Asynchronous Time Difference of Arrival Positioning System

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Abstract

This paper presents a newly proposed asynchronous time difference of arrival (A-TDOA) positioning system. Two major challenges for accurate location finding in indoor environments are the fine time resolution and high accuracy time synchronization. This paper proposes a new positioning method that resolves the time synchronization difficulty by measuring the time difference of arrival at a single anchor node. Unlike conventional time of arrival (TOA) and time difference of arrival (TDOA) based methods, no synchronization is required between a target node and any anchor node, nor is required among anchor nodes. Therefore, the A-TDOA system avoids stringent synchronization between target and anchor nodes, and it eliminates the need of back-bone cables for anchor nodes synchronization, which makes it highly practical and can be easily implemented. In addition, two novel algorithms are proposed to expand the application of the A-TDOA positioning system, one is a hybrid positioning algorithm that combines semidefinite programming (SDP) with a Taylor series method to achieve global convergence as well as superior estimation accuracy, and the other is a constrained least-squares method that has the advantage of low complexity and fast convergence while maintains relatively good performance.

Index Terms

Localization, positioning system, time difference of arrival (TDOA), semidefinite programming (SDP), Taylor series, constrained least-squares, CRLB

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I. Introduction

There has been a great demand to develop high accuracy positioning systems for many applications, including robot control, person and asset tracking, indoor navigation, as well as avalanche and earthquake victim rescue. By applying ultra-wideband (UWB) techniques, positioning systems can achieve up to millimeter accuracy, making it useful for many unique applications.

Global positioning system (GPS) is a satellite based positioning system providing navigation to military and civilian users. The system operates with 24 satellites and offers precise location and time information. Performing extremely well in outdoor environments, GPS barely works in indoor and dense urban environments because the signal from the GPS satellites is too weak to penetrate most buildings. Many positioning techniques have been developed to provide location information in indoor environments, such as infrared, ultrasound, and radio frequency (RF) techniques. However, both infrared and ultrasound require a line of sight (LOS) condition between the target and the anchor nodes, which limits their usage to a large extent. In a harsh indoor environment where various obstacles exist, RF based positioning systems can work effectively due to their inherent penetration property. Nearly all existing RF technologies such as RF identification, wireless local area network, cellular network, Zigbee and UWB can be used for localization. These techniques can achieve position accuracy from millimeter to meters, with different system complexity and robustness. A trade-off between performance and complexity has to be considered in designing a positioning system.

Positioning systems can be classified into three main categories: received signal strength (RSS), angle of arrival (AOA), and time (difference) of arrival (TOA/TDOA) based systems [1]. A combination of these techniques can also be used to improve accuracy. As the distance of a radio link increases, the received signal strength reduces accordingly. This relationship conveys information about the range between an anchor node and a target node. As many consumer electronic devices have the ability to measure received signal power, the RSS based localization system appears to be an economic solution because the existing infrastructure can be reused easily [2], [3]. However, the ranging accuracy of a RSS technique is highly dependent on the channel parameters and distance between the two nodes. Indoor wireless channels
are typically rather complicated that can cause significant fluctuations in RSS even over short distance. Therefore, the RSS based positioning method is usually not accurate enough for most indoor applications.

Time based localization method, including one-way TOA ranging, two-way TOA ranging, and TDOA, exploits the fine delay resolution property of wideband signals and has great potential for providing high accuracy location estimation. One-way TOA ranging requires the target node and any anchor node be perfectly synchronized, as the distance between the two is calculated based on the target node’s transmission time $t_1$ and anchor node’s receive time $t_2$. Synchronization error can significantly affect the ranging accuracy. The two-way ranging has been standardized in IEEE 802.15.4a [4] which is an amendment to IEEE 802.15.4 to provide high precision ranging/positioning capability. In two-way ranging, the system estimates round-trip time of the signal of interest without a common clock source. Therefore, although the target to anchor node synchronization is avoided, the clock offset between the two nodes’ crystal is the dominant source of the ranging error [5], [6]. In practice, the TDOA ranging technique is commonly used. It measures the time difference of flight between a target node and a pair of fixed anchor nodes. Therefore, the TDOA ranging technique only requires the anchor nodes synchronized to a common clock, and it is relatively easy to synchronize them by cables. This greatly improves the accuracy over the TOA and two-way TOA ranging with significantly reduced system complexity. From a geometric point of view, the position of the target node is given by hyperbolas with foci at the fixed anchor nodes. The location of the target node is at the intersection of the hyperbolas [7].

