Chapter 10
APPROXIMATIONS FOR ANALOG FILTERS

10.1 Introduction, 10.2 Realizability
10.3 to 10.7 Butterworth, Chebyshev, Inverse-Chebyshev, Elliptic, and Bessel-Thomson Approximations

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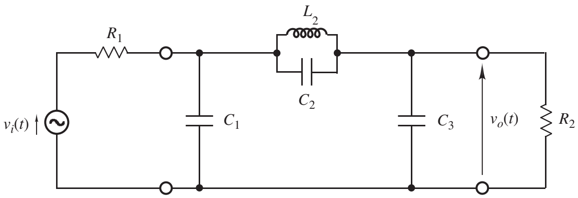
- This presentation deals with the basics of these approximations.

- An analog filter such as the one shown below can be represented by the equation

$$\frac{V_o(s)}{V_i(s)} = H(s) = \frac{N(s)}{D(s)}$$

where



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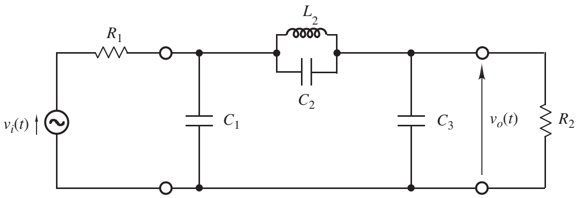

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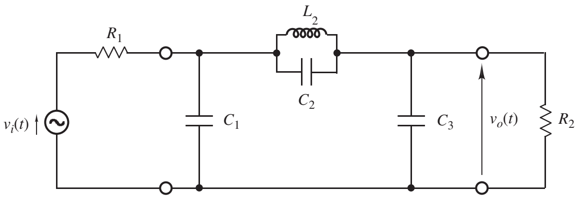

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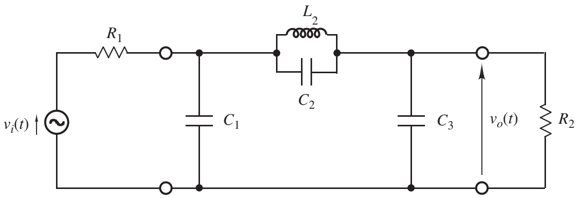

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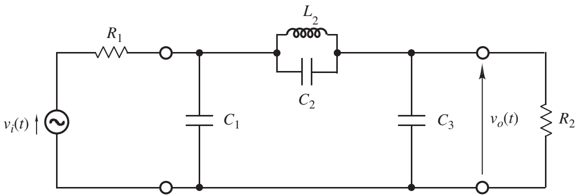
  - $N(s)$ and $D(s)$ are polynomials in complex variable s.



- The loss (or attenuation) is defined as

$$L(\omega^2) = \frac{|V_i(j\omega)|^2}{|V_o(j\omega)|^2} = \left|\frac{V_i(j\omega)}{V_o(j\omega)}\right|^2 = \frac{1}{|H(j\omega)|^2} = 10\log\frac{1}{H(j\omega)H(-j\omega)}$$

Hence the loss in dB is given by

$$\begin{aligned} A(\omega) &= 10\log L(\omega^2) = 10\log\frac{1}{|H(j\omega)|^2} \\ &= -20\log|H(j\omega)| \end{aligned}$$

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- As a function of ω, $A(\omega)$ is said to be the *loss characteristic*.

- The phase shift and group delay of analog filters are defined just as in digital filters, namely, the phase shift is the phase angle of the frequency response and the group delay is the negative of the derivative of the phase angle with respect to ω, i.e.,

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- As functions of ω, $\theta(\omega)$ and $\tau(\omega)$ are the *phase response* and *delay characteristic*, respectively.

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- Thus if the transfer function of an analog filter is known, its loss function can be readily deduced.

