# Chapter 3
# The $Z$ Transform




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# Introduction

- The Fourier series and Fourier transform can be used to obtain spectral representations for periodic and nonperiodic *continuous-time signals*, respectively (see Chap. 2).

  Analogous spectral representations can be obtained for discrete-time signals by using the $z$ transform.

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  Analogous spectral representations can be obtained for discrete-time signals by using the $z$ transform.

- The Fourier transform will convert a real continuous-time signal into a function of complex variable $j\omega$.

  Similarly, the $z$ transform will convert a real *discrete-time signal* into a function of complex variable $z$.

# Introduction *Cont'd*

- The $z$ transform, like the Fourier transform, comes along with an inverse transform, namely, the inverse $z$ transform.

  Consequently, a discrete-time signal can be readily recovered from its $z$ transform.

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  Consequently, a discrete-time signal can be readily recovered from its $z$ transform.

- The availability of an inverse makes the $z$ transform very useful for the representation of digital filters and discrete-time systems in general.

# Introduction *Cont'd*

- The most basic representation of discrete-time systems is in terms of difference equations (see Chap. 4) but through the use of the $z$ transform, difference equations can be reduced to algebraic equations which are much easier to handle.

# Objectives

- Definition of $Z$ Transform
- Convergence Properties
- The $Z$ Transform as a Laurent series
- Inverse $Z$ Transform
- Theorems and Properties
- Elementary Functions
- Examples

# The $Z$ Transform

- Consider a bounded discrete-time signal $x(nT)$ that satisfies the conditions

$$
\begin{array}{lll}
\text{(i)} & x(nT) = 0 & \text{for} \quad n < -N_1 \\
\text{(ii)} & |x(nT)| \leq K_1 & \text{for} \quad -N_1 \leq n < N_2 \\
\text{(iii)} & |x(nT)| \leq K_2 r^n & \text{for} \quad n \geq N_2
\end{array}
$$

where $N_1$ and $N_2$ are positive integers and $r$ is a positive constant.

# The $Z$ Transform *Cont'd*

- • •

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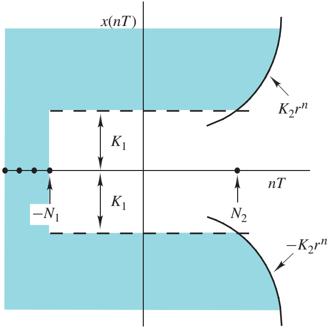


(a)

- The $z$ transform of a discrete-time signal $x(nT)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT)z^{-n}$$

for all $z$ for which $X(z)$ converges.

# The $Z$ Transform *Cont'd*

- Although the $z$ transform of a signal $x(nT)$ is an infinite series, in practice it can be represented in terms of a rational function as

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT)z^{-n}$$

$$= \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} a_i z^{M-i}}{z^N + \sum_{i=1}^{N} b_i z^{N-i}} = H_0 \frac{\prod_{i=1}^{M}(z - z_i)}{\prod_{i=1}^{N}(z - p_i)}$$

where $z_i$ and $p_i$ are the zeros and poles of the $z$ transform and $H_0$ is a multiplier constant.

# The $Z$ Transform *Cont'd*

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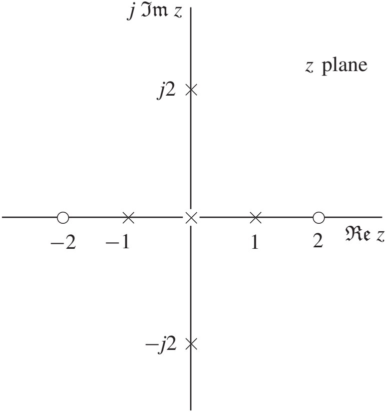
where $z_i$ and $p_i$ are the zeros and poles of the $z$ transform and $H_0$ is a multiplier constant.

- In effect, $z$ transforms can be represented by *zero-pole plots*.

