Digital Signal Processing Using MATLAB

z-Transform

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Outline

1. z
2. ROC
3. Functions
4. Properties
5. Conv
6. Sys
7. Poles
8. DFT
9. Inverse
10. Partial
11. Repeated
12. Equal

z-Transform

The One-Sided z-Transform Equation

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

1. z is a complex number
2. z-transform is really a polynomial in z^{-1}

$$\begin{aligned} x[n] &= x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots \\ X(z) &= x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \end{aligned}$$

ROC

The Region of Convergence (ROC)

1. We express z in Euler's notation as $z = r\ e^{j\widehat{\omega}}$ and the z-transform becomes

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2. Series converges when $|X(z)|$ is finite

$$|X(z)| = \left| \sum_{n=0}^{\infty} x[n]\ r^{-n}\ e^{-j\widehat{\omega}n} \right|$$

The Region of Convergence (ROC)

A more strict condition is when

$$|X(z)| \leq \sum_{n=0}^{\infty} \left| x[n]\, r^{-n}\, e^{-j\widehat{\omega} n} \right|$$

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The Region of Convergence (ROC)

A more strict condition is when

$$|X(z)| \leq \sum_{n=0}^{\infty} \left| x[n]\, r^{-n}\, e^{-j\widehat{\omega} n} \right| = \sum_{n=0}^{\infty} \left| x[n]\, r^{-n} \right|$$

In other words

$$|X(z)| \leq \sum_{n=0}^{\infty} \left| \frac{x[n]}{r^n} \right|$$

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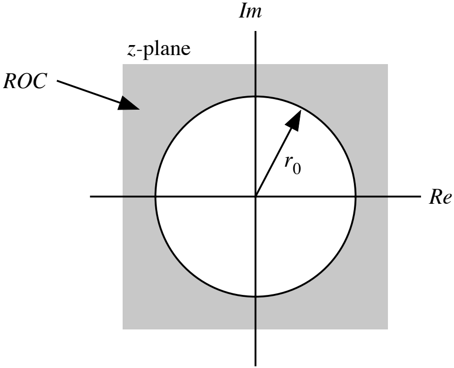
2. The absolute sum converges for all values of $|z| = r > r_0$
3. $|x[n]| / r^n$ is summable for $r > r_0$.



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Example

Assume the right-sided sequence $x[n] = a^n\, u[n]$.

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The ROC is when $|az^{-1}| < 1$. This could be stated as $|z| > a$.

The Region of Convergence (ROC) Observations

If the ROC is the exterior of a circle, then $x[n]$ is a right-sided sequence with $n \geq 0$.

The Region of Convergence (ROC) Observations

Example

Determine the z-transform of the following sequences and state the ROC for each signal.

1. $x_1[n] = \{\underset{\uparrow}{1},\ 2,\ 3,\ 4\}$
2. $x_2[n] = \delta[n]$
3. $x_3[n] = \delta[n-3]$
4. $x_4[n] = u[n]$
5. $x_5[n] = u[n-3]$

Simple Functions

z-Transform of Impulse Signal $\delta[n]$

$$\begin{aligned} \Delta(z) &= \sum_{n=0}^{\infty} \delta[n] z^{-n} \\ &= 1 \end{aligned}$$

ROC is the entire complex plane

$$\delta[n] \quad \stackrel{z}{\longleftrightarrow} \quad 1$$

z-Transform of Delayed Impulse $\delta[n - n_0]$

$$
\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} \delta[n - n_0] z^{-n} \\
&= z^{-n_0}
\end{aligned}
$$

Therefore time delay n_0 is equivalent to multiplication of the z-transform by the factor z^{-n_0}. The ROC is $z \neq 0$.

$$
\delta[n - n_0] \quad \stackrel{z}{\longleftrightarrow} \quad z^{-n_0}, \qquad z \neq 0
$$

z-Transform of Unit Step $u[n]$

$$
\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} u[n] z^{-n} \\
&= \frac{1}{1 - z^{-1}}
\end{aligned}
$$

The ROC is $|z| > 1$.

$$
u[n] \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{1 - z^{-1}}, \qquad |z| > 1
$$

z-Transform of Delayed Unit Step $u[n-n_0]$

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} u[n-n_0] z^{-n} \\ &= z^{-n_0} \sum_{m=0}^{\infty} u[m] z^{-m} \\ &= \frac{z^{-n_0}}{1-z^{-1}} \end{aligned}$$

The ROC is $|z| > 1$.

$$u[n] \quad \stackrel{z}{\longleftrightarrow} \quad \frac{z^{-n_0}}{1-z^{-1}}, \qquad |z| > 1$$

z-Transform of Length-L Pulse $r_L[n]$

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z-Transform of Exponential Signal: $x[n] = a^n\, u[n], \qquad 0 < a < 1$

$$X(z) = \sum_{n=0}^{\infty} a^n\, u[n] z^{-n}$$

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z-Transform of Complex Exponential Signal: $x[n] = a^n \, e^{j\widehat{\omega}_0 n} \, u[n]$