- If

$$H(s) = \frac{N(s)}{D(s)} = \frac{\prod_{i=1}^{M}(s - z_i)}{\prod_{i=1}^{N}(s - p_i)}$$

then

$$L(-s^2) = \frac{D(s)D(-s)}{N(s)N(-s)} = \frac{\prod_{i=1}^{N}(s - p_i)\prod_{i=1}^{N}(-s - p_i)}{\prod_{i=1}^{M}(s - z_i)\prod_{i=1}^{M}(-s - z_i)}$$

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- Therefore,
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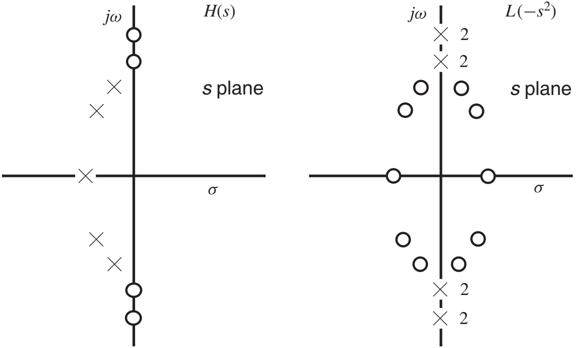
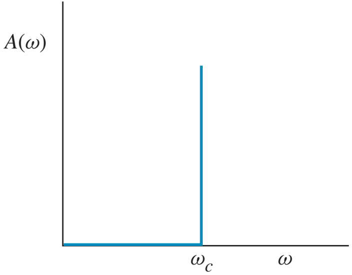
- Zero-pole plots for transfer function and loss function:



- An *ideal lowpass* filter is one that will pass only low-frequency components. Its loss characteristic assumes the form shown in the figure.



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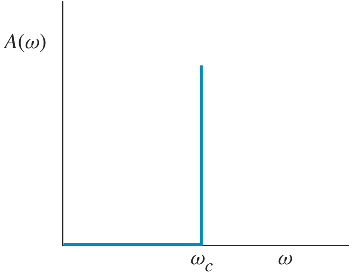
  - The frequency range 0 to ω_c is the *passband*.
  - The frequency range ω_c to ∞ is the *stopband*.
  - The boundary between the passband and stopband, namely, ω_c, is the *cutoff frequency*.



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- In the classical solutions of the approximation problem, an ideal *normalized* lowpass loss characteristic is assumed with a cutoff frequency of order unity, i.e., $\omega_c \approx 1$.
- A set of formulas are then derived that yield the *zeros and poles* or *coefficients* of the transfer function for a specified filter order.

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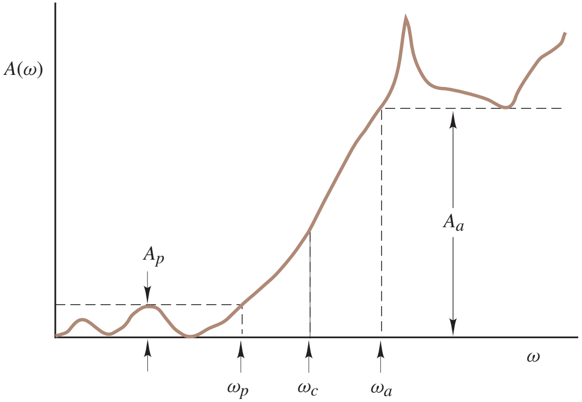
- Parameters ω_p and ω_a are the *passband* and *stoband* edges, A_p is the *maximum passband loss* (or *attenuation*), and A_a is the *minimum stopband loss* (or *attenuation*), respectively.

- The quality of an approximation depends on the values of A_p and A_a for a given filter order, i.e., a lower A_p and a larger A_a correspond to a better filter.



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- These transformations will be discussed in the next presentation.

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  - Minimum filter order to achieved prescribed specifications.
  - Formulas for the parameters of the transfer function (e.g., zeros, poles, coefficients, multiplier constant).

Butterworth Approximation

- The *Butterworth approximation* is derived on the assumption that the loss function $L(-s^2)$ is a polynomial. Since

$$\lim_{s\to\infty} L(-s^2) = \lim_{\omega\to\infty} L(\omega^2) = a_0 + a_2\omega^2 + \cdots + a_{2n}\omega^{2n} \to \infty$$