# Example

The following $z$ transform has the zero-pole plot shown.

$$X(z) = \frac{(z^2-4)}{z(z^2-1)(z^2+4)} = \frac{(z-2)(z+2)}{z(z-1)(z+1)(z-j2)(z+j2)}$$



# Theorem 3.1 Absolute Convergence

If

$$\begin{array}{lrl} \text{(i)} & x(nT) = 0 & \text{for} \quad n < -N_1 \\ \text{(ii)} & |x(nT)| \leq K_1 & \text{for} \quad -N_1 \leq n < N_2 \\ \text{(iii)} & |x(nT)| \leq K_2 r^n & \text{for} \quad n \geq N_2 \end{array}$$

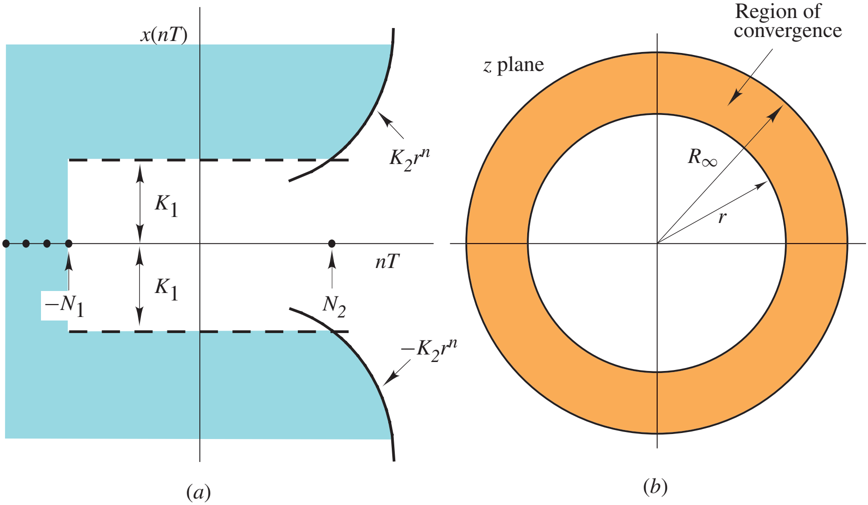
where $N_1$ and $N_2$ are positive constants and $r$ is the smallest positive constant that will satisfy condition (iii), then the $z$ transform of $x(nT)$, i.e.,

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT) z^{-n}$$

exists and converges absolutely if and only if

$$r < |z| < R_\infty \quad \text{with} \quad R_\infty \to \infty$$

# Absolute Convergence *Cont'd*



# Absolute Convergence *Cont'd*

The proofs of the Absolute Convergence Theorem and the theorems that follow can be found in the textbook.

# The $Z$ Transform as a Laurent Series

- The Laurent series of a function $X(z)$ about point $z = a$ assumes the form

$$X(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^{-n}$$

(See Appendix A.)

# The $Z$ Transform as a Laurent Series

- The Laurent series of a function $X(z)$ about point $z = a$ assumes the form

$$X(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^{-n}$$

(See Appendix A.)

- The $z$ transform is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT) z^{-n}$$

If we compare the above two series for $X(z)$, we conclude that the $z$ transform is a Laurent series of $X(z)$ about the origin, i.e., $a = 0$, with

$$a_n = x(nT)$$

# The $Z$ Transform as a Laurent Series *Cont'd*

- Since the $z$ transform is a specific Laurent series, it follows that *it inherits all the properties* of the Laurent series, which are stated in the Laurent theorem as detailed in the slides that follow.

# Laurent Theorem

(a) If $F(z)$ is an analytic and single-valued function on two concentric circles $C_1$ and $C_2$ with center $a$ and in the annulus between them, then it can be represented by the Laurent series

