Digital Signal Processing Using MATLAB

Sampling and Aliasing

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Outline

Sampling

Sampling of Analog Signals

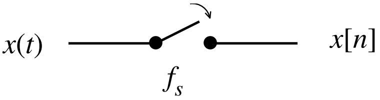
1. Discrete-time signal

$$x[n] = x(t = nT_s), \qquad n = 0, 1, 2, \cdots$$

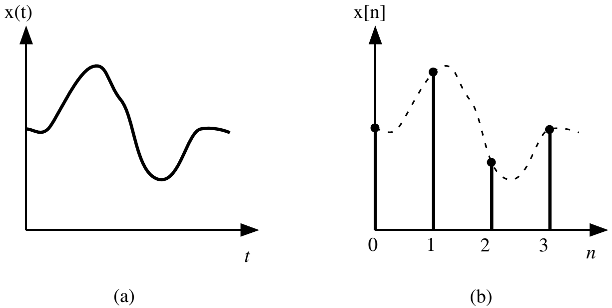
2. Sampling frequency

$$f_s = \frac{1}{T_s}$$

Taking a Snapshot of the Signal



Signal as a Sequence



Continuous-time

Discrete-time

Normalized Radian Frequency

Normalized Radian Frequency

1. $x(t) = \cos(\omega t + \phi)$
2. $x[n] = \cos(\omega n T_s + \phi)$
3. $x[n] = \cos(\widehat{\omega} n + \phi)$

$x(t)$ vs. $x[n]$

1. Analog signal

$$x(t) = \cos(\omega t + \phi)$$

2. Normalize time t in terms of T_s to get digital signal

$$\begin{aligned} x[n] &= \cos\left(\omega\ T_s \frac{t}{T_s} + \phi\right) \\ &= \cos(\widehat{\omega} n + \phi) \end{aligned}$$

with $\widehat{\omega} = \omega T_s$ and $n = t/T_s$

$x[n]$ Samples

Assume $x[n]$ is one-sided signal defined for $n \geq 0$. It is really a sequence of data vs. n:

n	0	1	2	3	$\cdots$
$x[n]$	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$\cdots$

Normalized Radian Frequency

$$\widehat{\omega} = \omega T_s = \omega / f_s \quad rad$$

Normalized Radian Frequency

Example

A sinusoidal signal

$$x(t) = \cos(2,000\pi t + \pi/3)$$

is being sampled at a rate $f_s = 3,000$ Hz.

1. What is the value of $\widehat{\omega}$?
2. Obtain an expression for $x[n]$

Normalized Radian Frequency

Example

A sinusoid is given by

$$x[n] = \cos 0.5\pi n$$

What is the equivalent continuous-time sinusoid $x(t)$ when the sampling rate is $f_1 = 1,000$ Hz and $f_2 = 5,000$ Hz?

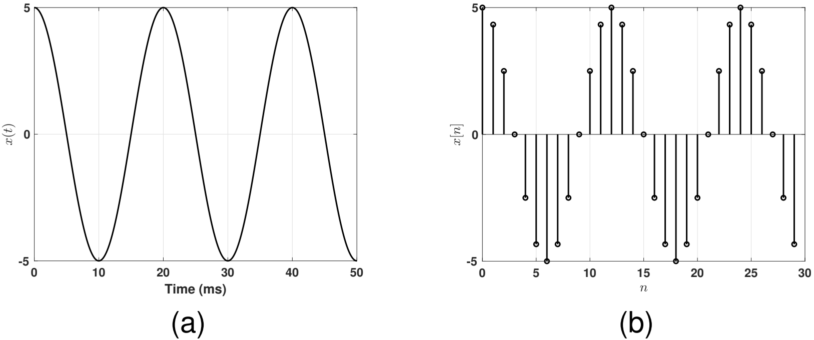
Example

For a sampling rate $f_s = 600$ Hz, the discrete-time sampled sinusoid below is obtained

$$x[n] = 5\cos \pi n/6$$

1. Identify the value of $\widehat{\omega}$
2. Find the values of ω and f
3. Sketch the signals $x(t)$ and $x[n]$
   MATLAB Code/Sampling/ex_stem.m

Continuous- and Discrete-Time Signals



How to relate *n* to *t*?