As stated earlier, a one-way TOA based ranging system requires stringent timing synchronization between the target and anchor nodes. Although two-way TOA ranging avoids synchronization, high performance and low offset crystal clocks are indispensable. In spite of less rigorous target-anchor synchronization, a TDOA system still requires a wired backbone to achieve precise anchor-anchor synchronization.

In this paper, we propose an asynchronous time difference of arrival (A-TDOA) positioning system. Unlike a conventional TDOA system using a pair of synchronized anchor nodes to measure time difference of arrival, an A-TDOA system achieves time difference measurement at a single anchor node. An asynchronous elliptical localization system employing 3 anchor nodes plus 1 transmitter node to achieve 2
dimension (2D) positioning (or 4 anchor nodes plus 1 transmitter node to achieve 3 dimension positioning) is proposed in [8]. The A-TDOA system proposed here saves cost by abandoning the additional transmitter node and using only 3 anchor nodes. We stress that the new A-TODA system is readily available for self configuring sensor networks since no synchronization is required among anchor nodes. In addition, two novel position estimation algorithms are proposed to achieve superior estimation performance and low complexity respectively. The rest of the paper is organized as follows. Section II describes in detail the system model for A-TDOA. Cramer-Rao lower bound (CRLB) for A-TDOA systems is derived in Section III. A hybrid SDP+Taylor algorithm and a constrained least-squares estimator for A-TDOA systems are presented in Section IV and V respectively. Simulation results are given in Section VI and the paper is concluded in Section VII.

The notation used in this paper is described as follows. Bold upper case symbols denote matrices and bold lower case symbols denote vectors. The $0_{m \times n}$ is the $m \times n$ zero matrix, $I_m$ is the $m \times m$ identity matrix. $(\cdot)^T$ denotes vector transpose operator and $\|x\|$ represents the 2-norm of a vector $x$. For two symmetric matrices $A$ and $B$, $A \succeq B$ means that $A - B$ is positive semidefinite.

II. SYSTEM MODEL

Let $x = [x, y]^T$ and $x_i = [x_i, y_i]^T, i = 1, 2, \ldots, M$ be the coordinate of the target node and anchor nodes respectively where $M$ is the number of anchor nodes with $M \geq 3$ for 2D positioning. The anchor node that initiates the pulse transmission is called anchor Tx, and the one that receives the pulse is called anchor Rx. Without loss of generality, let anchor Rx be the reference at coordinate $x_1$, and anchor Tx be positioned at $x_i, i = 2, 3, \ldots, M$.

Unlike a conventional TDOA system, the proposed A-TDOA system initiates pulse transmission by one of the anchor nodes. Fig. 1 demonstrates the signal flow and the system timing. At time $t_0$, anchor Tx transmits a pulse that is received at target node at time $t_1$ and at anchor Rx at time $t_2$. As soon as the target node receives the pulse from anchor Tx, it amplifies the received pulse and retransmits it immediately. The retransmitted signal reaches anchor Rx at time $t_3$. Ultimately, anchor Rx would receive two pulses in a row: one is from anchor Tx and the other is from the target node. It is noteworthy that
in practice a certain amount of delay is generated through the process of “amplify and retransmit” at the target node, this delay can be estimated and subtracted by system calibration.

The time difference measured at anchor Rx can be written as

\[(t_3 - t_2) \cdot c = \|x - x_i\| + \|x - x_1\| - \|x_i - x_1\| + n_i, \quad i = 2, 3, \ldots, M,\]

(1)

where \(c\) is the speed of light, and \(n_i\) is a zero mean Gaussian distributed measurement error. Eq. (1) exhibits the beauty of the A-TDOA system that the time difference \(t_3 - t_2\) is measured at and only at anchor Rx. Therefore, no clock synchronization is required among anchor Rx, anchor Tx and the target node. The use of the backbone cables which have to be employed in conventional TDOA positioning systems can be avoided. Rearranging (1) as

\[(t_3 - t_2) \cdot c + \|x_i - x_1\| = \|x - x_i\| + \|x - x_1\| + n_i, \quad i = 2, 3, \ldots, M,\]

(2)

demonstrates that the sum of the distances from \(x\) to two fixed anchor nodes \(x_i\) and \(x_1\) is a constant \(\|x_i - x_1\|\) plus a distance difference which is measurable. Therefore, the target node must lie on the trajectory of an ellipse with anchor Tx and anchor Rx as the two foci. Fig. 1 illustrates the ellipse trajectory on which a target node is located.

