Chapter 5
THE APPLICATION OF THE Z TRANSFORM

5.4 Time-Domain Analysis

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Introduction

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- ★ Although the induction method is rather intuitive, it runs into serious difficulties when the system order is increase to two or higher.
- ★ The state-space approach, on the other hand, yields solutions in the form of infinite summations rather than in terms of closed-form solutions.
- ★ The z transform approach overcomes these difficulties and it is, therefore, the preferred approach.

Time-Domain Analysis

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- The inverse z transform can be obtained by using any one of the standard inversion techniques described in Chap. 3.

Example

A discrete-time system is characterized by the transfer function

$$H(z) = \frac{z^2 - z + 1}{(z - p_1)(z - p_2)}$$

where

$$p_1,\ p_2 = \tfrac{1}{2} \pm j\tfrac{1}{2} = \frac{1}{\sqrt{2}} e^{\pm j\pi/4}$$

Find the unit-step response.

Solution The response of the system is given by

$$y(nT) = \mathcal{Z}^{-1}[H(z)X(z)]$$

The z transform of the input is given by

$$X(z) = \mathcal{Z}u(nT) = \frac{z}{z-1}$$

Expanding $H(z)X(z)/z$ into partial fractions gives

$$H(z)X(z) = \frac{R_0 z}{z-1} + \frac{R_1 z}{(z-p_1)} + \frac{R_2 z}{(z-p_2)}$$

where $R_0 = 2$, $R_1 = \frac{1}{\sqrt{2}}e^{-j5\pi/4}$, and $R_2 = R_1^* = \frac{1}{\sqrt{2}}e^{j5\pi/4}$.

Example *Cont'd*

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From the table of standard z transforms, we have

$$y(nT) = 2u(nT) + u(nT)\left(\frac{1}{\sqrt{2}}e^{j\pi/4}\right)^n \cdot \frac{1}{\sqrt{2}}e^{-j5\pi/4}$$

Example *Cont'd*

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$$y(nT) = 2u(nT) + u(nT)\left(\frac{1}{\sqrt{2}}e^{j\pi/4}\right)^n \cdot \frac{1}{\sqrt{2}}e^{-j5\pi/4}$$
$$+u(nT)\left(\frac{1}{\sqrt{2}}e^{-j\pi/4}\right)^n \cdot \frac{1}{\sqrt{2}}e^{j5\pi/4}$$

Example *Cont'd*

- ...

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$$+u(nT)\left(\frac{1}{\sqrt{2}}e^{-j\pi/4}\right)^n \cdot \frac{1}{\sqrt{2}}e^{j5\pi/4}$$
$$= 2u(nT) + \frac{1}{\left(\sqrt{2}\right)^{n+1}}u(nT)(e^{j(n-5)\pi/4} + e^{-j(n-5)\pi/4})$$

Example *Cont'd*

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$$
\begin{aligned}
y(nT) &= 2u(nT) + u(nT)\left(\frac{1}{\sqrt{2}}e^{j\pi/4}\right)^n \cdot \frac{1}{\sqrt{2}}e^{-j5\pi/4} \\
&\quad + u(nT)\left(\frac{1}{\sqrt{2}}e^{-j\pi/4}\right)^n \cdot \frac{1}{\sqrt{2}}e^{j5\pi/4} \\
&= 2u(nT) + \frac{1}{\left(\sqrt{2}\right)^{n+1}}u(nT)(e^{j(n-5)\pi/4} + e^{-j(n-5)\pi/4}) \\
&= 2u(nT) + \frac{1}{\left(\sqrt{2}\right)^{n-1}}u(nT)\cos\left[(n-5)\frac{\pi}{4}\right] \quad \blacksquare
\end{aligned}
$$

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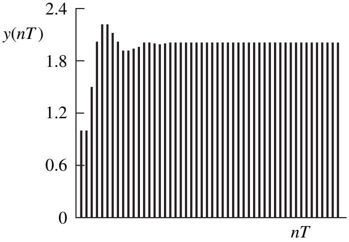

Unit-step response

Example

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where

$$p_1,\ p_2 = \tfrac{1}{2} \pm j\tfrac{1}{2} = \frac{1}{\sqrt{2}} e^{\pm j\pi/4}$$

Find the response of the system to a sinusoidal excitation

$$x(nT) = u(nT) \sin \omega nT$$

Solution The response of the system is given by

$$y(nT) = \mathcal{Z}^{-1}[H(z)X(z)]$$

The z transform of the input is given by

$$\begin{aligned} X(z) &= \mathcal{Z}[u(nT)\sin\omega nT] = \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1} \\ &= \frac{z\sin\omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})} \end{aligned}$$

and hence

$$\begin{aligned} H(z)X(z)z^{n-1} &= \frac{z^2 - z + 1}{(z - p_1)(z - p_2)} \cdot \frac{z\sin\omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})} \cdot z^{n-1} \\ &= \frac{z^2 - z + 1}{(z - p_1)(z - p_2)} \cdot \frac{\sin\omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})} \cdot z^{n} \end{aligned}$$

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Since the system is causal $y(nT) = 0$ for $n < 0$ and hence the general inversion formula gives

$$y(nT) = u(nT)[R_1 + R_2 + R_3 + R_4]$$

where R_1, R_2, R_3, and R_4 are the residues of $H(z)X(z)z^{n-1}$ at poles p_1, p_2, $p_3 = e^{j\omega T}$, and $p_4 = e^{j\omega T}$, respectively.

