

# Chapter 9

## DESIGN OF NONRECURSIVE (FIR) FILTERS

### 9.3 Design Using the Fourier Series

### 9.4 Use of Window Functions

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# Introduction

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- By using the Fourier series in conjunction with a class of functions known as *window functions*.

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- By using a multivariable optimization method known as the *weighted-Chebyshev method*.
- This presentation deals with the Fourier-series method.
- The weighted-Chebyshev method is described in another presentation. (See Chap. 15.)

# Fourier Series Method

- A fundamental property of digital filters in general is that they have a periodic frequency response with period equal to the sampling frequency  $\omega_s$ , i.e.,

$$H(e^{j(\omega+k\omega_s)T}) = H(e^{j\omega T})$$

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$$H(e^{j(\omega+k\omega_s)T}) = H(e^{j\omega T})$$

- Therefore, an arbitrary desired frequency response,  $H(e^{j\omega T})$ , can be represented by a Fourier series as

$$H(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} h(nT)e^{-j\omega nT}$$

where

$$h(nT) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(e^{j\omega T}) e^{j\omega nT} d\omega$$

are the Fourier series coefficients.

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$$H(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} h(nT)e^{-j\omega nT}$$

■ If we let  $e^{j\omega T} = z$ , we get

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- This is the transfer function of a nonrecursive filter with impulse response  $h(nT)$ .

## Fourier Series Method *Cont'd*

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  - The nonrecursive filter obtained is of infinite length.
  - The filter is noncausal because the impulse response is nonzero for negative time.

- A finite filter length can be achieved by truncating the impulse response such that

$$h(nT) = 0 \quad \text{for } |n| > M$$

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- On the other hand, a causal filter can be obtained by delaying the impulse response by a period  $MT$  seconds or by  $M$  sampling periods.

- Since delaying the impulse response by  $M$  sampling periods amounts to multiplying the transfer function by  $z^{-M}$ , the transfer function of the causal filter assumes the form

$$H'(z) = z^{-M} \sum_{n=-M}^M h(nT)z^{-n}$$

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- The frequency response of the causal filter is obtained by letting  $z = e^{j\omega T}$  in the transfer function, i.e.,

$$H'(e^{j\omega T}) = e^{-jM\omega T} \sum_{n=-M}^M h(nT)e^{-jn\omega T}$$

and since  $|e^{-jM\omega T}| = 1$ , delaying the impulse response by  $M$  sampling periods does not change the amplitude response.

## Example

Design a lowpass filter with a desired frequency response

$$H(e^{j\omega T}) \approx \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{for } \omega_c < |\omega| \leq \omega_s/2 \end{cases}$$

where  $\omega_s$  is the sampling frequency.

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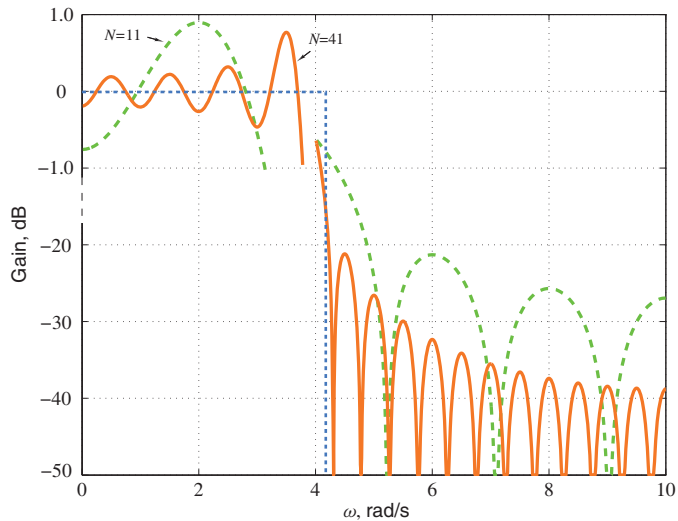
where  $\omega_s$  is the sampling frequency.

**Solution** The Fourier series gives the impulse response of the noncausal filter as

$$\begin{aligned} h(nT) &= \frac{1}{\omega_s} \int_{-\omega_c}^{\omega_c} e^{j\omega nT} d\omega = \frac{1}{n\pi} \frac{(e^{j\omega_c nT} - e^{-j\omega_c nT})}{2j} = \frac{1}{n\pi} \sin \omega_c nT \\ &= \begin{cases} \frac{2\omega_c}{\omega_s} & \text{for } n = 0 \\ \frac{1}{n\pi} \sin \omega_c nT & \text{otherwise} \end{cases} \end{aligned} \quad (\text{A})$$

Truncating and delaying the impulse response by  $M$  sampling periods immediately yields the required design.

# Example *Cont'd*



## Fourier Series Method *Cont'd*

- The amplitude response of the filter exhibits oscillations in the passband as well as the stopband, which are known as *Gibbs' oscillations*.

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They are caused by the *truncation* of the Fourier series.

- As the filter length is increased, the frequency of the oscillations increases but the amplitude stays constant.
- In other words, we do not seem to be able to reduce the passband and stopband errors below a certain limit by increasing the filter length.

Therefore, the filters that can be designed with the Fourier series method are of little practical usefulness.

## Fourier Series Method *Cont'd*

- The standard technique for the reduction of Gibbs' oscillations is to truncate the infinite-duration impulse response,  $h(nT)$ , through the use of a discrete-time *window function*  $w(nT)$ .

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- If we let

$$h_w(nT) = w(nT)h(nT)$$

then from the complex-convolution theorem (see Chap. 3) a *modified* transfer function can be obtained as

$$\begin{aligned} H_w(z) &= \mathcal{Z}[w(nT)h(nT)] \\ &= \frac{1}{2\pi j} \oint_{\Gamma} H(v)W\left(\frac{z}{v}\right)v^{-1}dv \end{aligned}$$

where  $H(z)$  is the original transfer function and  $W(z)$  is the  $z$  transform of the window function.

