

Chapter 4 DISCRETE-TIME SYSTEMS

4.6 Convolution Summation 4.7 Stability

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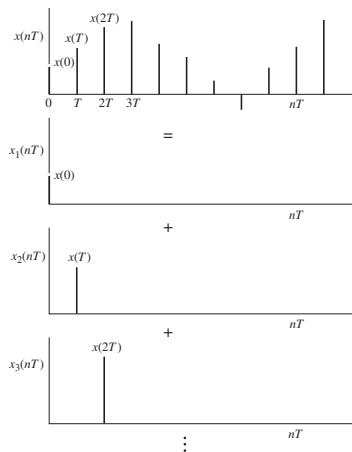
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Introduction

- ▲ The *convolution summation* is of considerable importance for the characterization, representation, analysis, and design of discrete-time systems.
- ▲ This presentation will deal with the derivation, properties, and applications of the convolution summation.

Derivation

- ▲ An arbitrary excitation $x(nT)$ can be considered to be made up of a series of impulses as shown:



- ▲ What has been done graphically can now be done in terms of algebra.

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- ▲ An arbitrary signal can be written as

$$x(nT) = \sum_{k=-\infty}^{\infty} x_k(nT)$$

where

$$x_k(nT) = \begin{cases} x(kT) & \text{for } n = k \\ 0 & \text{otherwise} \end{cases}$$
$$= x(kT)\delta(nT - kT)$$

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- ▲ Hence

$$x(nT) = \sum_{k=-\infty}^{\infty} x(kT)\delta(nT - kT) \quad (\text{A})$$

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$$h(nT) = \mathcal{R}\delta(nT)$$

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$$y(nT) = \mathcal{R}x(nT) = \mathcal{R} \sum_{k=-\infty}^{\infty} x(kT)\delta(nT - kT)$$

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- ▲ Since the system is *linear*,

$$y(nT) = \sum_{k=-\infty}^{\infty} x(kT)\mathcal{R}\delta(nT - kT)$$

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▲ The system is also *time-invariant* and hence we get

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- ▲ The system is also *time-invariant* and hence we get

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- ▲ This relation is known as the *convolution summation*.

Alternative Form

- ▲ If we let $k = n - k'$ in the convolution summation

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then $k' = n - k$.

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and if

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- ▲ Hence the convolution summation can also be expressed as

$$y(nT) = \sum_{k'=-\infty}^{-\infty} x(nT - k'T)h(k'T)$$



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- ▲ Dropping the primes and reversing the order of summation, we obtain the identity

$$y(nT) = \sum_{k=-\infty}^{\infty} x(kT)h(nT - kT) = \sum_{k=-\infty}^{\infty} h(kT)x(nT - kT)$$

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- ▲ If, in addition, $x(nT) = 0$ for $n < 0$, we have

$$y(nT) = \sum_{k=0}^n x(kT)h(nT - kT) = \sum_{k=0}^n h(kT)x(nT - kT)$$

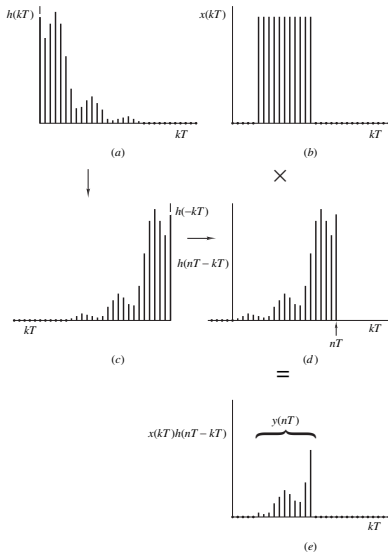
Important Property

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$$y(nT) = \sum_{k=0}^n x(kT)h(nT - kT) = \sum_{k=0}^n h(kT)x(nT - kT)$$

Evidently, *if the impulse response $h(nT)$ of a discrete-time system is known, its response to an arbitrary excitation can be readily determined* by using the convolution summation.

Graphical Representation



Example

Using the convolution summation, find the unit-step response of a discrete-time system characterized by the equation

$$y(nT) = x(nT) + py(nT - T)$$

The system has an impulse response

$$h(nT) = u(nT)p^n$$

and is initially relaxed (i.e., $y(nT) = 0$ for $n < 0$).

