Digital Signal Processing Using MATLAB

Discrete-Time Fourier Transform (DTFT)

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Outline

Discrete-Time Fourier Transform (DTFT)

Discrete-Time Fourier Transform (DTFT)

	Signal Periodicity	
	Periodic $X[k]$	**Non-Periodic** $X(\widehat{\omega})$
Continuous $x(t)$	Fourier Series: t continuous ω discrete	Fourier Integral: t continuous ω continuous
Discrete $x[n]$	DFT : t discrete ω discrete	DTFT: t discrete ω continuous

The Forward Discrete-Time Fourier Transform (DTFT)

$$X(\widehat{\omega}) = \sum_{n=-\infty}^{\infty} x[n]\; e^{-j\widehat{\omega}n}$$

When $x[n]$ is right-sided we can write

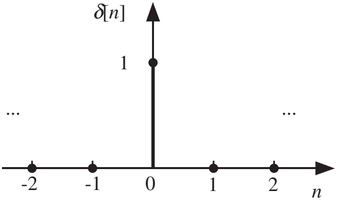
$$X(\widehat{\omega}) = \sum_{n=0}^{\infty} x[n]\; e^{-j\widehat{\omega}n}$$

Periodicity of DTFT

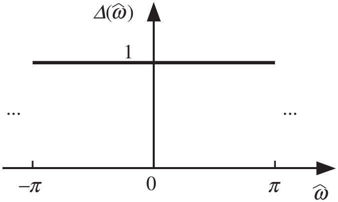
Example

Prove that the signal $X(\widehat{\omega})$ is periodic with period 2π.

DTFT of Unit Impulse



(a)



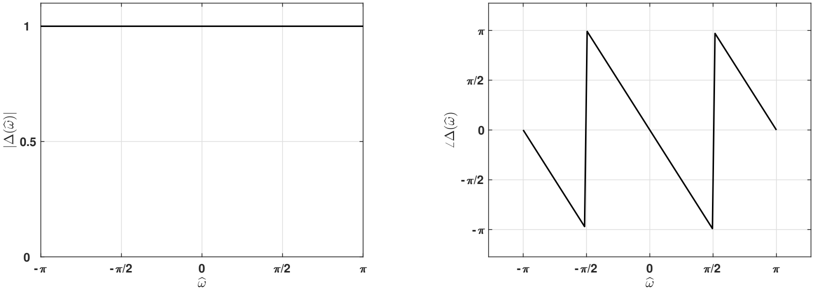
(b)

$$\Delta(\widehat{\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\widehat{\omega} n}$$
$$= 1, \qquad -\pi \leq \widehat{\omega} < \pi$$

DTFT of Delayed Unit Impulse

Example

Obtain the DTFT of the delayed impulse $\delta[n - n_0]$ when $n_0 = 2$.



DTFT of Length-L Pulse: $r_L[n]$

$$R_L(\widehat{\omega}) = \sum_{n=-\infty}^{\infty} r_L[n]\, e^{-j\widehat{\omega}n}$$

DTFT of Length-L Pulse: $r_L[n]$

$$
\begin{aligned}
R_L(\widehat{\omega}) &= \sum_{n=-\infty}^{\infty} r_L[n]\, e^{-j\widehat{\omega} n} \\
&= \sum_{n=0}^{L-1} e^{-j\widehat{\omega} n}
\end{aligned}
$$

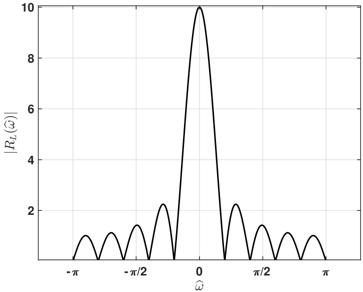
DTFT of Length-L Pulse: $r_L[n]$

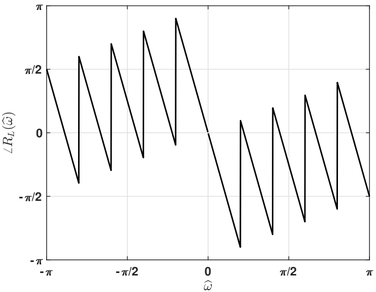
$$
\begin{aligned}
R_L(\widehat{\omega}) &= \sum_{n=-\infty}^{\infty} r_L[n]\, e^{-j\widehat{\omega} n} \\
&= \sum_{n=0}^{L-1} e^{-j\widehat{\omega} n} \\
&= \frac{1 - e^{-jL\widehat{\omega}}}{1 - e^{-j\widehat{\omega}}}, \qquad -\pi \le \widehat{\omega} < \pi
\end{aligned}
$$

DTFT of Length-L Pulse: $r_L[n]$

$$
\begin{aligned}
R_L(\widehat{\omega}) &= \sum_{n=-\infty}^{\infty} r_L[n]\, e^{-j\widehat{\omega}n} \\
&= \sum_{n=0}^{L-1} e^{-j\widehat{\omega}n} \\
&= \frac{1-e^{-jL\widehat{\omega}}}{1-e^{-j\widehat{\omega}}}, \qquad -\pi \le \widehat{\omega} < \pi \\
&= \frac{\sin(L\widehat{\omega}/2)}{\sin\widehat{\omega}/2}\, e^{-j(L-1)\widehat{\omega}/2}
\end{aligned}
$$

DTFT of Length-L Pulse: Case $L = 10$





Can you find peak value and location of zeros?