- Evaluating  $H_w(z)$  on the unit circle  $z = e^{j\omega T}$  gives the frequency response of the modified filter as

$$H_w(e^{j\omega T}) = \frac{T}{2\pi} \int_0^{2\pi/T} H(e^{j\varpi T}) W(e^{j(\omega-\varpi)T}) d\varpi \quad (\text{B})$$

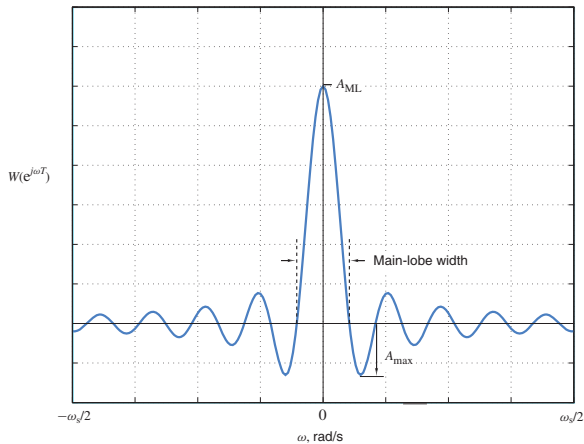
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- $W(e^{j\omega T})$  is the *frequency spectrum* of the window function and the integral at the right-hand side is a *convolution integral*.

# Window Functions

- Frequency spectrum of a typical window:



- Windows are characterized by their *main-lobe width*,  $B_{ML}$ , which is the bandwidth between the first negative and the first positive zero crossings, and by their *ripple ratio*,  $r$ , which is defined as

$$r = 100 \frac{A_{\max}}{A_{ML}} \% \quad \text{or} \quad R = 20 \log \frac{A_{\max}}{A_{ML}} \text{ dB}$$

where  $A_{\max}$  and  $A_{ML}$  are the maximum side-lobe and main-lobe amplitudes, respectively.

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- The main-lobe width and ripple ratio should be as low as possible, i.e., the spectral energy of the window should be concentrated as far as possible in the main lobe and the energy in the side lobes should be as low as possible.

- The way by which Gibbs' oscillations can be controlled by using a window is illustrated in the next few slides which are based on Eq. (B), i.e.,

$$H_w(e^{j\omega T}) = \frac{T}{2\pi} \int_0^{2\pi/T} H(e^{j\varpi T}) W(e^{j(\omega-\varpi)T}) d\varpi \quad (\text{B})$$

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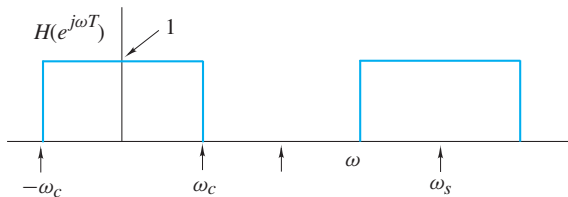
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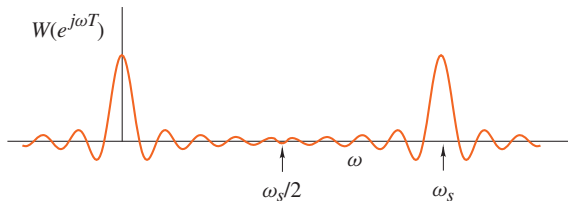
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- Let us consider the application of a generic window in the design of a lowpass filter, assuming that the area under the curve in the frequency spectrum of the window is  $2\pi/T$ .
- A window area of  $2\pi/T$  will ensure that the passband gain of the modified filter will be of order unity if the passband gain of the original filter is of order unity because the area of the window would cancel the scaling constant  $T/2\pi$  in Eq. (B).

# Window Functions *Cont'd*

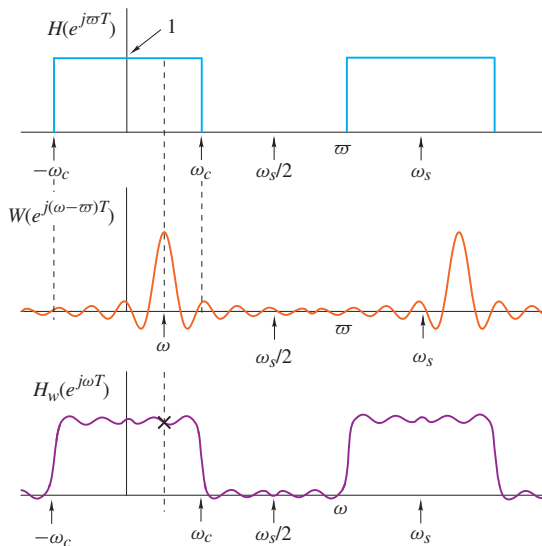


Frequency response of ideal lowpass filter

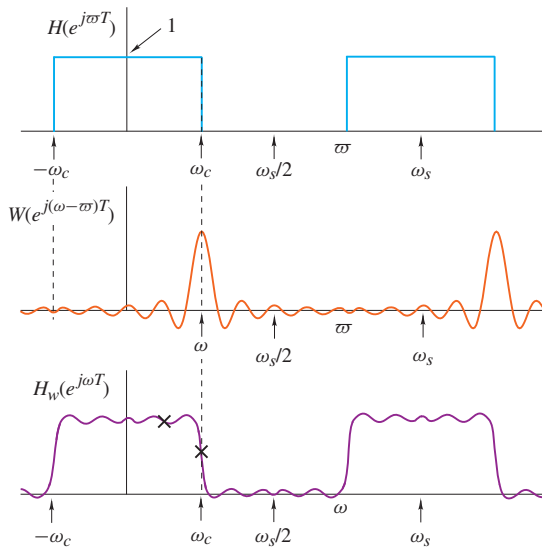


Frequency spectrum of window

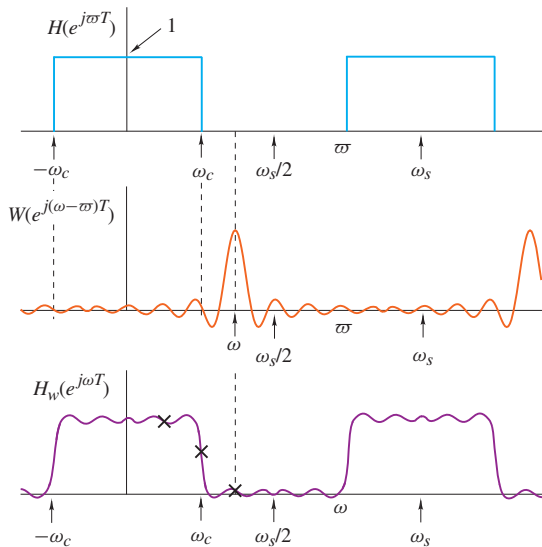
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- the amplitudes of the passband and stopband ripples depend on the ripple ratio of the window.

