0 t dt + \int_0^{\tau_0/2} x(t) \cos k\omega_0 t dt \right] \\
 &\quad - \frac{1}{\tau_0} \left[\int_{-\tau_0/2}^0 x(t) \sin k\omega_0 t dt + \int_0^{\tau_0/2} x(t) \sin k\omega_0 t dt \right]
 \end{aligned}$$

If $x(t)$ is even, then $x(t) \cos k\omega_0 t$ is an even function and $x(t) \sin k\omega_0 t$ is an odd function. Hence

$$\int_{-\tau_0/2}^0 x(t) \cos k\omega_0 t dt = \int_0^{\tau_0/2} x(t) \cos k\omega_0 t dt$$

and

$$\int_{-\tau_0/2}^0 x(t) \sin k\omega_0 t dt = - \int_0^{\tau_0/2} x(t) \sin k\omega_0 t dt$$

Therefore

$$X_k = \frac{2}{\tau_0} \int_0^{\tau_0/2} x(t) \cos k\omega_0 t dt \quad \blacksquare$$

(b) The given signal is symmetrical about the y axis and, therefore, it is an even function. Hence we have

$$\begin{aligned}
 X_k &= \frac{2}{\tau_0} \int_0^{\tau_0/2} x(t) \cos k\omega_0 t dt \\
 &= \frac{2}{\tau_0} \int_0^{\tau_0/4} \cos(\omega_0 t/2) \cos(k\omega_0 t) dt = \frac{2}{\tau_0} \int_0^{\tau_0/4} \cos(k\omega_0 t) \cos(\omega_0 t/2) dt
 \end{aligned}$$

From trigonometry, we have

$$\cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)]$$

and, therefore,

$$\begin{aligned}
 X_k &= \frac{2}{\tau_0} \int_0^{\tau_0/4} \frac{1}{2} [\cos(k\omega_0 t + \omega_0 t/2) + \cos(k\omega_0 t - \omega_0 t/2)] dt \\
 &= \frac{1}{\tau_0} \int_0^{\tau_0/4} \cos \left[\left(k + \frac{1}{2}\right) \omega_0 t \right] + \cos \left[\left(k - \frac{1}{2}\right) \omega_0 t \right] dt \\
 &= \frac{1}{\tau_0} \left[\frac{\sin \left[\left(k + \frac{1}{2}\right) \omega_0 t \right]}{\left(k + \frac{1}{2}\right) \omega_0} + \frac{\sin \left[\left(k - \frac{1}{2}\right) \omega_0 t \right]}{\left(k - \frac{1}{2}\right) \omega_0} \right]_0^{\tau_0/4} \\
 &= \frac{1}{\tau_0} \left[\frac{\sin \left[\left(k + \frac{1}{2}\right) \frac{2\pi}{\tau_0} \cdot \frac{\tau_0}{4} \right]}{\left(k + \frac{1}{2}\right) \frac{2\pi}{\tau_0}} + \frac{\sin \left[\left(k - \frac{1}{2}\right) \frac{2\pi}{\tau_0} \cdot \frac{\tau_0}{4} \right]}{\left(k - \frac{1}{2}\right) \frac{2\pi}{\tau_0}} \right] \\
 &= \frac{1}{2\pi} \left[\frac{\sin \left[\left(k + \frac{1}{2}\right) \frac{\pi}{2} \right]}{\left(k + \frac{1}{2}\right)} + \frac{\sin \left[\left(k - \frac{1}{2}\right) \frac{\pi}{2} \right]}{\left(k - \frac{1}{2}\right)} \right]
 \end{aligned}$$

The amplitude and phase spectrums of $\tilde{x}(t)$ are the magnitude and angle of X_k , i.e., $|X_k|$ and $\arg X_k$. Since X_k is real, the angle of X_k is 0 or π depending on whether X_k is positive or negative. ■

SA.3 A z transform is given by

$$X(z) = \frac{(z^2 + 1)(z + 1)}{(z^2 + z - 2)(z - 3)}$$

- (a) Construct the zero-pole plot of $X(z)$.
- (b) Function $X(z)$ is known to have as many Laurent series as there are annuli of convergence but only one of these series is a z transform that satisfies the absolute convergence theorem (Theorem 3.1). Identify the annulus of convergence of that series on the zero-pole plot obtained in part (a).
- (c) Through the use of partial fractions obtain a closed-form expression for $x(nT) = \mathcal{Z}^{-1}X(z)$.

Solution

- (a) $X(z)$ can be expressed as

$$\begin{aligned}
 X(z) &= \frac{(z^2 + 1)(z + 1)}{(z^2 + z - 2)(z - 3)} \\
 &= \frac{(z + j)(z - j)(z + 1)}{(z - 1)(z + 2)(z - 3)}
 \end{aligned}$$

Hence $X(z)$ has zeros at $z = \pm j, -1$ and poles at $z = 1, -2, 3$. The zero-pole plot is depicted in Fig. SA.5. ■

- (b) The correct annulus is the outer annulus which can be represented by

$$3 \leq |z| < R \quad \text{where } R \rightarrow \infty \quad \blacksquare$$

See Fig. SA.5.

- (c) Using Technique I, we can write

$$\begin{aligned}
 \frac{X(z)}{z} &= \frac{(z^2 + 1)(z + 1)}{z(z - 1)(z + 2)(z - 3)} \\
 &= \frac{R_0}{z} + \frac{R_1}{z - 1} + \frac{R_2}{z + 2} + \frac{R_3}{z - 3}
 \end{aligned} \tag{SA.1}$$

Since the poles are simple, we have

$$R_0 = \lim_{z \rightarrow 0} \frac{(z^2 + 1)(z + 1)}{(z - 1)(z + 2)(z - 3)} = \frac{1}{(-1) \times 2 \times (-3)} = \frac{1}{6}$$

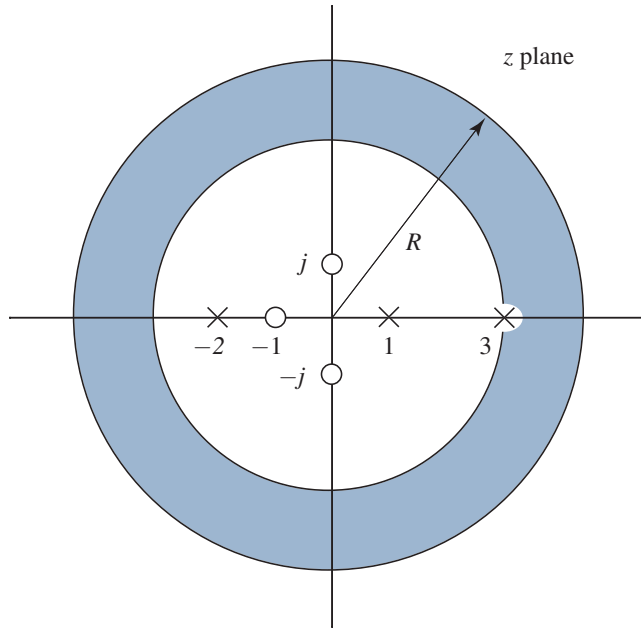


Figure SA.5

$$R_1 = \lim_{z \rightarrow 1} \frac{(z^2 + 1)(z + 1)}{z(z + 2)(z - 3)} = \frac{2 \times 2}{1 \times 3 \times (-2)} = -\frac{2}{3}$$

$$R_2 = \lim_{z \rightarrow -2} \frac{(z^2 + 1)(z + 1)}{z(z - 1)(z - 3)} = \frac{5 \times (-1)}{(-2) \times (-3) \times (-5)} = \frac{1}{6}$$

$$R_3 = \lim_{z \rightarrow 3} \frac{(z^2 + 1)(z + 1)}{z(z - 1)(z + 2)} = \frac{10 \times 4}{3 \times 2 \times 5} = \frac{4}{3}$$

From Eq. (SA.1), we can write

$$\begin{aligned} X(z) &= R_0 + \frac{R_1 z}{z - 1} + \frac{R_2 z}{z + 2} + \frac{R_3 z}{z - 3} \\ &= \frac{1}{6} - \frac{\frac{2}{3}z}{z - 1} + \frac{\frac{1}{6}z}{z + 2} + \frac{\frac{4}{3}z}{z - 3} \end{aligned}$$

Therefore, for $n \geq 0$, the use of Table 3.2 gives

$$\begin{aligned} x(nT) &= \frac{1}{6}\delta(nT) - \frac{2}{3}u(nT) + \frac{1}{6}u(nT)(-2)^n + \frac{4}{3}u(nT)(3)^n \\ &= \frac{1}{6}\delta(nT) + u(nT)\left[-\frac{2}{3} + \frac{1}{6}(-2)^n + \frac{4}{3}(3)^n\right] \quad \blacksquare \end{aligned}$$

Since the numerator degree of $X(z)$ is equal to the denominator degree, it follows from the corollary of the initial-value theorem (Theorem 3.8) that $x(nT) = 0$ for $n < 0$, i.e., the above solution applies for all values of n .

