

Chapter 9

DESIGN OF NONRECURSIVE (FIR) FILTERS

9.5 Design Based on Numerical-Analysis Formulas

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November 20, 2008

Introduction

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- ▶ Such mathematical operations can be performed by using many classical numerical-analysis formulas.
- ▶ Formulas of this type can be readily derived from the Taylor series.
- ▶ This presentation will show that numerical-analysis formulas can be used to design nonrecursive filters that can be used to perform interpolation, differentiation, and integration.

Interpolation Formulas

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- ▶ The value of $x(t)$ at $t = nT + pT$, where $0 \leq p < 1$, is given by the *forward* Gregory-Newton interpolation formula as

$$\begin{aligned}x(nT + pT) &= (1 + \Delta)^p x(nT) \\ &= \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \dots \right] x(nT)\end{aligned}$$

where

$$\Delta x(nT) = x(nT + T) - x(nT)$$

is commonly referred to as the *forward* difference.

- ▶ Similarly, the *backward* Gregory-Newton interpolation formula gives

$$\begin{aligned}x(nT + pT) &= (1 - \nabla)^{-p}x(nT) \\ &= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \dots \right] x(nT)\end{aligned}$$

where

$$\nabla x(nT) = x(nT) - x(nT - T)$$

is known as the *backward difference*.

Interpolation Formulas *Cont'd*

- ▶ Another interpolation formula known as the *Stirling formula* assumes the form

$$\begin{aligned}x(nT + pT) &= \left[1 + \frac{p^2}{2!}\delta^2 + \frac{p^2(p^2 - 1)}{4!}\delta^4 + \dots \right] x(nT) \\ &+ \frac{p}{2} [\delta x(nT - \frac{1}{2}T) + \delta x(nT + \frac{1}{2}T)] \\ &+ \frac{p(p^2 - 1)}{2(3!)} [\delta^3 x(nT - \frac{1}{2}T) + \delta^3 x(nT + \frac{1}{2}T)] \\ &+ \frac{p(p^2 - 1)(p^2 - 2^2)}{2(5!)} [\delta^5 x(nT - \frac{1}{2}T) + \delta^5 x(nT + \frac{1}{2}T)] \\ &+ \dots\end{aligned}$$

where $\delta x(nT + \frac{1}{2}T) = x(nT + T) - x(nT)$

is known as the *central difference*.

- ▶ The forward, backward, and central differences are linear operators.

Hence higher-order differences can be readily obtained, e.g.,

$$\begin{aligned}\delta^3 x(nT + \tfrac{1}{2}T) &= \delta^2 [\delta x(nT + \tfrac{1}{2}T)] = \delta^2 [x(nT + T) - x(nT)] \\ &= \delta [\delta x(nT + T) - \delta x(nT)] \\ &= \delta \left\{ x(nT + \tfrac{3}{2}T) - x(nT + \tfrac{1}{2}T) \right. \\ &\quad \left. - [x(nT + \tfrac{1}{2}T) - x(nT - \tfrac{1}{2}T)] \right\} \\ &= \delta x(nT + \tfrac{3}{2}T) - 2\delta x(nT + \tfrac{1}{2}T) + \delta x(nT - \tfrac{1}{2}T) \\ &= [x(nT + 2T) - x(nT + T)] - 2[x(nT + T) - x(nT)] \\ &\quad + [x(nT) - x(nT - T)] \\ &= x(nT + 2T) - 3x(nT + T) + 3x(nT) - x(nT - T)\end{aligned}$$

Differentiation Formulas

- ▶ The first derivative of $x(t)$ with respect to time at instant $t = nT + pT$ can be expressed as

$$\begin{aligned}\left. \frac{dx(t)}{dt} \right|_{t=nT+pT} &= \frac{dx(nT + pT)}{dp} \times \frac{dp}{dt} \\ &= \frac{1}{T} \frac{dx(nT + pT)}{dp}\end{aligned}$$

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- ▶ By differentiating each of the interpolation formulas considered with respect to p , corresponding differentiation formulas can be obtained.

Integration Formulas

- ▶ Integration formulas can be derived by writing

$$\int_{nT}^{t_2} x(t) dt = T \int_0^{p_2} x(nT + pT) dp$$

where

$$nT < t_2 \leq nT + T$$

and

$$t_2 = nT + Tp_2 \quad \text{or} \quad p_2 = \frac{t_2 - nT}{T}$$

with $0 < p_2 \leq 1$.