Since the time difference measured between two anchor nodes can only determine the target node on a locus of one ellipse, a third anchor node is necessary to pinpoint the unique position of the target node. Fig. 2 illustrates the geometric layout of the A-TDOA system in 2D, with the target position obtained from the intersection of 3 ellipses. In general, with \(M\) anchor nodes, there are \(M(M - 1)/2\) distinct A-TDOA measurements from all anchor node pairs. We shall call any set of \(M - 1\) measurement a non-redundant set. In the noiseless case, a non-redundant set is sufficient to determine the exact target node position. However, noise and estimation errors are unavoidable in real life problems. As will be seen in Section III, a proper selection of a non-redundant set can significantly improve the estimation accuracy in an A-TDOA system.
Cramer-Rao lower bound (CRLB) is commonly used for setting a lower bound on an estimator’s mean square error (MSE), and it states that the variance of any unbiased estimator is at least as high as the inverse of its Fisher information. Therefore, the CRLB sets a benchmark against which the performance of an unbiased estimation is compared. The CRLB can also be used to rule out infeasible estimators. Below we derive the CRLB for A-TDOA systems.

Denote
\[ \mathbf{d} = [d_{21}, d_{31}, \ldots, d_{M1}]^T \]
\[ \mathbf{n} = [n_2, n_3, \ldots, n_M]^T \]
\[ g(x) = [g_2(x), g_3(x), \ldots, g_M(x)]^T, \]
where \( d_{i1}, i = 2, 3, \ldots, M \) is the measured range difference \( d_{i1} = \|x - x_i\| + \|x - x_1\| - \|x_i - x_1\| + n_i, i = 2, 3, \ldots, M \). Also, \( g_i(x) \) can be expressed as \( g_i(x) = \|x - x_i\| + \|x - x_1\| - \|x_i - x_1\|, i = 2, 3, \ldots, M \).

Assume that the additive measurement errors have zero mean and are independent of the range difference observation, as well as the target coordinate \( x \). Thus the probability density function (PDF) of the measured range difference \( \mathbf{d} \) conditioned on the target node position \( x \) is given by [9]
\[
p(d|x) = \frac{\exp \left\{-\frac{1}{2}[\mathbf{d} - g(x)]^T \mathbf{C}_n^{-1} [\mathbf{d} - g(x)] \right\}}{\sqrt{(2\pi)^M \det(\mathbf{C}_n)}}, \tag{3}
\]
where \( \det(A) \) denotes the determinant of matrix \( A \) and \( \mathbf{C}_n \) is the covariance matrix of \( \mathbf{n} \),
\[
\mathbf{C}_n = \begin{bmatrix}
\sigma_2^2 & 0 & \cdots & 0 \\
0 & \sigma_3^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma_M^2 
\end{bmatrix}.
\]
The Fisher information matrix can be written as [10]
\[
\mathbf{F}(x) = \left[ \frac{\partial g(x)}{\partial x} \right]^T \mathbf{C}_n^{-1} \left[ \frac{\partial g(x)}{\partial x} \right], \tag{4}
\]
where $\frac{\partial g(x)}{\partial x}$ is the $(M - 1) \times 2$ Jacobian matrix defined as

$$\frac{\partial g(x)}{\partial x} = \begin{bmatrix} \frac{\partial g_2(x)}{\partial x} & \frac{\partial g_2(x)}{\partial y} \\ \frac{\partial g_3(x)}{\partial x} & \frac{\partial g_3(x)}{\partial y} \\ \vdots & \vdots \\ \frac{\partial g_M(x)}{\partial x} & \frac{\partial g_M(x)}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x-x_2}{||x-x_2||} + \frac{x-x_1}{||x-x_1||} & \frac{y-y_2}{||x-x_2||} + \frac{y-y_1}{||x-x_1||} \\ \frac{x-x_3}{||x-x_3||} + \frac{x-x_1}{||x-x_1||} & \frac{y-y_2}{||x-x_3||} + \frac{y-y_1}{||x-x_1||} \\ \vdots & \vdots \\ \frac{x-x_M}{||x-x_M||} + \frac{x-x_1}{||x-x_1||} & \frac{y-y_M}{||x-x_M||} + \frac{y-y_1}{||x-x_1||} \end{bmatrix}$$