An alternative but equivalent solution can be readily obtained by using Technique II (see p. 115) whereby we expand $X(z)$ instead of $X(z)/z$ into partial fractions. We can write

$$\begin{aligned} X(z) &= \frac{(z^2 + 1)(z + 1)}{(z - 1)(z + 2)(z - 3)} \\ &= R_0 + \frac{R_1}{z - 1} + \frac{R_2}{z + 2} + \frac{R_3}{z - 3} \end{aligned}$$

where

$$R_0 = \lim_{z \rightarrow \infty} \frac{(z^2 + 1)(z + 1)}{(z - 1)(z + 2)(z - 3)} = \lim_{z \rightarrow \infty} \frac{z^3}{z^3} = 1$$

$$R_1 = \lim_{z \rightarrow 1} \frac{(z^2 + 1)(z + 1)}{(z + 2)(z - 3)} = \frac{2 \times 2}{3 \times (-2)} = -\frac{2}{3}$$

Thus

$$R_2 = \lim_{z \rightarrow -2} \frac{(z^2 + 1)(z + 1)}{(z - 1)(z - 3)} = \frac{5 \times (-1)}{(-3) \times (-5)} = -\frac{1}{3}$$

$$R_3 = \lim_{z \rightarrow 3} \frac{(z^2 + 1)(z + 1)}{(z - 1)(z + 2)} = \frac{10 \times 4}{2 \times 5} = 4$$

Thus

$$X(z) = 1 - \frac{\frac{2}{3}}{z - 1} - \frac{\frac{1}{3}}{z + 2} + \frac{4}{z - 3}$$

and for $n \geq 0$, we have

$$\begin{aligned} x(nT) &= \delta(nT) - \frac{2}{3}u(nT - T) - \frac{1}{3}u(nT - T)(-2)^{n-1} + 4u(nT - T)(3)^{n-1} \\ &= \delta(nT) + u(nT - T)\left[-\frac{2}{3} - \frac{1}{3}(-2)^{n-1} + 4(3)^{n-1}\right] \quad \blacksquare \end{aligned}$$

SA.4 (a) Find the z transform of the following discrete-time signal

$$x(nT) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } 0 \leq n \leq 5 \\ 1 + (n - 5)T & \text{for } n > 5 \end{cases}$$

(b) The z transform of a discrete-time signal $x(nT)$ is given by

$$X(z) = \frac{z(3z^2 - 2z + 1)}{(z^2 + 1)(z - 1)}$$

Using the initial-value theorem (Theorem 3.8), show that $x(nT) = 0$ for $n < 0$. Then find $x(nT)$ for $n \geq 0$ using the general inversion formula

$$x(nT) = \frac{1}{2\pi j} \oint_{\Gamma} X(z)z^{n-1} dz.$$

Solution

(a) The signal can be expressed as

$$x(nT) = u(nT) + r(nT - 5T)$$

Hence

$$\begin{aligned} \mathcal{Z}x(nT) &= \mathcal{Z}u(nT) + \mathcal{Z}r(nT - 5T) \\ &= \mathcal{Z}u(nT) + z^{-5}\mathcal{Z}r(nT) \\ &= \frac{z}{z - 1} - \frac{Tz^{-4}}{(z - 1)^2} \end{aligned}$$

(a) The first nonzero value of $x(nT)$ occurs at $KT = (N - M)T$ where N is the denominator degree and M in the numerator degree in $X(z)$. Since $N = M = 3$, we have $K = 0$, i.e., the signal starts at $KT = 0$. Hence for $n < 0$, we have

$$x(nT) = 0 \quad \blacksquare$$

(b) For $n \geq 0$, we can write

$$\begin{aligned}
 x(nT) &= \sum_{i=1}^M \mathfrak{Re}_{s_{z=p_i}} [X(z)z^{n-1}] \\
 &= \sum_{i=1}^M \mathfrak{Re}_{s_{z=p_i}} \frac{z(3z^2 - 2z + 1)z^{n-1}}{(z^2 + 1)(z - 1)} \\
 &= \sum_{i=1}^M \mathfrak{Re}_{s_{z=p_i}} \frac{(3z^2 - 2z + 1)z^n}{(z - j)(z + j)(z - 1)} \\
 &= \frac{R_1}{z - j} + \frac{R_1^*}{z + j} + \frac{R_2}{z - 1} \tag{SA.2}
 \end{aligned}$$

where M is the number of poles and R_1^* is the complex conjugate of R_1 since the pole at $z = j = e^{j\pi/2}$ is the complex conjugate of the pole at $z = -j = e^{-j\pi/2}$. Since the poles are simple, we have

$$\begin{aligned}
 R_1 &= \lim_{z \rightarrow j} \left[(z - j) \frac{(3z^2 - 2z + 1)z^n}{(z - j)(z + j)(z - 1)} \right] \\
 &= \lim_{z \rightarrow j} \frac{(3z^2 - 2z + 1)z^n}{(z + j)(z - 1)} = \frac{(-3 - 2j + 1)j^n}{2j(j - 1)} \\
 &= \frac{-2 - 2j}{-2 - 2j} = j^n = e^{jn\pi/2}
 \end{aligned}$$

and

$$R_2 = \lim_{z \rightarrow 1} \frac{(3z^2 - 2z + 1)z^n}{(z^2 + 1)} = \frac{2}{2} = 1$$

Therefore, from Eq. (SA.2), we can write

$$\begin{aligned}
 x(nT) &= u(nT)e^{jn\pi/2} + u(nT)e^{-jn\pi/2} + u(nT) \\
 &= u(nT)(2 \cos n\pi/2 + 1) \quad \blacksquare
 \end{aligned}$$

SA.5 An initially relaxed discrete-time system can be represented by the equation

$$y(nT) = \mathcal{R}x(nT) = 2.5x(nT) + |e^{0.1(nT+2T)}|x(nT - T) + x(nT - 2T)$$

By using *appropriate tests*, check the system for

- (a) linearity,
- (b) time invariance, and
- (c) causality.

Solution

(a) **Linearity**

$$\begin{aligned}
 \mathcal{R}[\alpha x_1(nT) + \beta x_2(nT)] &= 2.5[\alpha x_1(nT) + \beta x_2(nT)] + |e^{0.1(nT+2T)}|[\alpha x_1(nT - T) \\
 &\quad + \beta x_2(nT - T)] + [\alpha x_1(nT - 2T) + \beta x_2(nT - 2T)] \\
 &= \alpha[2.5x_1(nT) + |e^{0.1(nT+2T)}|x_1(nT - T) + x_1(nT - 2T)] \\
 &\quad + \beta[2.5x_2(nT) + |e^{0.1(nT+2T)}|x_2(nT - T) + x_2(nT - 2T)] \\
 &= \alpha \mathcal{R}x_1(nT) + \beta \mathcal{R}x_2(nT)
 \end{aligned}$$

Therefore, the system is linear. \blacksquare

(b) **Time invariance**

The response to a delayed excitation is

$$\mathcal{R}x(nT - kT) = 2.5x(nT - kT) + |e^{0.1(nT+2T)}|x(nT - kT - T) + x(nT - kT - 2T)$$

The delayed response is

$$y(nT - kT) = 2.5x(nT - kT) + |e^{0.1(nT-kT+2T)}|x(nT - kT - T) + x(nT - kT - 2T)$$

For any $k \neq 0$, we have

$$|e^{0.1nT+2T}| \neq |e^{0.1(nT-kT+2T)}|$$

Thus

$$y(nT - kT) \neq \mathcal{R}x(nT - kT)$$

and, therefore, the system is time dependent. ■

(c) Let $x_1(nT)$ and $x_2(nT)$ be arbitrary discrete-time signals such that

$$\begin{aligned} x_1(nT) &= x_2(nT) & \text{for } n \leq k \\ x_1(nT) &\neq x_2(nT) & \text{for } n > k \end{aligned}$$

We have

$$\mathcal{R}x_1(nT) = 2.5x_1(nT) + |e^{0.1(nT+2T)}|x_1(nT - T) + x_1(nT - 2T) \quad (\text{SA.3})$$

and

$$\mathcal{R}x_2(nT) = 2.5x_2(nT) + |e^{0.1(nT+2T)}|x_2(nT - T) + x_2(nT - 2T) \quad (\text{SA.4})$$

Since

$$x_1(nT) = x_2(nT) \quad \text{for } n \leq k$$

then

$$x_1(nT - T) = x_2(nT - T) \quad \text{for } n \leq k$$

and

$$x_1(nT - 2T) = x_2(nT - 2T) \quad \text{for } n \leq k$$

Hence the right-hand side in Eq. (SA.3) is equal to the right-hand side in Eq. (SA.4) for $n \leq k$ and thus

$$\mathcal{R}x_1(nT) = \mathcal{R}x_2(nT) \quad \text{for } n \leq k$$

Therefore, the filter is causal. ■

- SA.6 (a) Derive a state-space representation for the filter shown in Fig. SA.6.
 (b) Using the state-space representation obtained in part (a), compute the impulse response of the filter at $nT = 5T$.