DTFT of Delayed Length-L Pulse

Example

Obtain the DTFT of the shifted length-L pulse $r_L[n - n_0]$.

DTFT of Exponential Signal: $a^n\, u[n]$

$$X(\widehat{\omega}) = \sum_{n=-\infty}^{\infty} a^n\, u[n]\, e^{-j\widehat{\omega} n}, \qquad a < 1$$

DTFT of Exponential Signal: $a^n\, u[n]$

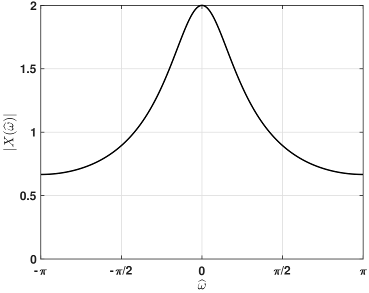
$$
\begin{aligned}
X(\widehat{\omega}) &= \sum_{n=-\infty}^{\infty} a^n\, u[n]\, e^{-j\widehat{\omega} n}, \qquad a < 1 \\
&= \sum_{n=0}^{\infty} a^n\, e^{-j\widehat{\omega} n}
\end{aligned}
$$

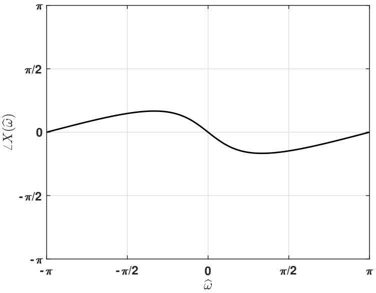
DTFT of Exponential Signal: $a^n\, u[n]$

$$
\begin{aligned}
X(\widehat{\omega}) &= \sum_{n=-\infty}^{\infty} a^n\, u[n]\, e^{-j\widehat{\omega} n}, \qquad a < 1 \\
&= \sum_{n=0}^{\infty} a^n\, e^{-j\widehat{\omega} n} \\
&= \frac{1}{1 - a\, e^{-j\widehat{\omega}}}
\end{aligned}
$$

DTFT of Decaying Exponential Signal: $a^n\ u[n]$

Case $a = 0.5$





DTFT of Complex Exponential Signal: $a^n \, e^{j\widehat{\omega}_0 n} \, u[n]$

$$X(\widehat{\omega}) = \sum_{n=-\infty}^{\infty} a^n \, e^{j\widehat{\omega}_0 n} \, u[n] \, e^{-j\widehat{\omega} n}$$

DTFT of Complex Exponential Signal: $a^n\ e^{j\widehat{\omega}_0 n}\ u[n]$

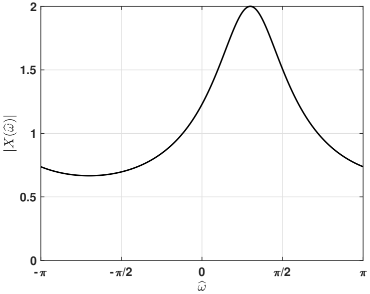
$$
\begin{aligned}
X(\widehat{\omega}) &= \sum_{n=-\infty}^{\infty} a^n\ e^{j\widehat{\omega}_0 n}\ u[n]\ e^{-j\widehat{\omega} n} \\
&= \sum_{n=0}^{\infty} a^n\ e^{-j(\widehat{\omega}-\widehat{\omega}_0)n}
\end{aligned}
$$

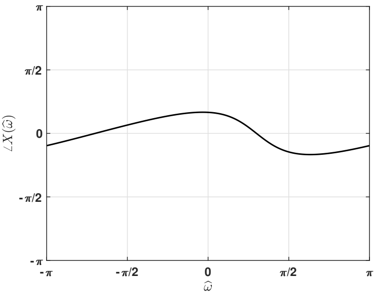
DTFT of Complex Exponential Signal: $a^n \, e^{j\widehat{\omega}_0 n} \, u[n]$

$$
\begin{aligned}
X(\widehat{\omega}) &= \sum_{n=-\infty}^{\infty} a^n \, e^{j\widehat{\omega}_0 n} \, u[n] \, e^{-j\widehat{\omega} n} \\
&= \sum_{n=0}^{\infty} a^n \, e^{-j(\widehat{\omega}-\widehat{\omega}_0)n} \\
&= \frac{1}{1 - a \, e^{-j(\widehat{\omega}-\widehat{\omega}_0)}}
\end{aligned}
$$

DTFT of Decaying Complex Exponential Signal: $a^n \, e^{j\widehat{\omega}_0 n} \, u[n]$

Case $a = 0.5$ and $\widehat{\omega}_0 = 0.3\pi$





Properties of the DTFT

Linearity Property of DTFT

1. Assume we are given two signals

$$x_1[n] \stackrel{DTFT}{\longleftrightarrow} X_1(\widehat{\omega}) \qquad x_2[n] \stackrel{DTFT}{\longleftrightarrow} X_2(\widehat{\omega})$$

Linearity Property of DTFT

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2. Now consider the DTFT of the signal $x[n] = x_1[n] + x_2[n]$

$$X(\widehat{\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\widehat{\omega}n}$$

Linearity Property of DTFT

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2. Now consider the DTFT of the signal $x[n] = x_1[n] + x_2[n]$

$$\begin{aligned} X(\widehat{\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\widehat{\omega}n} \\ &= \sum_{n=-\infty}^{\infty} (x_1[n] + x_2[n])\, e^{-j\widehat{\omega}n} \end{aligned}$$

