Digital Signal Processing Using MATLAB

Sinusoidal Signals

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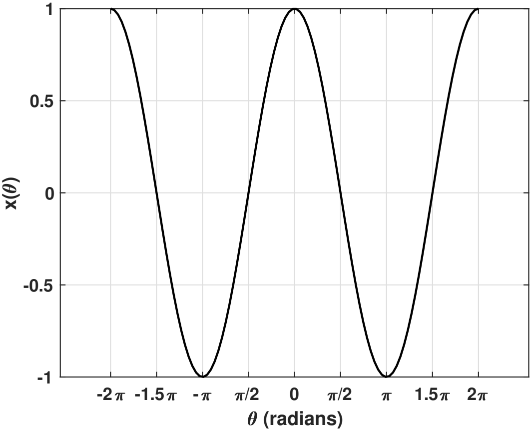
Outline

Cosine Function

Canonic Form of Sinusoidal Function

$$x(\theta) = \cos\theta$$



General Form of Sinusoidal Signal

$$x(t) = A\cos(\omega t + \phi) \quad = \quad A\sin[\omega(t + t_0)]$$

1. A is the signal amplitude with dimensions similar to those of the signal, e.g. volts (V) if the signal represents changes of voltage with time
2. ω is the radian frequency in radians/s
3. ϕ is the phase in radians
4. t_0 is time shift

Relation between ϕ and t_0

We have

$$\phi = \omega\, t_0$$

and we can write

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi) \\ &= A\cos(2\pi f t + \phi) \\ &= A\cos\omega(t + t_0) \end{aligned}$$

Time Shift	Advance ($t_0 > 0$)	Delay ($t_0 < 0$)
Phase Shfit	Lead ($\phi > 0$)	Lag ($\phi < 0$)

Example

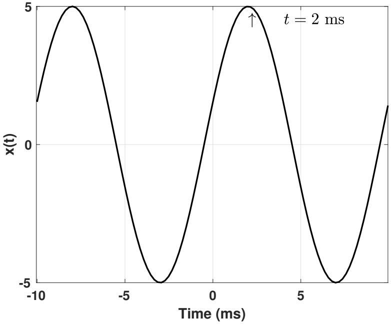
A sinusoidal current signal is given by

$$x(t) = 10^{-3} \cos(800\pi\, t - \pi/3) \quad A$$

Find the signal's parameters: A, ω, f, T, ϕ and t_0. And state whether this signal is delayed or advanced in time.

Example

Obtain the sinusoid parameters A, ω and the time shift value t_0



Euler's Formula

$$e^{j\pi} + 1 = 0$$

Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

For negative θ values

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Inverse Euler's Formula

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

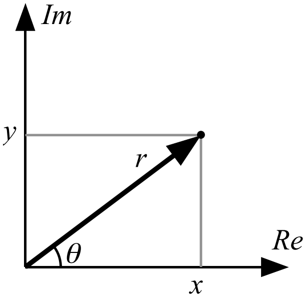
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Another simpler form for a sinusoid that does not have *j* and negative sign:

$$\sin\theta = \frac{e^{j(\theta-\pi/2)} + e^{-j(\theta-\pi/2)}}{2}$$

Complex Numbers

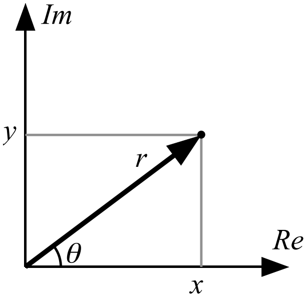
Complex Numbers: Cartesian



$$z = x + jy$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

Complex Numbers: Polar



$$z = r\, e^{j\theta}$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Sinusoid as Real Part of a Complex Exponential

1. The form $x(t) = A\cos(\omega t + \phi)$ is cumbersome
2. Define complex exponential signal $z(t) = Ae^{j(\omega t+\phi)}$

Using Euler's form we can write

$$
\begin{aligned}
z(t) &= A\cos(\omega t + \phi) + j\sin(\omega t + \phi) \\
x(t) &= \mathcal{R}\{z(t)\}
\end{aligned}
$$

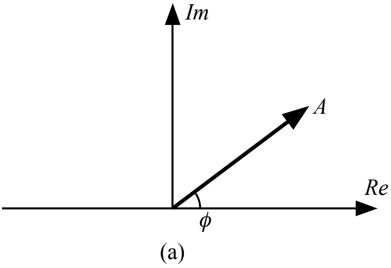
Phasors

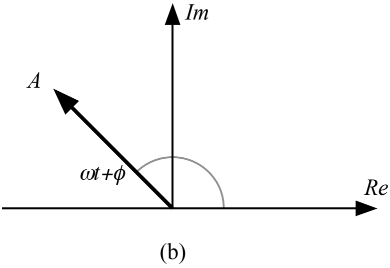
Phasor Representation of a Sinusoidal Signal

$$
\begin{aligned}
z(t) &= Ae^{j(\omega t+\phi)} \\
&= Ae^{j\omega t} \times e^{j\phi} \\
&= Ae^{j\phi} \times e^{j\omega t} \\
&= X\, e^{j\omega t}
\end{aligned}
$$

where X is the phasor representation of $z(t)$

$$X = Ae^{j\phi}$$





At time $t = 0$

At time $t > 0$

Spectrum of a Sinusoid

What is Spectrum?