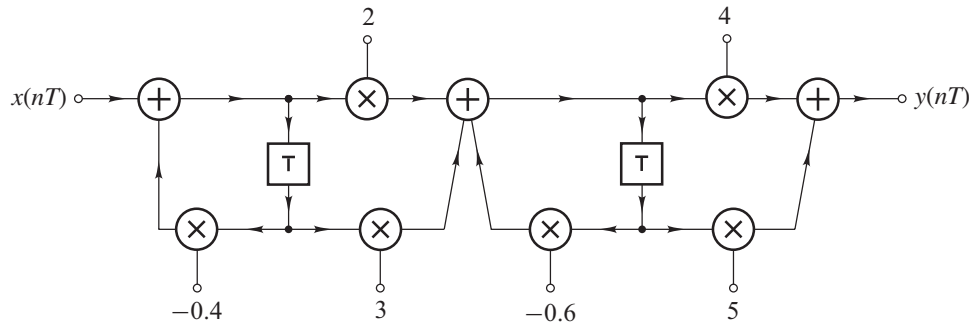


Figure SA.6

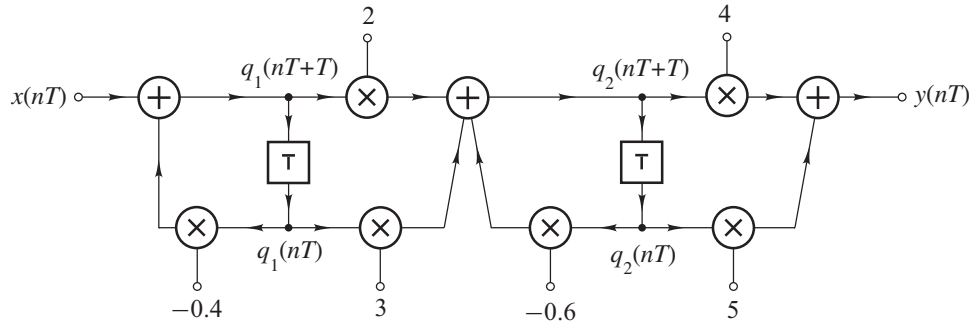


Figure SA.7

Solution

- (a) State variables can be assigned as shown in Fig. SA.7. Hence we can write

$$q_1(nT + T) = x(nT) - 0.4q_1(nT) \quad (\text{SA.5})$$

$$q_2(nT + T) = 2q_1(nT + T) + 3q_1(nT) - 0.6q_2(nT) \quad (\text{SA.6})$$

On eliminating $q_1(nT + T)$ in Eq. (SA.6) using Eq. (SA.5), we get

$$\begin{aligned} q_2(nT + T) &= 2x(nT) - 0.8q_1(nT) + 3q_1(nT) - 0.6q_2(nT) \\ &= 2.2q_1(nT) - 0.6q_2(nT) + 2x(nT) \end{aligned} \quad (\text{SA.7})$$

From Eqs. (SA.5) and (SA.7)

$$\begin{bmatrix} q_1(nT + T) \\ q_2(nT + T) \end{bmatrix} = \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix} \begin{bmatrix} q_1(nT) \\ q_2(nT) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x(nT) \quad (\text{SA.8})$$

The output is given by

$$y(nT) = 4q_2(nT + T) + 5q_2(nT) \quad (\text{SA.9})$$

and from Eqs. (SA.6) and (SA.9), we have

$$\begin{aligned} y(nT) &= 4 \times 2.2q_1(nT) - 4 \times 0.6q_2(nT) + 4 \times 2x(nT) + 5q_2(nT) \\ &= 8.8q_1(nT) + 2.6q_2(nT) + 8x(nT) \\ &= [8.8 \quad 2.6] \begin{bmatrix} q_1(nT) \\ q_2(nT) \end{bmatrix} + 8x(nT) \end{aligned} \quad (\text{SA.10})$$

Therefore, Eqs. (SA.8) and (SA.10) can be written as

$$\begin{aligned} \mathbf{q}(nT + T) &= \mathbf{A}\mathbf{q}(nT) + \mathbf{b}x(nT) \\ y(nT) &= \mathbf{c}^T \mathbf{q}(nT) + dx(nT) \end{aligned}$$

where

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix} & \mathbf{b} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \blacksquare \\ \mathbf{c}^T &= [8.8 \quad 2.6] & d &= 8 \quad \blacksquare \end{aligned}$$

- (b) The impulse response is given by

$$h(nT) = \begin{cases} a_1 & \text{for } n = 0 \\ \mathbf{c}^T \mathbf{A}^{n-1} \mathbf{b} & \text{otherwise} \end{cases}$$

For $n = 5$

$$h(5T) = \mathbf{c}^T \mathbf{A}^4 \mathbf{b} = [8.8 \quad 2.6] \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Since

$$\begin{aligned} \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix}^2 &= \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix} \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix} = \begin{bmatrix} 0.16 & 0.0 \\ -2.20 & 0.36 \end{bmatrix} \\ \begin{bmatrix} -0.4 & 0 \\ 2.2 & -0.6 \end{bmatrix}^4 &= \begin{bmatrix} 0.16 & 0.0 \\ -2.20 & 0.36 \end{bmatrix} \begin{bmatrix} 0.16 & 0.0 \\ -2.20 & 0.36 \end{bmatrix} = \begin{bmatrix} 0.0256 & 0.0 \\ -1.144 & 0.1296 \end{bmatrix} \end{aligned}$$

we get

$$h(5T) = [8.8 \quad 2.6] \begin{bmatrix} 0.0256 & 0.0 \\ -1.144 & 0.1296 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [8.8 \quad 2.6] \begin{bmatrix} 0.0256 \\ -0.8848 \end{bmatrix} = -2.0752 \quad \blacksquare$$

SA.7 Fig. SA.8 shows a recursive digital filter.

- Find its transfer function.
- By using the Jury-Marden stability criterion, determine whether the filter is stable or unstable.

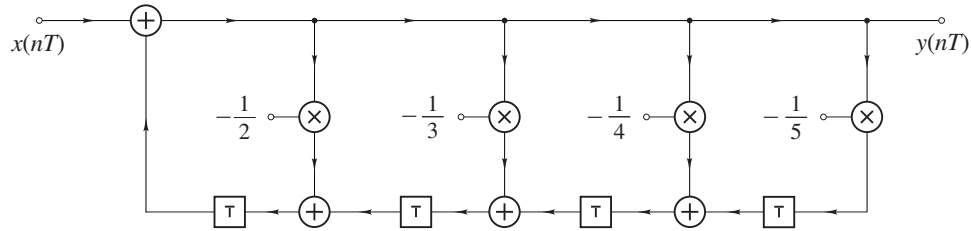


Figure SA.8

Solution

- From Fig. SA.8, we get

$$y(nT) = x(nT) - \frac{1}{2}y(nT - T) - \frac{1}{3}y(nT - 2T) - \frac{1}{4}y(nT - 3T) - \frac{1}{5}y(nT - 4T)$$

Hence the z transform gives

$$Y(z) = X(z) - \frac{1}{2}z^{-1}Y(z) - \frac{1}{3}z^{-2}Y(z) - \frac{1}{4}z^{-3}Y(z) - \frac{1}{5}z^{-4}Y(z)$$

or

$$Y(z) + \frac{1}{2}z^{-1}Y(z) + \frac{1}{3}z^{-2}Y(z) + \frac{1}{4}z^{-3}Y(z) + \frac{1}{5}z^{-4}Y(z) = X(z)$$

and so

$$\frac{Y(z)}{X(z)} = \frac{1}{1 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{5}z^{-4}}$$

In effect,

$$H(z) = \frac{N(z)}{D(z)} = \frac{z^4}{z^4 + \frac{1}{2}z^3 + \frac{1}{3}z^2 + \frac{1}{4}z + \frac{1}{5}}$$

where

$$D(z) = z^4 + \frac{1}{2}z^3 + \frac{1}{3}z^2 + \frac{1}{4}z + \frac{1}{5}$$

We note that

$$D(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = 2.283 > 0 \tag{SA.11a}$$

$$(-1)^4 D(-1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = 0.783 > 0 \tag{SA.11b}$$

(b) The Jury-Marden array can be constructed as shown in Table SA.4.

Table SA.4 The Jury-Marden array

	b_0	b_1	b_2	b_3	b_4
1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
2	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1
3	$\frac{24}{25}$	$\frac{9}{20}$	$\frac{4}{15}$	$\frac{3}{20}$	
4	$\frac{3}{20}$	$\frac{4}{15}$	$\frac{9}{20}$	$\frac{24}{25}$	
5	0.8991	0.392	0.1885		

From Eqs. (SA.11a) and (SA.11b) and Table SA.4, we have

$$\begin{aligned}
 D(1) &> 0 & (-1)^4 D(-1) &> 0 \\
 b_0 &= 1 > \frac{1}{5} = |b_4| \\
 |c_0| &= \frac{24}{25} > \frac{3}{20} = |c_3| \\
 |d_0| &= 0.8991 > 0.1885 = |d_2|
 \end{aligned}$$

Therefore, conditions (i) to (iii) of the Jury-Marden stability criterion (see p. 220) are satisfied and the filter is stable. ■

SA.8 The filter of Fig. SA.9 is subjected to an input

$$x(nT) = \begin{cases} 1 & \text{for } n = 0 \text{ and } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the time-domain response in *closed* form if $m_1 = -\frac{3}{4}$ and $m_2 = -\frac{1}{8}$. The filter is linear and time-invariant.