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Summary

Sequence	z-Transform	ROC
$\delta[n]$	1	Entire z-plane
$\delta[n-n_0]$	z^{-n_0}	$z \neq 0$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$u[n-n_0]$	$\frac{z^{-n_0}}{1-z^{-1}}$	$\|z\| > 1$
$r_L[n]$	$\frac{1-z^{-L}}{1-z^{-1}}$	$z \neq 0$
$a^n\, u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$e^{j\widehat{\omega}n}\, u[n]$	$\frac{1}{1-e^{j\widehat{\omega}}z^{-1}}$	$\|z\| > 1$

Properties

Linearity Property of the z-Transform

$$\alpha x_1[n] + \beta x_2[n] \quad \stackrel{z}{\longleftrightarrow} \quad \alpha X_1(z) + \beta X_2(z)$$

Time Shift Property of the z-Transform

Consider the z-transform of a delayed version of some signal

$$x[n - n_0] \quad \stackrel{z}{\longleftrightarrow} \quad X(z)\, z^{-n_0}$$

Convolution

Convolution and the z-Transform

Consider the convolution operation

$$y[n] = \sum_{k=0}^{N-1} h[k]x[n-k]$$

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Convolution and the z-Transform

Example

Assume $X(z) = 1+2z^{-1}+3z^{-2}$ $Y(z) = 4+5z^{-1}+6z^{-2}$

1. Find $x[n]$ and $y[n]$
2. Do the convolution $c[n] = x[n] * y[n]$
3. Find $C(z)$ and prove that $C(z) = X(z)Y(z)$

System Function

FIR System Function

$$y[n] = \sum_{k=0}^{N-1} b[k]x[n-k] = h[n] * x[n]$$

$$Y(z) = H(z)\, X(z)$$

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System function $H(z)$ for the FIR is the polynomial

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{N-1} b[k]\, z^{-k}$$

IIR System Function

$$y[n] = \sum_{k=1}^{M-1} a[k]y[n-k] + \sum_{k=0}^{N-1} b[k]x[n-k] = a[n] * y[n] + b[n] * x[n]$$

IIR System Function

$$y[n] = \sum_{k=1}^{M-1} a[k]y[n-k] + \sum_{k=0}^{N-1} b[k]x[n-k] = a[n]*y[n] + b[n]*x[n]$$

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$$Y(z) = A(z)\ Y(z) + B(z)\ X(z)$$

$$Y(z) = \frac{B(z)}{1 - A(z)}\ X(z) = H(z)\ X(z)$$

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The IIR system function is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{B(z)}{1-A(z)} = \frac{\sum_{k=0}^{N-1} b[k]z^{-k}}{1-\sum_{k=1}^{M-1} a[k]\ z^{-k}}$$

System Function

Example

Given the sequence

$$y[n] = 0.9y[n-1] + x[n]$$

1. Find $H(z)$
2. Determine the impulse response $h[n]$

$$Y(z) = 0.9\, z^{-1}\ Y(z) + X(z), \qquad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.9z^{-1}}$$

From tables we have $h[n] = 0.9^{n}\ u[n]$

Poles and Zeros

Poles and Zeros of the System Function $H(z)$

Must express $H(z)$ as positive powers of z.

$$H(z) = \frac{\sum_{k=0}^{N-1} b_k z^{-k}}{1 - \sum_{k=1}^{M-1} a_k z^{-k}}$$

$$\begin{aligned} H(z) &= b_0 \times \frac{(1 - q_0 z^{-1})}{(1 - p_0 z^{-1})} \times \frac{(1 - q_1 z^{-1})}{(1 - p_1 z^{-1})} \times \cdots \times \frac{(1 - q_{R-1} z^{-1})}{(1 - p_{R-1} z^{-1})} \\ &= b_0 \times \frac{(z - q_0)}{(z - p_0)} \times \frac{(z - q_1)}{(z - p_1)} \times \cdots \times \frac{(z - q_{R-1})}{(z - p_{R-1})} \end{aligned}$$

Poles and Zeros of the System Function $H(z)$

Example

Find the poles and zeros of the system function defined as a z-transform of a length-L pulse $r_L[n]$.

$$H(z) = \frac{1 - z^{-L}}{1 - z^{-1}} = \frac{z^L - 1}{z^{L-1}(z - 1)}$$

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We can write this as

$$z_k = e^{j2\pi k/L}, \qquad 0 \leq k < L$$

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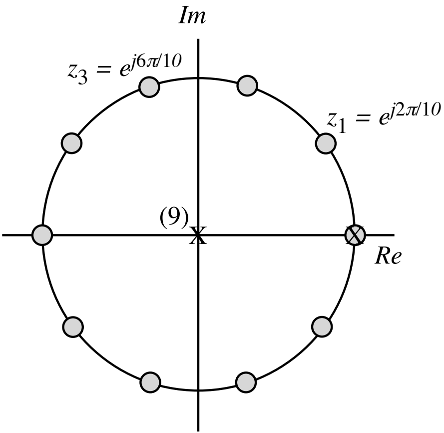
Those zeros are:

$$z_0 = 1,\ z_1 = e^{j2\pi/L},\ z_2 = e^{j4\pi/L},\ \cdots\ z_{L-1} = e^{j2\pi(L-1)/L}$$

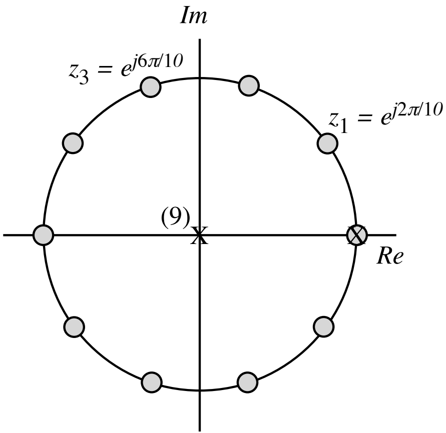
Poles & Zeros for Example: Finding the Zeros

Case L = 10 implies 10 zeros equally spaced around the unit circle in the complex plane. Zeros are shown as the circles.