(5)

The above CRLB derivation is based on the assumption that anchor node 1 ($x_1, y_1$) has been assigned as the reference node. Fig. 3(a) illustrates the corresponding CRLB measured at a 100 m × 100 m grid with a range measurement error (error standard deviation) of 0.1 m. Four anchor nodes’ coordinates are (0 m, 0 m), (0 m, 100 m), (100 m, 0 m) and (100 m, 100 m) respectively. We define dBm$^2$ as $10 \log_{10}(\sigma^2)$ where $\sigma^2$ is the error variance of the target position estimate in m$^2$. It is obvious from Fig. 3(a) that the position estimation error close to the reference anchor node (coordinate (0 m, 0 m)) is much higher than the other positions. In particular, the estimation error close to anchor node 4 is more than 10 dB less. Therefore, when multiple anchor nodes are available in the system, the reference node can be chosen to minimize the estimation error. This requires the system to have a-priori knowledge of an approximate target node position, so that the anchor node furthest apart from the target node can be selected as the reference. This a-priori knowledge can easily be obtained by using a localization algorithm that achieves global convergence to estimate an approximate coordinate of the target node. This position estimate can then be used to re-select the reference node. Given the updated reference node and an approximate target node coordinate as an initial guess, a high accuracy algorithm can be applied to give superb performance. Fig. 3(b) demonstrates the improved CRLB by selecting a proper reference node such that the CRLB becomes minimal. Comparison of CRLBs for TOA, TDOA, and A-TDOA systems is shown in Fig. 4,
and as illustrated in Fig. 4, A-TDOA outperforms both TDOA and TOA systems when reference re-
selection is applied.

IV. A HIGH-ACCURACY LOCALIZATION ALGORITHM

As argued earlier, the target node is localized at the intersection of a set of ellipses. However, finding
the intersection is a highly nonlinear optimization problem. Many techniques are available for solving
the problem at hand. The widely used approach is perhaps least-squares (LS) that provides a closed-form
solution. By squaring the set of nonlinear equations in (2) and introducing an extra variable that is a
function of the target coordinates, the original nonlinear optimization problem can be easily solved by
the LS approach.

The spherical-intersection method (SX) in [11] and spherical-interpolation (SI) method in [12] are
essentially LS approach, where a new variable is introduced and estimated along with the target node
with the dependence of the two ignored. As a result, the estimate obtained is non-optimum [13]. It is
also noted that for 2D positioning these methods cannot provide a position estimate by employing only 3
anchor nodes. Taking the aforementioned extra variable constraint into consideration, a linear-correction
least-squares method is proposed in [9]. However, the method requires an iterative procedure to find
a Lagrange multiplier, and convergence problems may occur. The aforementioned LS techniques also
appear to be vulnerable to noise, and reasonably accurate results may be achieved only when noise is
insignificant. The Taylor-series method is commonly employed to solve nonlinear equations to obtain
precise position estimate [14] [15] and is found to outperform the SX and SI methods [13]. A major
drawback of Taylor-series approach is that the initial guess must be close to the true position to avoid
local convergence. In addition to these conventional approaches, particle swarm optimization (PSO) is
also employed for position estimation [16]. In comparison with conventional approaches, however, PSO
involves many user-defined parameters that can affect the performance significantly. Also, when a massive
amount of particles are involved, the computing time is increased significantly.

In this section, we present a hybrid estimation algorithm that combines semidefinite programming
(SDP) and Taylor series methods to achieve high estimation accuracy. SDP is used to relax the non-
convex optimization problem to a convex optimization problem to provide approximate position estimation in a globally optimum fashion [17]. The proposed hybrid algorithm takes advantage of the SDP’s global convergence property as well as the Taylor series method’s high accuracy to achieve superior performance.