Solution The convolution summation gives

$$\begin{aligned}
 y(nT) &= \mathcal{R}u(nT) = \sum_{k=-\infty}^{\infty} u(kT)p^k u(nT - kT) \\
 &= \dots + \overbrace{u(-T)p^{-1}u(nT + T)}^{k=-1} + \overbrace{u(0)p^0u(nT)}^{k=0} + \overbrace{u(T)p^1u(nT - T)}^{k=1} \\
 &\quad + \dots + \overbrace{u(nT)p^n u(0)}^{k=n} + \overbrace{u(nT + T)p^{n+1}u(-T)}^{k=n+1} + \dots
 \end{aligned}$$

For $n < 0$, the unit step assumes a value of zero and hence we get $y(nT) = 0$ since all the terms are zero.

Example *Cont'd*

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$$\begin{aligned}y(nT) &= \mathcal{R}u(nT) = \sum_{k=-\infty}^{\infty} u(kT)p^k u(nT - kT) \\ &= \dots + \overbrace{u(-T)p^{-1}u(nT + T)}^{k=-1} + \overbrace{u(0)p^0u(nT)}^{k=0} + \overbrace{u(T)p^1u(nT - T)}^{k=1} \\ &\quad + \dots + \overbrace{u(nT)p^n u(0)}^{k=n} + \overbrace{u(nT + T)p^{n+1}u(-T)}^{k=n+1} + \dots\end{aligned}$$

For $n \geq 0$, we obtain

$$y(nT) = 1 + p^1 + p^2 + \dots + p^n = 1 + \sum_{n=1}^n p^n = \frac{1 - p^{(n+1)}}{1 - p}$$

since this is a geometric series with a common ratio p .

Example *Cont'd*

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For $n < 0$, $y(nT) = 0$.

For $n \geq 0$,

$$y(nT) = \frac{1 - p^{(n+1)}}{1 - p}$$

Therefore, the response can be expressed in closed form as

$$y(nT) = u(nT) \frac{1 - p^{(n+1)}}{1 - p} \quad \blacksquare$$

Example

$$y(nT) = \mathcal{R}u(nT) = u(nT) \frac{1 - p^{(n+1)}}{1 - p}$$

and is initially relaxed (i.e., $y(nT) = 0$ for $n < 0$).

Find the response produced by the excitation

$$x(nT) = \begin{cases} 1 & \text{for } 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Example *Cont'd*

Solution We observe that

$$x(nT) = \begin{cases} 1 & \text{for } 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases} = u(nT) - u(nT - 5T)$$

and so

$$y(nT) = \mathcal{R}x(nT) = \mathcal{R}u(nT) - \mathcal{R}u(nT - 5T)$$

Since

$$y(nT) = \mathcal{R}u(nT) = u(nT) \frac{1 - p^{(n+1)}}{1 - p}$$

we get

$$y(nT) = u(nT) \frac{1 - p^{(n+1)}}{1 - p} - u(nT - 5T) \frac{1 - p^{(n-4)}}{1 - p} \quad \blacksquare$$

Example

An initially relaxed causal nonrecursive system was tested with an input

$$x(nT) = \begin{cases} 0 & \text{for } n < 0 \\ n & \text{for } n \geq 0 \end{cases}$$

and found to have the response given by the following table:

n	0	1	2	3	4	5	6	7
$y(nT)$	0	1	4	10	20	30	40	50

- (a) Find the impulse response of the system for values of n over the range $0 \leq n \leq 5$.
- (b) Using the result in part (a), find the unit-step response for $0 \leq n \leq 5$.

Example *Cont'd*

Solution Since the system is causal and $x(nT) = 0$ for $n < 0$, the convolution summation assumes the form

$$\begin{aligned}y(nT) = \mathcal{R}x(nT) &= \sum_{k=0}^n x(kT)h(nT - kT) \\ &= x(0)h(nT) + x(T)h(nT - T) + \dots + h(0)x(nT)\end{aligned}$$

Example *Cont'd*

Solution Since the system is causal and $x(nT) = 0$ for $n < 0$, the convolution summation assumes the form

$$\begin{aligned}y(nT) &= \mathcal{R}x(nT) = \sum_{k=0}^n x(kT)h(nT - kT) \\ &= x(0)h(nT) + x(T)h(nT - T) + \dots + h(0)x(nT)\end{aligned}$$

