Digital Signal Processing Using MATLAB

Discrete-Time Systems

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Outline

1 Signals
2 E/P
3 MA
4 Causality
5 Linearity
6 Shift
7 FIR/IIR
8 Impulses
9 Response
10 Impulse
11 Convolution
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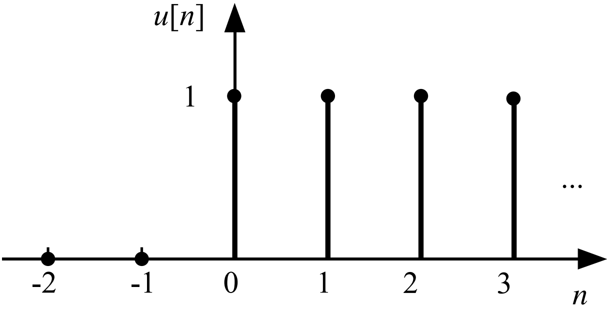
Special Signals

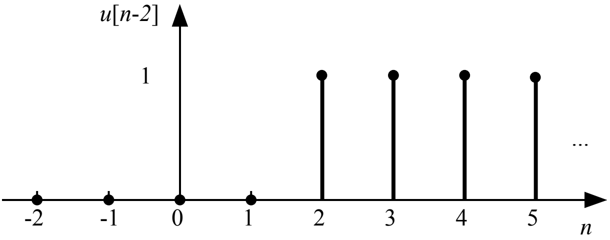
Unit Step $u[n]$

$$u[n] = \begin{cases} 1 & \text{when} \quad n \geq 0 \\ 0 & \qquad\quad\;\; n < 0 \end{cases}$$



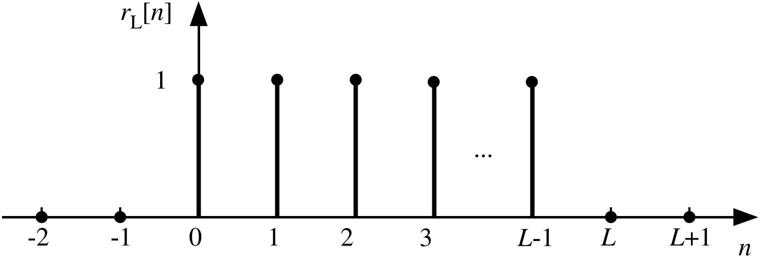
Delayed Unit Step $u[n - n_0]$: Case $n_0 = 2$

$$u[n - n_0] = \begin{cases} 1 & \text{when} \quad n \geq n_0 \\ 0 & \quad\quad\quad\;\; n < n_0 \end{cases}$$



Length L Rectangular Pulse $r_L[n]$

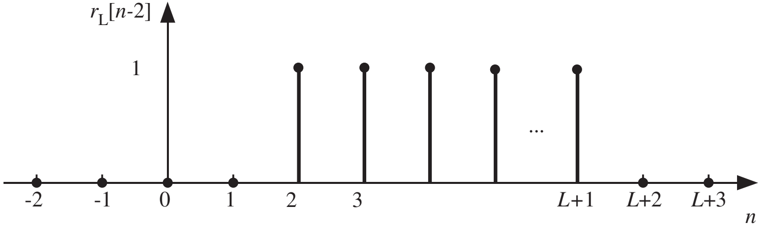
$$r_L[n] = \begin{cases} 0 & n < 0 \\ 1 \quad \text{when} & 0 \leq n < L \\ 0 & n \geq L \end{cases}$$



Length-L Pulse From Two Unit Step Functions

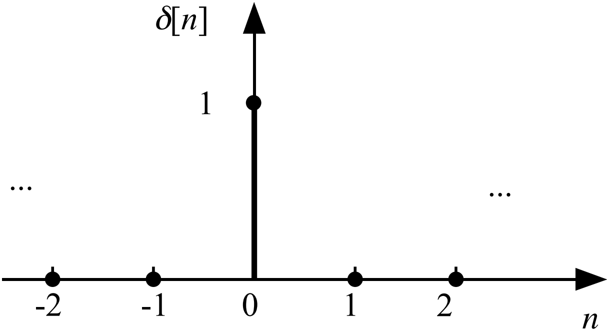
$$r_L[n] = u[n] - u[n-L]$$

Shifted (Delayed) Pulse $r_L[n - n_0]$: Case $n_0 = 2$



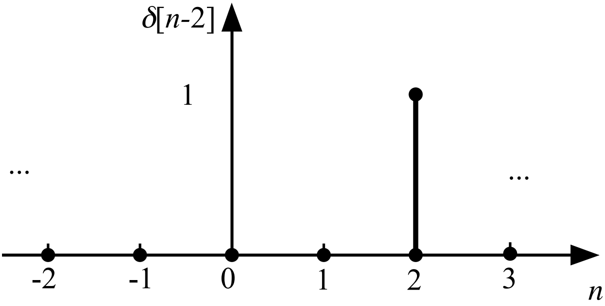
Unit Impulse (Kronecker Delta) $\delta[n]$

$$\delta[n] = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ 0, & n > 0 \end{cases}$$