Poles & Zeros for Example: Finding the Poles

$H(z)$ has L zeros and L poles. The locations of the poles are $L-1$ poles at $z=0$ and one pole at $z=1$.



z-Transform & DFT

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z-Transform & DTFT

1. $X(z) = \sum_{n=0}^{\infty} x[n]\; z^{-n}$
2. Use $z = re^{j\widehat{\omega}}$, $\qquad r > 0$ and $-\pi \leq \widehat{\omega} < \pi$
3. When $r = 1$, we get

$$\begin{aligned} X(z)|_{z=e^{j\widehat{\omega}}} &= \sum_{n=0}^{\infty} x[n]\; e^{-j\widehat{\omega}n} \\ &= X(\widehat{\omega}) \end{aligned}$$

z-Transform & DFT

1. Use discrete $\widehat{\omega}_k = \frac{2\pi k}{N}, \qquad -N/2 \leq k < N/2$

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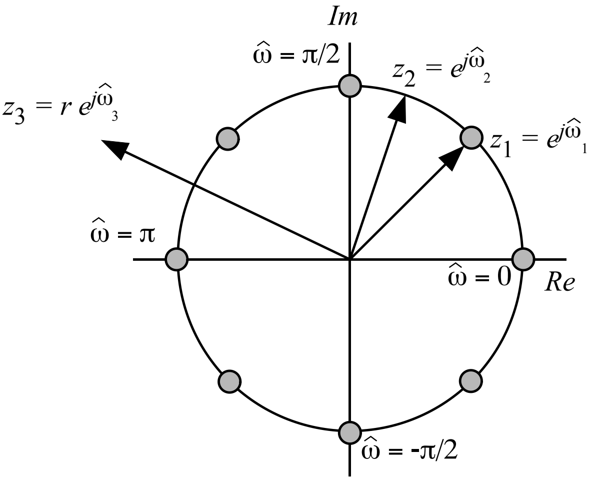
z-Transform & DTFT & DFT



Inverse z-Transform

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2. Need to find $h[n]$

Inverse z-Transform

1. We are given $H(z)$
2. Need to find $h[n]$
3. We consider special form of $H(z)$:

$$H(z) = \frac{B(z)}{A(z)}$$

Inverse z-Transform Methods

1. When $H(z) = B(z)$, we use direct inspection of the polynomial terms.

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1. When $H(z) = B(z)$, we use direct inspection of the polynomial terms.

2. When $H(z) = B(z)/A(z)$ we can use polynomial division then do inverse z-transform on the resulting quotient terms.

3. When $H(z) = B(z)/A(z)$ &

$$\deg[B(z^{-1})] < \deg[A(z^{-1})]$$

we can use the partial fraction approach then do inverse z-transform on the resulting terms.

Inverse z-Transform: Inspection of Polynomial

$$H(z) = b[0] + b[1]z^{-1} + b[2]z^{-2} + \cdots + b[N-1]z^{-N-1}$$

Using linearity property, we do inverse z-transform for each term

$$h[n] = b[0]\delta[n] + b[1]\delta[n-1] + \cdots + b[N-1]\delta[n-N+1]$$

Inverse z-Transform: Inspection of Polynomial

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Using linearity property, we do inverse z-transform for each term

$$\begin{aligned} h[n] &= b[0]\delta[n] + b[1]\delta[n-1] + \cdots + b[N-1]\delta[n-N+1] \\ &= \sum_{k=0}^{N-1} b[k]\delta[n-k] \end{aligned}$$

Inverse z-Transform: Inspection of Polynomial

Example

Find the inverse z-transform for the system

$$H(z) = \left(1 - z^{-1}\right)\left(1 - 2z^{-1}\right)\left(1 - 3z^{-1}\right), \qquad z \neq 0$$

Inverse z-Transform: Inspection of Polynomial

Example

Find the inverse z-transform for the system

$$H(z) = \left(1 - z^{-1}\right)\left(1 - 2z^{-1}\right)\left(1 - 3z^{-1}\right), \qquad z \neq 0$$

$$H(z) = 1 - 6z^{-1} + 11z^{-2} - 6z^{-3}, \qquad z \neq 0$$

Therefore we can write by inspection

Inverse z-Transform: Inspection of Polynomial

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$$H(z) = 1 - 6z^{-1} + 11z^{-2} - 6z^{-3}, \qquad z \neq 0$$

Therefore we can write by inspection

$$h[n] = \delta[n] - 6\delta[n-1] + 11\delta[n-2] - 6\delta[n-3]$$

Inverse z-Transform: Polynomial Division

Example

Find the inverse z-transform using polynomial division for

$$H(z) = \frac{1}{1 + z^{-1} - 0.5z^{-2}} = \frac{z^2}{z^2 + z - 0.5}, \qquad |z| > 1.36$$

$$\begin{array}{r|l} & 1 \quad -z^{-1} \quad +1.5z^{-2} \quad -2z^{-3} \quad \cdots \\ \hline z^2 \quad +z \quad -0.5 & z^2 \end{array}$$