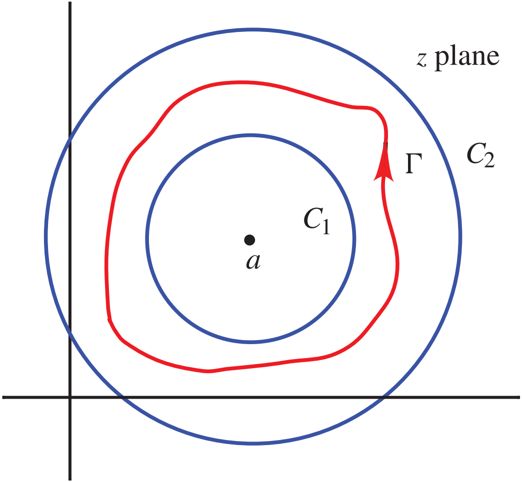
$$F(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^{-n}$$

where

$$a_n = \frac{1}{2\pi j} \oint_{\Gamma} F(z)(z-a)^{n-1}\, dz$$

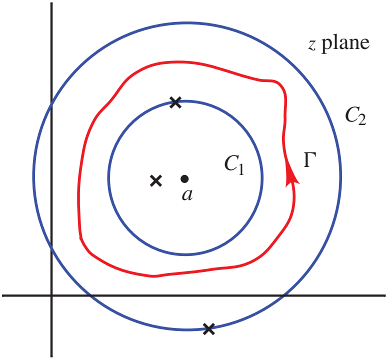
The contour of integration $\Gamma$ is a closed contour in the counterclockwise sense lying in the annulus between circles $C_1$ and $C_2$ and encircling the inner circle.

# Laurent Theorem *Cont'd*



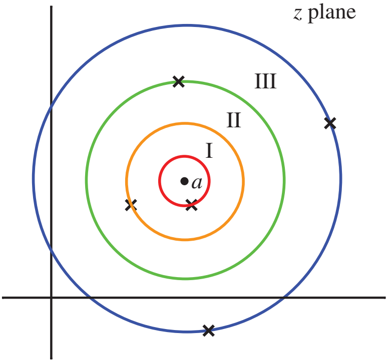

(a)

(b) The Laurent series converges and represents $F(z)$ in the open annulus obtained by continuously increasing the radius of $C_2$ and decreasing the radius of $C_1$ until each of $C_1$ and $C_2$ reaches a point where $F(z)$ is singular.



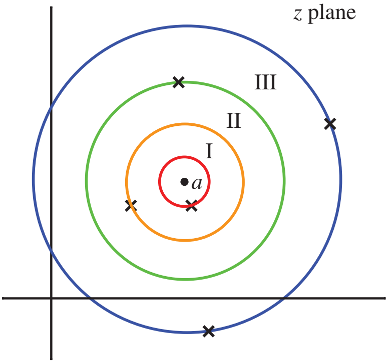

(b)

(c) A function $F(z)$ can have several, possibly many, annuli of convergence about a given point $z = a$ and for each one a Laurent series can be obtained.




(c)

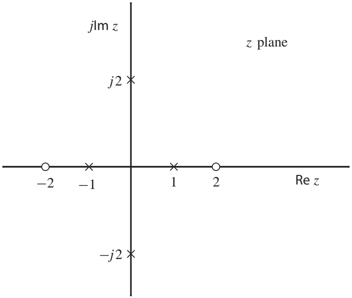
(d) The Laurent series for a given annulus of convergence is unique.



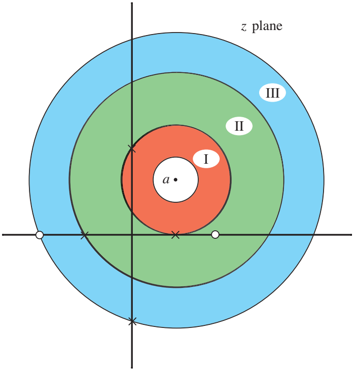

(c)

# Example

The function represented by the zero-pole plot at the left has three unique Laurent series as shown at the right.




(a)



# Inverse $Z$ Transform

- The absolute-convergence theorem states that the $z$ transform, $X(z)$, of a discrete-time signal $x(nT)$ satisfying the conditions

$$
\begin{array}{ll}
\text{(i)} & x(nT) = 0 \quad \text{for} \quad n < -N_1 \\
\text{(ii)} & |x(nT)| \leq K_1 \quad \text{for} \quad -N_1 \leq n < N_2 \\
\text{(iii)} & |x(nT)| \leq K_2 r^n \quad \text{for} \quad n \geq N_2
\end{array}
$$

exists and converges absolutely if and only if

$$r < |z| < R \quad \text{with} \quad R \to \infty$$

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- The Laurent theorem states that a function $X(z)$ has as many distinct Laurent series about the origin as there are annuli of convergence.