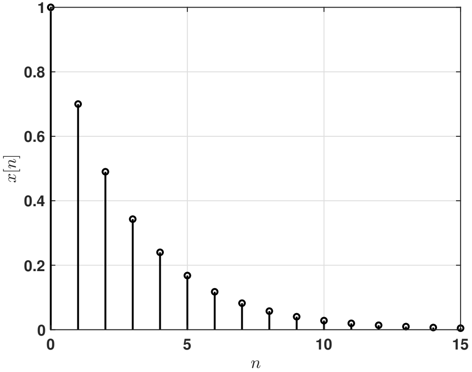
Shifted (Delayed) Impulse $\delta[n - n_0]$: Case $n_0 = 2$

$$\delta[n - n_0] = \begin{cases} 0 & & n < n_0 \\ 1 & \text{when} & n = n_0 \\ 0 & & n \geq n_0 \end{cases}$$



Real Exponential Signal: $x[n] = a^n \, u[n]$

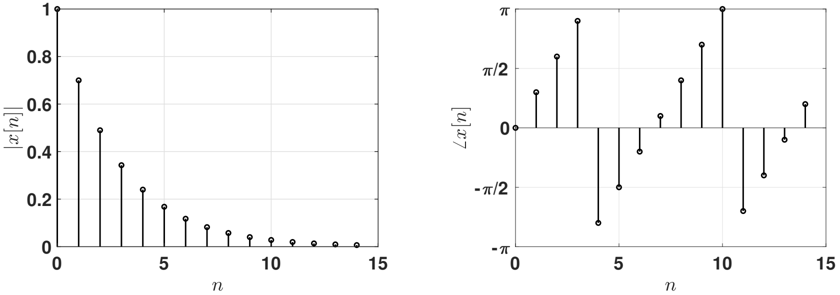
$$x[n] = a^n u[n], \qquad 0 < a < 1$$



$a = 0.7$

Complex Exponential Signal: $x[n] = a^n \, e^{j\widehat{\omega}_0 n} \, u[n]$

$$x[n] = a^n \, e^{j\widehat{\omega}_0 n} \, u[n], \qquad 0 < a < 1$$



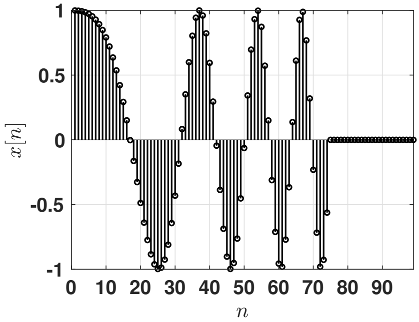
(a) (b)

$a = 0.7$ and $\widehat{\omega}_0 = 0.3\pi$

Linear Frequency-Modulated (LFM) Chirp Signal: $x[n] = e^{j\widehat{\omega}[n]\ n}\ r_L[n]$ case $\widehat{\omega}_0 = 0.01\pi$, $\widehat{\omega}_1 = 0.1\pi$ and $L = 75$

$$x[n] \quad = \quad e^{j\widehat{\omega}[n]\ n}\ r_L[n]$$

$$\widehat{\omega}[n] = \widehat{\omega}_0 + \widehat{\omega}_1 n, \qquad 0 \leq n < L$$



Linear Frequency-Modulated (LFM) Chirp Signal: case $\widehat{\omega}_0 = 0.01\pi$, $\widehat{\omega}_1 = 0.1\pi$ and $L = 75$

$$x[n] \quad = \quad e^{j\widehat{\omega}[n]\ n}\ r_L[n]$$

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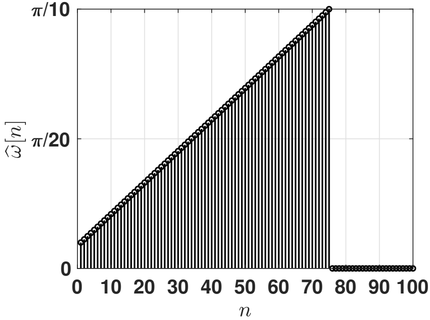


Signal Energy and Power

Signal Energy

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Energy Signal

Definition

A signal is called energy signal when its energy is nonzero and finite

$$0 < E < \infty$$

Signal Energy

Example

Determine the energy of a right-sided exponential signal.