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$$\begin{aligned} X(\widehat{\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\widehat{\omega}n} \\ &= \sum_{n=-\infty}^{\infty} (x_1[n] + x_2[n])\, e^{-j\widehat{\omega}n} \\ &= \sum_{n=-\infty}^{\infty} x_1[n]\, e^{-j\widehat{\omega}n} + \sum_{n=0}^{\infty} x_2[n]\, e^{-j\widehat{\omega}n} \end{aligned}$$

Linearity Property of DTFT

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$$x_1[n] \overset{DTFT}{\longleftrightarrow} X_1(\widehat{\omega}) \qquad\qquad x_2[n] \overset{DTFT}{\longleftrightarrow} X_2(\widehat{\omega})$$

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$$\begin{aligned}
X(\widehat{\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\widehat{\omega}n} \\
&= \sum_{n=-\infty}^{\infty} (x_1[n] + x_2[n])\, e^{-j\widehat{\omega}n} \\
&= \sum_{n=-\infty}^{\infty} x_1[n]\, e^{-j\widehat{\omega}n} + \sum_{n=0}^{\infty} x_2[n]\, e^{-j\widehat{\omega}n} \\
&= X_1(\widehat{\omega}) + X_2(\widehat{\omega})
\end{aligned}$$

Example

Find the DTFT of the signal $x[n]$:

$$x[n] = \{\underset{\uparrow}{1},\ 2,\ 4,\ 8\}$$

Time Shift Property of DTFT

Find $Y(\widehat{\omega})$ for

$$y[n] = x[n - n_0]$$

we have

$$Y(\widehat{\omega}) = \sum_{n=-\infty}^{\infty} x[n - n_0]\, e^{-j\widehat{\omega} n}$$

Time Shift Property of DTFT

Find $Y(\widehat{\omega})$ for

$$y[n] = x[n - n_0]$$

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$$\begin{aligned} Y(\widehat{\omega}) &= \sum_{n=-\infty}^{\infty} x[n - n_0]\, e^{-j\widehat{\omega} n} \\ &= \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\widehat{\omega}(m+n_0)} \end{aligned}$$

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Y(\widehat{\omega}) &= \sum_{n=-\infty}^{\infty} x[n - n_0]\, e^{-j\widehat{\omega} n} \\
&= \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\widehat{\omega}(m+n_0)} \\
&= e^{-j\widehat{\omega} n_0} \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\widehat{\omega} m} \\
&= X(\widehat{\omega})\, e^{-j\widehat{\omega} n_0}
\end{aligned}$$

Time delay $\equiv$ negative phase

Time Shift Property of a Rectangular Pulse

Example

Obtain the DTFT of the delayed length-L pulse $r_L[n-n_0]$ for $L = 10$ and $n_0 = 4$.

Periodicity Property of DTFT

We have

$$X(\widehat{\omega} + 2\pi) = X(\widehat{\omega})$$

This is because $\widehat{\omega} + 2\pi$ is an alias of $\widehat{\omega}$

Frequency Shift Property of DTFT

Find $Y(\widehat{\omega})$ for

$$y[n] = e^{j\widehat{\omega}_0 n} x[n]$$

we have

$$Y(\widehat{\omega}) = \sum_{n=-\infty}^{\infty} e^{j\widehat{\omega}_0 n} \, x[n] \, e^{-j\widehat{\omega} n}$$

Frequency Shift Property of DTFT

Find $Y(\widehat{\omega})$ for

$$y[n] = e^{j\widehat{\omega}_0 n} x[n]$$

we have

$$\begin{aligned} Y(\widehat{\omega}) &= \sum_{n=-\infty}^{\infty} e^{j\widehat{\omega}_0 n}\, x[n]\, e^{-j\widehat{\omega} n} \\ &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\widehat{\omega}-\widehat{\omega}_0) n} \end{aligned}$$

Frequency Shift Property of DTFT

Find $Y(\widehat{\omega})$ for

$$y[n] = e^{j\widehat{\omega}_0 n} x[n]$$

we have

$$\begin{aligned} Y(\widehat{\omega}) &= \sum_{n=-\infty}^{\infty} e^{j\widehat{\omega}_0 n} \, x[n] \, e^{-j\widehat{\omega} n} \\ &= \sum_{n=-\infty}^{\infty} x[n] \, e^{-j(\widehat{\omega}-\widehat{\omega}_0)n} \\ &= X(\widehat{\omega} - \widehat{\omega}_0) \end{aligned}$$

Conjugate Symmetry Property of DFT

We already know from two-sided spectrum of any signal:

$$X(-\widehat{\omega}) = X^*(\widehat{\omega})$$

DTFT & Convolution

Relationship Between DTFT and Convolution

1. Assume

$$y[n] = h[n] * x[n] = \sum_{k=0}^{N-1} h[k]\; x[n-k]$$

Relationship Between DTFT and Convolution

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$$y[n] = h[n] * x[n] = \sum_{k=0}^{N-1} h[k]\; x[n-k]$$

2. DTFT is

$$Y(\widehat{\omega}) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{N-1} h[k]\; x[n-k]\; e^{-j\widehat{\omega}n}$$

Relationship Between DTFT and Convolution

1. Reverse order of summations

$$Y(\widehat{\omega}) = \sum_{k=0}^{N-1} \sum_{n=-\infty}^{\infty} h[k]\, x[n-k]\, e^{-j\widehat{\omega} n}$$