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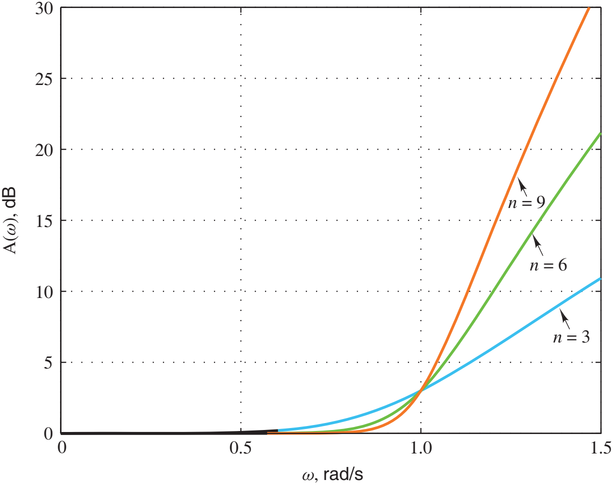
- Assuming that $L(1) = 2$, the loss function in dB can be expressed as

$$L(\omega^2) = 1 + \omega^{2n} \quad \text{and} \quad A(\omega) = 10\log(1 + \omega^{2n})$$

Butterworth Approximation *Cont'd*

- Typical loss characteristics:



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- Since $|z_k| = 1$, the zeros of $L(-s^2)$ are located on the *unit circle* $|s| = 1$.

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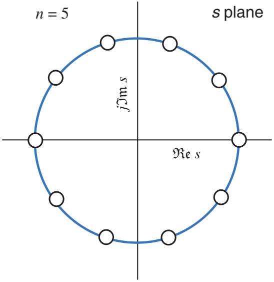
- Zero-pole plots for loss function:



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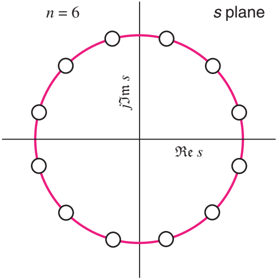

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  Therefore, they are identical with the zeros of the loss function located in the left-half s plane.

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  The *minimum* filter order that will satisfy the required specifications must be large enough to satisfy *both* of the following inequalities:

$$n \geq \frac{[-\log(10^{0.1A_p} - 1)]}{(-2\log\omega_p)} \quad \text{and} \quad n \geq \frac{\log(10^{0.1A_a} - 1)}{2\log\omega_a}$$

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  (See textbook for derivations and examples.)

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  As a result, the required specifications will usually be slightly oversatisfied.

• • •

$$n \geq \frac{[-\log(10^{0.1A_p} - 1)]}{(-2\log\omega_p)} \quad \text{and} \quad n \geq \frac{\log(10^{0.1A_a} - 1)}{2\log\omega_a}$$

- The right-hand sides in the above inequalities will normally yield a mixed number but since the filter order *must* be an integer, the value obtained must be *rounded up* to the next integer.

  As a result, the required specifications will usually be slightly oversatisfied.

- Once the required filter order is determined, the actual maximum passband loss and minimum stopband loss can be calculated as

  $$A_p = A(\omega_p) = 10\log(1 + \omega_p^{2n}) \quad \text{and} \quad A_a = A(\omega_a) = 10\log(1 + \omega_a^{2n})$$

  respectively.

Chebyshev Approximation

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- A more balanced characteristic with respect to the passband can be achieved by employing the *Chebyshev* approximation.

Chebyshev Approximation *Cont'd*

- As in the Butterworth approximation, the loss function in the Chebyshev approximation is assumed to be a polynomial in s, which would assure a lowpass characteristic.

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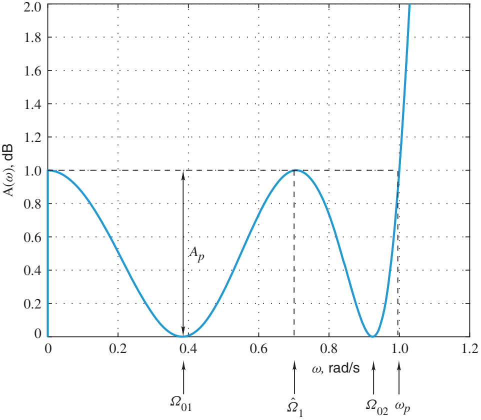
- Then by neglecting the zeros of the loss function in the right-half s plane, the poles of the transfer function can be obtained.