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- The Laurent theorem states that a function $X(z)$ has as many distinct Laurent series about the origin as there are annuli of convergence.

- One of these series converges in the outer annulus (i.e., the largest one) which is defined as

$$R_0 < |z| < R \quad \text{with} \quad R \to \infty$$

where $R_0$ is the radius of a circle passing through the most distant pole of $X(z)$ from the origin.

Summarizing:

- From the absolute convergence theorem, the $z$ transform converges in the annulus

$$r < |z| < R \quad \text{with} \quad R \to \infty$$

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- From the Laurent theorem, there is a unique Laurent series of $X(z)$ that converges in the outer annulus of convergence

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Summarizing:

- From the absolute convergence theorem, the $z$ transform converges in the annulus

$$r < |z| < R \quad \text{with} \quad R \to \infty$$

- From the Laurent theorem, there is a unique Laurent series of $X(z)$ that converges in the outer annulus of convergence

$$R_0 < |z| < R \quad \text{with} \quad R \to \infty$$

- Therefore, *the $z$ transform of $x(nT)$ is the unique Laurent series that converges in the outer annulus* and, furthermore, $r = R_0$.

- We conclude that signal $x(nT)$ can be obtained from its $z$ transform $X(z)$ by finding the coefficients of the Laurent series of $X(z)$ that converges in the outer annulus.

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- From the Laurent theorem, we have

$$x(nT) = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1} \, dz$$

where contour $\Gamma$ encloses all the poles of $X(z)z^{n-1}$.

# Inverse $Z$ Transform *Cont'd*

- We conclude that signal $x(nT)$ can be obtained from its $z$ transform $X(z)$ by finding the coefficients of the Laurent series of $X(z)$ that converges in the outer annulus.
- From the Laurent theorem, we have

$$x(nT) = \frac{1}{2\pi j} \oint_\Gamma X(z) z^{n-1}\, dz$$

where contour $\Gamma$ encloses all the poles of $X(z)z^{n-1}$.

- In DSP, this contour integral is said to be the *inverse z transform* of $X(z)$.

# Notation

- Like the Fourier transform and its inverse, the $z$ transform and its inverse are often represented in terms of operator notation as

$$X(z) = \mathcal{Z}x(nT) \quad \text{and} \quad x(nT) = \mathcal{Z}^{-1}X(z)$$

respectively.

# $Z$ Transform Theorems

- The general properties of the $z$ transform can be described in terms of a small number of theorems, as detailed in the slides that follow.

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- In these theorems

$$\mathcal{Z}x(nT) = X(z) \quad \mathcal{Z}x_1(nT) = X_1(z) \quad \mathcal{Z}x_2(nT) = X_2(z)$$

and $a$, $b$, $w$, and $K$ represent constants which may be complex.

# Theorem 3.3 Linearity

- The $z$ transform of a linear combination of discrete-time signals is given by

$$\mathcal{Z}[ax_1(nT) + bx_2(nT)] = aX_1(z) + bX_2(z)$$

# Theorem 3.3 Linearity

- The $z$ transform of a linear combination of discrete-time signals is given by

$$\mathcal{Z}[ax_1(nT) + bx_2(nT)] = aX_1(z) + bX_2(z)$$

- Similarly, the inverse $z$ transform of a linear combination of $z$ transforms is given by

$$\mathcal{Z}^{-1}[aX_1(z) + bX_2(z)] = ax_1(nT) + bx_2(nT)$$

# Theorem 3.4 Time Shifting

- For any positive or negative integer $m$,

$$\mathcal{Z}x(nT + mT) = z^m X(z)$$

In effect, multiplying the $z$ transform of a signal by a negative or positive power of $z$ will cause the signal to be *delayed or advanced* by $mT$ s.

# Theorem 3.5 Complex Scale Change

- For an arbitrary real or complex constant $w$

$$\mathcal{Z}[w^{-n}x(nT)] = X(wz)$$

Evidently, multiplying a discrete-time signal by $w^{-n}$ *is equivalent to replacing $z$ by $wz$* in its $z$ transform.