Relationship Between DTFT and Convolution

1. Reverse order of summations

$$
\begin{aligned}
Y(\hat{\omega}) &= \sum_{k=0}^{N-1} \sum_{n=-\infty}^{\infty} h[k]\, x[n-k]\, e^{-j\hat{\omega}n} \\
&= \sum_{k=0}^{N-1} h[k] \left(\sum_{n=-\infty}^{\infty} x[n-k]\, e^{-j\hat{\omega}n} \right)
\end{aligned}
$$

Relationship Between DTFT and Convolution

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&= \sum_{k=0}^{N-1} h[k] \left(\sum_{n=-\infty}^{\infty} x[n-k]\, e^{-j\widehat{\omega}n} \right)
\end{aligned}
$$

2. Change variables $m = n - k$

$$
Y(\widehat{\omega}) = \sum_{k=0}^{N-1} h[k]\, e^{-j\widehat{\omega}k} \left(\sum_{m=-\infty}^{\infty} x[m]\, e^{-j\widehat{\omega}m} \right)
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Relationship Between DTFT and Convolution

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\end{aligned}
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2. Change variables $m = n - k$

$$
\begin{aligned}
Y(\widehat{\omega}) &= \sum_{k=0}^{N-1} h[k]\, e^{-j\widehat{\omega}k} \left(\sum_{m=-\infty}^{\infty} x[m]\, e^{-j\widehat{\omega}m} \right) \\
&= X(\widehat{\omega}) \sum_{k=0}^{N-1} h[k]\, e^{-j\widehat{\omega}k}
\end{aligned}
$$

Relationship Between DTFT and Convolution

We can write

$$x[n] * h[n] \quad \overset{DTFT}{\longleftrightarrow} \quad H(\widehat{\omega})\, X(\widehat{\omega})$$

Inverse DTFT

The Inverse DTFT (IDTFT)

$$x[n] = \frac{1}{2\pi} \int_{\widehat{\omega}=-\pi}^{\pi} X(\widehat{\omega})\, e^{j\widehat{\omega} n}\, d\widehat{\omega}$$

Proving the Relation $\delta[n] \overset{DTFT}{\longleftrightarrow} 1$

$$x[n] = \frac{1}{2\pi} \int_{\widehat{\omega}=-\pi}^{\pi} \Delta(\widehat{\omega})\, e^{j\widehat{\omega} n}\, d\widehat{\omega}$$

Proving the Relation $\delta[n] \overset{DTFT}{\longleftrightarrow} 1$

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} \Delta(\widehat{\omega})\, e^{j\widehat{\omega}n}\, d\widehat{\omega} \\
&= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} e^{j\widehat{\omega}n}\, d\widehat{\omega}
\end{aligned}
$$

Proving the Relation $\delta[n] \overset{DTFT}{\longleftrightarrow} 1$

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} \Delta(\widehat{\omega})\, e^{j\widehat{\omega} n}\, d\widehat{\omega} \\
&= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} e^{j\widehat{\omega} n}\, d\widehat{\omega} \\
&= \frac{1}{2\pi} \times \frac{e^{j\pi n} - e^{-j\pi n}}{jn}
\end{aligned}
$$

Proving the Relation $\delta[n] \overset{DTFT}{\longleftrightarrow} 1$

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi} \int_{\widehat{\omega}=-\pi}^{\pi} \Delta(\widehat{\omega})\ e^{j\widehat{\omega} n}\ d\widehat{\omega} \\
&= \frac{1}{2\pi} \int_{\widehat{\omega}=-\pi}^{\pi} e^{j\widehat{\omega} n}\ d\widehat{\omega} \\
&= \frac{1}{2\pi} \times \frac{e^{j\pi n} - e^{-j\pi n}}{jn} \\
&= \frac{\sin(\pi n)}{\pi n}
\end{aligned}
$$

Proving the Relation $\delta[n] \overset{DTFT}{\longleftrightarrow} 1$

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi} \int_{\widehat{\omega}=-\pi}^{\pi} \Delta(\widehat{\omega})\, e^{j\widehat{\omega}n}\, d\widehat{\omega} \\
&= \frac{1}{2\pi} \int_{\widehat{\omega}=-\pi}^{\pi} e^{j\widehat{\omega}n}\, d\widehat{\omega} \\
&= \frac{1}{2\pi} \times \frac{e^{j\pi n} - e^{-j\pi n}}{jn} \\
&= \frac{\sin(\pi n)}{\pi n} \\
&= \delta[n]
\end{aligned}
$$

Proving the Relation $\delta[n-n_0] \overset{DTFT}{\longleftrightarrow} e^{-j\widehat{\omega}n_0}$

$$x[n] = \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} \Delta(\widehat{\omega})\, e^{j\widehat{\omega}n}\, d\widehat{\omega}$$

Proving the Relation $\delta[n - n_0] \overset{DTFT}{\longleftrightarrow} e^{-j\widehat{\omega} n_0}$

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi} \int_{\widehat{\omega}=-\pi}^{\pi} \Delta(\widehat{\omega})\, e^{j\widehat{\omega} n}\, d\widehat{\omega} \\
&= \frac{1}{2\pi} \int_{\widehat{\omega}=-\pi}^{\pi} e^{-j\widehat{\omega} n_0}\, e^{j\widehat{\omega} n}\, d\widehat{\omega}
\end{aligned}
$$