- In the case of a fourth-order Chebyshev filter the passband loss is assumed to be zero at $\omega = \Omega_{01}$, Ω_{02} and equal to A_p at $\omega = 0$, $\hat{\Omega}_1$, 1 as shown in the figure:



- On using all the information that can be extracted from the figure shown, a differential equation of the form

$$\left[\frac{dF(\omega)}{d\omega}\right]^2 = \frac{M_4[1 - F^2(\omega)]}{1 - \omega^2}$$

can be constructed.

Chebyshev Approximation *Cont'd*

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$$\left[\frac{dF(\omega)}{d\omega}\right]^2 = \frac{M_4[1 - F^2(\omega)]}{1 - \omega^2}$$

can be constructed.

- The solution of this differential equation gives the loss as

$$L(\omega^2) = 1 + \varepsilon^2 F^2(\omega)$$

where $\qquad \varepsilon^2 = 10^{0.1A_p} - 1$

and $\qquad F(\omega) = T_4(\omega) = \cos(4\cos^{-1}\omega)$

Chebyshev Approximation *Cont'd*

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- The function $\cos(4\cos^{-1}\omega)$ is actually a polynomial known as the *4th-order Chebyshev* polynomial.

Chebyshev Approximation *Cont'd*

- Similarly, for an nth-order Chebyshev approximation, one can show that

$$A(\omega) = 10\log L(\omega^2) = 10\log[1 + \varepsilon^2 T_n^2(\omega)]$$

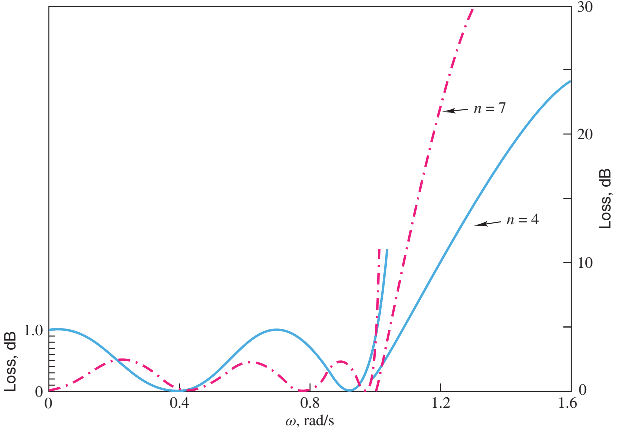
where $$\varepsilon^2 = 10^{0.1A_p} - 1$$

and $$T_n(\omega) = \begin{cases} \cos(n\cos^{-1}\omega) & \text{for } |\omega| \leq 1 \\ \cosh(n\cosh^{-1}\omega) & \text{for } |\omega| > 1 \end{cases}$$

is the *nth-order* Chebyshev polynomial.

Chebyshev Approximation *Cont'd*

- Typical loss characteristics for Chebyshev approximation:



(*a*)

Chebyshev Approximation *Cont'd*

- The zeros of the loss function for a *normalized* nth-order Chebyshev approximation ($\omega_p = 1$ rad/s) are given by $s_i = \sigma_i + j\omega_i$ where

$$\sigma_i = \pm \sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right)\sin\frac{(2i-1)\pi}{2n}$$

$$\omega_i = \cosh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right)\cos\frac{(2i-1)\pi}{2n}$$

for $i = 1, 2, \ldots, n$.

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for $i = 1, 2, \ldots, n$.

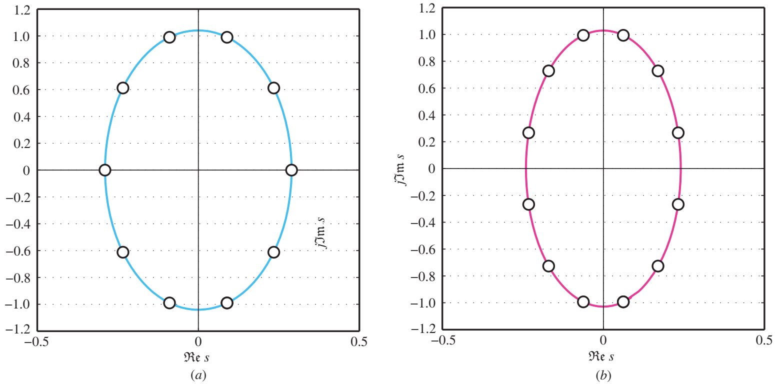
- From these equations, we note that

$$\frac{\sigma_i^2}{\sinh^2 u} + \frac{\omega_i^2}{\cosh^2 u} = 1 \quad \text{where} \quad u = \frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}$$

i.e., the zeros of $L(-s^2)$ are located on an *ellipse*.