We begin by writing the maximum likelihood (ML) formulation of the target estimation as

$$
\min_{\mathbf{x}} \sum_{i=2}^{M} \sum_{j=2}^{M} e_i(x)e_j(x),
$$

where $e_i(x)$ is the zero mean Gaussian measurement error and is defined as

$$
e_i(x) = d_{i1} - \|x - x_i\| - \|x - x_1\| + \|x_i - x_1\|.
$$

Denote $r_i = \|x - x_i\|$ and $\mathbf{r} = [r_1, r_2, \cdots, r_M]^T$. By expanding (6) and dropping the terms that have no effects on the minimization, the ML cost function can be expressed as a constrained optimization problem,

$$
\min_{\mathbf{x}, \mathbf{r}} \sum_{i=2}^{M} \sum_{j=2}^{M} \left\{ (r_ir_j + 2r_1r_j - 2(d_{i1}r_j + \|x_i - x_1\|r_j)) \\
- r_1(d_{i1} + d_{j1} + \|x_i - x_1\| + \|x_j - x_1\|) + r_1^2 \right\}
$$

subject to:

$$
r_i = \|x - x_i\|.
$$

The cost function in (8a) remains nonlinear because of the terms $r_ir_j$ and $r_1^2$. By introducing parameter $\mathbf{R} = \mathbf{rr}^T$ and letting $r_ir_j = r_{ij}$, the problem at hand becomes

$$
\min_{\mathbf{x}, \mathbf{r}, \mathbf{R}} \sum_{i=2}^{M} \sum_{j=2}^{M} \left\{ (r_{ij} + 2r_{1j} - 2(d_{i1}r_j + \|x_i - x_1\|r_j)) \\
- r_1(d_{i1} + d_{j1} + \|x_i - x_1\| + \|x_j - x_1\|) + r_{11} \right\}
$$

subject to:

$$\mathbf{R} = \mathbf{rr}^T$$

$$r_{ii} = \mathbf{x}_i^T \mathbf{x} + \mathbf{x}_i^T \mathbf{x}_i - 2\mathbf{x}_i^T \mathbf{x}, i = 1, 2, \ldots, M.$$
Furthermore, by introducing new variable \( z = x^T x \) to linearize constraint (9c), the above problem can be relaxed to a standard SDP problem as

\[
\begin{align*}
\min_{x, r, R, z} & \sum_{i=2}^{M} \sum_{j=2}^{M} \left\{ (r_{ij} + 2r_{1j} - 2(d_{i1}r_j + \|x_i - x_1\| r_j) \\
& - r_1(d_{i1} + d_{1j} + \|x_i - x_1\| + \|x_j - x_1\|) + r_{11} \right\} \\
\text{subject to:} & \\
\begin{bmatrix} R & r \\ r^T & 1 \end{bmatrix} & \succeq 0_{(m+1) \times (m+1)} \\
\begin{bmatrix} z & x^T \\ x & I_2 \end{bmatrix} & \succeq 0_{3 \times 3} \\
r_{ii} = z + x_i^T x_i - 2x_i^T x, i = 1, 2, \ldots, M, \end{align*}
\tag{10a-10d}
\]

The derivation of (8)-(10) is analogous to those found in [17]. In minimizing the objective function in (10a), \( r_{ij} \) tends to decrease while \( r_j \) tends to increase, hence the relaxation made above is not tight. In [17] this problem is resolved by assuming that the target node is closest to the reference node, which does not sound very practical. Nevertheless, (10) is a convex problem whose global solution can readily be computed. In addition, simulation studies have indicated that the approximate solution to problem (10) is typically close to the true location. Based on these, we propose a hybrid algorithm in that the SDP solution serves as an initial estimation to allow a Taylor-series based method step a quick convergence to an accurate location estimation.

To describe the Taylor-series based approach, we denote \( f_i(x) = \|x - x_i\| + \|x - x_1\| \), and express the error function in (7) as

\[
e_i(x) = d_{i1} + \|x_i - x_1\| - f_i(x), i = 2, 3, \ldots, M. \tag{11}
\]

Let \( x_0 = [x_0, y_0]^T \) be the initial guess of the target location and \( \Delta x = [\delta x, \delta y]^T \) be the small increment on \( x \).
By applying Taylor expansion to the equations in (11), these equations can be linearized as
\[ e_i(x) \approx d_{i1} + \|x_i - x_1\| - f_i(x_0) - \frac{\partial f_i(x)}{\partial x}\bigg|_{x_0} \cdot \delta x - \frac{\partial f_i(x)}{\partial y}\bigg|_{x_0} \cdot \delta y, \]
which can be expressed in vector form as
\[ e = b - A \cdot \Delta x, \]
where
\[ A \triangleq \begin{bmatrix} \frac{\partial f_2(x)}{\partial x} & \frac{\partial f_2(x)}{\partial y} \\ \frac{\partial f_3(x)}{\partial x} & \frac{\partial f_3(x)}{\partial y} \\ \vdots & \vdots \\ \frac{\partial f_M(x)}{\partial x} & \frac{\partial f_M(x)}{\partial y} \end{bmatrix}, \]
\[ b \triangleq \begin{bmatrix} d_{21} + \|x_2 - x_1\| - f_2(x_0) \\ d_{31} + \|x_3 - x_1\| - f_3(x_0) \\ \vdots \\ d_{M1} + \|x_M - x_1\| - f_M(x_0) \end{bmatrix}. \]
The least-squares estimate for (13) is given by
\[ \Delta x = (A^T A)^{-1} A^T b, \]
and the target location is updated to
\[ x = x_0 + \Delta x. \]
The updated target location is utilized in the next iteration until the magnitude of \( \Delta x \) becomes less than a prescribed tolerance.