Nonrecursive Filters as Interpolators, Differentiators, or Integrators

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Nonrecursive Filters as Interpolators, Differentiators, or Integrators

- ▶ Nonrecursive filters that can perform *interpolation*, *differentiation*, or *integration* can be obtained by expressing one of the available numerical formulas for these operations in the form of a difference equation.
- ▶ Let $x(nT)$ and $y(nT)$ be the input and output of a nonrecursive filter and assume that $y(nT)$ is equal to the required function of $x(t)$, i.e.,

$$y(nT) = f[x(t)] \Big|_{t=nT+pT}$$

Nonrecursive Filters as Interpolators, Differentiators, ...

- ▶ For interpolation, differentiation, or integration, we would have

$$y(nT) = x(t) \Big|_{t=nT+pT}$$

$$y(nT) = \frac{dx(t)}{dt} \Big|_{t=nT+pT}$$

or

$$y(nT) = \int_{nT}^{nT+pT} x(t) dt$$

as appropriate.

- ▶ By choosing an appropriate numerical formula for the operation of interest and then eliminating all the difference operators using their definitions, we can obtain a difference equation of the form

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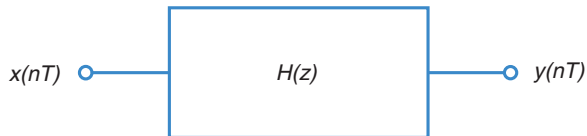
- ▶ Now by applying the z transform, a transfer function

$$H(z) = \sum_{n=-K}^M h(nT) z^{-n}$$

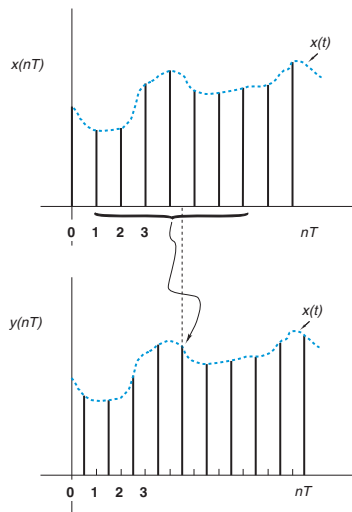
can be deduced.

• • •

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Interpolation:



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- ▶ For real-time applications it is necessary to convert a noncausal into a causal design.

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- ▶ For the case of a forward- or central-difference formula, the digital filter obtained turns out to be noncausal.
- ▶ For real-time applications it is necessary to convert a noncausal into a causal design.
- ▶ This is done by multiplying the transfer function by an appropriate negative power of z , which corresponds to delaying the impulse response of the filter to ensure that $h(nT) = 0$ for $n < 0$.

Example

- ▶ A signal $x(t)$ is sampled at a rate of $1/T$ Hz.

Design a sixth-order differentiator with a time-domain response

$$y(nT) = \left. \frac{dx(t)}{dt} \right|_{t=nT}$$

Use the Stirling formula.

- **Solution** From Stirling's formula for interpolation

$$\begin{aligned}
 y(nT) &= \left. \frac{dx(t)}{dt} \right|_{t=nT+pT} = \left. \frac{1}{T} \frac{dx(nT+pT)}{dp} \right|_{p=0} \\
 &= \frac{1}{2T} \left[\delta x\left(nT - \frac{1}{2}T\right) + \delta x\left(nT + \frac{1}{2}T\right) \right] \\
 &\quad - \frac{1}{12T} \left[\delta^3 x\left(nT - \frac{1}{2}T\right) + \delta^3 x\left(nT + \frac{1}{2}T\right) \right] \\
 &\quad + \frac{1}{60T} \left[\delta^5 x\left(nT - \frac{1}{2}T\right) + \delta^5 x\left(nT + \frac{1}{2}T\right) \right] + \dots
 \end{aligned}$$

Example *Cont'd*

- ▶ From the definition of the central difference, we get

$$\delta x(nT - \frac{1}{2}T) + \delta x(nT + \frac{1}{2}T) = x(nT + T) - x(nT - T)$$

$$\delta^3 x(nT - \frac{1}{2}T) + \delta^3 x(nT + \frac{1}{2}T) = x(nT + 2T) - 2x(nT + T) \\ + 2x(nT - T) - x(nT - 2T)$$

$$\delta^5 x(nT - \frac{1}{2}T) + \delta^5 x(nT + \frac{1}{2}T) = x(nT + 3T) - 4x(nT + 2T) \\ + 5x(nT + T) - 5x(nT - T) \\ + 4x(nT - 2T) - x(nT - 3T)$$