Chebyshev Approximation *Cont'd*

- Typical zero-pole plots for Chebyshev approximation: (a) $n = 5$ $A_p = 1$ dB; (b) $n = 6$ $A_p = 1$ dB.



- An nth-order normalized Chevyshev transfer function with a passband edge $\omega_p = 1$ rad/s and a maximum passband loss of A_p dB can be determined as follows:

$$\begin{aligned} H_N(s) &= \frac{H_0}{D_0(s)\prod_i^r (s-p_i)(s-p_i^*)} \\ &= \frac{H_0}{D_0(s)\prod_i^r [s^2 - 2\text{Re}\,(p_i)s + |p_i|^2]} \end{aligned}$$

where

$$r = \begin{cases} \frac{n-1}{2} & \text{for odd } n \\ \frac{n}{2} & \text{for even } n \end{cases} \quad \text{and} \quad D_0(s) = \begin{cases} s - p_0 & \text{for odd } n \\ 1 & \text{for even } n \end{cases}$$

Chebyshev Approximation *Cont'd*

- The poles and multiplier constant, H_0, can be calculated by using the following formulas in sequence:

$$\varepsilon = \sqrt{10^{0.1A_p} - 1}$$

$$p_0 = \sigma_{(n+1)/2} \quad \text{with} \quad \sigma_{(n+1)/2} = -\sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right)$$

$$p_i = \sigma_i + j\omega_i \qquad \text{for} \quad i = 1,\, 2,\, \ldots,\, r$$

$$\text{where} \quad \sigma_i = -\sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right)\sin\frac{(2i-1)\pi}{2n}$$

$$\omega_i = \cosh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right)\cos\frac{(2i-1)\pi}{2n}$$

$$H_0 = \begin{cases} -p_0 \prod_{i=1}^{r} |p_i|^2 & \text{for odd } n \\ 10^{-0.05A_p} \prod_{i=1}^{r} |p_i|^2 & \text{for even } n \end{cases}$$

Chebyshev Approximation *Cont'd*

- The minimum filter order required to achieve a maximum passband loss of A_p and a minimum stopband loss of A_a must be large enough to satisfy the inequality

$$n \geq \frac{\cosh^{-1} \sqrt{D}}{\cosh^{-1} \omega_a} \quad \text{where} \quad D = \frac{10^{0.1 A_a} - 1}{10^{0.1 A_p} - 1}$$

Chebyshev Approximation *Cont'd*

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$$A_a = A(\omega_a) = 10\log L(\omega_a^2) = 10\log[1+\varepsilon^2 T_n^2(\omega_a)]$$

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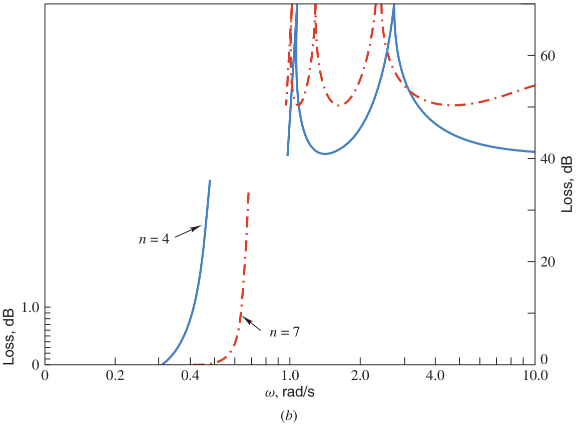
Inverse-Chebyshev Approximation

- The *inverse-Chebyshev* approximation is closely related to the Chebyshev approximation, as may be expected, and it is actually derived from the Chebyshev approximation.

Inverse-Chebyshev Approximation

- The *inverse-Chebyshev* approximation is closely related to the Chebyshev approximation, as may be expected, and it is actually derived from the Chebyshev approximation.
- The passband loss in the inverse-Chebyshev is very similar to that of the Butterworth approximation, i.e., it is an increasing monotonic function of ω, while the stopband loss oscillates between infinity and a prescribed minimum loss A_a.