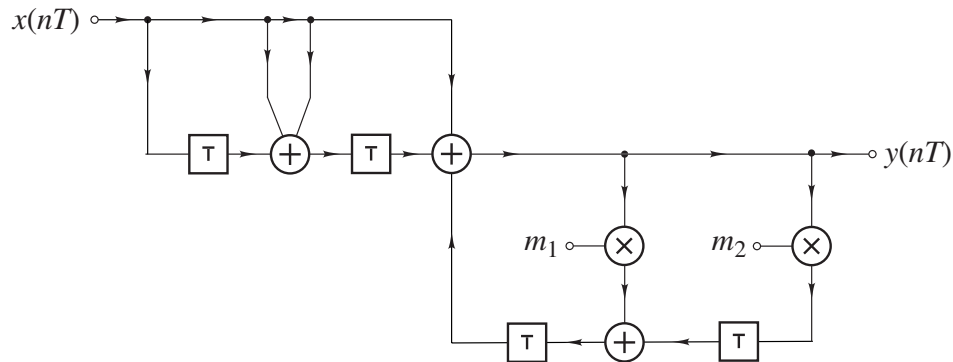


Figure SA.9

Solution

From Fig. SA.9

$$Y(z) = X(z) + 2z^{-1}X(z) + z^{-2}X(z) + m_1z^{-1}Y(z) + m_2z^{-2}Y(z)$$

Hence

$$Y(z)(1 - m_1z^{-1} - m_2z^{-2}) = (1 + 2z^{-1} + z^{-2})X(z)$$

or

$$Y(z) = \frac{(1 + 2z^{-1} + z^{-2})}{(1 - m_1z^{-1} - m_2z^{-2})}X(z)$$

Therefore,

$$\frac{Y(z)}{X(z)} = H(z) = \frac{z^2 + 2z + 1}{z^2 - m_1z - m_2} = \frac{z^2 + 2z + 1}{z^2 + \frac{3}{4}z + \frac{1}{8}} = \frac{z^2 + 2z + 1}{(z + \frac{1}{2})(z + \frac{1}{4})}$$

The time-domain response can be obtained in a number of ways, as detailed below.

Method 1:

The input is the sum of two impulse functions, i.e.,

$$x(nT) = \delta(nT) + \delta(nT - T)$$

Hence

$$\mathcal{R}x(nT) = \mathcal{R}[\delta(nT) + \delta(nT - T)]$$

Since the filter is linear, we have

$$\mathcal{R}x(nT) = \mathcal{R}\delta(nT) + \mathcal{R}\delta(nT - T)$$

and if $h(nT)$ is the impulse response, i.e.,

$$h(nT) = \mathcal{R}\delta(nT)$$

then by virtue of the fact that the filter is time-invariant, we get

$$y(nT) = h(nT) + h(nT - T)$$

In effect, all we have to do is find the impulse response. Expanding $H(z)/z$ into partial fractions gives

$$\frac{H(z)}{z} = \frac{z^2 + 2z + 1}{z(z + \frac{1}{2})(z + \frac{1}{4})} = \frac{R_1}{z} + \frac{R_2}{z + \frac{1}{2}} + \frac{R_2}{z + \frac{1}{4}}$$

where

$$\begin{aligned} R_1 &= \left. \frac{z^2 + 2z + 1}{(z + \frac{1}{2})(z + \frac{1}{4})} \right|_{z=0} = \frac{1}{\frac{1}{2} \times \frac{1}{4}} = 8 \\ R_2 &= \left. \frac{z^2 + 2z + 1}{z(z + \frac{1}{4})} \right|_{z=-\frac{1}{2}} = \frac{\frac{1}{4} - \frac{2}{2} + 1}{(-\frac{1}{2})(-\frac{1}{2} + \frac{1}{4})} = \frac{\frac{1}{4}}{(-\frac{1}{2})(-\frac{1}{4})} = 2 \\ R_3 &= \left. \frac{z^2 + 2z + 1}{z(z + \frac{1}{2})} \right|_{z=-\frac{1}{4}} = \frac{\frac{1}{16} - \frac{2}{4} + 1}{(-\frac{1}{4})(-\frac{1}{4} + \frac{1}{2})} = \frac{\frac{1}{16} + \frac{8}{16}}{-\frac{1}{16}} = -9 \end{aligned}$$

i.e.,

$$H(z) = R_1 + \frac{R_2z}{z + \frac{1}{2}} + \frac{R_3z}{z + \frac{1}{4}}$$

and

$$\begin{aligned} h(nT) &= R_1\delta(nT) + [R_2(-\frac{1}{2})^n + R_3(-\frac{1}{4})^n]u(nT) \\ &= 8\delta(nT) + [2(-\frac{1}{2})^n - 9(-\frac{1}{4})^n]u(nT) \end{aligned}$$

Now

$$y(nT) = h(nT) + h(nT - T)$$

Hence

$$\begin{aligned} y(nT) &= 8[\delta(nT) + \delta(nT - T)] + [2(-\frac{1}{2})^n - 9(-\frac{1}{4})^n]u(nT) \\ &\quad + [2(-\frac{1}{2})^{n-1} - 9(-\frac{1}{4})^{n-1}]u(nT - T) \quad \blacksquare \end{aligned}$$

Method 2

Since

$$x(nT) = \delta(nT) + \delta(nT - T)$$

the z transform gives

$$X(z) = 1 + z^{-1} = \frac{z+1}{z}$$

Hence

$$Y(z) = H(z)X(z) = \frac{(z^2 + 2z + 1)}{(z + \frac{1}{2})(z + \frac{1}{4})} \times \frac{(z+1)}{z}$$

Expanding $H(z)X(z)$ into partial fractions, gives

$$H(z)X(z) = R_1 + \frac{R_2}{z} + \frac{R_3}{z + \frac{1}{2}} + \frac{R_4}{z + \frac{1}{4}}$$

where

$$\begin{aligned} R_1 &= \lim_{z \rightarrow \infty} H(z)X(z) = 1 \\ R_2 &= \left. \frac{(z^2 + 2z + 1)(z + 1)}{(z + \frac{1}{2})(z + \frac{1}{4})} \right|_{z=0} = \frac{1}{\frac{1}{2} \times \frac{1}{4}} = 8 \\ R_3 &= \left. \frac{(z^2 + 2z + 1)(z + 1)}{(z + \frac{1}{4})z} \right|_{z=-\frac{1}{2}} = \frac{(\frac{1}{4} - 1 + 1)(-\frac{1}{2} + 1)}{(-\frac{1}{2} + \frac{1}{4})(-\frac{1}{2})} = \frac{(\frac{1}{4})(\frac{1}{2})}{(-\frac{1}{4})(-\frac{1}{2})} = 1 \\ R_4 &= \left. \frac{(z^2 + 2z + 1)(z + 1)}{(z + \frac{1}{2})z} \right|_{z=-\frac{1}{4}} = \frac{(\frac{1}{16} - \frac{2}{4} + 1)(-\frac{1}{4} + 1)}{(-\frac{1}{4} + \frac{1}{2})(-\frac{1}{4})} = \frac{(\frac{1}{16} + \frac{8}{16})(\frac{3}{4})}{(\frac{1}{4})(-\frac{1}{4})} \\ &= \frac{\frac{9}{16} \times \frac{3}{4}}{-\frac{1}{16}} = -\frac{27}{4} \end{aligned}$$

Therefore,

$$\begin{aligned} y(nT) &= R_1\delta(nT) + R_2\delta(nT - T) + R_3u(nT - T)\left(-\frac{1}{2}\right)^{n-1} + R_4u(nT - T)\left(-\frac{1}{4}\right)^{n-1} \\ &= \delta(nT) + 8\delta(nT - T) + u(nT - T)\left(-\frac{1}{2}\right)^{n-1} - \frac{27}{4}u(nT - T)\left(-\frac{1}{4}\right)^{n-1} \quad \blacksquare \end{aligned}$$

Method 3

The inverse of $Y(z)$ is obtained from first principles as

$$y(nT) = \sum_{\text{res}} Y_0(z)$$

where

$$Y_0(z) = Y(z)z^{n-1} = H(z)X(z)z^{n-1} = \frac{(z^2 + 2z + 1)(z + 1)}{z(z + \frac{1}{2})(z + \frac{1}{4})}z^{n-1}$$

However, watch out for pitfalls at the origin. In this case, we have a second-order pole at the origin if $n = 0$, a first-order pole at the origin if $n = 1$, and no poles at the origin if $n \geq 2$. Hence, we have to find $y(0)$ and $y(T)$ individually and then proceed to $y(nT)$ for $n \geq 2$. This would make this method quite long.