Example *Cont'd*

- ▶ Hence

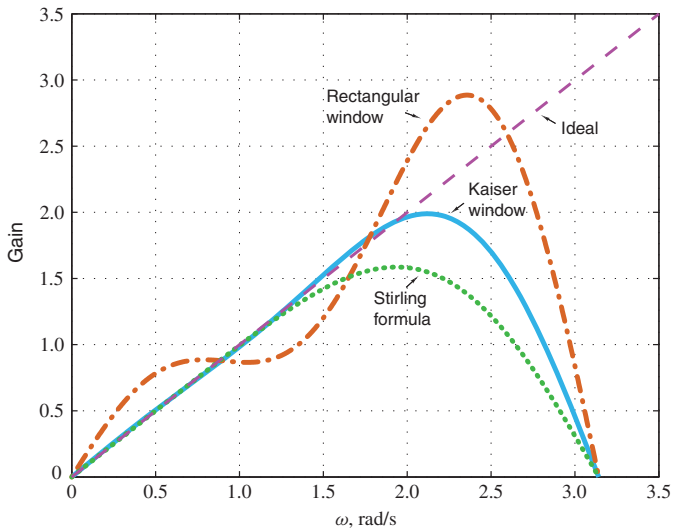
$$y(nT) = \frac{1}{60T} [x(nT + 3T) - 9x(nT + 2T) + 45x(nT + T) - 45x(nT - T) + 9x(nT - 2T) - x(nT - 3T)]$$

and, therefore

$$H(z) = \frac{1}{60T} (z^3 - 9z^2 + 45z - 45z^{-1} + 9z^{-2} - z^{-3})$$

- ▶ Note that the differentiator has an antisymmetrical impulse response, i.e., it has a constant group delay, and it is also noncausal.
- ▶ A causal filter can be obtained by multiplying $H(z)$ by z^{-3} .

Example *Cont'd*



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- ▶ Then on assuming a periodic frequency response, the appropriate impulse response can be determined.
- ▶ Gibbs' oscillations due to the transition in $H(e^{j\omega T})$ at $\omega = \omega_s/2$ can be reduced, as before, by using the window technique.

Example

- ▶ Design a sixth-order differentiator by employing the Fourier-series method.

Use (a) a rectangular window and (b) the Kaiser window with $\alpha = 3.0$.

- ▶ **Solution** Using the Fourier-series method, the impulse response of the differentiator can be obtained as

$$h(nT) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} j\omega e^{j\omega nT} d\omega = -\frac{1}{\omega_s} \int_0^{\omega_s/2} 2\omega \sin(\omega nT) d\omega$$

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- ▶ On integrating by parts, we get

$$h(nT) = \frac{1}{nT} \cos \pi n - \frac{1}{n^2 \pi T} \sin \pi n$$

or

$$h(nT) = \begin{cases} 0 & \text{for } n = 0 \\ \frac{1}{nT} \cos \pi n & \text{otherwise} \end{cases}$$

Example *Cont'd*

- ▶ If we now use the rectangular window with $N = 7$, we deduce

$$H_w(z) = \frac{1}{6T}(2z^3 - 3z^2 + 6z - 6z^{-1} + 3z^{-2} - 2z^{-3})$$

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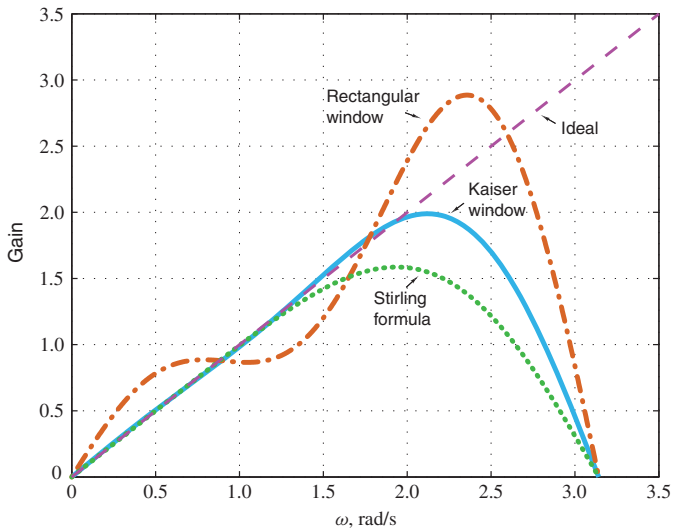
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- ▶ The parameter α in the Kaiser window can be increased to increase the in-band accuracy or decreased to increase the bandwidth.
- ▶ The design of digital differentiators that would satisfy prescribed specifications is considered in Chap. 15.

Example *Cont'd*



*This slide concludes the presentation.
Thank you for your attention.*