GLOBAL EDGE-CONDITIONED BASIS FUNCTIONS FROM LOCAL SOLUTIONS OF MAXWELL'S EQUATIONS

S. Amari, A. Motamedi, J. Bornemann and R. Vahldieck
Department of Electrical and Computer Engineering
University of Victoria, Victoria B.C., Canada V8W 3P6

Abstract

A new set of edge-conditioned basis functions for the moment method solutions of electromagnetic problems is introduced. The basis functions are themselves solutions to the differential forms of Maxwell's equations and satisfy the local boundary conditions at metallic wedges. Numerical results using this new set are presented and compared with available data for a ridged rectangular waveguide to demonstrate its adequacy. An efficient technique to compute integrals of rapidly oscillating and singular integrands will also be presented.

1 INTRODUCTION

The Method of Moments (MoM) is a frequently used approach to solving integral, differential, and integro-differential equations in electromagnetics; it has been applied to practically any imaginable structure [1].
From a computational point of view, the efficiency of the method depends on the availability of basis functions which approximate well the unknown function. When sharp metallic edges are present, the convergence of the technique is greatly improved by using basis functions which include the edge conditions. A variety of such functions were introduced and used by many researchers [2]. These basis functions succeed in including the edge conditions by using weighted elementary functions such as polynomials and trigonometric functions.
However, it is important to point out that the edge-conditioned basis functions encountered in the literature are not solutions to the differential forms of Maxwell's equations except in the limited vicinity of the metallic edges as direct differentiation and substitution show.
In this paper, we propose to show how the local solutions of Maxwell's equations in the vicinity of metallic wedges, which are known in terms of Bessel functions with a free parameter, can be used to construct general basis functions which include the edge conditions and satisfy the operational forms of Maxwell's equations. Admissible values of the free parameter are determined from the boundary conditions away from the metallic wedges.
In applying the moment method, it is necessary to calculate weighted integrals of these singular and oscillating basis functions. Accurate evaluation of these integrals requires a careful handling of the singularities. A technique to deal with these numerical pathologies consists in removing the singularity prior to numerical integration [3]. The oscillations in the integrands are handled using a partial substitutional technique where the slowly varying parts are replaced by splines or other interpolations to allow for the remaining integrals to be computed analytically.

2 THEORY

We consider the structure shown in Figure 1. It consists of a symmetric metallic ridge of width $2s$ and height d in a rectangular waveguide of cross section $2a \times b$. All metallic surfaces are assumed lossless in this analysis. We limit the analysis to the TE modes with a magnetic

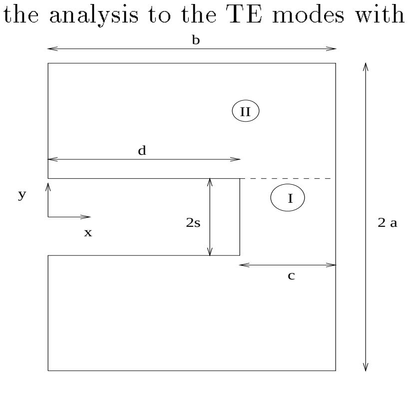

Figure 1. Cross section of ridged waveguide

wall along the x-axis. Other symmetries as well as the

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TM modes are obtained similarly. presentation.
The axial magnetic field of the TE modes with magnetic-wall symmetry is expanded in series of the form

$$H_z^I(x,y) = \sum_{n=0}^{\infty} A_n cos[n\pi \frac{x-d}{c}] \sinh[\gamma_{1n} y] \tag{1}$$

and

$$H_z^{II}(x,y) = \sum_{n=0}^{\infty} B_n cos[n\pi \frac{x}{b}] \cosh[\gamma_{2n}(y-a)] \tag{2}$$

Here, $\gamma_{1n}^2 = (n\frac{\pi}{c})^2 - k_c^2$, $\gamma_{2n} = (n\frac{\pi}{b})^2 - k_c^2$ where k_c is the unknown cutoff wavenumber. At cutoff, the only non-vanishing transverse components of the electromagnetic field are E_x and H_y. To enforce the boundary conditions of E_x at the interface, let us assume that it is given by a yet unknown function $X(x)$ which vanishes over the metallic part of the interface, i.e.