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$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} \Delta(\widehat{\omega})\, e^{j\widehat{\omega} n}\, d\widehat{\omega} \\
&= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} e^{-j\widehat{\omega} n_0}\, e^{j\widehat{\omega} n}\, d\widehat{\omega} \\
&= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} e^{j\widehat{\omega}(n-n_0)}\, d\widehat{\omega}
\end{aligned}
$$

Proving the Relation $\delta[n-n_0] \overset{DTFT}{\longleftrightarrow} e^{-j\widehat{\omega}n_0}$

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} \Delta(\widehat{\omega})\; e^{j\widehat{\omega}n}\; d\widehat{\omega} \\
&= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} e^{-j\widehat{\omega}n_0}\; e^{j\widehat{\omega}n}\; d\widehat{\omega} \\
&= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} e^{j\widehat{\omega}(n-n_0)}\; d\widehat{\omega} \\
&= \frac{1}{2\pi} \times \frac{e^{j\pi(n-n_0)} - e^{-j\pi(n-n_0)}}{j(n-n_0)}
\end{aligned}
$$

Proving the Relation $\delta[n-n_0] \overset{DTFT}{\longleftrightarrow} e^{-j\widehat{\omega}n_0}$

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} \Delta(\widehat{\omega})\, e^{j\widehat{\omega}n}\, d\widehat{\omega} \\
&= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} e^{-j\widehat{\omega}n_0}\, e^{j\widehat{\omega}n}\, d\widehat{\omega} \\
&= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} e^{j\widehat{\omega}(n-n_0)}\, d\widehat{\omega} \\
&= \frac{1}{2\pi} \times \frac{e^{j\pi(n-n_0)} - e^{-j\pi(n-n_0)}}{j(n-n_0)} \\
&= \frac{\sin \pi(n-n_0)}{\pi(n-n_0)}
\end{aligned}
$$

Proving the Relation $\delta[n-n_0] \overset{DTFT}{\longleftrightarrow} e^{-j\widehat{\omega}n_0}$

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} \Delta(\widehat{\omega})\, e^{j\widehat{\omega}n}\, d\widehat{\omega} \\
&= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} e^{-j\widehat{\omega}n_0}\, e^{j\widehat{\omega}n}\, d\widehat{\omega} \\
&= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} e^{j\widehat{\omega}(n-n_0)}\, d\widehat{\omega} \\
&= \frac{1}{2\pi} \times \frac{e^{j\pi(n-n_0)} - e^{-j\pi(n-n_0)}}{j(n-n_0)} \\
&= \frac{\sin \pi(n-n_0)}{\pi(n-n_0)} \\
&= \delta[n-n_0]
\end{aligned}
$$

DTFT of a Constant in the Time Domain

1. Find IDTFT of $\delta(\widehat{\omega})$

$$x[n] = \frac{1}{2\pi} \int_{\widehat{\omega}=-\pi}^{\pi} \delta(\widehat{\omega})\, e^{j\widehat{\omega} n}\, d\widehat{\omega}$$

DTFT of a Constant in the Time Domain

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x[n] &= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} \delta(\widehat{\omega})\, e^{j\widehat{\omega} n}\, d\widehat{\omega} \\
&= \frac{1}{2\pi}\int_{\widehat{\omega}=-\epsilon}^{\epsilon} \delta(\widehat{\omega})\, d\widehat{\omega}
\end{aligned}
$$

DTFT of a Constant in the Time Domain

1. Find IDTFT of $\delta(\widehat{\omega})$

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} \delta(\widehat{\omega})\, e^{j\widehat{\omega}n}\, d\widehat{\omega} \\ &= \frac{1}{2\pi}\int_{\widehat{\omega}=-\epsilon}^{\epsilon} \delta(\widehat{\omega})\, d\widehat{\omega} \\ &= \frac{1}{2\pi} \end{aligned}$$

DTFT of a Constant in the Time Domain

1. Find IDTFT of $\delta(\widehat{\omega})$

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{\widehat{\omega}=-\pi}^{\pi} \delta(\widehat{\omega})\, e^{j\widehat{\omega} n}\, d\widehat{\omega} \\ &= \frac{1}{2\pi} \int_{\widehat{\omega}=-\epsilon}^{\epsilon} \delta(\widehat{\omega})\, d\widehat{\omega} \\ &= \frac{1}{2\pi} \end{aligned}$$

2.

$$1 \quad \overset{DTFT}{\longleftrightarrow} \quad 2\pi\, \delta(\widehat{\omega})$$

Observation

$$\delta[n] \quad \overset{DTFT}{\longleftrightarrow} \quad 1$$

$$1 \quad \overset{DTFT}{\longleftrightarrow} \quad 2\pi\ \delta(\widehat{\omega})$$

DTFT of Double-Sided Complex Exponential $e^{j\widehat{\omega}_0 n}$ in the Time Domain

Use frequency shift property of DTFT

$$e^{j\widehat{\omega}_0 n} \quad \overset{DTFT}{\longleftrightarrow} \quad 2\pi\, \delta(\widehat{\omega} - \widehat{\omega}_0)$$

Observation

$$\delta[n - n_0] \quad \overset{DTFT}{\longleftrightarrow} \quad e^{-j\widehat{\omega} n_0} \quad \text{time shift property}$$

$$e^{j\widehat{\omega}_0 n} \quad \overset{DTFT}{\longleftrightarrow} \quad 2\pi\, \delta(\widehat{\omega} - \widehat{\omega}_0) \quad \text{frequency shift property}$$