Signal Power

1. When a signal has infinite energy, it has finite average power.

2. A signal is called power signal when its power is nonzero and finite

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Signal Power

Example

Determine the power of a unit step function $u[n]$.

Power of a Periodic Signal

1. Assume periodic signal wih period T_0
2. Assume sampling period is T_s
3. Number of samples in one period is

$$N_0 = \left\lfloor \frac{T_0}{T_s} \right\rfloor \qquad \text{or} \qquad N_0 = \left\lfloor \frac{2\pi}{\widehat{\omega}_0} \right\rfloor$$

4. Power is given by

$$P = \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2$$

Signal Energy & Power

Example

Determine the energy and power of a periodic length-L real exponential signal a^n with $a < 1$ whose period $T = L$.

Moving Average Filter (MA)

Moving Average (MA) Filter

$$
\begin{aligned}
y[n] &= \frac{1}{L}\sum_{k=0}^{L-1} x[n-k] \\
&= \frac{1}{L}\left(x[n] + x[n-1] + x[n-2] + \cdots + x[n-L+1]\right)
\end{aligned}
$$

MA Example

Example

Find the response of the MA filter when $L = 3$ and the input is given by

n	$n < 0$	0	1	2	3	4	5	6	$n > 6$
$x[n]$	0	1	3	1	5	1	7	1	0

Causal and Noncausal Systems

Causal and Noncausal Systems: Definitions

Assume current time index is n. We define:

1. Past sample $x[m]$ is when $m < n$
2. Current sample $x[m]$ is when $m = n$
3. Future sample $x[m]$ is when $m > n$

MA Filter

1. Causal MA filter

$$y[n] = \frac{1}{L} \sum_{k=0}^{L-1} x[n-k]$$

2. Noncausal MA filter

$$y[n] = \frac{1}{L} \sum_{k=-(L-1)/2}^{(L-1)/2} x[n-k]$$

Linear and Nonlinear Systems

Linear System Definition: Superposition Principle

Linear systems satisfy superposition principle: Scaling and addition

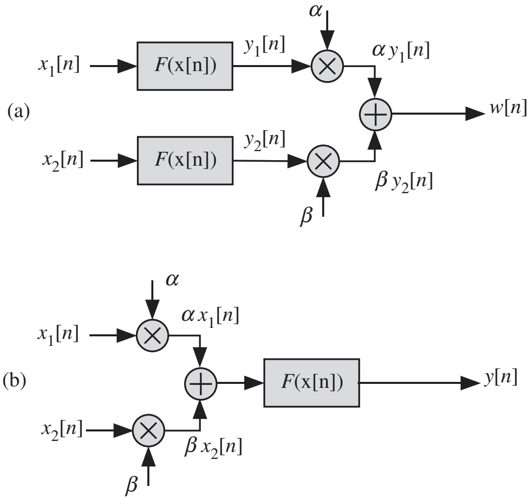
Scaling: $$y[n] = F(x[n]) \Longrightarrow ky[n] = F(kx[n])$$

Addition:

$$y_1[n] = F(x_1[n]) \quad and \quad y_2[n] = F(x_2[n])$$

$$y_1[n] + y_2[n] = F(x_1[n] + x_2[n])$$

Testing for Linearity



Testing for Linearity

Example

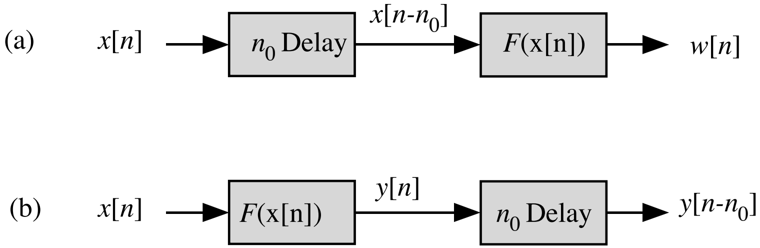
See whether the system

$$y[n] = x^2[n]$$

is linear or not.