Aliasing

Aliasing

Consider the two signals

$$x_0[n] = \cos \widehat{\omega}_0 n$$

$$x_1[n] = \cos(\widehat{\omega}_0 + 2\pi)n$$

The two sequences are identical.

Aliasing Example

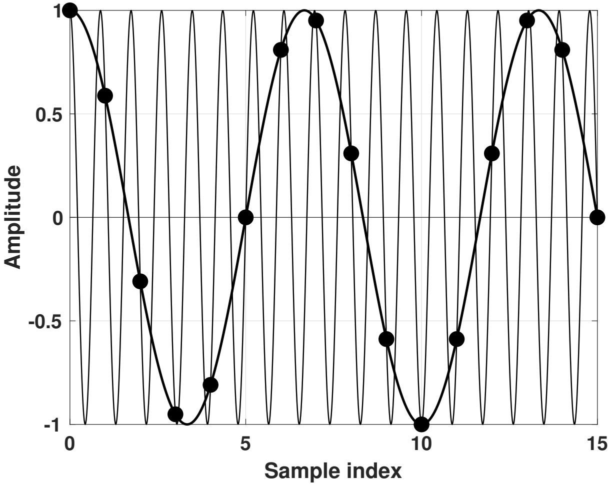
Example

Consider the two signals

$$\begin{aligned} x_1(t) &= \cos 300\pi t \\ x_2(t) &= \cos 2,300\pi t \end{aligned}$$

with $f_s = 1000$ Hz. Obtain the sequences $x_1[n]$ and $x_2[n]$.

Aliasing Example



MATLAB Code/Sampling/aliasing1.m

Range of Normalized Radian Frequency $\widehat{\omega}$

1. The range of $\widehat{\omega}$ is given by

$$0 \leq \widehat{\omega} < 2\pi$$

2. We can also choose the range to be

$$-\pi \leq \widehat{\omega} < \pi$$

Adding or subtracting $\pm 2\pi$ do not change signals

Spectrum of a Discrete-Time Sinusoid

$$x[n] = A\cos(\widehat{\omega}n + \phi)$$

Using inverse Euler's formula:

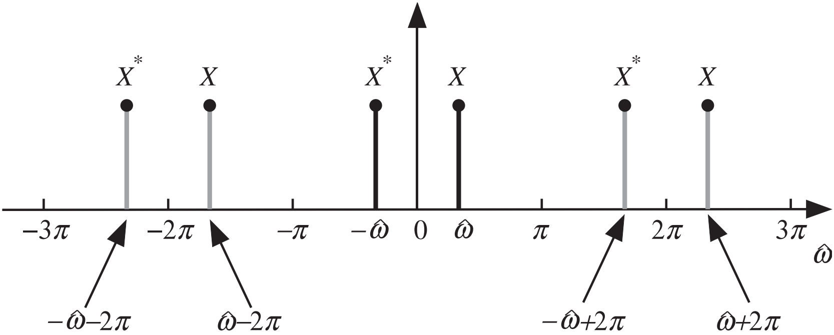
$$x[n] = Xe^{j\widehat{\omega}n} + X^* e^{-j\widehat{\omega}n}$$