- A variety of window functions have been described in the literature in recent years:
  - Rectangular
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  - Rectangular
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  - Blackman
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  - Kaiser
  - Ultraspherical window
- The first four are fixed windows whereas the last three are *adjustable* and are often referred to *parametric*.

# Rectangular window

- The rectangular window of length  $N$  is given by

$$w_R(nT) = \begin{cases} 1 & \text{for } -[(N-1)T]/2 \leq nT \leq [(N-1)T]/2 \\ 0 & \text{otherwise} \end{cases}$$

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- Multiplying a discrete-time signal that is defined over the range  $-\infty < nT < \infty$  by the rectangular window will simply cause the signal to become zero over the ranges

$$-\infty < nT < -[(N-1)T]/2 \quad \text{and} \quad [(N-1)T]/2 < nT < \infty$$

i.e., the window will *truncate* the discrete-time signal.

## Rectangular window *Cont'd*

- The most important property of a window function is its *spectrum*.

This can be determined by finding the window's  $z$  transform and then evaluating it on the unit circle of the  $z$  plane, i.e., by letting  $z = e^{j\omega T}$ .

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- From the definition of the  $z$  transform

$$W_R(z) = \sum_{n=-(N-1)/2}^{(N-1)/2} z^{-n}$$

- The above sum is a *geometric series* and, therefore, a closed-form expression can be readily obtained as

$$W_R(z) = \frac{z^{(N-1)/2} - z^{-(N+1)/2}}{1 - z^{-1}} = \frac{z^{N/2} - z^{-N/2}}{z^{1/2} - z^{-1/2}}$$

• • •

$$W_R(z) = \frac{z^{N/2} - z^{-N/2}}{z^{1/2} - z^{-1/2}}$$

■ If we now let  $z = e^{j\omega T}$ , we get

$$W_R(e^{j\omega T}) = \frac{e^{j\omega NT/2} - e^{-j\omega NT/2}}{e^{j\omega T/2} - e^{-j\omega T/2}} = \frac{\sin(\omega NT/2)}{\sin(\omega T/2)}$$

# von Hann and Hamming Windows

- The von Hann and Hamming Windows are given by

$$w_H(nT) = \begin{cases} \alpha + (1 - \alpha) \cos \frac{2\pi n}{N-1} & \text{for } |n| \leq \frac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

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where  $\alpha = 0.5$  in the von Hann window and  $\alpha = 0.54$  in the Hamming window.

- The small increase in the value of  $\alpha$  from 0.5 to 0.54 in the Hamming window has a beneficial effect: It reduces the ripple ratio by about 50 percent.

- By applying the z transform, the spectrums of the von Hann and Hamming Windows can be derived from the spectrum of the rectangular window as

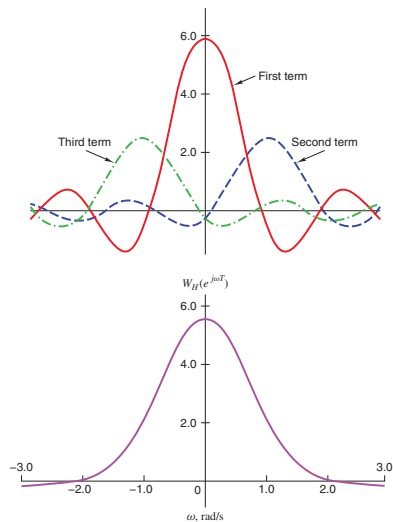
$$\begin{aligned}W_H(e^{j\omega T}) &= \alpha W_R(e^{j\omega T}) + \frac{1-\alpha}{2} W_R(e^{j[\omega T - 2\pi/(N-1)]}) \\ &\quad + \frac{1-\alpha}{2} W_R(e^{j[\omega T + 2\pi/(N-1)]}) \\ &= \frac{\alpha \sin(\omega NT/2)}{\sin(\omega T/2)} + \frac{1-\alpha}{2} \cdot \frac{\sin[\omega NT/2 - N\pi/(N-1)]}{\sin[\omega T/2 - \pi/(N-1)]} \\ &\quad + \frac{1-\alpha}{2} \cdot \frac{\sin[\omega NT/2 + N\pi/(N-1)]}{\sin[\omega T/2 + \pi/(N-1)]}\end{aligned}$$

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$$\begin{aligned}W_H(e^{j\omega T}) &= \alpha W_R(e^{j\omega T}) + \frac{1-\alpha}{2} W_R(e^{j[\omega T - 2\pi/(N-1)]}) \\ &\quad + \frac{1-\alpha}{2} W_R(e^{j[\omega T + 2\pi/(N-1)]}) \\ &= \frac{\alpha \sin(\omega NT/2)}{\sin(\omega T/2)} + \frac{1-\alpha}{2} \cdot \frac{\sin[\omega NT/2 - N\pi/(N-1)]}{\sin[\omega T/2 - \pi/(N-1)]} \\ &\quad + \frac{1-\alpha}{2} \cdot \frac{\sin[\omega NT/2 + N\pi/(N-1)]}{\sin[\omega T/2 + \pi/(N-1)]}\end{aligned}$$

- The second and third terms introduce ripples which reduce the oscillations in the spectrum of the rectangular window.

# von Hann and Hamming Windows *Cont'd*



# Blackman Window

- The Blackman window is as follows:

$$w_B(nT) = \begin{cases} 0.42 + 0.5 \cos \frac{2\pi n}{N-1} + 0.08 \cos \frac{4\pi n}{N-1} & \text{for } |n| \leq \frac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

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- The additional cosine term helps to reduce the ripple ratio relative to that of the Hamming window.