Inverse z-Transform: Polynomial Division

Example

Find the inverse z-transform using polynomial division for

$$H(z) = \frac{1}{1 + z^{-1} - 0.5z^{-2}} = \frac{z^2}{z^2 + z - 0.5}, \qquad |z| > 1.36$$

$$\begin{array}{rrr|lllll}
 & & & 1 & -z^{-1} & +1.5z^{-2} & -2z^{-3} & \cdots \\
\hline
z^2 & +z & -0.5 & z^2 & & & & \\
 & & & z^2 & +z & -0.5 & & \\
\hline
\end{array}$$

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 & 1 & -z^{-1} & +1.5z^{-2} & -2z^{-3} & \cdots \\ \hline
z^2 \quad +z \quad -0.5 & z^2 & & & & \\
 & z^2 & +z & -0.5 & & \\ \cline{2-4}
 & & -z & +0.5 & &
\end{array}$$

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 & 1 & -z^{-1} & +1.5z^{-2} & -2z^{-3} \quad \cdots \\
\hline
z^2 \quad +z \quad -0.5 & z^2 & & & \\
 & z^2 & +z & -0.5 & \\
\cline{2-4}
 & & -z & +0.5 & \\
 & & -z & -1 & +0.5z^{-1} \\
\cline{3-5}
\end{array}$$

Inverse z-Transform: Polynomial Division

Example

Find the inverse z-transform using polynomial division for

$$H(z) = \frac{1}{1 + z^{-1} - 0.5z^{-2}} = \frac{z^2}{z^2 + z - 0.5}, \qquad |z| > 1.36$$

$$
\begin{array}{ccc|ccccc}
 & & & 1 & -z^{-1} & +1.5z^{-2} & -2z^{-3} & \cdots \\
\hline
z^2 & +z & -0.5 & z^2 & & & & \\
 & & & z^2 & +z & -0.5 & & \\
\cline{4-6}
 & & & & -z & +0.5 & & \\
 & & & & -z & -1 & +0.5z^{-1} & \\
\cline{5-7}
 & & & & & 1.5 & -0.5z^{-1} &
\end{array}
$$

Inverse z-Transform: Polynomial Division

Example

Find the inverse z-transform using polynomial division for

$$H(z) = \frac{1}{1 + z^{-1} - 0.5z^{-2}} = \frac{z^2}{z^2 + z - 0.5}, \qquad |z| > 1.36$$

$$\begin{array}{ccc|ccccc}
 & & & 1 & -z^{-1} & +1.5z^{-2} & -2z^{-3} & \cdots \\
\hline
z^2 & +z & -0.5 & z^2 & & & & \\
 & & & z^2 & +z & -0.5 & & \\
\cline{4-6}
 & & & & -z & +0.5 & & \\
 & & & & -z & -1 & +0.5z^{-1} & \\
\cline{5-7}
 & & & & & 1.5 & -0.5z^{-1} & \\
 & & & & & 1.5 & 1.5z^{-1} & -0.75z^{-2} \\
\cline{6-8}
\end{array}$$

Inverse z-Transform: Polynomial Division

Example

Find the inverse z-transform using polynomial division for

$$H(z) = \frac{1}{1 + z^{-1} - 0.5z^{-2}} = \frac{z^2}{z^2 + z - 0.5}, \qquad |z| > 1.36$$

$$
\begin{array}{r|lllll}
 & 1 & -z^{-1} & +1.5z^{-2} & -2z^{-3} & \cdots \\
\hline
z^2 \quad +z \quad -0.5 & z^2 & & & & \\
 & z^2 & +z & -0.5 & & \\
\cline{2-4}
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\cline{4-6}
 & & & & -2z^{-1} & +0.75z^{-2}
\end{array}
$$

Inverse z-Transform: Polynomial Division

Example

Find the inverse z-transform using polynomial division for

$$H(z) = \frac{1}{1 + z^{-1} - 0.5z^{-2}} = \frac{z^2}{z^2 + z - 0.5}, \qquad |z| > 1.36$$

	1	$-z^{-1}$	$+1.5z^{-2}$	$-2z^{-3}$	$\cdots$
$z^2 \quad +z \quad -0.5$	z^2				
	z^2	$+z$	-0.5		
		$-z$	$+0.5$		
		$-z$	-1	$+0.5z^{-1}$	
			1.5	$-0.5z^{-1}$	
			1.5	$1.5z^{-1}$	$-0.75z^{-2}$
				$-2z^{-1}$	$+0.75z^{-2}$
				$-2z^{-1}$	$-2z^{-2}$	$+z^{-3}$