V. A Low Complexity Localization Algorithm

The SDP + Taylor algorithm presented in Sec. IV provides accurate solutions at a cost of considerable computational complexity, thus it may not be an ideal approach for applications where computational resources are limited. In this section we present a constrained least-squares estimator that provides good solution accuracy with reduced complexity for A-TDOA positioning systems.
We start by rewriting the error functions in (7) as

\[ \hat{e}(x) = Ar - p, \]  

(18a)

where

\[
A = \begin{bmatrix}
1 & 1 & 0 & \ldots & 0 \\
1 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \ldots & 1 \\
\end{bmatrix},
\]

(18b)

\[
r = \begin{bmatrix}
r_1 \\
r_2 \\
\vdots \\
r_M \\
\end{bmatrix},
\]

(18c)

\[
p = \begin{bmatrix}
p_2 \\
p_3 \\
\vdots \\
p_M \\
\end{bmatrix},
\]

(18d)

with

\[ r_i = \|x - x_i\|; \]  

(18e)

\[ p_i = d_{i1} + \|x_i - x_1\|; \]  

(18f)

and \(d_{i1}\) representing the measured range differences.

The localization problem at hand can be formulated as a constrained least-squares problem

\[ \min_{x,r} \|Ar - p\|^2 \]  

(19a)

subject to:

\[ r_i = \|x - x_i\|, \quad i = 1, 2, \ldots, M. \]  

(19b)
Below we propose to solve (19) in two steps: the first step treats \( r \) as the variable of (19a) only and finds all minimizers \( r \) that are parameterized in terms of a free parameter \( \phi \) (see (22)); the second step then deal with the constraints in (19a).

Let the singular value decomposition [18] of matrix \( A \) be given by

\[
A = U \Sigma V^T, \tag{20}
\]

where \( U \in \mathbb{R}^{(M-1) \times (M-1)} \) and \( V \in \mathbb{R}^{M \times M} \) are orthogonal and \( \Sigma = [S \ 0] \) with \( S = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_{M-1}\} > 0 \). Using (20), we can write

\[
\|Ar - p\| = \|\Sigma z - \tilde{p}\| \tag{21}
\]

where \( z = V^T r \) and \( \tilde{p} = U^T p \). If we denote

\[
z = \begin{pmatrix} \hat{z} \\ \phi \end{pmatrix} \tag{22}
\]

then (21) becomes \( \|Ar - p\| = \|S \hat{z} - \tilde{p}\| \), hence \( \|Ar - p\| \) reaches its minimum if \( \hat{z} = S^{-1} \tilde{p} \), and the optimal \( z \) is given by

\[
z^* = \begin{pmatrix} S^{-1} \tilde{p} \\ \phi \end{pmatrix} \tag{23}
\]

with \( \phi \) as a free scaler parameter. Therefore the optimal \( r \) for (19a) is given by

\[
r^* = Vz^* = V_1 S^{-1} U^T p + v_M \phi \triangleq r_s + v_M \phi \tag{24}
\]

where parameter \( \phi \) will be optimally tuned in the next step in dealing with (19b).