$$X = \frac{A}{2}e^{j\phi}, \qquad X^* = \frac{A}{2}e^{-j\phi}$$

$$\widehat{\omega}_k = \widehat{\omega} \pm 2\pi k$$

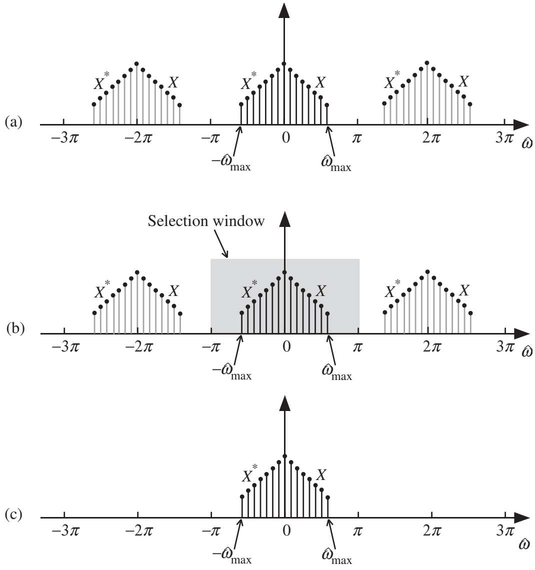
$$\widehat{\omega}_k = -\widehat{\omega} \pm 2\pi k$$

Aliases

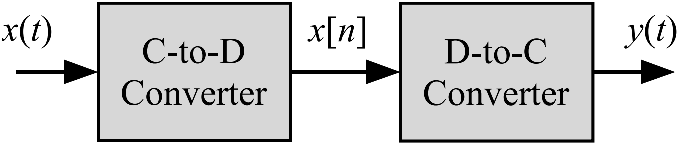


Definition

If f_{max} is the highest frequency of a continuous-time signal $x(t)$ and the signal is sampled at the rate $f_s > 2f_{max}$, then $x(t)$ can be perfectly reconstructed from $x[n]$ by taking only the principal components in the range $-\pi \leq \widehat{\omega} < \pi$.


(a)
(b)
(c)
X*
X
Selection window
−3π
−2π
−π
0
π
2π
3π
ω̂
−ω̂max
ω̂max

Converting Continuous-Time to Discrete-Time Signals and Vice Versa



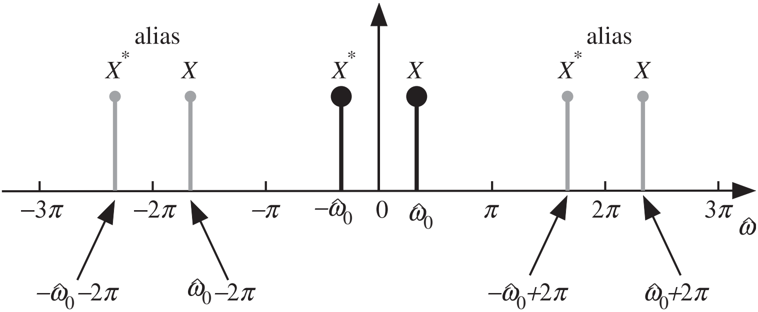
Oversampling

Conditions for Oversampling

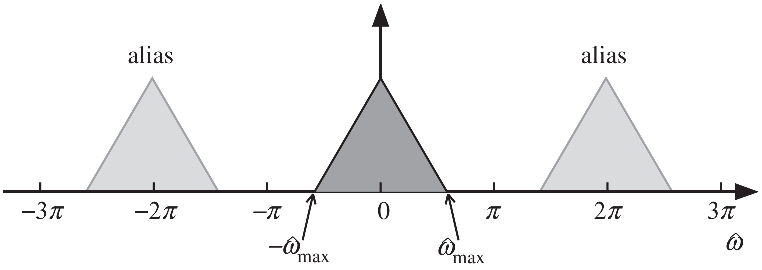
We must have

1. $0 \le 2f_0 < f_s$
2. $0 \le \omega_0 T_s < \pi$
3. $0 \le \widehat{\omega} < \pi$