# Blackman Window

- Using the same method as before, the spectrum of the Blackman window can be obtained as

$$\begin{aligned}W_B(e^{j\omega T}) = & 0.42 \frac{\sin(\omega NT/2)}{\sin(\omega T/2)} + 0.25 \frac{\sin[\omega NT/2 - N\pi/(N-1)]}{\sin[\omega T/2 - \pi/(N-1)]} \\ & + 0.25 \frac{\sin[\omega NT/2 + N\pi/(N-1)]}{\sin[\omega T/2 + \pi/(N-1)]} \\ & + 0.04 \frac{\sin[\omega NT/2 - N\pi/(N-1)]}{\sin[\omega T/2 - \pi/(N-1)]} \\ & + 0.04 \frac{\sin[\omega NT/2 + N\pi/(N-1)]}{\sin[\omega T/2 + \pi/(N-1)]}\end{aligned}$$

# Comparison of Fixed Windows

Window	Main-lobe width	Ripple ratio, %		
		N=11	N=21	N=101
Rectangular	$\frac{2\omega_s}{N}$	22.34	21.89	21.70
von Hann	$\frac{4\omega_s}{N}$	2.62	2.67	2.67
Hamming	$\frac{4\omega_s}{N}$	1.47	0.93	0.74
Blackman	$\frac{6\omega_s}{N}$	0.08	0.12	0.12

## Example

Design a lowpass filter with a desired frequency response

$$H(e^{j\omega T}) \approx \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{for } \omega_c < |\omega| \leq \omega_s/2 \end{cases}$$

where  $\omega_s$  is the sampling frequency using the von Han, Hamming, and Blackman windows.

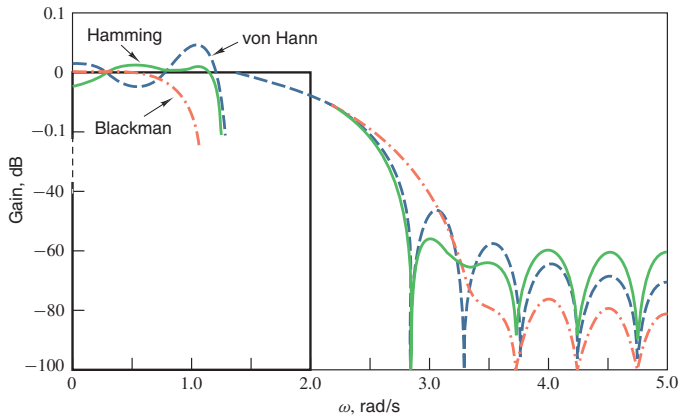
**Solution** The impulse response is  $h(nT) = \frac{1}{n\pi} \sin \omega_c nT$

If we multiply  $h(nT)$  by the appropriate window function, we obtain

$$H'_w(z) = z^{-(N-1)/2} \sum_{n=0}^{(N-1)/2} \frac{a'_n}{2} (z^n + z^{-n})$$

where  $a'_0 = w(0)h(0)$  and  $a'_n = 2w(nT)h(nT)$

# Example *Cont'd*



# Dolph-Chebyshev Window

- The Dolph-Chebyshev window is given by

$$w_{DC}(nT) = \frac{1}{N} \left[ \frac{1}{r} + 2 \sum_{i=1}^{(N-1)/2} T_{N-1} \left( x_0 \cos \frac{i\pi}{N} \right) \cos \frac{2n\pi i}{N} \right]$$

for  $n = 0, 1, 2, \dots, (N-1)/2$  where  $r$  is the required ripple ratio as a fraction,

$$x_0 = \cosh \left( \frac{1}{N-1} \cosh^{-1} \frac{1}{r} \right)$$

and

$$T_k(x) = \begin{cases} \cos(k \cos^{-1} x) & \text{for } |x| \leq 1 \\ \cosh(k \cosh^{-1} x) & \text{for } |x| > 1 \end{cases}$$

is the *kth-order Chebyshev polynomial*.

The Dolph-Chebyshev window has three properties of interest:

- The ripple ratio can be *adjusted* to suit the application.

## Dolph-Chebyshev Window *Cont'd*

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- The ripple ratio can be *adjusted* to suit the application.
- With  $N$  fixed, the main-lobe width is the *smallest* that can be achieved for a given ripple ratio.
- All the side lobes have the same amplitude.

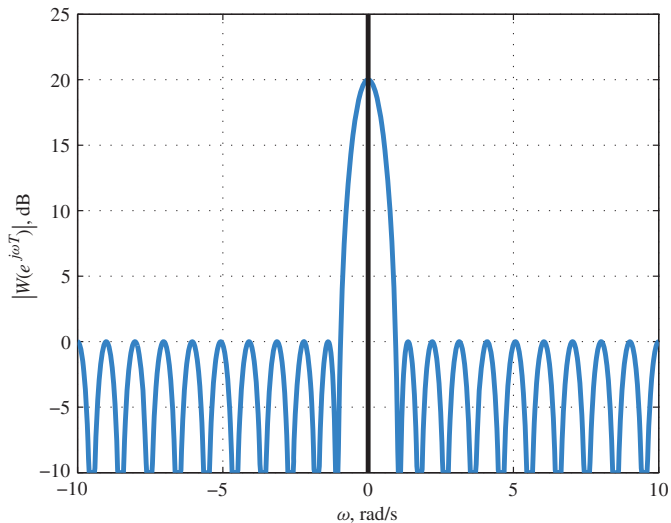
In other words, the amplitude spectrum is *equiripple*.

- A close-form formula for the frequency spectrum of the Dolph-Chebyshev Window is difficult to obtain.