With the optimal \( r \) determined in (24), the constraints in (19b) can be written as

\[
h(x) - v_M \phi - r_s = 0, \tag{25}
\]
where

\[ h(x) = \begin{bmatrix} \|x - x_1\| \\ \|x - x_2\| \\ \vdots \\ \|x - x_M\| \end{bmatrix} \]  \hfill (26)

The \( L_2 \)-optimal approximate solution of (25) can be obtained by solving

\[
\min_{x, \phi} b(x, \phi) = \frac{1}{2} \|h(x) - v_M \phi - r_s\|^2
\]  \hfill (27)

The Gauss-Newton (GN) iteration [19] for minimizing \( b(x, \phi) \) is given by

\[
\begin{bmatrix} x^{k+1} \\ \phi^{k+1} \end{bmatrix} = \begin{bmatrix} x^k \\ \phi^k \end{bmatrix} - \alpha_k \cdot H^{-1}(x_i) \cdot \nabla b(x^k, \phi^k)
\]  \hfill (28)

where \( \alpha_k \) is determined by an inexact line search, and

\[
\nabla b(x^k, \phi^k) = J^T(x^k) \cdot (h(x^k) - v_M \phi^k - r_s)
\]  \hfill (29)

\[
H(x^k) = J^T(x^k)J(x^k) + \epsilon \text{ with } \epsilon \text{ a small positive constant, and}
\]  \hfill (30)

\[
J(x) = \begin{bmatrix} \frac{x - x_1}{\|x - x_1\|} \\ \vdots \\ -v_M \\ \frac{x - x_M}{\|x - x_M\|} \end{bmatrix}
\]  \hfill (31)

We remark that matrix \( A \) in (18b) is independent of measurements, hence \( V_1, S, U \) and \( v_M \) can be pre-calculated; and for 2-dimensional location problems \( H(x^k) \) is of size \( 3 \times 3 \), hence the complexity of computing \( H^{-1}(x^k) \) as required in (28) is insignificant. The algorithm is found insensitive to its initial point \([x_0^T \phi_0]^T\) as long as it is a reasonable one e.g. \( x_0 = \frac{1}{M} \sum_{i=1}^{M} x_i \) and \( \phi_0 = 0 \). Typically the algorithm converges in less than five iterations.

VI. SIMULATION RESULTS

The hybrid SDP+Taylor algorithm and constrained LS algorithm proposed above were applied to a system with 4 anchor nodes that are located at (0 m, 0 m), (0 m, 100 m), (100 m, 0 m), and (100 m, 100 m) with anchor node 1 at (0 m, 0 m) as the reference node. The convex solver CVX [20] was applied
to solve the SDP problem involved. The algorithms’ performance was evaluated for the above system with the target assigned to each integer grid point \((i, j)\) over the region \(0 \leq i \leq 100 \text{ m}, 0 \leq j \leq 100 \text{ m}\). Measurement noise was assumed to be Gaussian white with zero mean and standard deviation \(\sigma = 0.1\). Mean-square error (MSE), namely

\[
\text{MSE} = \mathbb{E}\{\|\hat{x}_k - x_k\|^2\},
\]

was employed as the performance measure, with \(x_k\) the \(k_{th}\) true target position and \(\hat{x}_k\) its estimation.

Fig. 5 and Fig. 6 demonstrate the estimated target node MSE of the hybrid and constrained LS algorithm respectively. The results indicate that the hybrid estimator outperforms the constrained LS estimator. Nevertheless, the constrained LS consumes less CPU time and requires lower computational complexity. Therefore, there is a trade-off between the accuracy and complexity.

As discussed in Section III, the estimation error is significantly higher when the target node is located close to the reference node. This is also observed from Figs. 5 and 6 where positions around anchor node 4 (100 m, 100 m) have over 10 dB smaller estimation error than positions around the reference node 1 (0 m, 0 m). Intuitively, choosing a reference node far apart from the target node can result in better performance. This can be accomplished by employing a two step estimation method, where the target node coarse position is obtained by a first step estimator, and the reference node can then be re-selected based on the coarse target coordinate. As the reference node is re-selected, another non-redundant A-TDOA measurement set has to be taken, and measurement results are in turn supplied to a second step estimator to achieve more precise estimate. Having said that, it is not necessary to perform another set of measurement if the re-selected reference node does not change from step one. It is noteworthy that the first step estimator should be a global minimizer, and its estimate can be used as an initial guess for the second step estimator. In addition, the second step algorithm should be able to provide precise estimation, since the ultimate performance is limited by it. Fig. 7 gives an example demonstrating the MSE performance of the SDP + Taylor algorithm with reference node re-selection employed after SDP optimization. It is obvious that the MSE is reduced significantly, and therefore, estimation accuracy is greatly improved.
Monte Carlo computer simulations have been conducted to corroborate the theoretical development and to evaluate the performance of the proposed hybrid estimator and the constrained least-squares estimator by comparing against the LS method [12], the Taylor series method [14], and the Cramer-Rao lower bound (CRLB). For each of the estimation algorithms, a total of \( N = 10000 \) Monte Carlo runs are performed.