Time-Invariant and Time-Variant Systems

Testing for Time Invariance



Step 1: $w[n]$: delay input by n_0 in the input signal only, i.e. $x[n - n_0]$

Step 2: $y[n]$: delay output by n_0, i.e. $n \to n - n_0$ in all occurrences on RHS

Testing for Time Invariance

Example

Test whether the system

$$y[n] = x^2[n]$$

is time-invariant or not.

Testing for Time Invariance

Example

Check time invariance property of the system

$$y[n] = nx[n]$$

Obtain $w[n]$

Step 1: Delay $x[n]$ by n_0 (subtract n_0 in argument of $x[n]$ only):

$$w[n] = n\,x[n - n_0]$$

n	0	1	2	3	4
$x[n]$	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$
$x[n-2]$	0	0	$x[0]$	$x[1]$	$x[2]$
$w[n]$	0	0	$2x[0]$	$3x[1]$	$4x[2]$

Delay Output

Step 2: Replace all n's in expression for $y[n]$:

$$y[n - n_0] = (n - n_0)\, x[n - n_0]$$

n	0	1	2	3	4
$x[n]$	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$
$y[n]$	$0x[0]$	$x[1]$	$2x[2]$	$3x[3]$	$4x[4]$
$y[n-2]$	0	0	$0x[0]$	$x[1]$	$2x[2]$

Recursive and Nonrecursive Systems

Nonrecursive Systems: FIR Filter Example

A nonrecursive LSI system has difference equation where output $y[n]$ depends on current and past input samples $x[n]$

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{N-1} h[k]x[n-k] \\
&= h[0]x[n] + h[1]x[n-1] + \cdots h[N-1]x[n-N+1]
\end{aligned}
$$

Recursive Systems: IIR Filter Example

A recursive LSI system has difference equation where output $y[n]$ depends on current and past input samples $x[n]$ as well as past output samples

$$
\begin{aligned}
y[n] &= \sum_{k=1}^{M} a[k]y[n-k] + \sum_{k=0}^{N} b[k]x[n-k] \\
&= a[1]y[n-1] + a[2]y[n-2] + \cdots a[M]y[n-M] \\
&+ b[0]x[n] + b[1]x[n-1] + \cdots b[N]x[n-N]
\end{aligned}
$$

Expressing a Discrete-Time Signal as a Sequence of Scaled and Delayed Impulses

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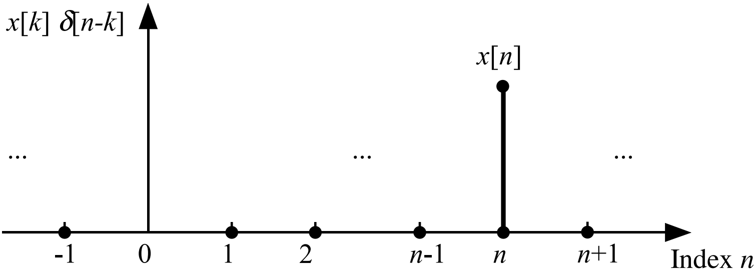
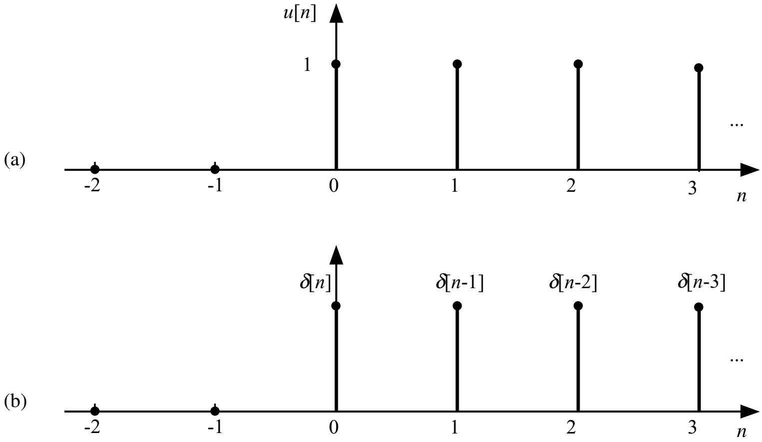