The residues can be evaluated as shown in the next three slides.

Example *Cont'd*

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$$H(z)X(z)z^{n-1} = \frac{z^2 - z + 1}{(z - p_1)(z - p_2)} \cdot \frac{\sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})} \cdot z^n$$

$$\begin{aligned}
R_1 &= \lim_{z=p_1} \left[\frac{z^2 - z + 1}{(z - p_2)} \cdot \frac{\sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})} \cdot z^n \right] \\
&= \left[\frac{p_1^2 - p_1 + 1}{(p_1 - p_2)} \cdot \frac{\sin \omega T}{(p_1 - e^{j\omega T})(p_1 - e^{-j\omega T})} \cdot p_1^n \right] \\
&= \rho(\omega) e^{j\psi(\omega)} \left(\frac{1}{\sqrt{2}} \right)^n e^{jn\pi/4} = \rho(\omega) \left(\frac{1}{\sqrt{2}} \right)^n e^{j[n\pi/4 + \psi(\omega)]}
\end{aligned}$$

where $$\rho(\omega) = \left| \frac{p_1^2 - p_1 + 1}{(p_1 - p_2)} \cdot \frac{\sin \omega T}{(p_1 - e^{j\omega T})(p_1 - e^{-j\omega T})} \right|$$

$$\psi(\omega) = \arg \left[\frac{p_1^2 - p_1 + 1}{(p_1 - p_2)} \cdot \frac{\sin \omega T}{(p_1 - e^{j\omega T})(p_1 - e^{-j\omega T})} \right]$$

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$$H(z)X(z)z^{n-1} = \frac{z^2 - z + 1}{(z-p_1)(z-p_2)} \cdot \frac{\sin \omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})} \cdot z^n$$

$$\begin{aligned}
R_2 &= R_1^* = \rho(\omega)\left(\frac{1}{\sqrt{2}}\right)^n e^{-j[n\pi/4+\psi(\omega)]} \\
R_3 &= \lim_{z=e^{j\omega T}} [H(z)X(z)z^{n-1}] \\
&= H(e^{j\omega T}) \cdot \frac{\sin \omega T}{(e^{j\omega T} - e^{-j\omega T})} \cdot e^{jn\omega T} \\
&= \frac{1}{2j} H(e^{j\omega T}) e^{jn\omega T} \\
R_4 &= R_3^* = -\frac{1}{2j} H(e^{-j\omega T}) e^{-jn\omega T}
\end{aligned}$$

Example *Cont'd*

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$$R_1 = \rho(\omega)\left(\frac{1}{\sqrt{2}}\right)^n e^{j[n\pi/4+\psi(\omega)]}, \quad R_2 = \rho(\omega)\left(\frac{1}{\sqrt{2}}\right)^n e^{-j[n\pi/4+\psi(\omega)]}$$

$$R_3 = \frac{1}{2j}H(e^{j\omega T})e^{jn\omega T}, \quad R_4 = -\frac{1}{2j}H(e^{-j\omega T})e^{-jn\omega T}$$

If we now let

$$H(e^{j\omega T}) = M(\omega)e^{j\theta(\omega)} \quad \text{then} \quad H(e^{-j\omega T}) = M(\omega)e^{-j\theta(\omega)}$$

and so

$$y(nT) = u(nT)\Big[\rho(\omega)\left(\frac{1}{\sqrt{2}}\right)^n e^{j[n\pi/4+\psi(\omega)]} + \rho(\omega)\left(\frac{1}{\sqrt{2}}\right)^n e^{-j[n\pi/4+\psi(\omega)]}$$
$$+\frac{1}{2j}M(\omega)e^{j\theta(\omega)}e^{jn\omega T} - \frac{1}{2j}M(\omega)e^{-j\theta(\omega)}e^{-jn\omega T}\Big]$$

Example *Cont'd*

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$$
\begin{aligned}
y(nT) &= u(nT)\Big[\rho(\omega)\left(\tfrac{1}{\sqrt{2}}\right)^n e^{j[n\pi/4+\psi(\omega)]} + \rho(\omega)\left(\tfrac{1}{\sqrt{2}}\right)^n e^{-j[n\pi/4+\psi(\omega)]} \\
&\quad + \frac{1}{2j}M(\omega)e^{j\theta(\omega)}e^{jn\omega T} - \frac{1}{2j}M(\omega)e^{-j\theta(\omega)}e^{-jn\omega T}\Big] \\
&= u(nT)\Big\{\rho(\omega)\left(\tfrac{1}{\sqrt{2}}\right)^n \Big[e^{j[n\pi/4+\psi(\omega)]} + e^{-j[n\pi/4+\psi(\omega)]}\Big] \\
&\quad + M(\omega)\frac{1}{2j}\Big[e^{j[n\omega T+\theta(\omega)]} - e^{-j[n\omega T+\theta(\omega)]}\Big]\Big\} \\
&= u(nT)\Big\{\rho(\omega)\left(\tfrac{1}{\sqrt{2}}\right)^{n-2}\cos[\tfrac{n\pi}{4}+\psi(\omega)] \\
&\quad + M(\omega)\sin[n\omega T+\theta(\omega)]\Big\} \quad \blacksquare
\end{aligned}
$$

The cosine term is a *transient* component that tends to zero as $n \to \infty$ whereas the sine term represents the *steady-state* response of the system.

This slide concludes the presentation.
Thank you for your attention.