DTFT of Sinusoid

Example

Find DTFT of double-sided sinusoid

$$x[n] = A \cos \widehat{\omega}_0 n$$

DTFT of Sinusoid

Example

Find DTFT of double-sided sinusoid

$$x[n] = A \sin \widehat{\omega}_0 n$$

The Frequency Response $H(\widehat{\omega})$

The $H(\widehat{\omega})$

$$\text{Impulse Response} \quad \overset{DTFT}{\longleftrightarrow} \quad \text{Frequency Response}$$

$$h[n] \quad \overset{DTFT}{\longleftrightarrow} \quad H(\widehat{\omega})$$

DTFT of the FIR Filter

$$y[n] = \sum_{k=0}^{N} h[k]x[n-k] = h[n] * x[n], \qquad n > 0$$

DTFT of the FIR Filter

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$$Y(\widehat{\omega}) = H(\widehat{\omega})\ X(\widehat{\omega}), \qquad -\pi \leq \widehat{\omega} < \pi$$

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The frequency response is

$$H(\widehat{\omega}) = \frac{Y(\widehat{\omega})}{X(\widehat{\omega})}$$

DTFT of the IIR Filter

$$y[n] = \sum_{k=1}^{M} a[k]y[n-k] + \sum_{k=0}^{N} b[k]x[n-k]$$

DTFT of the IIR Filter

$$
\begin{aligned}
y[n] &= \sum_{k=1}^{M} a[k]y[n-k] + \sum_{k=0}^{N} b[k]x[n-k] \\
&= a[n] * y[n] + b[n] * x[n]
\end{aligned}
$$

DTFT of the IIR Filter

$$
\begin{aligned}
y[n] &= \sum_{k=1}^{M} a[k]y[n-k] + \sum_{k=0}^{N} b[k]x[n-k] \\
&= a[n] * y[n] + b[n] * x[n]
\end{aligned}
$$

We can write

$$
Y(\widehat{\omega}) = \frac{B(\widehat{\omega})}{1 - A(\widehat{\omega})}\, X(\widehat{\omega})
$$

DTFT of the IIR Filter

$$\begin{aligned} y[n] &= \sum_{k=1}^{M} a[k]y[n-k] + \sum_{k=0}^{N} b[k]x[n-k] \\ &= a[n] * y[n] + b[n] * x[n] \end{aligned}$$

We can write

$$Y(\widehat{\omega}) = \frac{B(\widehat{\omega})}{1 - A(\widehat{\omega})} X(\widehat{\omega})$$

The frequency response is

$$H(\widehat{\omega}) = \frac{Y(\widehat{\omega})}{X(\widehat{\omega})} = \frac{B(\widehat{\omega})}{1 - A(\widehat{\omega})}$$

Frequency Response of FIR Filter $H(\widehat{\omega})$

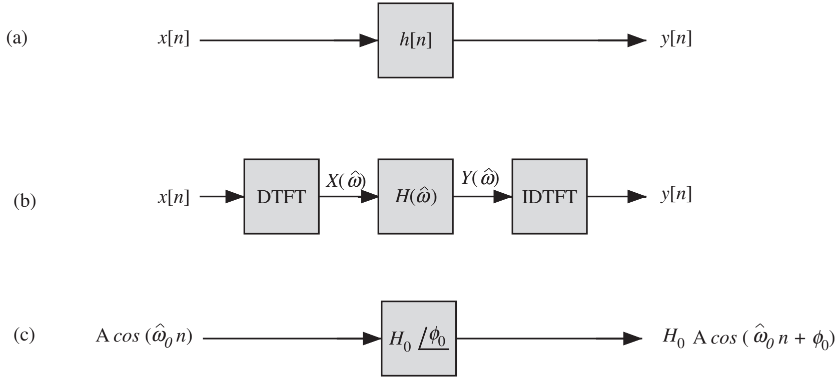
Example

Given the sequence for a recursive IIR filter

$$y[n] = 0.9y[n-1] + x[n]$$

1. Find the frequency response $H(\widehat{\omega})$
2. Determine the impulse response $h[n]$ and verify with direct evaluation of $y[n]$ when an impulse is applied
3. Plot $H(\widehat{\omega})$

Output of FIR Filter Given its Frequency Response



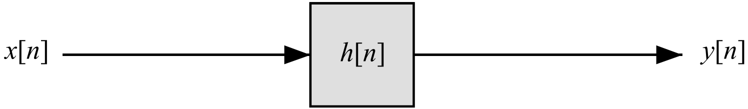
Method 1: Evaluating $y[n]$ when Impulse Response $h[n]$ is Given

Example

Given an FIR difference equation:

$$y[n] = x[n] + x[n-1]$$

1. Find the frequency response for the filter
2. Find the filter response when the input signal is $x[n] = 4\cos(0.3\pi n + 0.2\pi)$.



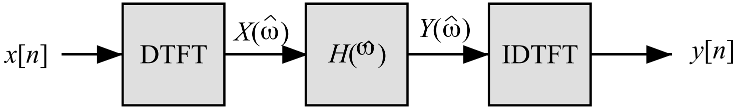
Method 2: Evaluating $y[n]$ when Frequency Response $H(\hat{\omega})$ is Given

Example

Given an FIR difference equation:

$$y[n] = x[n] + x[n-1]$$

1. Find the frequency response for the filter
2. Find the filter response when the input signal is $x[n] = 4\cos(0.3\pi n + 0.2\pi)$.



