Digital Signal Processing Using MATLAB

Discrete Fourier Transform (DFT)

F. Gebali

EOW 433

Office Phone: 250-721-6509

https://ece.engr.uvic.ca/~fayez/

Outline

DFT

Discrete Fourier Transform (DFT)

	Signal Periodicity	
	Periodic $X[k]$	**Non-Periodic** $X(\widehat{\omega})$
Continuous $x(t)$	Fourier Series: t continuous ω discrete	Fourier Integral: t continuous ω continuous
Discrete $x[n]$	DFT/FFT: t discrete ω discrete	DTFT: t discrete ω continuous

The Forward DFT

$$X[\widehat{\omega}_k] = \sum_{n=0}^{N-1} x[n]\; e^{-j\widehat{\omega}_k n}$$

where

$$\widehat{\omega}_k = \frac{2\pi k}{N}, \qquad 0 \le k < N \quad \text{and} \quad 0 \le \widehat{\omega}_k < 2\pi$$

The Forward DFT

$$X[\widehat{\omega}_k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\widehat{\omega}_k n}$$

where

$$\widehat{\omega}_k = \frac{2\pi k}{N}, \qquad 0 \le k < N \quad \text{and} \quad 0 \le \widehat{\omega}_k < 2\pi$$

Just as time is indexed by n, $X[\widehat{\omega}_k]$ is indexed by k:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N}$$

The Range of $\widehat{\omega}_k$ and k

Bounds on $\widehat{\omega}_k$ are given, as usual, by

$$-\pi \leq \quad \widehat{\omega}_k = \frac{2\pi k}{N} \quad < \pi$$

This implies bounds on k depending whether N is even or odd

$$-N/2 \leq k < N/2 \qquad N \text{ is even}$$

$$-(N-1)/2 \leq k \leq (N-1)/2 \qquad N \text{ is odd}$$

The Range of k: Case $N = 6$ Even

k	-3	-2	-1	0	1	2
$\widehat{\omega}_k$	$-\frac{2\pi\times 3}{6}$	$-\frac{2\pi\times 2}{6}$	$-\frac{2\pi}{6}$	0	$\frac{2\pi}{6}$	$\frac{2\pi\times 2}{6}$

The Range of k: Case $N = 7$ Odd

k	-3	-2	-1	0	1	2	3
$\widehat{\omega}_k$	$-\frac{2\pi\times 3}{7}$	$-\frac{2\pi\times 2}{7}$	$-\frac{2\pi}{7}$	0	$\frac{2\pi}{7}$	$\frac{2\pi\times 2}{7}$	$\frac{2\pi\times 3}{7}$

DFT of Simple Signals

DFT of Periodic Impulse Signal $\delta[n]$

$$\Delta[k] = \sum_{n=0}^{N-1} \delta[n] \, e^{-j2\pi kn/N}$$

DFT of Periodic Impulse Signal $\delta[n]$

$$
\begin{aligned}
\Delta[k] &= \sum_{n=0}^{N-1} \delta[n]\, e^{-j2\pi kn/N} \\
&= 1, \qquad -N/2 \leq k < N/2
\end{aligned}
$$

DFT of Delayed Periodic Impulse Signal $\delta[n - n_0]$

$$\Delta[k] = \sum_{n=0}^{N-1} \delta[n - n_0] \, e^{-j2\pi kn/N}$$

DFT of Delayed Periodic Impulse Signal $\delta[n - n_0]$

$$
\begin{aligned}
\Delta[k] &= \sum_{n=0}^{N-1} \delta[n - n_0]\, e^{-j2\pi kn/N} \\
&= e^{-j2\pi kn_0/N}, \qquad -N/2 \le k < N/2
\end{aligned}
$$

DFT of Periodic Length-L Pulse $r_L[n]$, $L < N$

$$R_L[k] = \sum_{n=0}^{N-1} r_L[n]\, e^{-j2\pi kn/N}$$

DFT of Periodic Length-L Pulse $r_L[n]$, $L < N$

$$
\begin{aligned}
R_L[k] &= \sum_{n=0}^{N-1} r_L[n]\, e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{L-1} e^{-j2\pi kn/N}
\end{aligned}
$$

DFT of Periodic Length-L Pulse $r_L[n]$, $L < N$

$$
\begin{aligned}
R_L[k] &= \sum_{n=0}^{N-1} r_L[n]\, e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{L-1} e^{-j2\pi kn/N} \\
&= \frac{1 - e^{-j2\pi kL/N}}{1 - e^{-j2\pi k/N}}, \qquad -N/2 \le k < N/2
\end{aligned}
$$