Method 4

We can express $Y(z)$ into partial fractions as

$$Y(z) = \frac{(z^2 + 2z + 1)(z + 1)}{z(z + \frac{1}{2})(z + \frac{1}{4})} = R_1 + \frac{R_2}{z} + \frac{R_3z}{z + \frac{1}{2}} + \frac{R_4z}{z + \frac{1}{4}}$$

where

$$\begin{aligned}
 R_2 &= \lim_{z=0} zY(z) = \frac{z(z^2 + 2z + 1)(z + 1)}{z(z + \frac{1}{2})(z + \frac{1}{4})} \Big|_{z=0} = \frac{1 \times 1}{\frac{1}{2} \times \frac{1}{4}} = 8 \\
 R_3 &= \lim_{z=-\frac{1}{2}} \frac{z + \frac{1}{2}}{z} \times Y(z) = \frac{(z^2 + 2z + 1)(z + 1)}{z^2(z + \frac{1}{4})} \Big|_{z=-\frac{1}{2}} \\
 &= \frac{\left[(-\frac{1}{2})^2 + 2(-\frac{1}{2}) + 1\right](-\frac{1}{2} + 1)}{(-\frac{1}{2})^2(-\frac{1}{2} + \frac{1}{4})} = \frac{(\frac{1}{4} - 1 + 1)(\frac{1}{2})}{(\frac{1}{4})(-\frac{1}{4})} = \frac{\frac{1}{8}}{-\frac{1}{16}} = -2 \\
 R_4 &= \lim_{z=-\frac{1}{4}} \frac{z + \frac{1}{4}}{z} \times Y(z) = \frac{(z^2 + 2z + 1)(z + 1)}{z^2(z + \frac{1}{2})} \Big|_{z=-\frac{1}{4}} \\
 &= \frac{\left[(-\frac{1}{4})^2 + 2(-\frac{1}{4}) + 1\right](-\frac{1}{4} + 1)}{(-\frac{1}{4})^2(-\frac{1}{4} + \frac{1}{2})} = \frac{(\frac{1}{16} - \frac{2}{4} + 1)(\frac{3}{4})}{\frac{1}{16} \times \frac{1}{4}} = \frac{\frac{9}{16} \times \frac{3}{4}}{\frac{1}{64}} = 27
 \end{aligned}$$

Constant A can be obtained by noting that

$$\lim_{z \rightarrow \infty} Y(z) = R_1 + R_3 + R_4 = \frac{z^3}{z^3} = 1$$

Thus

$$R_1 = 1 - R_3 - R_4 = 1 - (-2) - 27 = 3 - 27 = -24$$

Hence

$$Y(z) = -24 + \frac{8}{z} - \frac{2z}{z + \frac{1}{2}} + \frac{27z}{z + \frac{1}{4}}$$

Therefore,

$$y(nT) = -24\delta(nT) + 8\delta(nT - T) + [27(-\frac{1}{4})^n - 2(-\frac{1}{2})^n] u(nT) \quad \blacksquare$$

Method 5

One could expand $Y(z)/z$ into partial fractions as

$$\frac{Y(z)}{z} = \frac{R_1}{z} + \frac{R_2}{z^2} + \frac{R_3}{z + \frac{1}{2}} + \frac{R_4}{z + \frac{1}{4}}$$

However, this is essentially the same as method 4.

SA.9 A discrete-time system has a transfer function

$$H(z) = \frac{z^2 + 2}{z^2 - (2r \cos \theta)z + r^2}$$

- Find the unit-step response in closed form.
- Using MATLAB, D-Filter, or similar software, plot the unit-step response for $r = 0.3$ and $\theta = \pi/4$.
- Repeat part (b) for $r = 0.6$ and $\theta = \pi/4$.
- Repeat part (b) for $r = 0.9$ and $\theta = \pi/4$.
- Compare the unit-step responses in parts (b) to (d).

Solution

- The z transform of the output of the system is given by

$$Y(z) = H(z)X(z) \tag{SA.12}$$

where $H(z)$ is the transfer function and $X(z)$ is the z transform of the input. Since the input is a unit step, we have

$$X(z) = \mathcal{Z}u(nT) = \frac{z}{z-1} \quad (\text{SA.13})$$

Thus Eqs. (SA.12) and (SA.13) give

$$Y(z) = \frac{z^2 + 2}{z^2 - (2r \cos \theta)z + r^2} \cdot \frac{z}{z-1} = \frac{z^3 + 2z}{(z-1)[z^2 - (2r \cos \theta)z + r^2]}$$

The general inversion formula gives

$$y(nT) = \frac{1}{2\pi j} \oint_{\Gamma} Y(z)z^{n-1}dz = \sum_{i=1}^M \mathfrak{Res}_{z=p_i} Y_0(z) \quad (\text{SA.14})$$

where

$$\begin{aligned} Y_0(z) &= Y(z)z^{n-1} = \frac{(z^3 + 2z)z^{n-1}}{(z-1)[z^2 - (2r \cos \theta)z + r^2]} \\ &= \frac{(z^3 + 2z)z^{n-1}}{(z-1)[z^2 - r(e^{j\theta} + e^{-j\theta})z + r^2]} \\ &= \frac{(z^2 + 2)z^n}{(z-1)(z - re^{j\theta})(z - re^{-j\theta})} \end{aligned}$$

and $M = 3$. The residues of $Y_0(z)$ can be obtained as follows:

$$\begin{aligned} R_1 &= \lim_{z=1} (z-1)Y_0(z) = \frac{(z^2 + 2)z^n}{z^2 - (2r \cos \theta)z + r^2} \Big|_{z=1} \\ &= \frac{3}{1 - (2r \cos \theta) + r^2} \end{aligned} \quad (\text{SA.15})$$

$$\begin{aligned} R_2 &= \lim_{z=re^{j\theta}} (z - re^{j\theta})Y_0(z) = \frac{(z^2 + 2)z^n}{(z-1)(z - re^{-j\theta})} \Big|_{z=re^{j\theta}} \\ &= \frac{(r^2 e^{j2\theta} + 2)r^n e^{jn\theta}}{(re^{j\theta} - 1)(re^{j\theta} - re^{-j\theta})} = \frac{(r^2 e^{j2\theta} + 2)r^{n-1} e^{jn\theta}}{(re^{j\theta} - 1)2j \sin \theta} \\ &= \frac{(r^2 \cos 2\theta + 2 + jr^2 \sin 2\theta)r^{n-1} e^{jn\theta}}{(r \cos \theta - 1 + jr \sin \theta)2j \sin \theta} \\ &= \frac{(r^2 \cos 2\theta + 2 + jr^2 \sin 2\theta)r^{n-1} e^{jn\theta}}{[(r \cos \theta - 1) + jr \sin \theta]2e^{j\pi/2} \sin \theta} \\ &= Me^{j\phi} r^{n-1} e^{jn\theta} = Mr^{n-1} e^{j(n\theta + \phi)} \end{aligned} \quad (\text{SA.16})$$

where

$$\begin{aligned} M &= \left| \frac{r^2 \cos 2\theta + 2 + jr^2 \sin 2\theta}{[j(r \cos \theta - 1) - r \sin \theta]2 \sin \theta} \right| \\ &= \sqrt{\frac{(r^2 \cos 2\theta + 2)^2 + (r^2 \sin 2\theta)^2}{4[(r \cos \theta - 1)^2 + (r \sin \theta)^2] \sin^2 \theta}} \end{aligned}$$

and

$$\begin{aligned} \phi &= \tan^{-1} \frac{r^2 \sin 2\theta}{r^2 \cos 2\theta + 2} - \tan^{-1} \frac{(r \cos \theta - 1)}{-r \sin \theta} \\ &= \tan^{-1} \frac{r^2 \sin 2\theta}{r^2 \cos 2\theta + 2} - \tan^{-1} \frac{r \sin \theta}{(r \cos \theta - 1)} - \frac{\pi}{2} \end{aligned}$$

Similarly,

$$R_3 = R_2^* \quad (\text{SA.17})$$

since complex conjugate poles give complex conjugate residues. We note that there is no additional pole at the origin when $n = 0$ and hence for $n \geq 0$ Eq. (SA.14) gives

$$y(nT) = R_1 + R_2 + R_2^* \quad (\text{SA.18})$$

Since the numerator degree in $Y(z)$ does not exceed the denominator degree, we have

$$y(nT) = 0 \quad \text{for } n < 0$$

Therefore, for any n , Eqs. (SA.15)–(SA.18) give

$$\begin{aligned} y(nT) &= u(nT) \left[R_1 + Mr^{n-1}e^{j(n\theta+\phi)} + Mr^{n-1}e^{-j(n\theta+\phi)} \right] \\ &= u(nT) \left[R_1 + 2Mr^{n-1} \cos(n\theta + \phi) \right] \quad \blacksquare \end{aligned}$$

(b) and (d) The step responses for the three cases are illustrated in Fig. SA.10a to c. \blacksquare

(e) On the basis of the step responses obtained, the system in part (b) is what they call *overdamped*, the one in part (c) is *critically damped*, and the one in part (d) is referred to as *underdamped*. \blacksquare

SA.10 A second-order digital filter has zeros $z_1 = e^{j\pi/3}$ and $z_2 = e^{-j\pi/3}$ and poles $p_1 = 0.5e^{j\pi/4}$ and $p_2 = 0.5e^{-j\pi/4}$ and its multiplier constant is 2.

(a) Obtain the transfer function of the filter.

(b) Obtain an expression for the gain.

(c) Assuming that the sampling frequency is 2π , calculate the gain at $\omega = 0, \pi/4, \pi/3$, and π .