Similarly, multiplying a discrete-time signal by $v^n$ is *equivalent to replacing $z$ by $z/v$* in its $z$ transform.

# Theorem 3.6 Complex Differentiation

- The $z$ transform of an arbitrary signal $nT_1x(nT)$ is given by

$$\mathcal{Z}[nT_1x(nT)] = -T_1z\frac{dX(z)}{dz}$$

Complex differentiation provides a simple way of obtaining the $z$ transform of a discrete-time signal that can be expressed as a product $nT_1x(nT)$.

# Theorem 3.7 Real Convolution

- The $z$ transform of the real convolution summation of two signals $x_1(kT)$ and $x_2(nT)$ is given by

$$\mathcal{Z} \sum_{k=-\infty}^{\infty} x_1(kT)x_2(nT - kT) = \mathcal{Z} \sum_{k=-\infty}^{\infty} x_1(nT - kT)x_2(kT)$$

$$= X_1(z)X_2(z)$$

The real convolution summation is used frequently for the representation of digital filters and discrete-time systems (see Chap. 4).

# Theorem 3.8 Initial-Value Theorem

- The initial value of a signal $x(nT)$ represented by a $z$ transform of the form

$$X(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} a_i z^{M-i}}{\sum_{i=0}^{N} b_i z^{N-i}}$$

occurs at

$$KT = (N - M)T$$

and its value at $nT = KT$ is given by

$$x(KT) = \lim_{z \to \infty} [z^K X(z)]$$

...

$$X(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} a_i z^{M-i}}{\sum_{i=0}^{N} b_i z^{N-i}}$$

- Corollary: If the degree of the numerator polynomial, $N(z)$, in a $z$ transform is equal to or less than the degree of the denominator polynomial $D(z)$, then we have

$$x(nT) = 0 \quad \text{for} \quad n < 0$$

i.e., the signal is right-sided.

# Theorem 3.9 Final-Value Theorem

- The value of $x(nT)$ as $n \to \infty$ is given by

$$x(\infty) = \lim_{z \to 1} [(z-1)X(z)]$$

The final-value theorem can be used to determine the steady-state response of a discrete-time system.

# Theorem 3.10 Complex Convolution

- If the $z$ transforms of two discrete-time signals $x_1(nT)$ and $x_2(nT)$ are available, then the $z$ transform of their product, $X_3(z)$, can be obtained as

$$X_3(z) = \mathcal{Z}[x_1(nT)x_2(nT)] = \frac{1}{2\pi j}\oint_{\Gamma_1} X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}\,dv$$
$$= \frac{1}{2\pi j}\oint_{\Gamma_2} X_1\left(\frac{z}{v}\right)X_2(v)v^{-1}\,dv$$

where $\Gamma_1$ $(or \Gamma_2)$ is a contour in the common region of convergence of $X_1(v)$ and $X_2(z/v)$ (or $X_1(z/v)$ and $X_2(v)$).

# Theorem 3.10 Complex Convolution *Cont'd*

- The complex convolution theorem can be used to obtain the $z$ transform of a product of discrete-time signals whose $z$ transforms are available.

# Theorem 3.10 Complex Convolution *Cont'd*

- The complex convolution theorem can be used to obtain the $z$ transform of a product of discrete-time signals whose $z$ transforms are available.

- It is also the basis of the window method for the design of nonrecursive digital filters (see Chap. 9).

# Theorem 3.11 Parseval's Discrete-Time Formula

- If $X(z)$ is the $z$ transform of a discrete-time signal $x(nT)$, then

$$\sum_{n=-\infty}^{\infty} |x(nT)|^2 = \frac{1}{\omega_s} \int_0^{\omega_s} |X(e^{j\omega T})|^2 \, d\omega$$

where $\omega_s = 2\pi/T$.

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$$\sum_{n=-\infty}^{\infty} |x(nT)|^2 = \frac{1}{\omega_s} \int_0^{\omega_s} |X(e^{j\omega T})|^2 \, d\omega$$

where $\omega_s = 2\pi/T$.