DFT of Periodic Length-L Pulse $r_L[n]$, $L < N$

$$
\begin{aligned}
R_L[k] &= \sum_{n=0}^{N-1} r_L[n]\, e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{L-1} e^{-j2\pi kn/N} \\
&= \frac{1 - e^{-j2\pi kL/N}}{1 - e^{-j2\pi k/N}}, \qquad -N/2 \leq k < N/2 \\
&= \frac{\sin \frac{\pi kL}{N}}{\sin \frac{\pi k}{N}}\, e^{-j\pi k(L-1)/N}
\end{aligned}
$$

DFT of Periodic Length-N Pulse $r_N[n]$

$$X[k] = \frac{1 - e^{-j2\pi k}}{1 - e^{-j2\pi k/N}}, \qquad -N/2 \leq k < N/2$$

DFT of Periodic Length-N Pulse $r_N[n]$

$$\begin{aligned} X[k] &= \frac{1 - e^{-j2\pi k}}{1 - e^{-j2\pi k/N}}, \qquad -N/2 \leq k < N/2 \\ &= N\delta[k] \end{aligned}$$

DFT of Periodic Exponential Signal a^n

We have $x[n] = a^n u[n], \qquad 0 < |a| < 1.$

DFT of Periodic Exponential Signal a^n

We have $x[n] = a^n u[n], \qquad 0 < |a| < 1.$

$$X[k] = \sum_{n=0}^{N-1} a^n \, u[n] \, e^{-j2\pi kn/N}$$

DFT of Periodic Exponential Signal a^n

We have $x[n] = a^n u[n], \qquad 0 < |a| < 1.$

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} a^n \; u[n] \; e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{N-1} a^n \; e^{-j2\pi kn/N}
\end{aligned}
$$

DFT of Periodic Exponential Signal a^n

We have $x[n] = a^n u[n], \qquad 0 < |a| < 1.$

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} a^n \; u[n] \; e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{N-1} a^n \; e^{-j2\pi kn/N} \\ &= \frac{1 - a^N e^{-j2\pi k}}{1 - a \; e^{-j2\pi k/N}} \end{aligned}$$

DFT of Periodic Exponential Signal a^n

We have $x[n] = a^n u[n], \qquad 0 < |a| < 1.$

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} a^n \, u[n] \, e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{N-1} a^n \, e^{-j2\pi kn/N} \\
&= \frac{1 - a^N e^{-j2\pi k}}{1 - a \, e^{-j2\pi k/N}} \\
&= \frac{1 - a^N}{1 - a \, e^{-j2\pi k/N}}, \qquad -N/2 \le k < N/2
\end{aligned}
$$

DFT of Periodic Complex Exponential Signal $a^n\ e^{j2\pi k_0 n/N}, \quad 0 \leq k < N$

We have $x[n] = a^n\ e^{j2\pi k_0 n/N}$.

DFT of Periodic Complex Exponential Signal $a^n\ e^{j2\pi k_0 n/N}, \quad 0 \le k < N$

We have $x[n] = a^n\ e^{j2\pi k_0 n/N}$.

$$X[k] = \sum_{n=0}^{N-1} a^n\ e^{j2\pi k_0 n/N}\ e^{-j2\pi kn/N}$$

DFT of Periodic Complex Exponential Signal $a^n\ e^{j2\pi k_0 n/N}, \quad 0 \leq k < N$

We have $x[n] = a^n\ e^{j2\pi k_0 n/N}$.

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} a^n\ e^{j2\pi k_0 n/N}\ e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{N-1} a^n\ e^{-j2\pi(k-k_0)n/N}
\end{aligned}
$$

DFT of Periodic Complex Exponential Signal

$a^n \, e^{j2\pi k_0 n/N}, \quad 0 \le k < N$

We have $x[n] = a^n \, e^{j2\pi k_0 n/N}$.

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} a^n \, e^{j2\pi k_0 n/N} \, e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{N-1} a^n \, e^{-j2\pi (k-k_0) n/N} \\
&= \frac{1 - a^N}{1 - a \, e^{-j2\pi (k-k_0)/N}}, \qquad -N/2 \le k < N/2
\end{aligned}
$$

DFT of Periodic Complex Exponential Signal $e^{j2\pi k_0 n/N}, \quad 0 \leq k < N$

We have $x[n] = e^{j2\pi k_0 n/N}$

DFT of Periodic Complex Exponential Signal $e^{j2\pi k_0 n/N}, \quad 0 \le k < N$