Solution

(a) Since we have the zeros, poles, and multiplier constant of the filter, the transfer function can be readily constructed as

$$\begin{aligned} H(z) &= \frac{2(z - e^{j\pi/3})(z - e^{-j\pi/3})}{(z - \frac{1}{2}e^{j\pi/4})(z - \frac{1}{2}e^{-j\pi/4})} = \frac{2[z^2 - (e^{j\pi/3} + e^{-j\pi/3})z + 1]}{z^2 - \frac{1}{2}(e^{j\pi/4} + e^{-j\pi/4}) + \frac{1}{4}} \\ &= \frac{2[z^2 - 2(\cos \pi/3)z + 1]}{z^2 - (\cos \pi/4)z + \frac{1}{4}} = \frac{2[z^2 - z + 1]}{z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}} \quad \blacksquare \end{aligned}$$

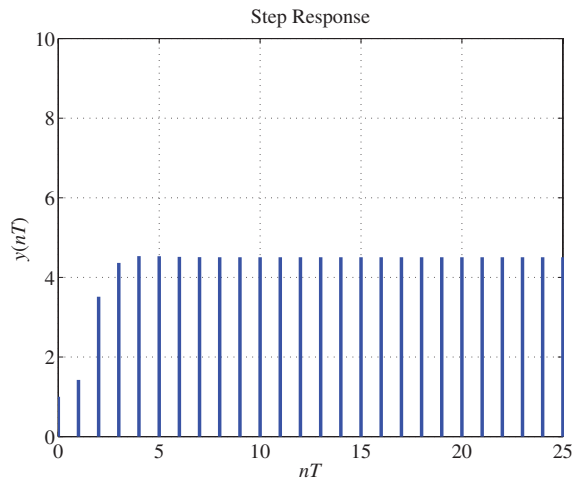
(b) The gain is given by

$$\begin{aligned} M(\omega) &= |H(e^{j\omega T})| = \left| \frac{2[e^{2j\omega T} - e^{j\omega T} + 1]}{e^{j2\omega T} - \frac{\sqrt{2}}{2}e^{j\omega T} + \frac{1}{4}} \right| \\ &= 2 \left| \frac{\cos 2\omega T + j \sin 2\omega T - \cos \omega T - j \sin \omega T + 1}{\cos 2\omega T + j \sin 2\omega T - \frac{\sqrt{2}}{2}(\cos \omega T + j \sin \omega T) + \frac{1}{4}} \right| \\ &= 2 \sqrt{\frac{(\cos 2\omega T - \cos \omega T + 1)^2 + (\sin 2\omega T - \sin \omega T)^2}{(\cos 2\omega T - \frac{\sqrt{2}}{2} \cos \omega T + \frac{1}{4})^2 + (\sin 2\omega T - \frac{\sqrt{2}}{2} \sin \omega T)^2}} \quad \blacksquare \end{aligned}$$

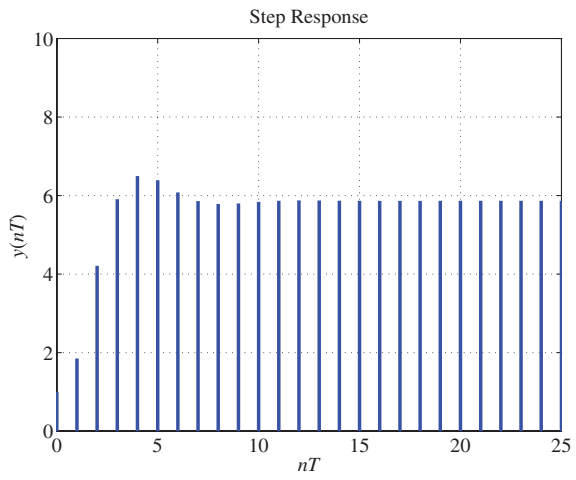
(c) Since $\omega_s = 2\pi$, we have $2\pi f_s = 2\pi/T = 2\pi$. Hence $T = 1$ s. For the frequencies given, the numerical values in Table SA.5 can be readily calculated.

Table SA.5

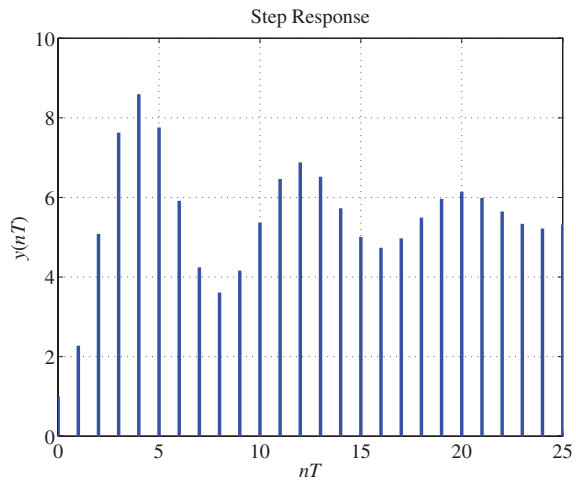
ω	$\cos \omega T$	$\sin \omega T$	$\cos 2\omega T$	$\sin 2\omega T$
0	1	0	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	1
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
π	-1	0	1	0



(a)



(b)



(c)

Figure SA.10

Thus

$$M(0) = 2\sqrt{\frac{(1-1+1)^2 + (0-0)^2}{(1-\frac{\sqrt{2}}{2}+\frac{1}{4})^2 + (0-0)^2}} = \frac{2}{\frac{5-2\sqrt{2}}{4}} = 3.6840 \quad \blacksquare$$

$$\begin{aligned} M\left(\frac{\pi}{4}\right) &= 2\sqrt{\frac{(0-\frac{\sqrt{2}}{2}+1)^2 + (1-\frac{\sqrt{2}}{2})^2}{(0-\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2}+\frac{1}{4})^2 + (1-\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2})^2}} \\ &= 2\sqrt{\frac{(2-\sqrt{2})^2 + (2-\sqrt{2})^2}{(\frac{1}{2}-1)^2 + 1}} = 2\sqrt{\frac{2(2-\sqrt{2})^2}{(-\frac{1}{2})^2 + 1}} = 1.489 \quad \blacksquare \end{aligned}$$

$$\begin{aligned} M\left(\frac{\pi}{3}\right) &= 2\sqrt{\frac{(-\frac{1}{2}-\frac{1}{2}+1)^2 + (\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2})^2}{(-\frac{1}{2}-\frac{\sqrt{2}}{2}\times\frac{1}{2}+\frac{1}{4})^2 + (\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2})^2}} \\ &= 2\sqrt{\frac{0+0}{\left(\frac{1-2-\sqrt{2}}{4}\right)^2 + \dots}} = 0 \quad \blacksquare \end{aligned}$$

$$\begin{aligned} M(\pi) &= 2\sqrt{\frac{(1+1+1)^2 + (0-0)}{(1+\frac{\sqrt{2}}{2}+\frac{1}{4})^2 + (0-0)}} = 2\sqrt{\frac{3^2 \times 16}{(4+2\sqrt{2}+1)^2}} \\ &= \frac{2 \times 3 \times 4}{4+2\sqrt{2}+1} = 3.0658 \quad \blacksquare \end{aligned}$$

SA.11 A digital filter characterized by the transfer function

$$H(z) = \frac{2(z^2 - z + 1)}{z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}}$$

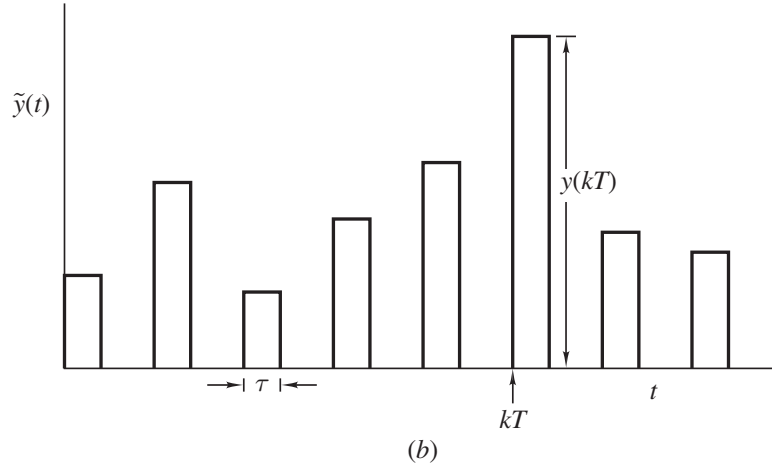
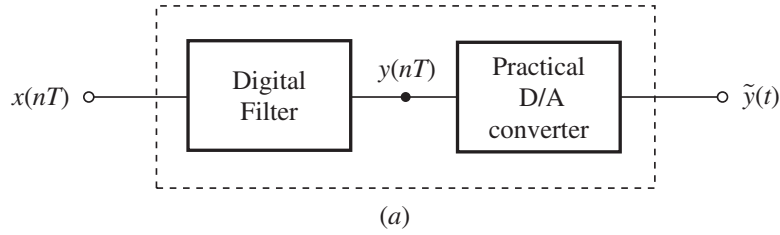
and a practical D/A converter are connected in cascade as shown in Fig. SA.11a. The output waveform of the D/A converter is of the form illustrated in Fig. S11b where $\tau = 0.01T$ s and T is the sampling period. The sampling frequency is 2π rad/s.