Inverse z-Transform: Polynomial Division

Example

Find the inverse z-transform using polynomial division for

$$H(z) = \frac{1}{1 + z^{-1} - 0.5z^{-2}} = \frac{z^2}{z^2 + z - 0.5}, \qquad |z| > 1.36$$

$$\begin{array}{r|llllll}
 & 1 & -z^{-1} & +1.5z^{-2} & -2z^{-3} & \cdots & \\
\hline
z^2 \quad +z \quad -0.5 & z^2 & & & & & \\
 & z^2 & +z & -0.5 & & & \\
\cline{2-4}
 & & -z & +0.5 & & & \\
 & & -z & -1 & +0.5z^{-1} & & \\
\cline{3-5}
 & & & 1.5 & -0.5z^{-1} & & \\
 & & & 1.5 & 1.5z^{-1} & -0.75z^{-2} & \\
\cline{4-6}
 & & & & -2z^{-1} & +0.75z^{-2} & \\
 & & & & -2z^{-1} & -2z^{-2} & +z^{-3} \\
\cline{5-7}
 & & & & & \vdots &
\end{array}$$

Inverse z-Transform: Polynomial Division, Continued

Therefore our system function is

$$H(z) = 1 - z^{-1} + 1.5z^{-2} - 2z^{-3} + 2.75z^{-4} \cdots$$

and the impulse response is

$$h[n] = \delta[n] - \delta[n-1] + 1.5\delta[n-2] - 2\delta[n-3] + 2.75\delta[n-5] + \cdots$$

Inverse z-Transform: Polynomial Division

Example

Find the inverse z-transform using polynomial division for

$$H(z) = \frac{1 - 0.1z^{-1} - z^{-2} - 0.5z^{-3}}{1 - 0.5z^{-1} - 0.2z^{-2} - 0.1z^{-3}}, \qquad |z| > 0.8649$$

$$\begin{array}{cccc|cccc}
 & & & & 1 & +0.4z^{-1} & -0.6z & -0.62z^{-3} \\
\cline{5-8}
z^3 & -0.5z^2 & -0.2z & -0.1 & z^3 & -0.1z^2 & -z & -0.5 \\
\cline{1-4}
\end{array}$$

Inverse z-Transform: Polynomial Division

Example

Find the inverse z-transform using polynomial division for

$$H(z) = \frac{1 - 0.1z^{-1} - z^{-2} - 0.5z^{-3}}{1 - 0.5z^{-1} - 0.2z^{-2} - 0.1z^{-3}}, \qquad |z| > 0.8649$$

$$\begin{array}{rrrr|rrrr}
 & & & & 1 & +0.4z^{-1} & -0.6z & -0.62z^{-3} \\
\hline
z^3 & -0.5z^2 & -0.2z & -0.1 & z^3 & -0.1z^2 & -z & -0.5 \\
 & & & & z^3 & -0.5z^2 & -0.2z & -0.1 \\
\hline
\end{array}$$

Inverse z-Transform: Polynomial Division

Example

Find the inverse z-transform using polynomial division for

$$H(z) = \frac{1 - 0.1z^{-1} - z^{-2} - 0.5z^{-3}}{1 - 0.5z^{-1} - 0.2z^{-2} - 0.1z^{-3}}, \qquad |z| > 0.8649$$

$$
\begin{array}{cccc|cccc}
 & & & & 1 & +0.4z^{-1} & -0.6z & -0.62z^{-3} \\
\cline{5-8}
z^3 & -0.5z^2 & -0.2z & -0.1 & z^3 & -0.1z^2 & -z & -0.5 \\
\cline{1-4}
 & & & & z^3 & -0.5z^2 & -0.2z & -0.1 \\
\cline{5-8}
 & & & & & 0.4z^2 & -0.8z & -0.4
\end{array}
$$

Inverse z-Transform: Polynomial Division

Example

Find the inverse z-transform using polynomial division for

$$H(z) = \frac{1 - 0.1z^{-1} - z^{-2} - 0.5z^{-3}}{1 - 0.5z^{-1} - 0.2z^{-2} - 0.1z^{-3}}, \qquad |z| > 0.8649$$

$$
\begin{array}{cccc|cccc}
 & & & & 1 & +0.4z^{-1} & -0.6z & -0.62z^{-3} \\
\cline{5-8}
z^3 & -0.5z^2 & -0.2z & -0.1 & z^3 & -0.1z^2 & -z & -0.5 \\
\cline{1-4}
 & & & & z^3 & -0.5z^2 & -0.2z & -0.1 \\
\cline{5-8}
 & & & & & 0.4z^2 & -0.8z & -0.4 \\
 & & & & & 0.4z^2 & -0.2z & -0.08 \\
\cline{6-8}
\end{array}
$$

Inverse z-Transform: Polynomial Division

Example

Find the inverse z-transform using polynomial division for

$$H(z) = \frac{1 - 0.1z^{-1} - z^{-2} - 0.5z^{-3}}{1 - 0.5z^{-1} - 0.2z^{-2} - 0.1z^{-3}}, \qquad |z| > 0.8649$$

				1	$+0.4z^{-1}$	$-0.6z$	$-0.62z^{-3}$
z^3	$-0.5z^2$	$-0.2z$	-0.1	z^3	$-0.1z^2$	$-z$	-0.5
				z^3	$-0.5z^2$	$-0.2z$	-0.1
					$0.4z^2$	$-0.8z$	-0.4
					$0.4z^2$	$-0.2z$	-0.08
						$-0.6z$	-0.32