We have $x[n] = e^{j2\pi k_0 n/N}$

$$X[k] = \sum_{n=0}^{N-1} e^{j2\pi k_0 n/N} \, e^{-j2\pi kn/N}$$

DFT of Periodic Complex Exponential Signal $e^{j2\pi k_0 n/N}, \quad 0 \leq k < N$

We have $x[n] = e^{j2\pi k_0 n/N}$

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} e^{j2\pi k_0 n/N} \, e^{-j2\pi kn/N} \\ &= \frac{1 - e^{-j2\pi(k-k_0)}}{1 - \ e^{-j2\pi(k-k_0)/N}}, \qquad -N/2 \leq k < N/2 \end{aligned}$$

DFT of Periodic Complex Exponential Signal $e^{j2\pi k_0 n/N}, \quad 0 \le k < N$

We have $x[n] = e^{j2\pi k_0 n/N}$

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} e^{j2\pi k_0 n/N} \, e^{-j2\pi kn/N} \\
&= \frac{1 - e^{-j2\pi(k-k_0)}}{1 - e^{-j2\pi(k-k_0)/N}}, \qquad -N/2 \le k < N/2 \\
&= N\delta[k - k_0]
\end{aligned}
$$

DFT of Periodic Cosine Signal

$$x[n] = A\cos(\widehat{\omega}_0 n) \qquad = A\cos(2\pi k_0 n/N)$$

$$\widehat{\omega}_0 = 2\pi k_0/N$$

$$A\cos 2\pi k_0/N \quad \overset{DFT}{\longleftrightarrow} \quad \frac{AN}{2}\delta[k-k_0] + \frac{AN}{2}\delta[k+k_0]$$

Relating DFT and DTFT

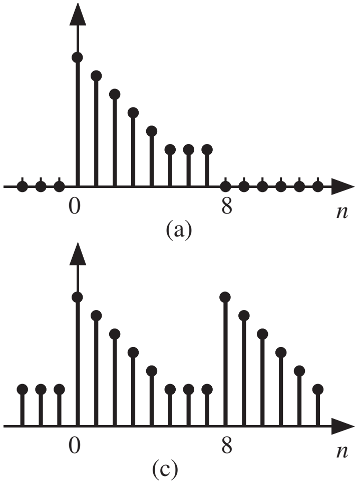
$$\widehat{\omega} = \frac{2\pi k}{N}$$

DFT Properties

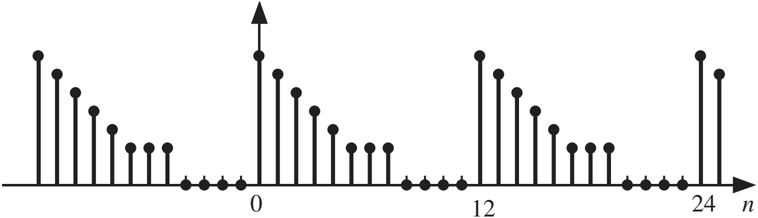
Linearity Property of DFT

$$\alpha x_1[n] + \beta x_2[n] \quad \stackrel{DFT}{\longleftrightarrow} \quad \alpha X_1[k] + \beta X_2[k]$$

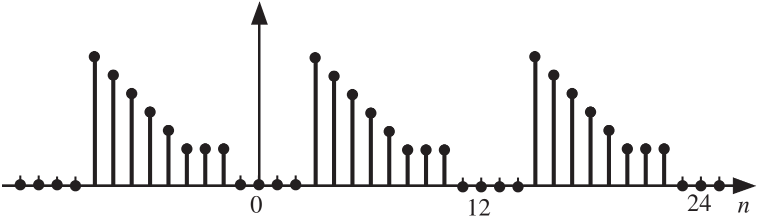
Time Shift of DFT: Circular Shift when $N = 8$



Padding Periodic Signals: $N = 8$ and $N' = 12$



(a)



(b)

Circular Shift Transform Pair

$$x[n - n_0] \quad \overset{DFT}{\longleftrightarrow} \quad e^{-j2\pi k n_0/N}\, X[k]$$

Periodicity Property of DFT

$$x[n] = x[n+N] \quad \overset{DFT}{\longleftrightarrow} \quad X[k] = X[k+N]$$

Frequency Shift Property of DFT

$$e^{j2\pi k_0 n/N}\, x[n] \quad \overset{DFT}{\longleftrightarrow} \quad X[k-k_0]$$