- Typical loss characteristics for inverse-Chebyshev approximation:



- The loss for the inverse-Chebyshev approximation is given by

$$A(\omega) = 10 \log \left[1 + \frac{1}{\delta^2 T_n^2(1/\omega)} \right]$$

where

$$\delta^2 = \frac{1}{10^{0.1A_a} - 1}$$

and the stopband extends from $\omega = 1$ to ∞.

- The *normalized* transfer function for a specified order, n, stopband edge of $\omega_a = 1$ rad/s, and minimum stopband loss, A_a, is given by

$$H_N(s) = \frac{H_0}{D_0(s)} \prod_{i=1}^{r} \frac{(s - 1/z_i)(s - 1/z_i^*)}{(s - 1/p_i)(s - 1/p_i^*)}$$
$$= \frac{H_0}{D_0(s)} \prod_{i=1}^{r} \frac{s^2 + \frac{1}{|z_i|^2}}{s^2 - 2\text{Re}\left(\frac{1}{p_i}\right) s + \frac{1}{|p_i|^2}}$$

where

$$r = \begin{cases} \frac{n-1}{2} & \text{for odd } n \\ \frac{n}{2} & \text{for even } n \end{cases} \quad \text{and} \quad D_0(s) = \begin{cases} s - \frac{1}{p_0} & \text{for odd } n \\ 1 & \text{for even } n \end{cases}$$

Inverse-Chebyshev Approximation *Cont'd*

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- The parameters of the transfer function can be calculated by using the formulas in the next slide.

Inverse-Chebyshev Approximation *Cont'd*

$$\delta = \frac{1}{\sqrt{10^{0.1A_a}-1}}, \quad z_i = j\cos\frac{(2i-1)\pi}{2n} \qquad \text{for } 1,\, 2,\, \ldots,\, r$$

$$p_0 = \sigma_{(n+1)/2} \quad \text{with} \quad \sigma_{(n+1)/2} = -\sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\delta}\right)$$

$$p_i = \sigma_i + j\omega_i \qquad \text{for } 1,\, 2,\, \ldots,\, r$$

with

$$\sigma_i = -\sinh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\delta}\right)\sin\frac{(2i-1)\pi}{2n}$$

$$\omega_i = \cosh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\delta}\right)\cos\frac{(2i-1)\pi}{2n}$$

and

$$H_0 = \begin{cases} \frac{1}{-p_0}\prod_{i=1}^{r}\frac{|z_i|^2}{|p_i|^2} & \text{for odd } n \\ \prod_{i=1}^{r}\frac{|z_i|^2}{|p_i|^2} & \text{for even } n \end{cases}$$

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- In this approximation, the actual minimum stopband loss will be exactly as specified, i.e., A_a.

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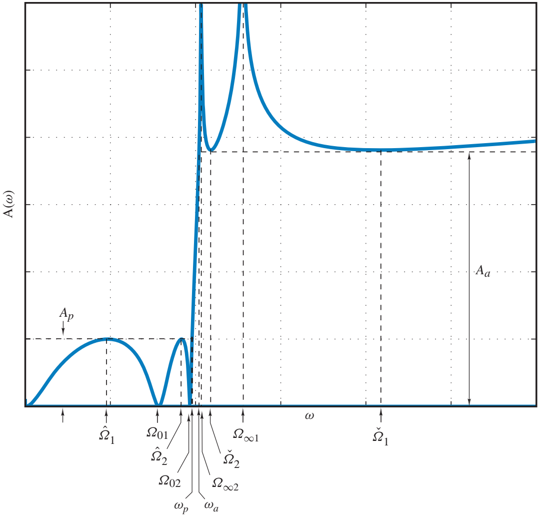
- The elliptic approximation is more efficient that all the other analog-filter approximations in that the transition between passband and stopband is steeper for a given approximation order.

  In fact, this is the *optimal* approximation for a given piecewise constant approximation.

- Loss characteristic for a 5th-order elliptic approximation:



Elliptic Approximation *Cont'd*

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- On the basis of the assumed structure of the loss characteristic, a differential equation is derived, as in the case of the Chebyshev approximation.