Inverse z-Transform: Polynomial Division, Continued

Therefore our system function is

$$H(z) = 1 + 0.4z^{-1} - 0.6z^{-2} - 0.62z^{-3} - 0.39z^{-4} + \cdots$$

and the impulse response is

$$h[n] = \delta[n] + 0.4\delta[n-1] - 0.6\delta[n-2] - .62\delta[n-3] - 0.39\delta[n-4] + \cdots$$

Partial Fractions

Useful Formulas Often Seen in Partial Fractions Method

1. $\delta[n] \overset{z}{\longleftrightarrow} 1$
2. $u[n] \overset{z}{\longleftrightarrow} 1/(1-z^{-1}), \quad |z| > 1$
3. $n\,u[n] \overset{z}{\longleftrightarrow} z^{-1}/(1-z^{-1})^2, \quad |z| > 1$
4. $a^n\,u[n] \overset{z}{\longleftrightarrow} 1/(1-az^{-1}), \quad |z| > a$
5. $na^n\,u[n] \overset{z}{\longleftrightarrow} az^{-1}/(1-az^{-1})^2, \quad |z| > a$
6. $(n+1)a^n\,u[n] \overset{z}{\longleftrightarrow} \frac{1}{(1-az^{-1})^2}, \quad |z| > a$
7. $x[n-n_0] \overset{z}{\longleftrightarrow} X(z)\,z^{-n_0}$
8. z-transform is linear

Partial Fraction Steps

1. Ensure $H(z)$ is expressed in powers of z^{-1}

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Partial Fraction Steps

1. Ensure $H(z)$ is expressed in powers of z^{-1}
2. Ensure $H(z)$ is proper: $\deg[B(z^{-1})] < \deg[A(z^{-1})]$
3. Use polynomial division if $\deg[B(z^{-1})] \geq \deg[A(z^{-1})]$
4. Express $H(z)$ as a sum

$$H(z) = \frac{r_1}{1 - p_1 z^{-1}} + \frac{r_2}{1 - p_2 z^{-1}} + \cdots$$

Partial Fraction Steps

1. Ensure $H(z)$ is expressed in powers of z^{-1}
2. Ensure $H(z)$ is proper: $\deg[B(z^{-1})] < \deg[A(z^{-1})]$
3. Use polynomial division if $\deg[B(z^{-1})] \geq \deg[A(z^{-1})]$
4. Express $H(z)$ as a sum

$$H(z) = \frac{r_1}{1 - p_1 z^{-1}} + \frac{r_2}{1 - p_2 z^{-1}} + \cdots$$

5. Find inverse z-transform of the sum

Inverse z-Transform: Partial Fraction Expansion

$$H(z) = \frac{B(z)}{A(z)}$$

Inverse z-Transform: Partial Fraction Expansion

$$
\begin{aligned}
H(z) &= \frac{B(z)}{A(z)} \\
&= \frac{(1-q_0 z^{-1})}{(1-p_0 z^{-1})} \times \frac{(1-q_1 z^{-1})}{(1-p_1 z^{-1})} \times \cdots \times \frac{(1-q_{R-1} z^{-1})}{(1-p_{R-1} z^{-1})}
\end{aligned}
$$

Inverse z-Transform: Partial Fraction Expansion

$$
\begin{aligned}
H(z) &= \frac{B(z)}{A(z)} \\
&= \frac{(1 - q_0 z^{-1})}{(1 - p_0 z^{-1})} \times \frac{(1 - q_1 z^{-1})}{(1 - p_1 z^{-1})} \times \cdots \times \frac{(1 - q_{R-1} z^{-1})}{(1 - p_{R-1} z^{-1})} \\
&= \frac{r_1}{1 - p_1 z^{-1}} + \frac{r_2}{1 - p_2 z^{-1}} + \cdots
\end{aligned}
$$

The residue r_k is given by

$$r_k = H(z) \times (1 - p_k z^{-1})|_{z=p_k}$$

Example

Use partial fraction expansion to find $h[n]$ for

$$H(z) = \frac{1 + 2z^{-1}}{1 - 1.5z^{-1} + 0.5z^{-2}}, \qquad |z| > 1$$

$r_1 = -5$ with pole at z = 0.5, $r_2 = 6$ with pole at z=1

Partial Fraction Expansion: Alternative Method

Example

Find $h[n]$ for the system function

$$H(z) = \frac{1 - 3z^{-1}}{(1 - 4z^{-1})(1 - 8z^{-1})}, \qquad |z| > 8$$

Partial Fraction Expansion: Alternative Method

Example

Find $h[n]$ for the system function

$$H(z) = \frac{1 - 3z^{-1}}{(1 - 4z^{-1})(1 - 8z^{-1})}, \qquad |z| > 8$$

Since $N < M$, we can write

$$\frac{1 - 3z^{-1}}{(1 - 4z^{-1})(1 - 8z^{-1})} = \frac{r_1}{1 - 4z^{-1}} + \frac{r_2}{1 - 8z^{-1}}$$