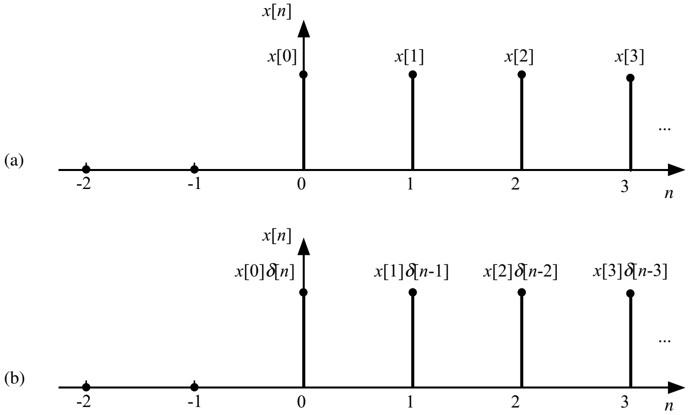
$$\delta[k]x[n-k] \equiv x[k]\delta[n-k] \equiv x[n]$$

Two Ways to Express Unit Step Function



$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots \qquad = \sum_{k=0}^{\infty} \delta[n-k]$$

Expressing any Sequence as a Sum of Scaled and Delayed Impulses



$$
\begin{aligned}
x[n] &= \{\underset{\uparrow}{x[0]},\ x[1],\ x[2],\ \cdots\} \\
&= x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots
\end{aligned}
$$

Expressing a Discrete-Time Signal as a Sequence of Scaled and Delayed Impulses

Example

Express the finite duration signal

$$x[n] = \{\underset{\uparrow}{1},\ 2,\ 4, 8, 16\}$$

as the sum of weighted/scaled and delayed impulses.

Impulse Response of Nonrecursive Systems

What is Impulse Response?

(a) (b)

Impulse Response of Nonrecursive Systems

Example

Obtain the impulse response of the system

$$y[n] = \sum_{k=0}^{N-1} h[k]x[n-k]$$

where $h[k]$ are the FIR filter coefficients

Impulse Response as a Time Sequence

$$h[n] = \sum_{k=0}^{N-1} h[k]\delta[n-k]$$

n	$n < 0$	0	1	2	$\cdots$	$N-2$	$N-1$	$n \geq N$
$h[n]$	0	$h[0]$	$h[1]$	$h[2]$	$\cdots$	$h[N-2]$	$h[N-1]$	0

$$h[n] = \left\{ \underset{\uparrow}{h[0]},\ h[1],\ h[2],\ \cdots,\ h[N-1] \right\}$$

Nonrecursive System Properties

Nonrecurisive systems are linear and shift/time invariant

Impulse Response of Recursive Systems

Impulse Response of Recursive Systems

1. The response to an impulse at $n = 0$ produces infinite output sequence
2. Such systems are called infinite impulse response (IIR)
3. We study simplest possible IIR system in next frame

Impulse Response of Simplest Recursive Systems

Assume initially relaxed system (i.e. $y[n] = 0$ for $n < 0$)

$$y[n] = ay[n-1] + bx[n]$$

and impulse response is given by

$$y[n] = ay[n-1] + b\delta[n]$$

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
$x[n]$	1	0	0	0	0	0	0	0	0	0	$\cdots$
$y[n]$	b	ab	a^2b	a^3b	a^4b	a^5b	a^6b	a^7b	a^8b	a^9b	$\cdots$

Recursive System Properties

Recurisive systems are linear and shift/time invariant

Convolution

Convolution Operation

$$
\begin{aligned}
y[n] &= h[n] * x[n] \\
&= \sum_{k=0}^{\infty} h[k]x[n-k] \\
&= \sum_{k=0}^{\infty} x[k]h[n-k]
\end{aligned}
$$

Convolution Properties

1. Linearity property
2. Commutative property
3. Distributive property
4. Associative property

Convolution Linearity Property

$$h[n] * (\alpha x_1[n] + \beta x_2[n]) = \alpha h[n] * x_1[n] + \beta h[n] * x_2[n]$$