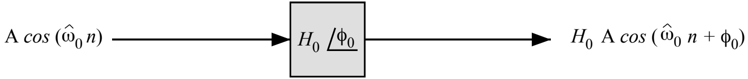
Method 3: Evaluating $y[n]$ when Input is Double-Sided Sinusoid

Example

Given an FIR difference equation:

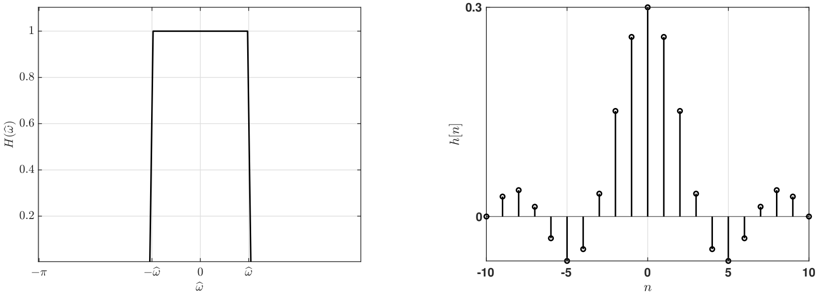
$$y[n] = x[n] + x[n-1]$$

1. Find the frequency response for the filter
2. Find the filter response when the input signal is $x[n] = 4\cos(0.3\pi n + 0.2\pi)$.



DTFT of Ideal Filters

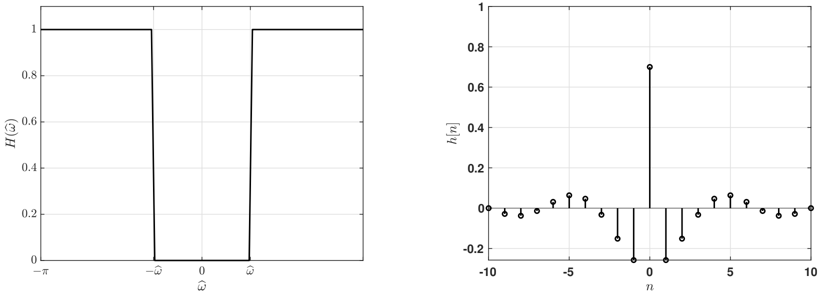
DTFT of Ideal Lowpass Filter: $\widehat{\omega}_B = 0.3\pi$



$$h[n] = \frac{1}{2\pi}\int_{\widehat{\omega}=-\pi}^{\pi} H(\widehat{\omega})\, e^{j\widehat{\omega}n}\, d\widehat{\omega}$$

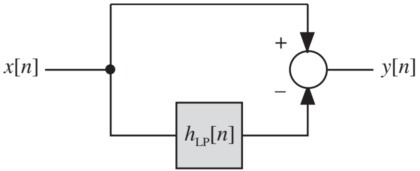
$$= \frac{\sin \widehat{\omega}_B n}{\pi n}, \qquad -\infty < n < \infty$$

DTFT of Ideal Highpass Filter: $\widehat{\omega}_B = 0.3\pi$

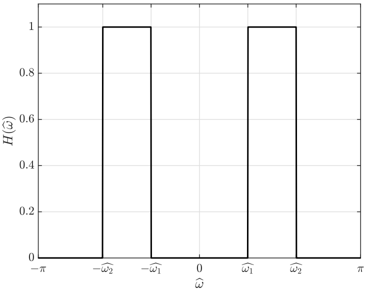


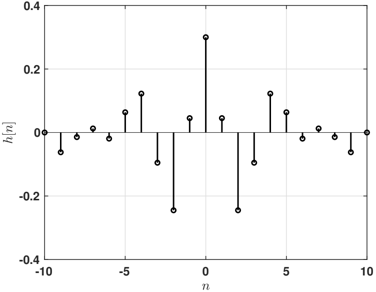
$$
\begin{aligned}
h[n] &= \delta[n] - h_{LP}[n] \\
&= \delta[n] - \frac{\sin \widehat{\omega}_B n}{\pi n}
\end{aligned}
$$

Block Diagram of Ideal Highpass Filter



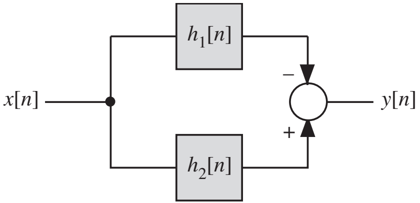
DTFT of Ideal Bandpass Filter: $\widehat{\omega}_1 = 0.3\pi$, $\widehat{\omega}_2 = 0.6\pi$



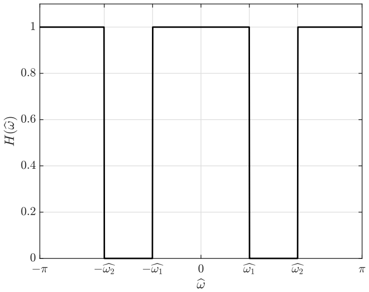


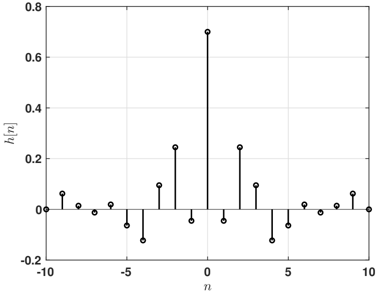
$$
\begin{aligned}
h[n] &= h_2[n] - h_1[n] \\
&= \frac{\sin \widehat{\omega}_2 n}{\pi n} - \frac{\sin \widehat{\omega}_1 n}{\pi n}
\end{aligned}
$$