1. Spectrum shows the amplitude & phase vs. frequency of a signal
2. Helps in studying frequencies of a composite signal
3. In effect we move from time-domain description to frequency-domain description

Use Inverse Euler's Formula

$$
\begin{aligned}
x(t) &= A\cos(\omega t + \phi) \\
&= \frac{A}{2} e^{j(\omega t+\phi)} + \frac{A}{2} e^{-j(\omega t+\phi)}
\end{aligned}
$$

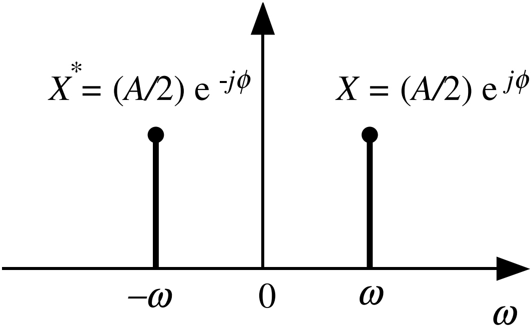
We can write the above equation in phasor form as

$$x(t) = Xe^{j\omega t} + X^* e^{-j\omega t}$$

where $X = (A/2)e^{j\phi}$.

You must note first appearance of negative frequency!

Two-Sided Spectrum of a Sinusoid



Radian frequency (ω)	$-\omega$	ω
Phasor	$\frac{A}{2}e^{-j\phi}$	$\frac{A}{2}e^{j\phi}$

Valid result for real signals only

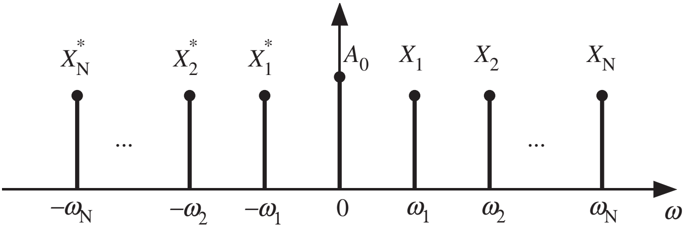
Two-Sided Spectrum of a Sinusoid

Example

Find the two-sided spectrum of the sinusoidal signal

$$x(t) = 4\cos(500\pi t + \pi/3)$$

General Two-Sided Spectrum of a Multi-Frequency Signal



Two-Sided Spectrum of a Signal

Example

Find the two-sided spectrum of the composite signal

$$x(t) = 5 + 8\cos(20\pi t + \pi/3) + 6\cos(50\pi t - \pi/5)$$

Two-Sided Spectrum of a Signal

Example

Find the two-sided spectrum of the signal

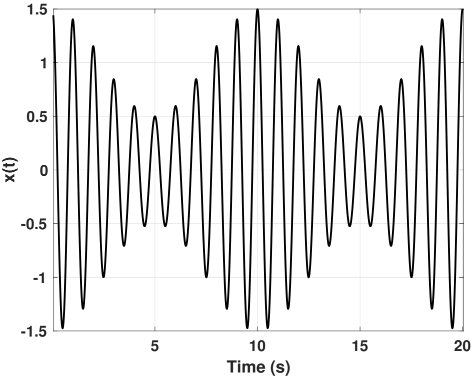
$$x(t) = \cos 3\pi t \times \cos 5\pi t$$

Amplitude Modulation

Consider the product of two sinusoids

$$x(t) = x_2(t)[1 + m\, x_1(t)] = \cos\omega_2 t(1 + m \cos\omega_1 t)$$

$x_1(t)$ is the signal and $x_2(t)$ is the carrier



Amplitude Modulation

Our low-frequency signal $x_1(t)$ is now replaced by three high-frequency signals:

$$x(t) = x_2(t) + 0.5m\ \cos(\omega_2 + \omega_1)t + 0.5m\ \cos(\omega_2 - \omega_1)t$$

We now have the carrier signal plus two high-frequency sidebands

Superheterodyning (Opposite of amplitude modulation)

1. Objective is to extract the low-frequency signal from the modulated high-frequency signal

2. Assume a high-frequency signal $x(t) = A\cos\omega_1 t$

3. We can mix with a lower frequency signal

$$\begin{aligned} z(t) &= A\cos\omega_1 t \times \cos\omega_2 t \\ &= \frac{A}{2}\cos(\omega_1 - \omega_2)t + \frac{A}{2}\cos(\omega_1 + \omega_2)t \end{aligned}$$

4. We can use a filter to obtain the low-frequency signal

$$y(t) = \frac{A}{2}\cos(\omega_1 - \omega_2)t$$