- Parseval's formula is often used to solve a problem known as *scaling* which is associated with the design of recursive digital filters in hardware form (see Chap. 14).

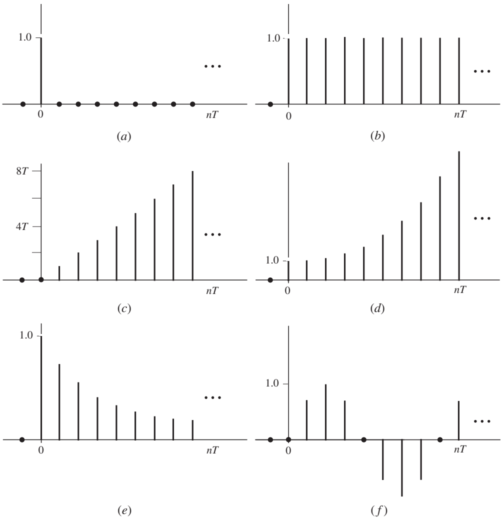
- If $T$ is normalized to 1 s, Parceval's formula simplifies to:

$$\sum_{n=-\infty}^{\infty} |x(nT)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |X(e^{j\omega T})|^2 \, d\omega$$

# Elementary Discrete-Time Signals

| Function | Definition |
|---|---|
| Unit impulse | $\delta(nT) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$ |
| Unit step | $u(nT) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$ |
| Unit ramp | $r(nT) = \begin{cases} nT & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$ |
| Exponential | $u(nT)e^{\alpha nT},\ (\alpha > 0)$ |
| Exponential | $u(nT)e^{\alpha nT},\ (\alpha < 0)$ |
| Sinusoid | $u(nT)\sin \omega nT$ |

# Elementary Discrete-Time Signals *Cont'd*




($a$) Unit impulse, ($b$) unit step, ($c$) unit ramp, ($d$) increasing exponential ($e$) decreasing exponential, ($c$) sinusoid.

# Examples

Find the $z$ transforms of the following signals:

(a) unit-impulse $\delta(nT)$

(b) unit-step $u(nT)$

(c) delayed unit-step $u(nT - kT)K$

(d) signal $u(nT)Kw^n$

(e) exponential signal $u(nT)e^{-\alpha nT}$

(f) unit-ramp $r(nT)$

(g) sinusoidal signal $u(nT)\sin\omega nT$

**Solutions**

(*a*) From the definitions of the $z$ transform and $\delta(nT)$, we have

$$\mathcal{Z}\delta(nT) = \delta(0) + \delta(T)z^{-1} + \delta(2T)z^{-2} + \cdots = 1 \quad \blacksquare$$

# Examples

## Solutions

($a$) From the definitions of the $z$ transform and $\delta(nT)$, we have

$$\mathcal{Z}\delta(nT) = \delta(0) + \delta(T)z^{-1} + \delta(2T)z^{-2} + \cdots = 1 \quad \blacksquare$$

($b$) As in part ($a$)

$$\begin{aligned} \mathcal{Z}u(nT) &= u(0) + u(T)z^{-1} + u(2T)z^{-2} + \cdots \\ &= 1 + z^{-1} + z^{-2} + \cdots = (1 - z^{-1})^{-1} \\ &= \frac{z}{z-1} \quad \blacksquare \end{aligned}$$

# Examples

### Solutions

($a$) From the definitions of the $z$ transform and $\delta(nT)$, we have

$$\mathcal{Z}\delta(nT) = \delta(0) + \delta(T)z^{-1} + \delta(2T)z^{-2} + \cdots = 1 \quad \blacksquare$$

($b$) As in part ($a$)

$$\begin{aligned}\mathcal{Z}u(nT) &= u(0) + u(T)z^{-1} + u(2T)z^{-2} + \cdots \\ &= 1 + z^{-1} + z^{-2} + \cdots = (1 - z^{-1})^{-1} \\ &= \frac{z}{z-1} \quad \blacksquare\end{aligned}$$

($c$) From the time-shifting theorem (Theorem 3.4) and part ($b$), we have

$$\mathcal{Z}[u(nT - kT)K] = Kz^{-k}\mathcal{Z}u(nT) = \frac{Kz^{-(k-1)}}{z-1} \quad \blacksquare$$