$$X(x) = 0, \quad 0 \leq x \leq d. \tag{3}$$

Using the expansions (1) and (2) in the boundary conditions of E_x, the modal coefficients A_n and B_n are expressed in terms of X as follows

$$\begin{aligned} A_n &= \tfrac{1}{\gamma_{1n} cosh[\gamma_{1n} s]} \tfrac{2}{c(1+\delta_{n0})} \int_d^b X(x) cos[n\pi \tfrac{x-d}{c}] dx \\ &= \tfrac{1}{\gamma_{1n} cosh[\gamma_{1n} s]} \tilde{X}^I(n) \end{aligned} \tag{4}$$

and

$$\begin{aligned} B_n &= \tfrac{1}{\gamma_{2n} sinh[\gamma_{2n}(s-a)]} \tfrac{2}{b(1+\delta_{n0})} \int_d^b X(x) cos[n\pi \tfrac{x}{b}] dx \\ &= \tfrac{1}{\gamma_{2n} sinh[\gamma_{2n}(s-a)]} \tilde{X}^{II}(n) \end{aligned} \tag{5}$$

From the continuity of the H_z at the interface, we obtain an integral equation for $X(x)$ [4].

$$\begin{aligned} &\sum_{n=0}^{\infty} \quad \tilde{X}^I(n) \tfrac{tanh[\gamma_{1n} s]}{\gamma_{1n}} cos[n\pi \tfrac{x-d}{c}] \\ &+ \sum_{n=0}^{\infty} \quad \tfrac{\tilde{X}^{II}(n)}{\gamma_{2n} tanh[\gamma_{2n}(a-s)]} cos[n\pi \tfrac{x}{b}] = 0 \end{aligned} \tag{6}$$

To solve this integral equation, we expand the function $X(x)$ in a series of basis functions

$$X(x) = \sum_{i=1}^{M} c_i B_i(x), \tag{7}$$

and apply Galerkin's method to obtain a homogeneous set of equations in the coefficients c_i.

$$[A][c] = 0. \tag{8}$$

Here, the entries of the matrix $[A]$ are given by

$$\begin{aligned} [A]_{ij} = &\sum_{n=0}^{\infty} \quad \tilde{B}_i^I(n) \tilde{B}_j^I(n) \tfrac{tanh[\gamma_{1n} s]}{\gamma_{1n}} (1+\delta_{n0}) \\ &+ \frac{b}{c} \sum_{n=0}^{\infty} \quad \tfrac{\tilde{B}_i^{II}(n) \tilde{B}_j^{II}(n)}{\gamma_{2n} tanh[\gamma_{2n}(a-s)]} (1+\delta_{n0}). \end{aligned} \tag{9}$$

The cutoff wavenumbers are given by the zeros of the determinant of the matrix $[A]$, or equivalently, by the zeros of the minimal singular value of $[A]$ [5]. Similar equations can be derived for TM and TE modes with an electric wall symmetry.

3 BASIS FUNCTIONS

For modes which are TE to the axis of a 90° metallic wedge, the axial magnetic field has solutions of the form [6]

$$\begin{aligned} J_\nu(\eta\rho) cos[\nu(\phi - \pi/2)], \quad \nu = \frac{2n}{3}, \\ n = 0, 1, 2 \ldots \end{aligned} \tag{10}$$

where ρ is the radial distance from the metallic wedge. Therefore, in a given angular direction, the behavior of the axial magnetic field is proportional to $J_\nu(\eta\rho)$, and only the value $\nu = 2/3$ leads to singular fields.
At the interface I-II, E_x, in the vicinity of the wedge, is proportional to the radial derivative of the axial magnetic field

$$E_x \propto J'_{2/3}(\eta(x-d)). \tag{11}$$

The acceptable values of η are determined from observing that E_x is an even function about the electric wall located at $x = b$, or

$$J''_{2/3}(\eta b) = 0 \tag{12}$$

Let r''_i denote a root of the second derivative of $J_{2/3}$, then the following basis functions are used in this work

$$B_i(x) = J'_{2/3}[r''_i \frac{x-d}{c}] \tag{13}$$

The basis functions for TM modes are constructed following similar steps to those of the previous section. Since E_z vanishes at the metallic edge at $x = d$ as well as at the metallic wall at $x = b$, the following basis functions are used

$$B_i(x) = J_{2/3}[r_i \frac{x-d}{c}] \tag{14}$$

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Here, r_i is a root of

$$J_{2/3}(r_i) = 0 \tag{15}$$

The computation of the entries of the matrix $[A]$ in equation (9) requires the evaluation of weighted integrals of the basis functions. For example, a generic element of $[A]$ contains integrals of the form

$$I_i(n) = \int_d^b J'_{2/3}[r''_i \frac{x-d}{c}] cos[\frac{n\pi}{b}] dx \tag{16}$$