However, a formula in the form of a finite summation can be easily obtained by evaluating its  $z$  transform on the unit circle of the  $z$  plane, as follows:

$$W_{DC}(e^{j\omega T}) = w_{DC}(0) + 2 \sum_{n=1}^M w_{DC}(nT) \cos \omega nT$$

# Dolph-Chebyshev Window *Cont'd*



# Kaiser Window

- The Kaiser window function is given by

$$w_K(nT) = \begin{cases} \frac{I_0(\beta)}{I_0(\alpha)} & \text{for } |n| \leq (N-1)/2 \\ 0 & \text{otherwise} \end{cases}$$

where  $\beta = \alpha \sqrt{1 - \left(\frac{2n}{N-1}\right)^2}$ ,  $I_0(x) = 1 + \sum_{k=1}^{\infty} \left[ \frac{1}{k!} \left(\frac{x}{2}\right)^k \right]^2$

$\alpha$  is an independent parameter, and  $I_0(x)$  is a *zeroth-order modified Bessel function* of the first kind.

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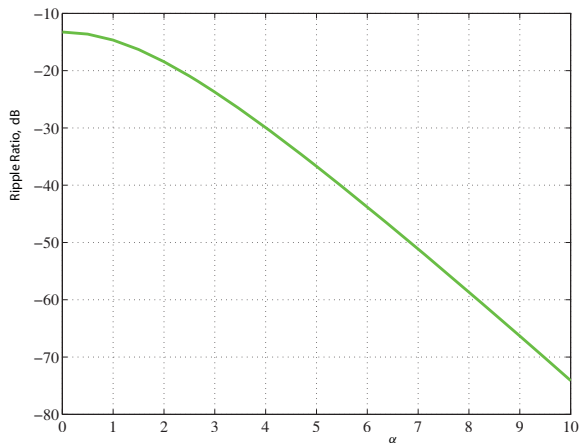
where  $\beta = \alpha \sqrt{1 - \left(\frac{2n}{N-1}\right)^2}$ ,  $I_0(x) = 1 + \sum_{k=1}^{\infty} \left[ \frac{1}{k!} \left(\frac{x}{2}\right)^k \right]^2$

$\alpha$  is an independent parameter, and  $I_0(x)$  is a *zeroth-order modified Bessel function* of the first kind.

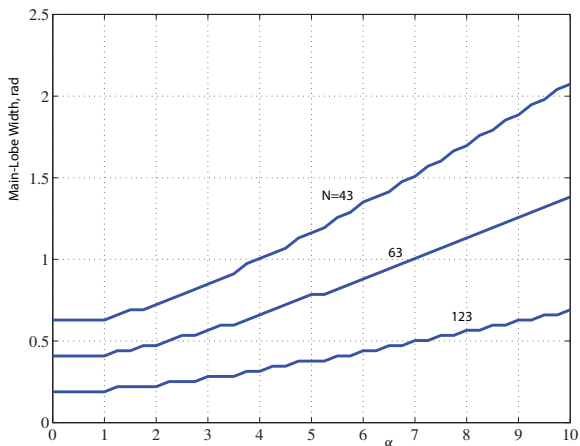
- As for the Dolph-Chebyshev window, the frequency spectrum is given by

$$W_K(e^{j\omega T}) = w_K(0) + 2 \sum_{n=1}^{(N-1)/2} w_K(nT) \cos \omega nT$$

Ripple ratio versus  $\alpha$ :



Main-lobe width versus  $\alpha$ :



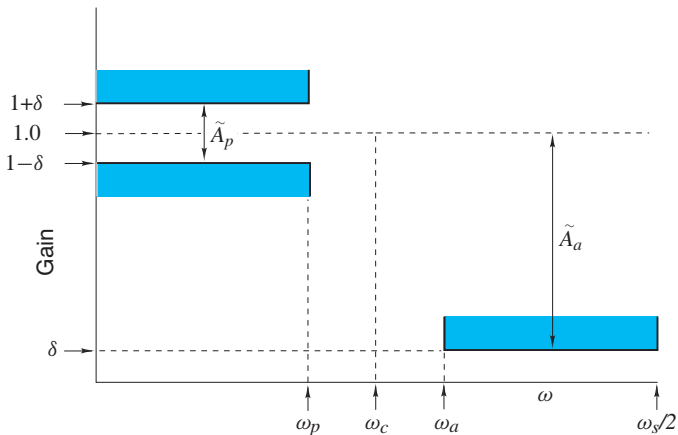
- As will be shown, a filter-design method is available, also *proposed by Kaiser*, that can be used to design nonrecursive filters that would satisfy arbitrary *prescribed specifications*.

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- As will be shown, a filter-design method is available, also *proposed by Kaiser*, that can be used to design nonrecursive filters that would satisfy arbitrary *prescribed specifications*.
- In this method, the value of  $\alpha$  and the filter length,  $N$ , that would yield the required design are estimated using certain empirical formulas.
- Kaiser's method can be used to design lowpass (LP), highpass (HP), bandpass (BP), and bandstop (BS) filters.

# Design of Nonrecursive Lowpass Filters

- A nonrecursive LP filter that would satisfy the specifications shown can be designed by using the procedure described in the next three slides:



## Design of Nonrecursive Lowpass Filters *Cont'd*

1. Determine the impulse response  $h(nT)$  using the Fourier series assuming an idealized frequency response

$$H(e^{j\omega T}) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{for } \omega_c < |\omega| \leq \omega_s/2 \end{cases} \quad \text{where } \omega_c = \frac{1}{2}(\omega_p + \omega_a)$$

This is given by

$$\begin{aligned} h(nT) &= \frac{1}{\omega_s} \int_{-\omega_c}^{\omega_c} e^{j\omega nT} d\omega = \frac{1}{n\pi} \frac{(e^{j\omega_c nT} - e^{-j\omega_c nT})}{2j} = \frac{1}{n\pi} \sin \omega_c nT \\ &= \begin{cases} \frac{2\omega_c}{\omega_s} & \text{for } n = 0 \\ \frac{1}{n\pi} \sin \omega_c nT & \text{otherwise} \end{cases} \end{aligned}$$

- Choose  $\delta$  such that the actual passband ripple,  $A_p$ , is equal to or less than specified passband ripple,  $\tilde{A}_p$ , and the actual minimum stopband attenuation,  $A_a$ , is equal or greater than the specified minimum stopband attenuation,  $\tilde{A}_a$ .