(*d*) From the complex-scale-change theorem (Theorem 3.5) and part (*b*), we get

$$\mathcal{Z}[u(nT)Kw^n] = K\mathcal{Z}\left[\left(\frac{1}{w}\right)^{-n} u(nT)\right]$$

$$= K\mathcal{Z}u(nT)\big|_{z\to z/w} = \frac{Kz}{z-w} \quad \blacksquare$$

# Examples *Cont'd*

(*d*) From the complex-scale-change theorem (Theorem 3.5) and part (*b*), we get

$$\mathcal{Z}[u(nT)Kw^n] = K\mathcal{Z}\left[\left(\frac{1}{w}\right)^{-n} u(nT)\right]$$

$$= K\mathcal{Z}u(nT)\big|_{z\to z/w} = \frac{Kz}{z-w} \quad \blacksquare$$

(*e*) By letting $K = 1$ and $w = e^{-\alpha T}$ in part (*d*), we obtain

$$\mathcal{Z}[u(nT)e^{-\alpha nT}] = \frac{z}{z - e^{-\alpha T}} \quad \blacksquare$$

($f$) From the complex-differentiation theorem (Theorem 3.6) and part ($b$), we have

$$\mathcal{Z}r(nT) = \mathcal{Z}[nTu(nT)] = -Tz\frac{d}{dz}[\mathcal{Z}u(nT)]$$

$$= -Tz\frac{d}{dz}\left[\frac{z}{(z-1)}\right] = \frac{Tz}{(z-1)^2} \quad \blacksquare$$

(*g*) From part (*e*), we deduce

$$\begin{aligned}
\mathcal{Z}[u(nT)\sin\omega nT] &= \mathcal{Z}\left[\frac{u(nT)}{2j}\left(e^{j\omega nT} - e^{-j\omega nT}\right)\right] \\
&= \frac{1}{2j}\mathcal{Z}[u(nT)e^{j\omega nT}] - \frac{1}{2j}\mathcal{Z}\left[u(nT)e^{-j\omega nT}\right] \\
&= \frac{1}{2j}\left(\frac{z}{z-e^{j\omega T}} - \frac{z}{z-e^{-j\omega T}}\right) \\
&= \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1} \quad \blacksquare
\end{aligned}$$

# Standard $Z$ transforms

| $x(nT)$ | $X(z)$ |
|---|---|
| $\delta(nT)$ | $1$ |
| $u(nT)$ | $\dfrac{z}{z-1}$ |
| $u(nT-kT)K$ | $\dfrac{Kz^{-(k-1)}}{z-1}$ |
| $u(nT)Kw^n$ | $\dfrac{Kz}{z-w}$ |
| $u(nT-kT)Kw^{n-1}$ | $\dfrac{K(z/w)^{-(k-1)}}{z-w}$ |
| $u(nT)e^{-\alpha nT}$ | $\dfrac{z}{z-e^{-\alpha T}}$ |
| $r(nT)$ | $\dfrac{Tz}{(z-1)^2}$ |

# Standard $Z$ transforms *Cont'd*

| $x(nT)$ | $X(z)$ |
|---|---|
| $r(nT)e^{-\alpha nT}$ | $\dfrac{Te^{-\alpha T}z}{(z-e^{-\alpha T})^2}$ |
| $u(nT)\sin\omega nT$ | $\dfrac{z\sin\omega T}{z^2-2z\cos\omega T+1}$ |
| $u(nT)\cos\omega nT$ | $\dfrac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$ |
| $u(nT)e^{-\alpha nT}\sin\omega nT$ | $\dfrac{ze^{-\alpha T}\sin\omega T}{z^2-2ze^{-\alpha T}\cos\omega T+e^{-2\alpha T}}$ |
| $u(nT)e^{-\alpha nT}\cos\omega nT$ | $\dfrac{z(z-e^{-\alpha T}\cos\omega T)}{z^2-2ze^{-\alpha T}\cos\omega T+e^{-2\alpha T}}$ |

*This slide concludes the presentation.*
*Thank you for your attention.*