Evaluating $y(nT)$ for $n = 1, 2, \dots$, we get

$$\begin{aligned}y(T) &= x(0)h(T) + x(T)h(0) = 0 \cdot h(T) + 1 \cdot h(0) = 1 \quad \text{or} \quad h(0) = 1 \\ y(2T) &= x(0)h(2T) + x(T)h(T) + x(2T)h(0) \\ &= 0 \cdot h(2T) + 1 \cdot h(T) + 2 \cdot h(0) \\ &= 0 + h(T) + 2 = 4 \quad \text{or} \quad h(T) = 2 \\ y(3T) &= x(0)h(3T) + x(T)h(2T) + x(2T)h(T) + x(3T)h(0) \\ &= 0 \cdot h(3T) + 1 \cdot h(2T) + 2 \cdot h(T) + 3 \cdot h(0) \\ &= h(2T) + 2 \cdot 2 + 3 \cdot 1 = 10 \quad \text{or} \quad h(2T) = 3\end{aligned}$$

Example *Cont'd*

$$y(4T) = x(0)h(4T) + x(T)h(3T) + x(2T)h(2T) + x(3T)h(T) \\ + x(4T)h(0)$$

$$= 0 \cdot h(4T) + 1 \cdot h(3T) + 2 \cdot h(2T) + 3 \cdot h(T) + 4 \cdot h(0)$$

$$= h(3T) + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1 = 20 \quad \text{or} \quad h(3T) = 4$$

$$y(5T) = x(0)h(5T) + x(T)h(4T) + x(2T)h(3T) + x(3T)h(2T) \\ + x(4T)h(T) + x(5T)h(0)$$

$$= 0 \cdot h(5T) + 1 \cdot h(4T) + 2 \cdot h(3T) + 3 \cdot h(2T) + 4 \cdot h(T) + 5 \cdot h(0)$$

$$= 0 + h(4T) + 2 \cdot 4 + 3 \cdot 3 + 4 \cdot 2 + 5 \cdot 1 = 30 \quad \text{or} \quad h(4T) = 0$$

$$y(6T) = x(0)h(6T) + x(T)h(5T) + x(2T)h(4T) + x(3T)h(3T) \\ + x(4T)h(2T) + x(5T)h(T) + x(6T)h(0)$$

$$= 0 \cdot h(6T) + 1 \cdot h(5T) + 2 \cdot h(4T) + 3 \cdot h(3T) + 4 \cdot h(2T)$$

$$+ 5 \cdot h(T) + 6 \cdot h(0)$$

$$= h(5T) + 2 \cdot 0 + 3 \cdot 4 + 4 \cdot 3 + 5 \cdot 2 + 6 \cdot 1$$

$$= 40 \quad \text{or} \quad h(5T) = 0$$

Example *Cont'd*

Summarizing the results so far, we have

$$\begin{aligned} h(0) &= 1 & h(T) &= 2 & h(2T) &= 3 \\ h(3T) &= 4 & h(4T) &= 0 & h(5T) &= 0 \quad \blacksquare \end{aligned}$$

Example *Cont'd*

(b) Using the convolution summation again, we obtain the unit-step response as follows:

$$y(nT) = \mathcal{R}x(nT) = \sum_{k=0}^n u(kT)h(nT - kT) = \sum_{k=0}^n h(nT - kT)$$

Hence

$$y(0) = h(0) = 1$$

$$y(T) = h(T) + h(0) = 2 + 1 = 3$$

$$y(2T) = h(2T) + h(T) + h(0) = 3 + 2 + 1 = 6$$

$$y(3T) = h(3T) + h(2T) + h(T) + h(0) = 10$$

$$y(4T) = h(4T) + h(3T) + h(2T) + h(T) + h(0) = 15$$

$$y(5T) = h(5T) + h(4T) + h(3T) + h(2T) + h(T) + h(0) = 21 \quad \blacksquare$$

Alternative Classification of Discrete-Time Systems

Discrete-time systems can also be classified on the basis of the duration of the impulse response as

- finite-duration impulse response (FIR) systems
- infinite-duration impulse response (IIR) systems

- ▲ If the impulse response of a discrete-time system is of finite duration such that $h(nT) = 0$ for $n > N$, then the convolution summation gives

$$y(nT) = \sum_{k=0}^N h(kT)x(nT - kT)$$

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$$y(nT) = \sum_{k=0}^N h(kT)x(nT - kT)$$

- ▲ This equation is of the same form as the difference equation of a nonrecursive system, i.e.,

$$y(nT) = \sum_{i=0}^N a_i x(nT - iT)$$

with

$$h(0) = a_0, \quad h(T) = a_1, \quad \dots, \quad h(NT) = a_N$$

Thus we conclude that

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Thus we conclude that

- ▲ the impulse response of a nonrecursive system is always of finite duration, and
- ▲ given an impulse response of finite duration, a nonrecursive system can be obtained.