A suitable value is

$$\delta = \min(\tilde{\delta}_p, \tilde{\delta}_a)$$

where 
$$\tilde{\delta}_p = \frac{10^{0.05\tilde{A}_p} - 1}{10^{0.05\tilde{A}_p} + 1} \quad \text{and} \quad \tilde{\delta}_a = 10^{-0.05\tilde{A}_a}$$

## Design of Nonrecursive Lowpass Filters *Cont'd*

3. With the required  $\delta$  defined, the actual stopband attenuation  $A_a$  can be calculated as

$$A_a = -20 \log \delta$$

## Design of Nonrecursive Lowpass Filters *Cont'd*

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$$A_a = -20 \log \delta$$

4. Choose parameter  $\alpha$  as

$$\alpha = \begin{cases} 0 & \text{for } A_a \leq 21 \\ 0.5842(A_a - 21)^{0.4} + 0.07886(A_a - 21) & \text{for } 21 < A_a \leq 50 \\ 0.1102(A_a - 8.7) & \text{for } A_a > 50 \end{cases}$$

5. Choose parameter  $D$  as

$$D = \begin{cases} 0.9222 & \text{for } A_a \leq 21 \\ \frac{A_a - 7.95}{14.36} & \text{for } A_a > 21 \end{cases}$$

Then select the lowest odd value of  $N$  that would satisfy the inequality

$$N \geq \frac{\omega_s D}{B_t} + 1 \quad \text{where } B_t = \omega_a - \omega_p$$

6. Form  $w_K(nT)$  using the following equations:

$$w_K(nT) = \begin{cases} \frac{I_0(\beta)}{I_0(\alpha)} & \text{for } |n| \leq (N-1)/2 \\ 0 & \text{otherwise} \end{cases}$$

where

$$\beta = \alpha \sqrt{1 - \left(\frac{2n}{N-1}\right)^2}, \quad I_0(x) = 1 + \sum_{k=1}^{\infty} \left[ \frac{1}{k!} \left(\frac{x}{2}\right)^k \right]^2$$

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7. Form

$$H'_w(z) = z^{-(N-1)/2} H_w(z) \quad \text{where } H_w(z) = \mathcal{Z}[w_K(nT)h(nT)]$$

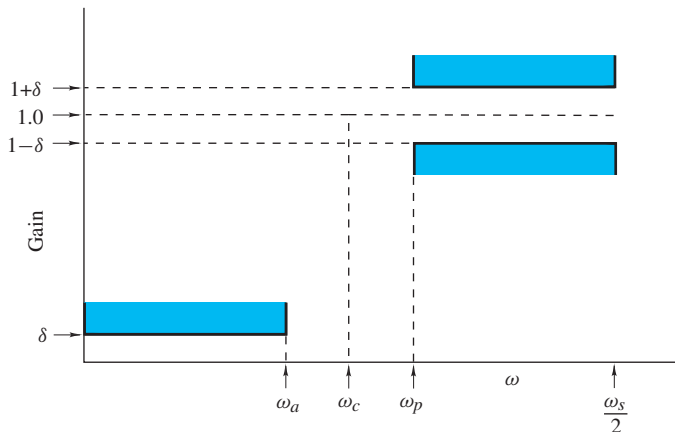
# Design of Nonrecursive Highpass Filters

- The method presented can also be applied for the design of nonrecursive HP filters.

# Design of Nonrecursive Highpass Filters

- The method presented can also be applied for the design of nonrecursive HP filters.
- Consider the case where a HP filter is required that would satisfy the following specifications:
  - Passband ripple  $\leq \tilde{A}_p$
  - Minimum stopband attenuation  $\geq \tilde{A}_a$
  - Passband edge  $\omega_p$
  - Stopband edge  $\omega_a$
  - Sampling frequency  $\omega_s$

# Design of Nonrecursive Highpass Filters *Cont'd*



## Design of Nonrecursive Highpass Filters *Cont'd*

- The transition width and idealized frequency response in Step 1 of the design procedure can be taken as

$$B_t = \omega_p - \omega_a \quad \text{and} \quad H(e^{j\omega T}) = \begin{cases} 1 & \text{for } -\omega_s/2 \leq \omega \leq -\omega_c \\ 1 & \text{for } \omega_c \leq \omega \leq \omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

respectively.

## Design of Nonrecursive Highpass Filters *Cont'd*

- The transition width and idealized frequency response in Step 1 of the design procedure can be taken as

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respectively.

- Applying the Fourier series, the impulse response of the ideal filter can be obtained as

$$h(nT) = \begin{cases} 1 - \frac{2\omega_c}{\omega_s} & \text{for } n = 0 \\ -\frac{1}{n\pi} \sin \omega_c nT & \text{otherwise} \end{cases}$$

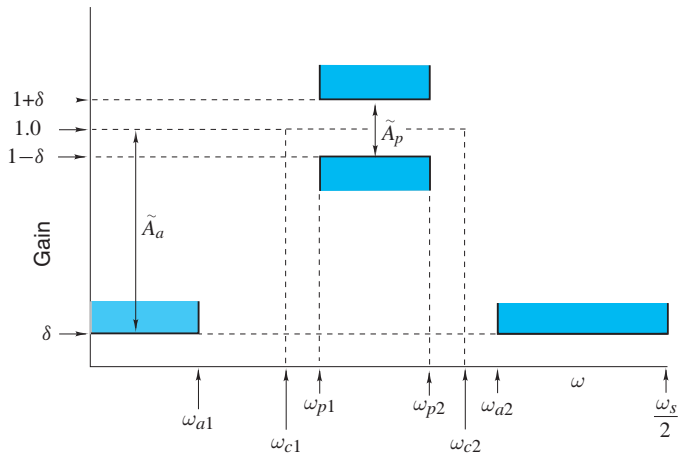
# Design of Nonrecursive Bandpass Filters

- The design method presented can be easily extended to nonrecursive BP and BS filters.