Case when $0 \leq \widehat{\omega}_0 < \pi$: Oversampling



Case when $0 \leq \hat{\omega}_0 < \pi$: Oversampling



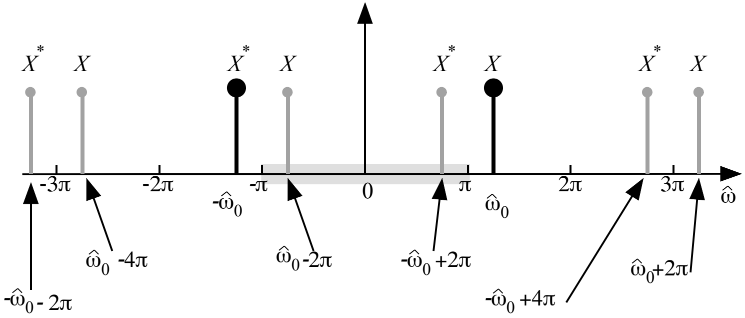
Undersampling with Folding

Conditions for Undersampling with Folding

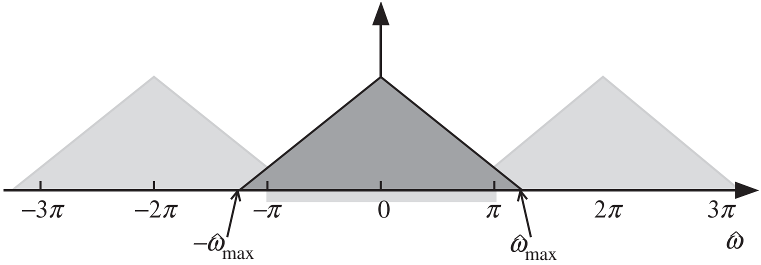
We must have

1. $f_0 < f_s$ and $2f_0 > f_s$
2. $\widehat{\omega}_0 < 2\pi$ and $\widehat{\omega}_0 > \pi$

Case when $\pi \leq \hat{\omega} < 2\pi$: Undersampling with Folding



Case when $\pi \leq \widehat{\omega}_0 < 2\pi$: Undersampling with Folding



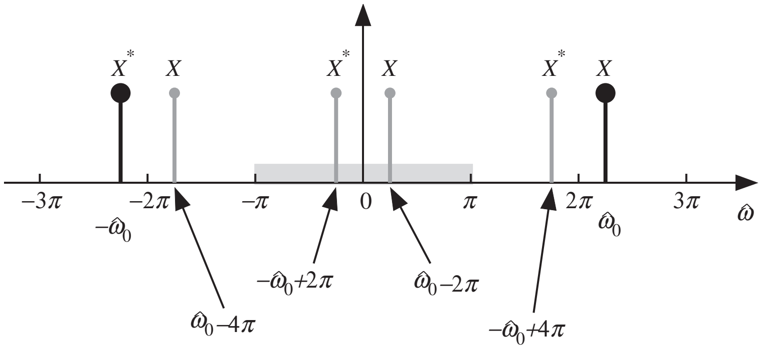
Undersampling when $\widehat{\omega} \geq 2\pi$

Conditions for Undersampling

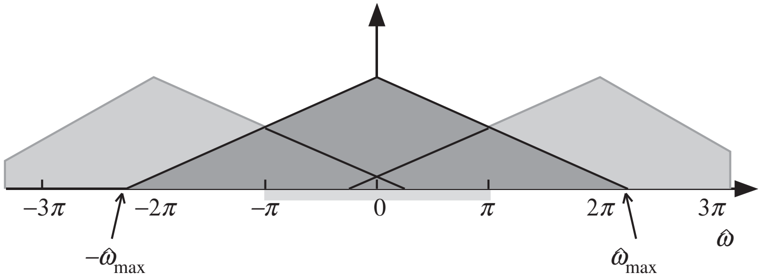
We must have

1. $f_0 > f_s$
2. $\widehat{\omega}_0 > 2\pi$

Case when $\widehat{\omega} \geq 2\pi$: Undersampling



Case when $\widehat{\omega} \geq 2\pi$: Undersampling



Sampling Example

Example

Consider the signal

$$x(t) = \cos(200\pi t + \pi/3)$$

that is being sampled with sampling frequency $f_s = 80$ Hz.
Write down an expression for the reconstructed signal $y(t)$.