Convolution Commutative Property

$$h[n] * x[n] = x[n] * h[n]$$

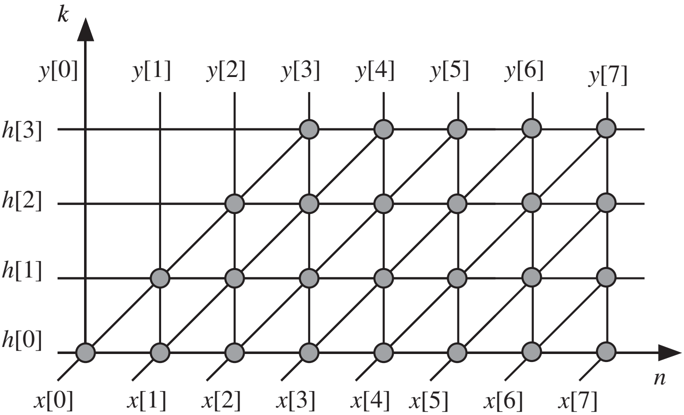
Convolution Distributive Property

$$h[n] * (x_1[n] + x_2[n]) = h[n] * x_1[n] + h[n] * x_2[n]$$

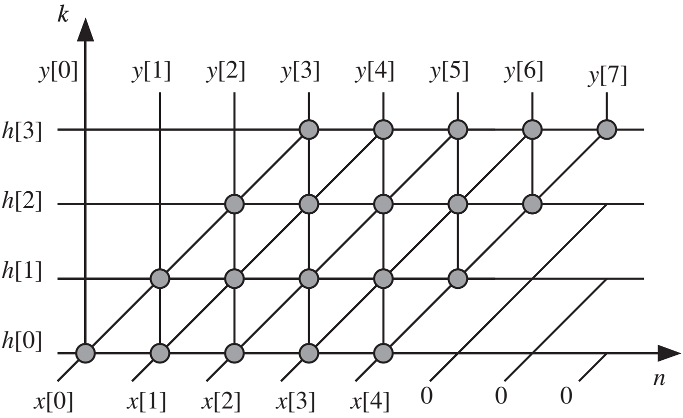
Convolution Associative Property

$$(f[n] * g[n]) * h[n] = f[n] * (g[n]) * h[n])$$

Convolution Dependence Graph

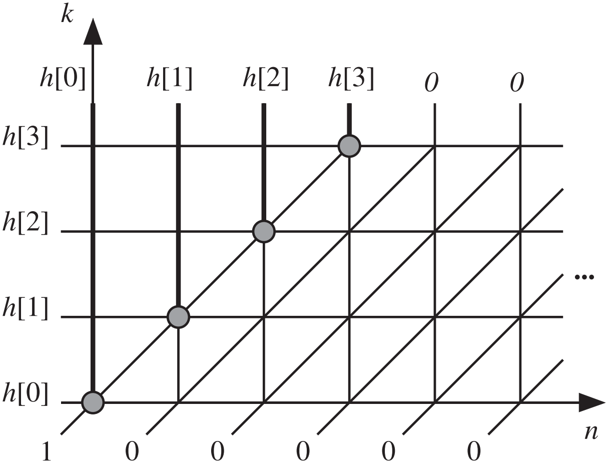


Length of Convolution



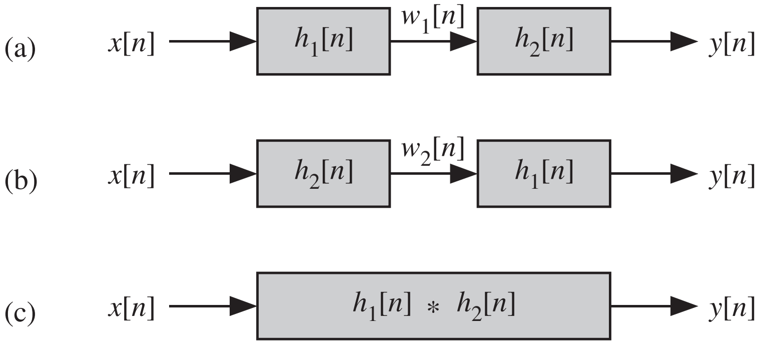
$$L_y = L_x + L_h - 1$$

Extracting Impulse Response Using Convolution



Cascaded LTI Systems

Cascaded LTI Systems



Cascaded LTI Systems

Example

Consider the cascaded systems. Assume that the inputs are

$$x[n] = \{\underset{\uparrow}{1},\ 1,\ 1,\ 1\} \qquad h_1[n] = \{\underset{\uparrow}{1},\ -1\} \qquad h_2[n] = \{\underset{\uparrow}{1},\ 0,\ -1\}$$

1. Determine the response $w_1[n] = x[n] * h_1[n]$
2. Determine the response $w_2[n] = x[n] * h_2[n]$
3. Determine the response $y_1[n] = w_1 * h_2[n]$
4. Determine the response $y_2[n] = w_2 * h_1[n]$
5. Determine the impulse response $h[n] = h_1[n] * h_2[n]$
6. Determine the response $y[n] = x[n] * h[n]$ and prove that $y[n] = y_1[n] = y_2[n]$