Since $J'_{2/3}(z) \propto z^{-1/3}$ as $z \to 0$, the integrand is singular at $x = d$. This singularity is removed by integrating by parts the derivative of the Bessel function leading to

$$I_i(n) = \frac{c}{r''_i}[J_{2/3}(r''_i) + \frac{n\pi}{b} \int_d^b J_{2/3}[r''_i \frac{x-d}{c}] sin[\frac{n\pi}{b}] dx \tag{17}$$

The remaining integrand is no longer singular and lends itself to straightforward numerical integration except for large values of n where the integrand oscillates rapidly. To circumvent this problem we introduce a partial substitution in the integrand of the last equation in such a way that the thus obtained integrals can be evaluated analytically [7]. Assume that the function $J_{2/3}[r''_i \frac{x-d}{c}]$ is approximated by a set of interpolating functions $L_i(x)$ in the interval $d \leq x \leq b$ such that

$$J_{2/3}[r''_i \frac{x-d}{c}] = \sum_j p_j L_j(x) \tag{18}$$

We also require that the integrals

$$\tilde{L}_j(n) = \int_d^b L_j(x) sin[\frac{n\pi}{b}] dx \tag{19}$$

be known analytically. Using this equation in the expression of $I_i(n)$, we get

$$I_i(n) = \frac{c}{r''_i}[J_{2/3}(r''_i) + \frac{n\pi}{b} \sum_j p_j \tilde{L}_j(n)] \tag{20}$$

In this work, cubic splines, whose weighted integrals against sines and cosines are known analytically, are used for interpolation.

4 RESULTS

The new set of basis functions is used to compute the TE and TM cutoff wavenumbers of the ridge waveguide analyzed in [8].

Table I summarizes our results for the first 8 TE modes with a magnetic wall symmetry along with those of reference [8]. Excellent agreement is seen from the two sets of data. It is worth mentioning that for each mode, one basis function is dominant while the remaining ones have much smaller coefficients reflecting the fact that the field distributions of the modes are well represented by a single term in the expansion in equation (7). For these modes, it was also necessary to add a term corresponding to J_0 in addition to the basis functions originating from $J_{2/3}$ and its derivative. The additional basis function is needed because the basis functions obtained from Bessel functions of fractional order following the scheme outlined here, all change sign. The following additional term, was therefore used

$$B_1^{TE}(x) = \frac{1}{[(2 + \frac{d-x}{c})(\frac{x-d}{c})]^{1/3}} \tag{21}$$

Note that this functions includes the presence of an image singularity in the metallic wall at $x = b$. The integrals of this function against cosines can be expressed in terms of Bessel function of order 1/6 [9].

TABLE I

Cutoff wavenumbers (rad/mm) of the first eight odd TE modes of a ridged waveguide

Mode	1	2	3	4	5	6	7	8
Present Method	0.0926	0.3332	0.3811	0.5263	0.6653	0.6916	0.7453	0.8295
Ref.[6]	0.0930	0.3332	0.3881	0.5265	0.6654	0.6913	0.7456	0.8298

a = b = 9.5 mm, s = 0.15 mm, d = 1.7 mm

Table II shows the cutoff wavenumbers of the first 8 TM modes, good agreement is observed between our results and those corresponding to the magnetic wall symmetry presented in [8]. For TM modes, and with the present dimensions of the ridge, the first basis function is dominant for all the modes reported here. The coefficients of the second and higher bases functions are 50 times smaller than that of $B_1(x)$, thus leading to a rapidly converging numerical solution.

5 CONCLUSIONS

A new set of global and edge-conditioned basis functions obtained from local solutions of Maxwell's equations was introduced. Numerical results from a moment method solution of a ridged rectangular waveguide using

TABLE II

Cutoff wavenumbers (rad/mm) of the first eight TM modes of a ridge waveguide

Mode	1	2	3	4	5	6	7	8
Present Method	0.4711	0.4714	0.7410	0.7416	0.7481	0.7487	0.9400	0.9422
Ref.[6]	0.4665		0.7358					0.9427

a = b = 9.5 mm, s = 0.15 mm, d = 1.7 mm

the new basis functions are in excellent agreement with data available in the literature. Rapid convergence for all the modes is observed; typically one basis function for TM modes and two for TE modes are sufficient.

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