Partial Fraction Expansion: Alternative Method

Example

Find $h[n]$ for the system function

$$H(z) = \frac{1 - 3z^{-1}}{(1 - 4z^{-1})(1 - 8z^{-1})}, \qquad |z| > 8$$

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Equating denominators on both sides we have

$$1 - 3z^{-1} = r_1(1 - 8z^{-1}) + r_2(1 - 4z^{-1})$$

Partial Fraction Expansion: Alternative Method

Example

Find $h[n]$ for the system function

$$H(z) = \frac{1 - 3z^{-1}}{(1 - 4z^{-1})(1 - 8z^{-1})}, \qquad |z| > 8$$

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$$\frac{1 - 3z^{-1}}{(1 - 4z^{-1})(1 - 8z^{-1})} = \frac{r_1}{1 - 4z^{-1}} + \frac{r_2}{1 - 8z^{-1}}$$

Equating denominators on both sides we have

$$1 - 3z^{-1} = r_1(1 - 8z^{-1}) + r_2(1 - 4z^{-1})$$

Hence $r_1 = -0.25$ and $r_2 = 1.25$.

$$h[n] = -0.25 \times 4^n u[n] + 1.25 \times 8^n u[n]$$

Inverse z-Transform: Partial Fraction Expansion

Example

Find $h[n]$ for the system function

$$H(z) = \frac{1}{(1 - 0.25z^{-1})(1 - 0.5z^{-1})}, \qquad |z| > 0.5$$

$r_1 = 4$ for z = 0.5 and $r_2 = -4$ for z = 0.25

Inverse z-Transform: Partial Fraction Expansion

Example

Find $h[n]$ for the system function

$$H(z) = \frac{1 - z^{-1}}{(1 - 0.2z^{-1})(1 - 0.3z^{-1})}$$

$r_1 = -7$ for z = 0.3 and $r_2 = 8$ for z = 0.2

Inverse z-Transform: Partial Fraction Expansion

Example

Assume the system function is given by

$$H(z) = \frac{1}{(1 - z^{-1})(1 + 2z^{-1})}, \qquad |z| > 2$$

Express this function in partial fractions and determine its time-domain description.

$r_1 = 2/3$ for z = -2 and $r_2 = 1/3$ for z = 1

Inverse z-Transform: Partial Fraction Expansion

Example

Assume the system function is given by

$$H(z) = \frac{1}{(1 - 0.2z^{-1})(1 - 0.5z^{-1})}, \qquad |z| > 0.5$$

Express this function in partial fractions and determine its time-domain description.

Inverse z-Transform: Partial Fraction Expansion

Example

Assume the system function is given by

$$H(z) = \frac{1 - z^{-1}}{1 - 0.5z^{-1} + 0.06z^{-2}}, \qquad |z| > 0.3$$

Express this function in partial fractions and determine its time-domain description.

Review of Exponential Signal z Transform

We remember the following facts:

1. Assume we have $X(z) = \frac{a}{1-pz^{-1}} = \frac{az}{z-p}$

Review of Exponential Signal z Transform

We remember the following facts:

1. Assume we have $X(z) = \frac{a}{1-pz^{-1}} = \frac{az}{z-p}$

2. The inverse z-transform is given by

$$x[n] = a\, p^n\, u[n]$$

Repeated Poles

Inverse z-Transform: Repeated Poles

Example

Assume the system function is given by

$$H(z) = \frac{1}{(1 - z^{-1})^2(1 - 2z^{-1})}, \qquad |z| > 2$$

Express this function in partial fractions and determine its time-domain description.

Inverse z-Transform: Repeated Poles

Example

Assume the system function is given by

$$H(z) = \frac{1}{(1 - z^{-1})^2(1 - 2z^{-1})}, \qquad |z| > 2$$

Express this function in partial fractions and determine its time-domain description.

$$H(z) = \frac{1}{(1 - z^{-1})^2(1 - 2z^{-1})} = \frac{r_1}{1 - z^{-1}} + \frac{r_2}{(1 - z^{-1})^2} + \frac{r_3}{1 - 2z^{-1}}$$

Inverse z-Transform: Repeated Poles

Example

Assume the system function is given by

$$H(z) = \frac{1}{(1-z^{-1})^2(1-2z^{-1})}, \qquad |z| > 2$$

Express this function in partial fractions and determine its time-domain description.

$$H(z) = \frac{1}{(1-z^{-1})^2(1-2z^{-1})} = \frac{r_1}{1-z^{-1}} + \frac{r_2}{(1-z^{-1})^2} + \frac{r_3}{1-2z^{-1}}$$

Nominators on both sides give

$$1 = r_1(1-z^{-1})(1-2z^{-1}) + r_2(1-2z^{-1}) + r_3(1-z^{-1})^2$$

Inverse z-Transform: Repeated Poles, Continued

$$r_1 = -2, \qquad r_2 = -1, \qquad r_3 = 4$$

$$H(z) = \frac{-2}{1 - z^{-1}} + \frac{-1}{(1 - z^{-1})^2} + \frac{4}{1 - 2z^{-1}}$$

Inverse z-Transform: Repeated Poles, Continued

$$
\begin{aligned}
r_1 &= -2, \qquad r_2 = -1, \qquad r_3 = 4 \\
H(z) &= \frac{-2}{1 - z^{-1}} + \frac{-1}{(1 - z^{-1})^2} + \frac{4}{1 - 2z^{-1}}
\end{aligned}
$$

We have the pair:

$$
(n+1)\, u[n] \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{(1 - z^{-1})^2}
$$