- Obtain an expression for the gain of just the digital filter.
- Obtain an expression for the overall gain of the digital filter in cascade with the D/A converter.
- Calculate the gain of just the digital filter at $\omega = 0, \pi/4, \pi/3$, and π .
- Calculate the overall gain of the digital filter in cascade with the D/A converter at $\omega = 0, \pi/4, \pi/3$, and π .
- Sketch (i) the gain of just the digital filter and (ii) the overall gain of the digital filter in cascade with the D/A converter, and explain the effect of the D/A converter on the amplitude response.

Solution

(a) The gain is given by

$$\begin{aligned} M(\omega) &= |H(e^{j\omega T})| = \left| \frac{2[e^{2j\omega T} - e^{j\omega T} + 1]}{e^{j2\omega T} - \frac{\sqrt{2}}{2}e^{j\omega T} + \frac{1}{4}} \right| \\ &= 2 \left| \frac{\cos 2\omega T + j \sin 2\omega T - \cos \omega T - j \sin \omega T + 1}{\cos 2\omega T + j \sin 2\omega T - \frac{\sqrt{2}}{2}(\cos \omega T + j \sin \omega T) + \frac{1}{4}} \right| \\ &= 2\sqrt{\frac{(\cos 2\omega T - \cos \omega T + 1)^2 + (\sin 2\omega T - \sin \omega T)^2}{(\cos 2\omega T - \frac{\sqrt{2}}{2}\cos \omega T + \frac{1}{4})^2 + (\sin 2\omega T - \frac{\sqrt{2}}{2}\sin \omega T)^2}} \quad \blacksquare \end{aligned}$$


Figure SA.11

(b) The practical D/A converter has a gain

$$|H_p(j\omega)| = \tau \left| \frac{\sin(\omega\tau/2)}{\omega\tau/2} \right| \quad (\text{SA.19})$$

where $\tau = 0.01T$ and $T = 2\pi/\omega_s = 2\pi/2\pi = 1$ s (see Eq. (6.60) in textbook). Hence the overall gain of the digital filter in cascade with the D/A converter is given by

$$M_T(\omega) = 2 \sqrt{\frac{(\cos 2\omega T - \cos \omega T + 1)^2 + (\sin 2\omega T - \sin \omega T)^2}{(\cos 2\omega T - \frac{\sqrt{2}}{2} \cos \omega T + \frac{1}{4})^2 + (\sin 2\omega T - \frac{\sqrt{2}}{2} \sin \omega T)^2}} \cdot \tau \left| \frac{\sin \omega\tau/2}{\omega\tau/2} \right|$$

(c) Since $\omega_s = 2\pi$, we have $2\pi f_s = 2\pi/T = 2\pi$. Hence $T = 1$ s. For the frequencies given, the numerical values in Table SA.6 can be readily calculated.

Table SA 6

ω	$\cos \omega T$	$\sin \omega T$	$\cos 2\omega T$	$\sin 2\omega T$
0	1	0	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	1
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
π	-1	0	1	0

Thus

$$\begin{aligned}
 M(0) &= 2\sqrt{\frac{(1-1+1)^2 + (0-0)^2}{(1-\frac{\sqrt{2}}{2} + \frac{1}{4})^2 + (0-0)^2}} = \frac{2}{\frac{5-2\sqrt{2}}{4}} = 3.6840 \quad \blacksquare \\
 M\left(\frac{\pi}{4}\right) &= 2\sqrt{\frac{(0-\frac{\sqrt{2}}{2} + 1)^2 + (1-\frac{\sqrt{2}}{2})^2}{(0-\frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} + \frac{1}{4})^2 + (1-\frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2})^2}} \\
 &= 2\sqrt{\frac{(2-\sqrt{2})^2 + (2-\sqrt{2})^2}{(\frac{1}{2}-1)^2 + 1}} = 2\sqrt{\frac{2(2-\sqrt{2})^2}{(-\frac{1}{2})^2 + 1}} = 1.4890 \quad \blacksquare \\
 M\left(\frac{\pi}{3}\right) &= 2\sqrt{\frac{(-\frac{1}{2}-\frac{1}{2} + 1)^2 + (\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2})^2}{(-\frac{1}{2}-\frac{\sqrt{2}}{2} \times \frac{1}{2} + \frac{1}{4})^2 + (\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2})^2}} \\
 &= 2\sqrt{\frac{0+0}{\left(\frac{1-2-\sqrt{2}}{4}\right)^2 + \dots}} = 0 \quad \blacksquare \\
 M(\pi) &= 2\sqrt{\frac{(1+1+1)^2 + (0-0)}{(1+\frac{\sqrt{2}}{2} + \frac{1}{4})^2 + (0-0)}} = 2\sqrt{\frac{3^2 \times 16}{(4+2\sqrt{2}+1)^2}} \\
 &= \frac{2 \times 3 \times 4}{4+2\sqrt{2}+1} = 3.0657 \quad \blacksquare
 \end{aligned}$$

(d) Since $T = 1\text{s}$, Eq. (SA.19) gives

$$\begin{aligned}
 |H_P(j\omega)| &= 0.01T \left| \frac{\sin(0.01T\omega/2)}{0.01T\omega/2} \right| = 0.01 \left| \frac{\sin(0.01\omega/2)}{0.01\omega/2} \right| \\
 &= p0.01
 \end{aligned}$$

Hence the overall gain of the digital filter in cascade with the D/A converter is obtained from the above numerical values as follows:

$$\begin{aligned}
 M_T(0) &= 3.6840 \times 0.01 = 3.6840 \times 10^{-2} \quad \blacksquare \\
 M_T\left(\frac{\pi}{4}\right) &= 1.4890 \times 0.01 = 1.4890 \times 10^{-2} \quad \blacksquare \\
 M_T\left(\frac{\pi}{3}\right) &= 0 \times 0.0100 = 0.0 \quad \blacksquare \\
 M_T(\pi) &= 3.0557 \times 0.01 = 3.0557 \times 10^{-2} \quad \blacksquare
 \end{aligned}$$

SA.12 Realize the transfer function

$$H(z) = \left(\frac{z}{z-0.5+j0.3} + \frac{z}{z-0.5-j0.3} \right) \cdot \left(\frac{3z}{z+0.4} + \frac{5z}{z+0.5} \right)$$

using two second-order canonic filter sections in cascade.

Solution

The transfer function can be expressed as

$$\begin{aligned}
 H(z) &= \frac{2z^2 - 0.5z - 0.5z}{z^2 - z + 0.5^2 + 0.3^2} \cdot \frac{3z^2 + 1.5z + 5z^2 + 2z}{z^2 + 0.9z + 0.2} \\
 &= \frac{2z^2 - z}{z^2 - z + 0.34} \cdot \frac{8z^2 + 3.5z}{z^2 + 0.9z + 0.2} \\
 &= \frac{2 - z^{-1}}{1 - z^{-1} + 0.34z^{-2}} \cdot \frac{8 + 3.5z^{-1}}{1 + 0.9z^{-1} + 0.2z^{-2}} \\
 &= H_1(z) \cdot H_2(z)
 \end{aligned}$$

where

$$H_1(z) = \frac{2 - z^{-1}}{1 - z^{-1} + 0.34z^{-2}} \quad \text{and} \quad H_2(z) = \frac{8 + 3.5z^{-1}}{1 + 0.9z^{-1} + 0.2z^{-2}}$$

Now if we realize $H_1(z)$ and $H_2(z)$ in terms of direct canonic sections, the cascade realization shown in Fig. SA.12 can be readily obtained. ■

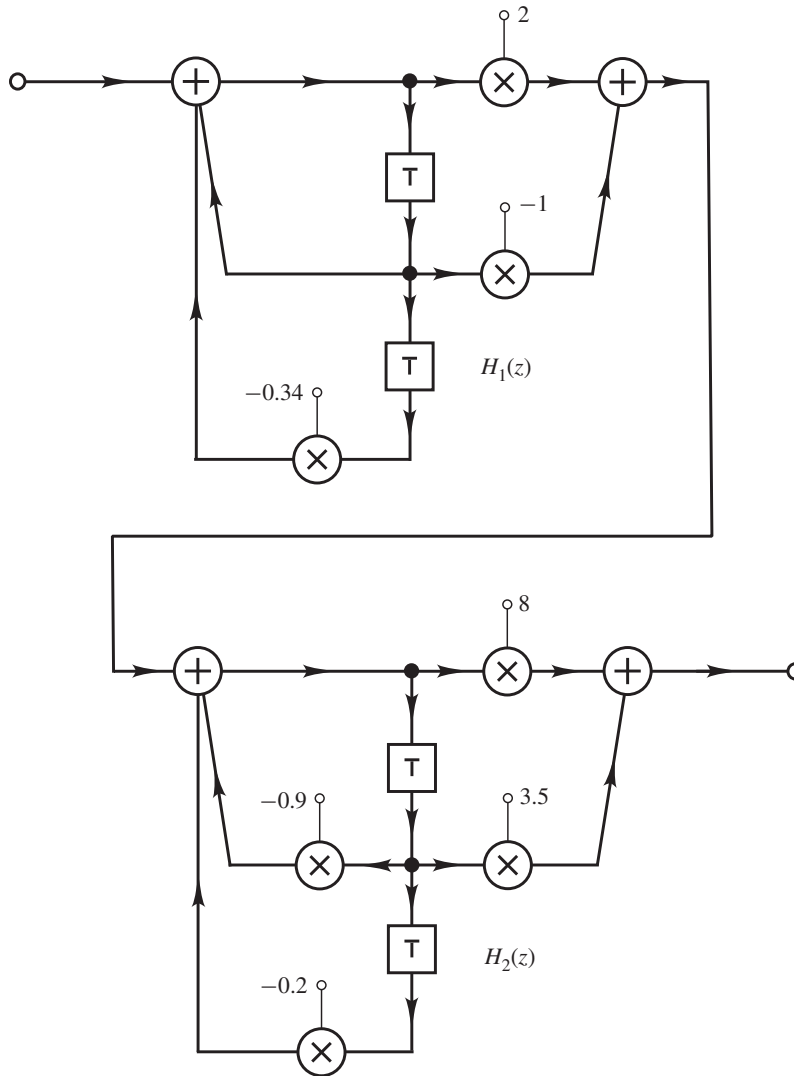


Figure SA.12

SA.13 An analog elliptic lowpass filter with a cutoff frequency of 1 rad/s has a transfer function of the form