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- After considerable mathematical complexity, the differential equation obtained is solved through the use of *elliptic functions*, and the parameters of the transfer function are deduced.

Elliptic Approximation *Cont'd*

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- On the basis of the assumed structure of the loss characteristic, a differential equation is derived, as in the case of the Chebyshev approximation.
- After considerable mathematical complexity, the differential equation obtained is solved through the use of *elliptic functions*, and the parameters of the transfer function are deduced.

  The approximation owes its name to the use of elliptic functions in the derivation.

Elliptic Approximation *Cont'd*

- The passband and stopband edges and cutoff frequency of a *normalized* elliptic approximation are defined as follows:

$$\omega_p = \sqrt{k}, \quad \omega_a = \frac{1}{\sqrt{k}}, \quad \omega_c = \sqrt{\omega_a \omega_p} = 1$$

  Constants k and k_1 given by

$$k = \frac{\omega_p}{\omega_a} \quad \text{and} \quad k_1 = \left(\frac{10^{0.1A_p} - 1}{10^{0.1A_a} - 1} \right)^{1/2}$$

  are known as the *selectivity* and *discrimination* constants.

- A normalized elliptic lowpass filter with a selectivity factor k, passband edge $\omega_p = \sqrt{k}$, stopband edge $\omega_a = 1/\sqrt{k}$, a maximum passband loss of A_p dB, and a minimum stopband loss equal to or in excess of A_a dB has a transfer function of the form

$$H_N(s) = \frac{H_0}{D_0(s)} \prod_{i=1}^{r} \frac{s^2 + a_{0i}}{s^2 + b_{1i}s + b_{0i}}$$

where
$$r = \begin{cases} \frac{n-1}{2} & \text{for odd } n \\ \frac{n}{2} & \text{for even } n \end{cases}$$

and
$$D_0(s) = \begin{cases} s + \sigma_0 & \text{for odd } n \\ 1 & \text{for even } n \end{cases}$$

- A normalized elliptic lowpass filter with a selectivity factor k, passband edge $\omega_p = \sqrt{k}$, stopband edge $\omega_a = 1/\sqrt{k}$, a maximum passband loss of A_p dB, and a minimum stopband loss equal to or in excess of A_a dB has a transfer function of the form

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and
$$D_0(s) = \begin{cases} s + \sigma_0 & \text{for odd } n \\ 1 & \text{for even } n \end{cases}$$

- The parameters of the transfer function can be obtained by using the formulas in the next three slides in sequence in the order shown.

Elliptic Approximation *Cont'd*

$$
\begin{aligned}
k' &= \sqrt{1-k^2}\\
q_0 &= \frac{1}{2}\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)\\
q &= q_0 + 2q_0^5 + 15q_0^9 + 150q_0^{13}\\
D &= \frac{10^{0.1A_a}-1}{10^{0.1A_p}-1}\\
n &\geq \frac{\log 16D}{\log(1/q)} \quad \text{(round up to the next integer)}\\
\Lambda &= \frac{1}{2n}\ln\frac{10^{0.05A_p}+1}{10^{0.05A_p}-1}\\
\sigma_0 &= \left|\frac{2q^{1/4}\sum_{m=0}^{\infty}(-1)^m q^{m(m+1)}\sinh[(2m+1)\Lambda]}{1+2\sum_{m=1}^{\infty}(-1)^m q^{m^2}\cosh 2m\Lambda}\right|
\end{aligned}
$$

Elliptic Approximation *Cont'd*

$$W = \sqrt{\left(1 + k\sigma_0^2\right)\left(1 + \frac{\sigma_0^2}{k}\right)}$$

$$\Omega_i = \frac{2q^{1/4}\sum_{m=0}^{\infty}(-1)^m q^{m(m+1)} \sin\frac{(2m+1)\pi\mu}{n}}{1 + 2\sum_{m=1}^{\infty}(-1)^m q^{m^2} \cos\frac{2m\pi\mu}{n}}$$

$$\text{where} \quad \mu = \begin{cases} i & \text{for odd } n \\ i - \frac{1}{2} & \text{for even } n \end{cases} \qquad i = 1, 2, \ldots, r$$

$$V_i = \sqrt{\left(1 - k\Omega_i^2\right)\left(1 - \frac{\Omega_i^2}{k}\right)}$$