Inverse z-Transform: Repeated Poles, Continued

$$\begin{aligned} r_1 &= -2, \qquad r_2 = -1, \qquad r_3 = 4 \\ H(z) &= \frac{-2}{1 - z^{-1}} + \frac{-1}{(1 - z^{-1})^2} + \frac{4}{1 - 2z^{-1}} \end{aligned}$$

We have the pair:

$$(n+1)\, u[n] \quad \stackrel{z}{\longleftrightarrow} \quad \frac{1}{(1 - z^{-1})^2}$$

and

$$h[n] = -(n+1)u[n] - 2u[n] + 4 \times 2^n\, u[n]$$

Partial Fractions: Repeated Roots

Example

Find inverse of

$$X(z) = \frac{1 - 0.6z^{-1} + 0.06z^{-2}}{1 - 1.7z^{-1} + 0.96z^{-2} - 0.18z^{-3}}, \qquad |z| > 0.6$$

where $z_1 = 0.5$ and $z_{2,3} = 0.6$

```
> roots([1 -1.7 0.96 -0.18])
```

$x = 1$ for $z = 0.5$

$r_1 = -1$ for $z = 0.6$

$r_2 = 1$ for $z = 0.6$

General Formula to find a Residue

Assume $H(z)$ is a function of z^{-1} and pole p is repeated m times

$$H(z) = \left[\frac{x_1}{(1 - pz^{-1})} + \cdots \frac{x_m}{(1 - pz^{-1})^m}\right]$$

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The residue x_i for pole p of order m is given in general by

$$x_i = \frac{1}{(m-i)!(-p)^{m-i}} \lim_{w \to p^{-1}} \left[\frac{d^{m-i}}{dw^{m-i}} (1 - pw)^m H(w)\right], \quad i = 1, 2, \cdots$$

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Find the inverse z-transform for the system function

$$H(z) = \frac{1 - z^{-1} - z^{-2}}{1 - 5z^{-1} + 6z^{-2}} = \frac{z^2 - z - 1}{z^2 - 5z + 6}, \qquad |z| > 3$$

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$$\begin{array}{r|rrr} & 1 & & \\ \hline z^2 \quad -5z \quad +6 & z^2 & -z & -1 \\ & z^2 & -5z & +6 \\ \hline & & 4z & -7 \end{array}$$

Case deg$[B(z^{-1})] \geq$ deg$[A(z^{-1})]$ Continued

$$
\begin{aligned}
H(z) &= 1 + \frac{4z - 7}{z^2 - 5z + 6} = 1 + \frac{4z^{-1} - 7z^{-2}}{1 - 5z^{-1} + 6z^{-2}} \\
&= 1 + z^{-1} \times \frac{4 - 7z^{-1}}{1 - 5z^{-1} + 6z^{-2}}
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Notice that now we have an expression where $N < M$ as desired.

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$r_1 = -1$ and $r_2 = 5$.

$$
h_1[n] = \delta[n] - 2^{n-1} u[n-1] + 5 \times 3^{n-1} u[n-1]
$$

Case $\deg[B(z^{-1})] \geq \deg[A(z^{-1})]$: Alternative Method (Ex. 7.14)

1. Consider case when $N = M$

$$H(z) = \frac{1 - z^{-1} - z^{-2}}{1 - 5z^{-1} + 6z^{-2}} = \frac{z^2 - z - 1}{z^2 - 5z + 6}, \qquad |z| > 3$$

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$$H'(z) = \frac{H(z)}{z} = \frac{B(z)}{z\,A(z)}$$

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5. Proceed with partial fractions on $H'(z) = H(z)/z$.

Case deg$[B(z^{-1})] \geq$ deg$[A(z^{-1})]$: Alternative Method Continued

$$\frac{H(z)}{z} = \frac{z^2 - z - 1}{z(z-2)(z-3)}, \qquad |z| > 3$$

Case $\deg[B(z^{-1})] \geq \deg[A(z^{-1})]$: Alternative Method Continued

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\begin{aligned}
\frac{H(z)}{z} &= \frac{z^2 - z - 1}{z(z-2)(z-3)}, \qquad |z| > 3 \\
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Therefore we can write

$$\frac{H(z)}{z} = -\frac{1/6}{z} - \frac{0.5}{z-2} + \frac{5/3}{z-3}$$

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and we can write

$$h_2[n] = -\frac{1}{6}\delta[n] - 0.5 \times 2^n\, u[n] + \frac{5}{3} \times 3^n\, u[n]$$

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$$\begin{aligned} h_1[n] &= \delta[n] - 2^{n-1}u[n-1] + 5 \times 3^{n-1}u[n-1] \\ h_2[n] &= -\frac{1}{6}\delta[n] - 0.5 \times 2^n\, u[n] + \frac{5}{3} \times 3^n\, u[n] \end{aligned}$$

The two solutions are identical.

Transient & Steady State Response of an FIR Filter

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Transient, Steady State Response of an IIR Filter

Assume

$$y[n] = ay[n-1] + bx[n] \qquad Y(z) = \frac{b}{1 - az^{-1}} \times X(z), \qquad |z| > a$$

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We have

$$Y(z) = \frac{b}{(1 - az^{-1})(1 - e^{j\widehat{\omega}_0} z^{-1})} \quad y[n] = r_1\, a^n\, u[n] + r_2\, e^{j\widehat{\omega}_0 n}\, u[n]$$