- ▲ An impulse response of infinite duration could be achieved with a nonrecursive system of infinite order or with a recursive system.

Alternative Classification *Cont'd*

- ▲ An impulse response of infinite duration could be achieved with a nonrecursive system of infinite order or with a recursive system.
- ▲ Since infinite-order systems are not feasible, an infinite-duration impulse response can only be achieved with a recursive system.

Alternative Classification *Cont'd*

- ▲ An impulse response of infinite duration could be achieved with a nonrecursive system of infinite order or with a recursive system.
- ▲ Since infinite-order systems are not feasible, an infinite-duration impulse response can only be achieved with a recursive system.
- ▲ To confuse the issue somewhat, it is possible to construct a recursive system that has a finite-duration impulse response!

An FIR Recursive System

- ▲ To illustrate that an FIR system can be represented by a recursive equation, or by a network with feedback, consider an FIR system represented by the equation

$$y(nT) = x(nT) + 3x(nT - T)$$

An FIR Recursive System

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$$y(nT) = x(nT) + 3x(nT - T)$$

- ▲ If we premultiply both sides of the equation by the operator $(1 + 4\mathcal{E}^{-1})$, we get

$$(1 + 4\mathcal{E}^{-1})y(nT) = (1 + 4\mathcal{E}^{-1})[x(nT) + 3x(nT - T)]$$

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- ▲ After simplification, we have

$$\begin{aligned}y(nT) + 4y(nT - T) &= x(nT) + 3x(nT - T) \\ &\quad + 4x(nT - T) + 12x(nT - 2T)\end{aligned}$$

• • •

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- ▲ Thus the FIR system can be represented by the recursive equation

$$y(nT) = x(nT) + 7x(nT - T) + 12x(nT - 2T) - 4y(nT - T)$$

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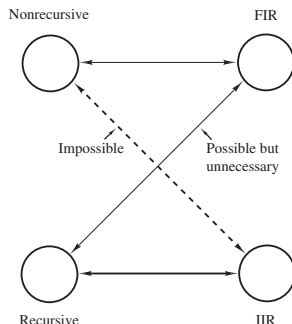
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- ▲ Evidently, the manipulation has actually increased the order of the difference equation and, therefore, no obvious advantage is gained by converting an FIR system into a recursive one.
- ▲ For most practical purposes *nonrecursive systems are FIR systems and recursive systems are IIR systems.*



Note: An IIR system *cannot* be a nonrecursive system and vice-versa. However, a recursive system can be constructed that is also an FIR system but such a system would serve no useful purpose.

- ▲ A discrete-time system is said to be *stable* if and only if any bounded excitation results in a bounded response, i.e.,

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- ▲ Hence

$$|y(nT)| = \left| \sum_{k=-\infty}^{\infty} h(kT)x(nT - kT) \right| \leq \sum_{k=-\infty}^{\infty} |h(kT) \cdot x(nT - kT)|$$

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▲ For example,

$$\begin{aligned} \left| \sum 2 \cdot 3 + (-1) \cdot 4 + 2 \cdot (-2) + (-3) \cdot (-3) \right| &= 7 \\ &\leq \sum |2 \cdot 3| + |(-1) \cdot 4| + |2 \cdot (-2)| + |(-3) \cdot (-3)| = 23 \end{aligned}$$

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▲

If $|x(nT)| \leq P < \infty$ for all n

we have $|y(nT)| \leq P \sum_{k=-\infty}^{\infty} |h(kT)|$

• • •

$$|y(nT)| \leq P \sum_{k=-\infty}^{\infty} |h(kT)|$$

▲ Clearly, if

$$\sum_{k=-\infty}^{\infty} |h(kT)| < \infty \quad (\text{B})$$

then $|y(nT)| < \infty$ for all n

• • •

$$|y(nT)| \leq P \sum_{k=-\infty}^{\infty} |h(kT)|$$

▲ Clearly, if

$$\sum_{k=-\infty}^{\infty} |h(kT)| < \infty \quad (\text{B})$$

then $|y(nT)| < \infty$ for all n

▲ Therefore, Eq. (B) constitutes a *sufficient* condition for stability.

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- ▲ Let us consider a bounded excitation of the form

$$x(nT - kT) = \begin{cases} P & \text{if } h(kT) \geq 0 \\ -P & \text{if } h(kT) < 0 \end{cases}$$

where P is a positive constant.