Fig. 8-Fig. 10 show the MSE of the position estimation versus noise variance \( \sigma^2 \) with the target node located at \((90, 20)\) m, \((55, 80)\) m and \((50, 150)\) m respectively. The reference node is fixed at anchor node 1. It is obvious that the LS estimator performs the worst as compared to the other estimators. In addition, during the simulation, it was observed that the selection of the initial guess point for Taylor estimator is extremely critical. A bad selection could result in a dramatic position offset from the real target position. To guarantee global convergence such that the performance comparison among various estimators is valid, the initial point is restricted to a small area close to the target node. Having said that, it is worthwhile to note that for the hybrid estimator there is no need to have the initial guess close to the real target position due to the global convergence feature of the SDP method. Ultimately, as can be seen from Fig. 8-Fig. 10, the hybrid estimator closely follows the CRLB and outperforms all the other estimators.

In addition, the proposed constrained least-squares algorithm achieves better accuracy than the LS algorithm but performs slightly worse than the hybrid and Taylor estimator. The greatest advantage of this estimator is its simplicity and relatively good performance. Battery life and the computational capability of a mobile device is rather limited, and the low computational complexity feature of this algorithm can close the gap. Also, fast convergence makes the real-time tracking applicable. Thus, in real-time applications, and in cases where computing module resides in a mobile device, the constrained least-squares algorithm is a good fit.

VII. CONCLUSIONS

An A-TDOA positioning system has been proposed in this paper. The distinct advantage of the A-TDOA system is that no clock synchronization is needed, neither for anchor-target nor anchor-anchor synchronization. Thus, the complexity of the system can be reduced significantly. The asynchronous
property is achieved by adding an “amplify and retransmit” capability on the target node. Once the target node receives a signal from a Tx anchor node, it amplifies the received signal and then retransmits it immediately. Afterwards, the Rx anchor nodes measure the time difference of the two received signals to estimate the target position. Without synchronization requirements, the A-TDOA system can be built with lower cost and lower complexity. Besides, by properly selecting the reference, the A-TDOA system can achieve superior performance.

Additionally, two novel localization algorithms, namely the hybrid (SDP+Taylor) and constrained LS algorithm have been proposed to estimate the target position in the A-TDOA system. The hybrid estimation algorithm combines the SDP and Taylor series methods to achieve global convergence and superior estimation accuracy. Reference re-selection can readily be employed by the SDP + Taylor method to further improve the performance. In addition, its robustness is demonstrated via providing good accuracy regardless of the noise power. The hybrid algorithm can be applied in applications where accuracy is the most critical. The constrained LS algorithm obtains relatively good performance while keeps the computational complexity low, and the convergence speed is fast. These properties are very useful in real-time systems and mobile devices where battery life and computational capability is limited. Simulations have been performed to corroborate the theoretical development and to compare the relative localization accuracy for different methods as well as the CRLB. Results have shown that the proposed hybrid estimation algorithm performs better than the LS and Taylor series methods, and is close to the CRLB, while the constrained LS algorithm performs slightly worse than the hybrid algorithm with the advantage of low complexity and fast convergence.

REFERENCES


Fig. 1. Illustration of the timing diagram of the elliptical positioning system.
Fig. 2. Three anchor nodes locate a target on a 2D area.
Fig. 3. (a) CRLB of the A-TDOA system with anchor node 1 being the reference. (b) CRLB of the A-TDOA system with optimum reference selection.
Fig. 4. Comparison of the TOA, TDOA, and A-TDOA system CRLB.
Fig. 5. The MSE of the A-TDOA system using the hybrid algorithm with Anchor node 1 as the reference.
Fig. 6. The MSE of the A-TDOA system using the constrained least-squares algorithm with Anchor node 1 as the reference.
Fig. 7. The MSE of the A-TDOA system using the hybrid (SDP+Taylor) algorithm with reference node selection optimized.
Fig. 8. Algorithm comparison in terms of mean square position error measured at (90 m, 20 m).
Fig. 9. Algorithm comparison in terms of mean square position error measured at (55 m, 80 m).
Fig. 10. Algorithm comparison in terms of mean square position error measured at (50 m, 150 m).