$$H(s) = \frac{0.075(s^2 + 2.6)}{(s + 0.38)(s^2 + 0.31s + 0.51)}$$

(a) By applying the lowpass-to-highpass transformation

$$s = \frac{1}{s}$$

get a continuous-time highpass transfer function.

(b) Construct the zero-pole plot of the continuous-time highpass transfer function.

(c) Using the zeros and poles obtained in part (b), get a corresponding discrete-time highpass transfer function using the matched- z -transformation method. The sampling frequency is $\omega_s = 20$ rad/s.

- (d) How does the matched- z -transformation method compare with the invariant-impulse-response method?

Solution

The transfer function has zeros at

$$z = z_1, z_1^*$$

where

$$z_1 = 0 + j1.6125$$

and poles at

$$z = p_0, p_1, p_1^*$$

where

$$p_0 = -0.3800 + j0.0000$$

$$p_1 = -0.1550 + j0.6971$$

- (a) The highpass transfer function is obtained as

$$\begin{aligned} H_{HP}(\bar{s}) &= H(s) \Big|_{s \rightarrow \frac{1}{\bar{s}}} = \frac{0.075(\bar{s}^2 + 2.6)}{(s + 0.38)(s^2 + 0.31s + 0.51)} \Big|_{s=1/\bar{s}} \\ &= \frac{0.075(\frac{1}{\bar{s}^2} + 2.6)}{(\frac{1}{\bar{s}} + 0.38)(\frac{1}{\bar{s}^2} + 0.31\frac{1}{\bar{s}} + 0.51)} \\ &= \frac{0.075\bar{s}(1 + 2.6\bar{s}^2)}{(1 + 0.38\bar{s})(0.51\bar{s}^2 + 0.31\bar{s} + 1)} \\ &= \frac{0.075 \times 2.6}{0.38 \times 0.51} \times \frac{\bar{s}(\bar{s}^2 + \frac{1}{2.6})}{(\bar{s} + \frac{1}{0.38})(\bar{s}^2 + \frac{0.31}{0.51}\bar{s} + \frac{1}{0.51})} \\ &= 1.0063 \times \frac{\bar{s}(\bar{s}^2 + 0.3846)}{(\bar{s} + 2.6316)(\bar{s}^2 + 0.6079\bar{s} + 1.9609)} \quad \blacksquare \end{aligned}$$

Therefore, the highpass filter has zeros at

$$\bar{s} = \bar{s}_0, \bar{s}_1, \bar{s}_2$$

where

$$\bar{s}_0 = 0, \quad \bar{s}_1, \bar{s}_2 = 0 \pm j0.6202$$

and poles at

$$\bar{s} = \bar{p}_0, \bar{p}_1, \bar{p}_2$$

where

$$\bar{p}_0 = -2.6316, \quad \bar{p}_1, \bar{p}_2 = -0.3040 \pm j1.3669$$

- (b) The transfer function of the digital filter is given by

$$H_D(z) = (z + 1)^L \frac{H_0 \prod_{i=1}^M (z - e^{\bar{s}_i T})}{\prod_{i=1}^N (z - e^{\bar{p}_i T})}$$

where $L = 0$ for a highpass filter. Hence

$$H_D(z) = \frac{\prod_{i=1}^3 (z - \tilde{z}_i)}{\prod_{i=1}^3 (z - \tilde{p}_i)} \quad \blacksquare$$

where

$$\begin{aligned}\tilde{z}_0 &= e^{0T} = 1, & \tilde{z}_1, \tilde{z}_2 &= e^{\pm j0.6202T} \\ \tilde{p}_0 &= e^{-2.6316T}, & \tilde{p}_1, \tilde{p}_2 &= e^{(-0.3040 \pm j1.3669)T}\end{aligned}$$

with

$$T = \frac{2\pi}{\omega_s} = \frac{2\pi}{20} = \frac{\pi}{10}$$

- (c) The method works with all types of filters, i.e., LP, HP, BP, and BS, and is easy to apply. However, it tends to increase the passband ripple. ■

SA.14 A lowpass analog filter has a transfer function

$$H_A(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

- (a) Assuming a sampling frequency of 10π , design a digital filter using the bilinear transformation method.
 (b) Find the 1-dB and 30-dB frequencies of the analog filter.
 (c) Find the 1-dB and 30-dB frequencies of the digital filter.
 (d) What should be the 3-dB frequency of the analog filter to get a 3-dB frequency in the digital filter at 1 rad/s?

Solution

- (a) The sampling period is give by

$$T = \frac{1}{f_s} = \frac{2\pi}{\omega_s} = \frac{2\pi}{10\pi} = \frac{1}{5}$$

The discrete-time transfer function is given by

$$\begin{aligned}H_D(z) &= H_A(s) \Big|_{s=\frac{2}{T} \frac{z-1}{z+1}} \\ &= \frac{1}{s^2 + \sqrt{2}s + 1} \Big|_{s=\frac{10(z-1)}{z+1}} \\ &= \frac{1}{\left[\frac{10(z-1)}{z+1}\right]^2 + \sqrt{2}\left[\frac{10(z-1)}{z+1}\right] + 1} \\ &= \frac{z^2 + 2z + 1}{100(z^2 - 2z + 1) + 10\sqrt{2}(z^2 - 1) + (z^2 + 2z + 1)} \\ &= \frac{z^2 + 2z + 1}{b_2z^2 + b_1z + b_0} \quad \blacksquare\end{aligned}$$

where

$$\begin{aligned}b_0 &= 100 - 10\sqrt{2} + 1 = 86.86 \\ b_1 &= -200 + 2 = -198.00 \\ b_2 &= 100 + 10\sqrt{2} + 1 = 115.14\end{aligned}$$

Alternatively, by factorizing constant b_2 in the denominator, the transfer function can be expressed as

$$H_D(z) = H_0 \frac{z^2 + 2z + 1}{z^2 + b'_1z + b'_0} \quad \blacksquare$$

where

$$H_0 = 8.685 \times 10^{-3}, \quad b'_0 = 0.7544, \quad b'_1 = -1.720$$

(b) The frequency response of the analog filter

$$H_A(j\omega) = \frac{1}{-\omega^2 + j\sqrt{2}\omega + 1}$$

Hence the loss is given by

$$\begin{aligned} A(\omega) &= 20 \log \sqrt{(1 - \omega^2)^2 + 2\omega^2} \\ &= 10 \log(1 - 2\omega^2 + \omega^4 + 2\omega^2) \end{aligned}$$

Thus

$$1 + \omega^4 = 10^{0.1 \times A(\omega)}$$

By letting $A(\omega) = 1$ dB, the 1-dB frequency is obtained as

$$\omega_1 = (10^{0.1} - 1)^{1/4} = 0.7133 \text{ rad/s} \quad \blacksquare$$

By letting $A(\omega) = 30$ dB, the 30-dB frequency is obtained as

$$\omega_2 = (10^{0.1 \times 30} - 1)^{1/4} = 5.622 \text{ rad/s} \quad \blacksquare$$

(c) A frequency ω_i in the analog filter transforms into a frequency Ω_i given by

$$\Omega_i = \frac{2}{T} \tan^{-1} \frac{\omega_i T}{2} \tag{SA.20}$$

Hence the 1- and 30-dB frequencies in the digital filter are obtained as

$$\Omega_1 = 10 \times \tan^{-1} \frac{0.7133}{10} = 0.7121 \text{ rad/s} \quad \blacksquare$$

and

$$\Omega_2 = 10 \times \tan^{-1} \frac{5.622}{10} = 5.122 \text{ rad/s} \quad \blacksquare$$

respectively.

(d) Now if the 3-dB frequency in the digital filter is required to be 1 rad/s, then according to Eq. (SA.20) the 3-dB frequency in the analog filter should be

$$\omega_3 = \frac{2}{T} \tan \frac{\Omega_3 T}{2} = 10 \times \tan \frac{1}{10} = 1.003 \text{ rad/s} \quad \blacksquare$$