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- ▲ From the convolution summation, we get

$$|y(nT)| = \left| \sum_{k=-\infty}^{\infty} x(nT - kT)h(kT) \right|$$

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- ▲ Hence

$$|y(nT)| = \sum_{k=-\infty}^{\infty} P \cdot |h(kT)| = P \sum_{k=-\infty}^{\infty} |h(kT)|$$



$$|y(nT)| = P \sum_{k=-\infty}^{\infty} |h(kT)|$$

- ▲ Evidently, at least for the type of signal under consideration, the response will be bounded if and only if

$$\sum_{k=-\infty}^{\infty} |h(kT)| < \infty$$

which implies that this condition is also a *necessary* condition for stability.

- ▲ Summarizing, the condition

$$\sum_{k=-\infty}^{\infty} |h(kT)| < \infty$$

is both a *necessary and sufficient* condition for stability.

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- ▲ *Note:* In nonrecursive systems, the impulse response is both finite and of finite duration and hence the above condition is always satisfied, i.e., *nonrecursive systems are always stable*.

Example

A first-order system is characterized by the equation

$$y(nT) = x(nT) + py(nT - T)$$

has an impulse response

$$h(nT) = u(nT)p^n$$

Check the stability of the system.

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$$y(nT) = x(nT) + py(nT - T)$$

has an impulse response

$$h(nT) = u(nT)p^n$$

Check the stability of the system.

Solution We can write

$$\sum_{k=-\infty}^{\infty} |h(kT)| = 1 + |p| + \dots + |p^k| + \dots$$

Example

A first-order system is characterized by the equation

$$y(nT) = x(nT) + py(nT - T)$$

has an impulse response

$$h(nT) = u(nT)p^n$$

Check the stability of the system.

Solution We can write

$$\sum_{k=-\infty}^{\infty} |h(kT)| = 1 + |p| + \dots + |p^k| + \dots$$

This is a geometric series and has a sum

$$\sum_{k=-\infty}^{\infty} |h(kT)| = \lim_{n \rightarrow \infty} \frac{1 - |p|^{(n+1)}}{1 - |p|}$$

Example *Cont'd*

If $p > 1$,

$$\sum_{k=-\infty}^{\infty} |h(kT)| = \lim_{n \rightarrow \infty} \frac{1 - |p|^{(n+1)}}{1 - |p|} \rightarrow \infty$$

Example *Cont'd*

If $p > 1$,

$$\sum_{k=-\infty}^{\infty} |h(kT)| = \lim_{n \rightarrow \infty} \frac{1 - |p|^{(n+1)}}{1 - |p|} \rightarrow \infty$$

and if $p = 1$,

$$\sum_{k=-\infty}^{\infty} |h(kT)| = 1 + 1 + 1 + \dots = \infty$$

Example *Cont'd*

If $p > 1$,

$$\sum_{k=-\infty}^{\infty} |h(kT)| = \lim_{n \rightarrow \infty} \frac{1 - |p|^{(n+1)}}{1 - |p|} \rightarrow \infty$$

and if $p = 1$,

$$\sum_{k=-\infty}^{\infty} |h(kT)| = 1 + 1 + 1 + \dots = \infty$$

On the other hand, if $p < 1$,

$$\sum_{k=-\infty}^{\infty} |h(kT)| = \lim_{n \rightarrow \infty} \frac{1 - |p|^{(n+1)}}{1 - |p|} \rightarrow \frac{1}{1 - |p|} = K < \infty$$

where K is a positive constant. Therefore, the system is stable if and only if

$$|p| < 1 \quad \blacksquare$$

Example

A discrete-time system has an impulse response

$$h(nT) = u(nT)e^{0.1nT} \sin \frac{n\pi}{6}$$

Check the stability of the system.

Example

A discrete-time system has an impulse response

$$h(nT) = u(nT)e^{0.1nT} \sin \frac{n\pi}{6}$$

Check the stability of the system.

Solution We can write

$$\begin{aligned} \sum_{k=0}^{\infty} |h(nT)| &= \sum_{k=0}^{\infty} \left| u(kT)e^{0.1kT} \sin \frac{k\pi}{6} \right| \\ &= \sum_{k=3,9,15,\dots}^{\infty} |e^{0.1kT}| + \sum_{k \neq 3,9,15,\dots}^{\infty} \left| e^{0.1kT} \sin \frac{k\pi}{6} \right| \rightarrow \infty \end{aligned}$$

Therefore, the system is *unstable*. ■

*This slide concludes the presentation.
Thank you for your attention.*