# Design of Nonrecursive Bandpass Filters

- The design method presented can be easily extended to nonrecursive BP and BS filters.
- Consider the design of a BP filter that would satisfy the following specifications:
  - Passband ripple  $\leq \tilde{A}_p$
  - Minimum stopband attenuation  $\geq \tilde{A}_a$
  - Lower stopband edge  $\omega_{a1}$
  - Lower passband edge  $\omega_{p1}$
  - Upper passband edge  $\omega_{p2}$
  - Upper stopband edge  $\omega_{a2}$
  - Sampling frequency  $\omega_s$

# Design of Nonrecursive Bandpass Filters *Cont'd*



## Design of Nonrecursive Bandpass Filters *Cont'd*

The only differences in the design of BP filters are to

- use the more critical of the two transition widths for the design, and

## Design of Nonrecursive Bandpass Filters *Cont'd*

The only differences in the design of BP filters are to

- use the more critical of the two transition widths for the design, and
- then use the idealized frequency response of a BP filter for the determination of the initial impulse response.

- The more critical transition width is

$$B_t = \min[(\omega_{p1} - \omega_{a1}), (\omega_{a2} - \omega_{p2})]$$

and hence the idealized frequency response for a BP filter is deduced as

$$H(e^{j\omega T}) = \begin{cases} 1 & \text{for } -\omega_{c2} \leq \omega \leq -\omega_{c1} \\ 1 & \text{for } \omega_{c1} \leq \omega \leq \omega_{c2} \\ 0 & \text{otherwise} \end{cases}$$

where  $\omega_{c1} = \omega_{p1} - \frac{B_t}{2}$ ,  $\omega_{c2} = \omega_{p2} + \frac{B_t}{2}$

• • •

$$H(e^{j\omega T}) = \begin{cases} 1 & \text{for } -\omega_{c2} \leq \omega \leq -\omega_{c1} \\ 1 & \text{for } \omega_{c1} \leq \omega \leq \omega_{c2} \\ 0 & \text{otherwise} \end{cases}$$

- Applying the Fourier series to the idealized frequency response, we get

$$h(nT) = \begin{cases} \frac{2}{\omega_s}(\omega_{c2} - \omega_{c1}) & \text{for } n = 0 \\ \frac{1}{n\pi}(\sin \omega_{c2} nT - \sin \omega_{c1} nT) & \text{otherwise} \end{cases}$$

## Design of Nonrecursive Bandstop Filters *Cont'd*

The design of BS filters is similar to that of BP filters.

- We use the more critical of the two transition widths for the design (as before),

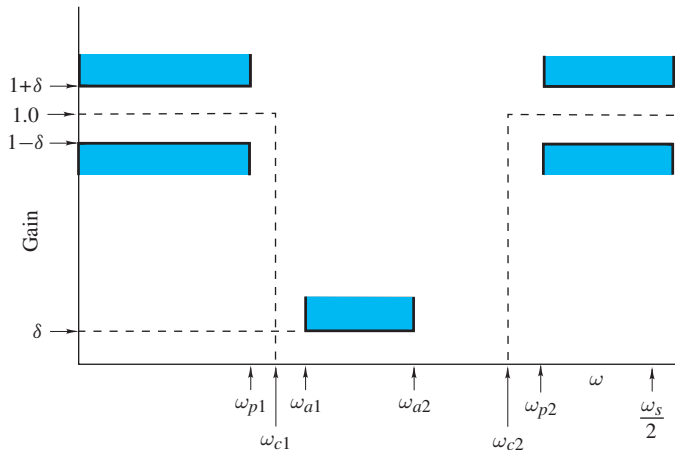
## Design of Nonrecursive Bandstop Filters *Cont'd*

The design of BS filters is similar to that of BP filters.

- We use the more critical of the two transition widths for the design (as before),
- then we use the idealized frequency response of a BS filter for the determination of the initial impulse response.

- Consider the design of a BS filter that would satisfy the following specifications:
  - Passband ripple  $\leq \tilde{A}_p$
  - Minimum stopband attenuation  $\geq \tilde{A}_a$
  - Lower passband edge  $\omega_{p1}$
  - Lower stopband edge  $\omega_{a1}$
  - Upper stopband edge  $\omega_{a2}$
  - Upper passband edge  $\omega_{p2}$
  - Sampling frequency  $\omega_s$

# Design of Nonrecursive Bandstop Filters *Cont'd*



- From the specifications, we can write

$$B_t = \min [(\omega_{a1} - \omega_{p1}), (\omega_{p2} - \omega_{a2})]$$

$$H(e^{j\omega T}) = \begin{cases} 1 & \text{for } 0 \leq |\omega| \leq \omega_{c1} \\ 0 & \text{for } \omega_{c1} < |\omega| < \omega_{c2} \\ 1 & \text{for } \omega_{c2} \leq |\omega| \leq \omega_s/2 \end{cases}$$

where

$$\omega_{c1} = \omega_{p1} + \frac{B_t}{2}, \quad \omega_{c2} = \omega_{p2} - \frac{B_t}{2}$$

• • •

$$H(e^{j\omega T}) = \begin{cases} 1 & \text{for } 0 \leq |\omega| \leq \omega_{c1} \\ 0 & \text{for } \omega_{c1} < |\omega| < \omega_{c2} \\ 1 & \text{for } \omega_{c2} \leq |\omega| \leq \omega_s/2 \end{cases}$$

- Using the Fourier series, the impulse response of the ideal bandstop filter can be obtained as

$$h(nT) = \begin{cases} 1 + \frac{2(\omega_{c1} - \omega_{c2})}{\omega_s} & \text{for } n = 0 \\ \frac{1}{n\pi}(\sin \omega_{c1} nT - \sin \omega_{c2} nT) & \text{otherwise} \end{cases}$$

## Example

Design a nonrecursive BP filter satisfying the following specifications:

- Minimum attenuation for  $0 \leq \omega \leq 200$ : 45 dB
- Maximum passband ripple for  $400 < \omega < 600$ : 0.2 dB
- Minimum attenuation for  $700 \leq \omega \leq 1000$ : 45 dB
- Sampling frequency: 2000 rad/s