Conjugate Symmetry of DFT: Case $-N/2 \leq k < N/2$

$$X[-k] = X^*[k]$$

1. This is the situation when $-\pi \leq \widehat{\omega}_k < \pi$
2. Only for real signals

Conjugate Symmetry of DFT: Case $0 \leq k < N$

$$X[k] = X^*[N - k]$$

1. This is the situation when $0 \leq \widehat{\omega} < 2\pi$
2. Only for real signals

The Inverse DFT

The Inverse DFT

$$
\begin{aligned}
x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j2\pi kn/N} \\
&= \frac{1}{N} \sum_{k=-N/2}^{N/2-1} X[k]\, e^{j2\pi kn/N}
\end{aligned}
$$

IDFT of a Constant in Frequency Domain

$$X[k] = 1, \qquad 0 \leq k < N$$

we can write

$$\begin{aligned} x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} e^{j2\pi kn/N} \\ &= \frac{1}{N} \times \frac{1 - e^{j2\pi n}}{1 - e^{j2\pi n/N}} \\ &= \delta[n] \end{aligned}$$

Summary So Far

$$\delta[n] \overset{DFT}{\longleftrightarrow} 1$$

$$\delta[n - n_0] \overset{DFT}{\longleftrightarrow} e^{-j2\pi k n_0/N}$$

$$e^{j2\pi k_0 n/N} \overset{DFT}{\longleftrightarrow} N\,\delta[k - k_0]$$

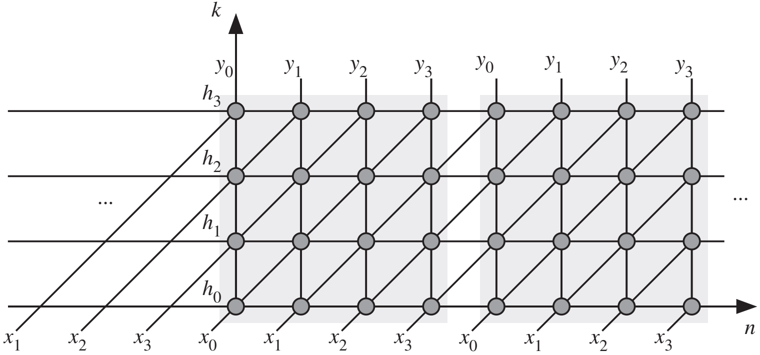
$$r_L[n] \overset{DFT}{\longleftrightarrow} \left[1 - e^{-j\pi kL/N}\right] / (1 - e^{j\pi k/N})$$

$$1 \overset{DFT}{\longleftrightarrow} N\,\delta[k]$$

$$A \cos 2\pi k_0/N \overset{DFT}{\longleftrightarrow} \frac{AN}{2}\delta[k - k_0] + \frac{AN}{2}\delta[k + k_0]$$

Circular Convolution

Circular Convolution Dependence Graph: Case $N = 4$



Circular Convolution Example

Example

Perform the linear and circular convolution operation on the two length-4 sequences

$$\begin{aligned} x[n] &= \{\underset{\uparrow}{1},\ 2,\ 3,\ 4\} \\ h[n] &= \{\underset{\uparrow}{1},\ -1,\ 2,\ -2\} \end{aligned}$$

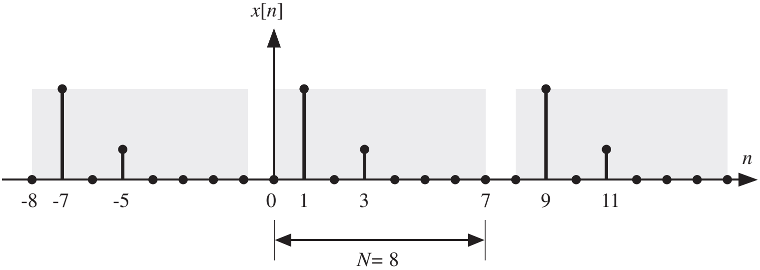
Parseval's Theorem

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=-N/2}^{N/2-1} |X[k]|^2$$

Example

Find the DFT for this signal and find its energy from time and frequency domain representations.

$$x[n] = \{\underset{\uparrow}{0},\ 3,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0\}$$



MATLAB code

```
N = 8;      x = [0 3 0 1 0 0 0 0];
X1 = fft(x,N)      X2 = fftshift(X1)
```

$$X1 = \begin{matrix} 4 & 1.4142 - 2.8284i & -2i & -1.4142 - 2.8284i \\ -4 & -1.4142 + 2.8284i & 2i & 1.4142 + 2.8284i \end{matrix}$$

$$X2 = \begin{matrix} -4 & -1.4142 + 2.8284i & 2i & 1.4142 + 2.8284i \\ 4 & 1.4142 - 2.8284i & -2i & -1.4142 - 2.8284i \end{matrix}$$