Block Diagram of Ideal Bandpass Filter



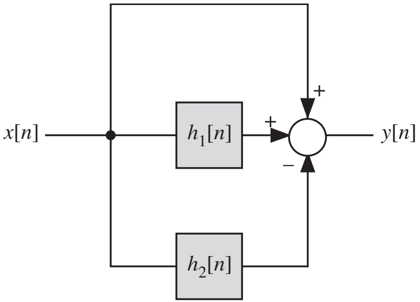
DTFT of Ideal Bandstop Filter: $\widehat{\omega}_1 = 0.3\pi$, $\widehat{\omega}_2 = 0.6\pi$





$$
\begin{aligned}
h[n] &= \delta[n] - h_2[n] + h_1[n] \\
&= \delta[n] - \frac{\sin \widehat{\omega}_2 n}{\pi n} + \frac{\sin \widehat{\omega}_2 n}{\pi n}
\end{aligned}
$$

Block Diagram of Ideal Bandstop Filter



Parseval's Theorem

Parseval's Theorem

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\widehat{\omega})|^2 \; d\widehat{\omega}$$

Parseval's Theorem

Example

Find the energy of the signal

$$x[n] = e^{-an}\, u[n], \qquad a > 0$$

Gibbs Phenomenon

Gibbs Phenomenon in DTFT

Gibbs phenomenon occurs due to imperfect evaluation of the forward- or inverse-DTFT operations.

Gibbs Phenomenon in DTFT: In Forward DTFT

In the forward DTFT, $X(\widehat{\omega})$ is evaluated using a summation over the index $0 \leq n < N$

$$X(\widehat{\omega}) = \sum_{i=0}^{N-1} x[n] \; e^{-j\widehat{\omega} n}$$

Gibbs Phenomenon in DTFT: In Forward DTFT

In the forward DTFT, $X(\widehat{\omega})$ is evaluated using a summation over the index $0 \leq n < N$

$$X(\widehat{\omega}) = \sum_{i=0}^{N-1} x[n]\ e^{-j\widehat{\omega}n}$$

Sometimes we truncate the summation to deal with fewer terms $M < N$

$$X'(\widehat{\omega}) = \sum_{i=0}^{M-1} x[n]\ e^{-j\widehat{\omega}n}, \qquad M < N$$

Gibbs Phenomenon in DTFT: In Inverse DTFT

$$x[n] = \frac{1}{2\pi} \int_{\widehat{\omega}=-\pi}^{\pi} X(\widehat{\omega})\, e^{j\widehat{\omega} n}\, d\widehat{\omega}$$

Gibbs Phenomenon in DTFT: In Inverse DTFT

$$x[n] = \frac{1}{2\pi} \int_{\widehat{\omega}=-\pi}^{\pi} X(\widehat{\omega})\; e^{j\widehat{\omega} n}\; d\widehat{\omega}$$

Integration is evaluated numerically using the trapezoidal rule

$$x[n] = \frac{1}{2\pi} \sum_{i=0}^{N_s-1} \left[\frac{X(\widehat{\omega}_i) + X(\widehat{\omega}_{i+1})}{2} \right] \; \Delta\widehat{\omega}$$

Gibbs Phenomenon in DTFT: In Inverse DTFT

$$x[n] = \frac{1}{2\pi} \int_{\widehat{\omega}=-\pi}^{\pi} X(\widehat{\omega})\, e^{j\widehat{\omega} n}\, d\widehat{\omega}$$

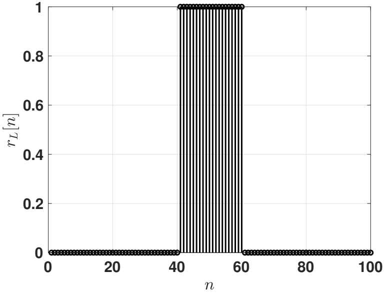
Integration is evaluated numerically using the trapezoidal rule

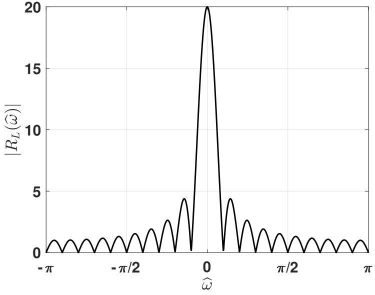
$$x[n] = \frac{1}{2\pi} \sum_{i=0}^{N_s-1} \left[\frac{X(\widehat{\omega}_i) + X(\widehat{\omega}_{i+1})}{2} \right] \Delta\widehat{\omega}$$

The quantities $\Delta\widehat{\omega}$ and $\widehat{\omega}_i$ are given by

$$\begin{aligned} \Delta\widehat{\omega} &= 2\pi/(N_s - 1) \\ \widehat{\omega}_i &= -\pi + i\, \Delta\widehat{\omega} \end{aligned}$$

Gibbs Phenomenon in DTFT: In Inverse DTFT: Ideal Case





Gibbs Phenomenon in DTFT: In Inverse DTFT when $N_s = 1000,\ 100$