## Example *Cont'd*

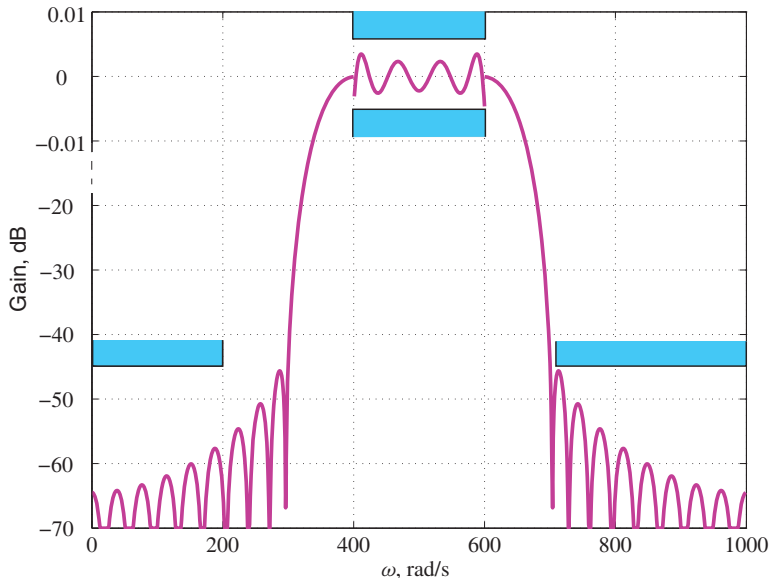
The idealized impulse response is obtained as

$$h(nT) = \begin{cases} \frac{2}{\omega_s}(\omega_{c2} - \omega_{c1}) & \text{for } n = 0 \\ \frac{1}{n\pi}(\sin \omega_{c2}nT - \sin \omega_{c1}nT) & \text{otherwise} \end{cases}$$

The application of the design procedure described will give

$$\alpha = 3.9754, \quad D = 2.580, \quad \text{and} \quad N = 53$$

# Example *Cont'd*

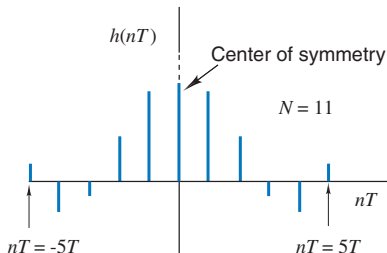


# Design of Even-Length Filters

- The application of the Fourier series method for the design of an LP, HP, BP, or BS filter will yield a noncausal filter with an impulse response defined at

$$nT = \dots, -(N-1)T/2, \dots -T, 0, T, \dots, (N-1)T/2$$

- The impulse response turns out to be symmetrical about  $nT = 0$  and, furthermore, the length of the filter turns out to be *odd*.



## Design of Even-Length Filters *Cont'd*

- We then proceed to obtain a causal filter by delaying the impulse response by  $(N - 1)/2$  sampling periods.
- For a filter of length 11, we would delay the impulse response by

$$\frac{N - 1}{2} = 5 \text{ samples}$$

## Design of Even-Length Filters *Cont'd*

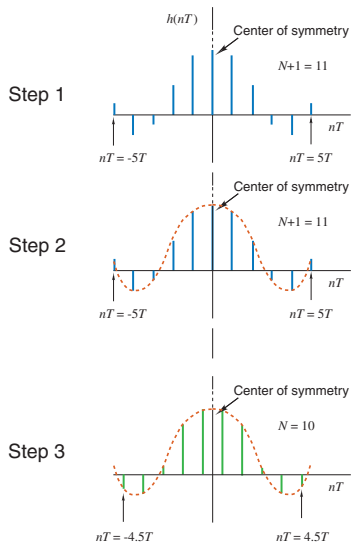
- The Fourier series method as described *cannot* be used to design filters of even length.
- However, through the use of a simple algebraic technique as detailed in the next three slides, even-length filters can be easily designed.

## Design of Even-Length Filters *Cont'd*

The steps involved are as follows:

1. Assuming that  $N$  is the required even filter length, design a filter of length  $N + 1$  using the Fourier series method.
2. Convert the impulse response,  $h(nT)$ , of the filter into a continuous-time function  $h(t)$  by letting  $nT = t$ .
3. Resample  $h(t)$  at  $t = \pm(n - 0.5)T$  for  $n = 1, 2, \dots, N/2$ .
4. In the case of a real-time application, delay the impulse response by  $(N - 1)T/2$  s to get a causal filter.

# Design of Even-Length Filters *Cont'd*



## Design of Even-Length Filters *Cont'd*

- To get a causal filter, we need to delay the impulse response by

$$\frac{N-1}{2} \quad \text{e.g., 4.5, if } N = 10$$

- Delaying the impulse response by  $4T$  is simple: we just use 4 unit delays.
- Delaying the impulse response by  $0.5T$  corresponds to half a sampling period and a special device would be required, which would increase the cost of the implementation.
- For this reason, even-order nonrecursive filters are usually avoided for real-time applications.

# Use of Ultraspherical Window Function

- A more recent method for the design of nonrecursive filters that parallels the method described is one based on the *ultraspherical* window function proposed by Bergen and Antoniou (see References of Chap. 9).

# Use of Ultraspherical Window Function

- A more recent method for the design of nonrecursive filters that parallels the method described is one based on the *ultraspherical* window function proposed by Bergen and Antoniou (see References of Chap. 9).
- For certain specifications, this new approach tends to give more efficient designs, i.e., the minimum filter length that will achieve the required specifications is somewhat lower.

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- Arbitrary specifications can be achieved by using a method proposed by Kaiser.
- Easy to apply.
- The design entails a relatively insignificant amount of computation.

## Disadvantages of Fourier-Series Method

- Designs are suboptimal, i.e., the filter order needed to satisfy a given set of prescribed specifications is not the lowest, i.e., other methods are available that yield a lower filter order (e.g., the weighted-Chebyshev method, see Chap. 15).

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- In a hardware implementation, higher filter order means more unit delays, adders, and multipliers, which implies a more expensive design.

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- In a hardware implementation, higher filter order means more unit delays, adders, and multipliers, which implies a more expensive design.
- In a software implementation, higher filter order means more computations per sample, which implies a less efficient design.

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- D-Filter can be downloaded from

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*This slide concludes the presentation.  
Thank you for your attention.*