Elliptic Approximation *Cont'd*

$$a_{0i} = \frac{1}{\Omega_i^2}$$

$$b_{0i} = \frac{(\sigma_0 V_i)^2 + (\Omega_i W)^2}{\left(1 + \sigma_0^2 \Omega_i^2\right)^2}$$

$$b_{1i} = \frac{2\sigma_0 V_i}{1 + \sigma_0^2 \Omega_i^2}$$

$$H_0 = \begin{cases} \sigma_0 \prod_{i=1}^{r} \frac{b_{0i}}{a_{0i}} & \text{for odd } n \\ 10^{-0.05 A_p} \prod_{i=1}^{r} \frac{b_{0i}}{a_{0i}} & \text{for even } n \end{cases}$$

- Because of the fact that the filter order is *rounded up* to the next integer, the minimum stopband loss is usually oversatisfied.

Elliptic Approximation *Cont'd*

- Because of the fact that the filter order is *rounded up* to the next integer, the minimum stopband loss is usually oversatisfied.
- The actual minimum stopband loss is given by the following formula:

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(See textbook for details.)

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- The last approximation in Chap. 10, namely, the *Bessel-Thomson approximation*, is derived on the assumption that the group delay is *maximally flat* at zero frequency.

- As in the Butterworth and Chebyshev approximations, the loss function is a polynomial. Hence the Bessel-Thomson approximation is essentially a *lowpass* approximation.

Bessel-Thomson Approximation *Cont'd*

- The transfer function for a *normalized* Bessel-Thomson approximation is give by

$$H(s) = \frac{b_0}{\sum_{i=0}^{n} b_i s^i} = \frac{b_0}{s^n B(1/s)}$$

where
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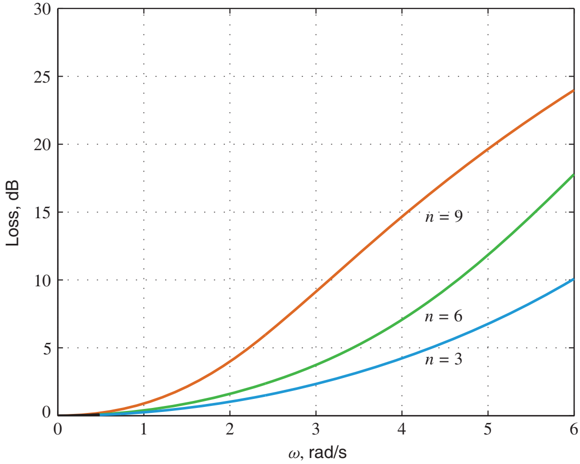
- The group-delay is 1 s. An arbitrary delay can be obtained by replacing s by $\tau_0 s$ where τ_0 is a constant.

- Function $B(\cdot)$ is a *Bessel polynomial*, and $s^n B(1/s)$ can be shown to have zeros in the left-half s plane, i.e., *the Bessel-Thomson approximation represents stable analog filters*.

Bessel-Thomson Approximation *Cont'd*

- Typical loss characteristics:



Bessel-Thomson Approximation *Cont'd*

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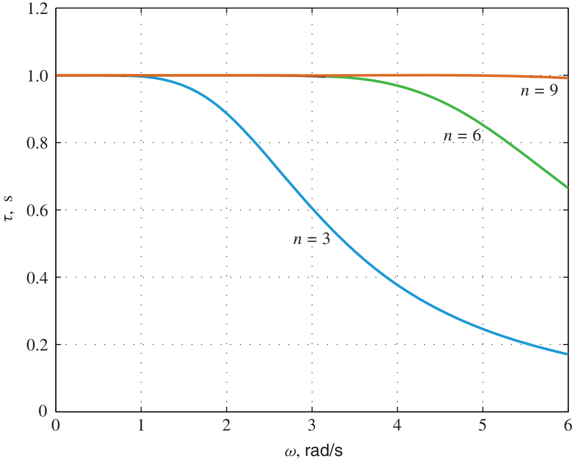

This slide concludes the presentation.
Thank you for your attention.