# Comparative study of two integral-equation formulations for inductive irises in rectangular waveguides

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**Abstract:** A comparative study of two formulations based on a combination of the mode-matching technique and the method of moments is presented. Basis functions, which include the edge conditions as well as mirror images in the walls of the waveguide, are used to accurately determine the properties of an infinitely thin inductive iris. For small irises, $d/a < 0.1$, the formulation based on the induced surface current converges with only one basis function. Two basis functions are found sufficient for all values of the width of the iris. However, the formulation based on the tangential electric field at the gap of the iris converges with only one basis function for $d/a > 0.1$. Two parametric functions involving Bessel functions of the first kind of order 1/2, $J_{1/2}$ and $J_1$, which approximate the iris's surface current and the electric field in the gap are introduced. Numerical results for the susceptance of the iris are compared with the analytical results; excellent agreement is documented.

## 1 Introduction

Thin irises are frequently encountered in waveguide filters and matching systems. These structures have been analysed using both analytic and numerical methods [1]. The mode-matching technique (MMT) was applied by Masterman *et al.* to the inductive iris problem [2]. The phenomenon of relative convergence in MMT in relation to the iris problem was investigated by Lee *et al.* [3] and in detail by Mittra *et al.* [4].

MMT is readily applied to these structures because of their simple geometry. The MMT solution, however, converges slowly because of its failure to include the edge conditions. Recently, a set of basis functions, which include the edge conditions, were used in a method-of-moments (MoM) solution of the infinitely thin iris problem by Yang and Omar [5]. In their formulation, the tangential electric field at the gap of the iris is expanded in a series of basis functions which include the edge conditions and are mirror-imaged in the plane containing the iris. In addition, an extensive investigation of the numerical properties on the MMT in relation to the infinitely thin iris problem was presented by Sorrentino *et al.* [6].

In this paper, a different set of basis functions, which contain the edge conditions but are mirror-imaged in the wall which is not in contact with the iris, are used in a mode-matching-method-of-moments technique (MMMoMT).

An alternative formulation consists in deriving an integral equation for the induced current density on the iris [6]. The MoM is then used to solve such an equation by expanding the unknown current density in a series of appropriate basis functions which include the edge condition and are mirror-imaged in the plane containing the iris.

## 2 Tangential electric field formulation (TEFF)

The structure under consideration is shown in Fig. 1. The walls of the rectangular waveguide and the iris of width $d$ are assumed to be perfectly conducting. We also assume that only the fundamental mode $TE_{10}$, with amplitude equal to unity, is incident on the iris.

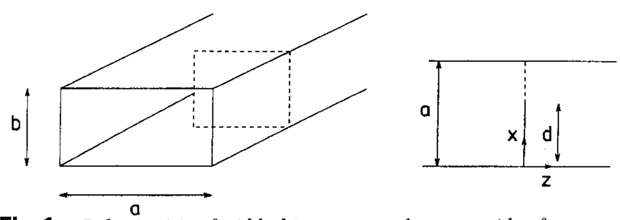


**Fig. 1** *Inductive iris of width d in a rectangular waveguide of cross section a times b*
a = 2b = 19.5mm

Following the mode-matching technique, the tangential electric and magnetic fields on each side of the iris are expanded in the $TE_{n0}$ modes of the waveguide as these are the only modes excited at the iris. The details are not presented here but can be found in [7], for example.

$$E_y^I(x, z=0) = \sum_{m=1}^{\infty} (\delta_{m1} + B_m) \sin\left[m\pi \frac{x}{a}\right] \quad (1a)$$



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$$E_y^{II}(x, z=0) = \sum_{m=1}^{\infty} F_m \sin\left[m\pi\frac{x}{a}\right] \quad (1b)$$

$$H_x^{I}(x, z=0) = \sum_{m=1}^{\infty} Y_m(-\delta_{m1} + B_m) \sin\left[m\pi\frac{x}{a}\right] \quad (2a)$$

$$H_x^{II}(x, z=0) = -\sum_{m=1}^{\infty} Y_m F_m \sin\left[m\pi\frac{x}{a}\right] \quad (2b)$$

The wave admittances $Y_m$ are given in [7], $B_m$ and $F_m$ are unknown expansion coefficients.

The tangential electric field at the gap $E_{gap}(x)$, is easily shown to satisfy the following integral equation [4, 6, 8]:

$$\sum_{m=1}^{\infty} Y_m \left\{\frac{a}{2}\int_{gap} E_{gap}(y) \sin\left[m\pi\frac{y}{a}\right] dy\right\} \sin\left[m\pi\frac{x}{a}\right] = Y_1 \sin\left[\pi\frac{x}{a}\right], \quad d \le x \le a \quad (3)$$

By concentrating on the tangential electric field at the gap, we are able to include in the theory whatever *a priori* information we have about this quantity, such as the edge conditions, for example.

Therefore, we expand $E_{gap}$ in a series of basis functions $E_i(x)$ such that

$$E_{gap}(x) = \sum_{i=1}^{M} c_i E_i(x) \quad (4)$$

The value of $M$ is increased until convergence is reached. The expansion coefficients are determined using the standard Galerkin's method [9]. By taking the inner product of eqn. 3 against the basis functions we get a linear set of equations in the coefficients $c_i$

$$[A][c] = [T] \quad (5)$$

The entries of the square and symmetric matrix **[A]** and the column vector **T** are given by

$$[\mathbf{A}]_{ij} = \sum_{m=1}^{\infty} Y_m \tilde{E}_i(m)\tilde{E}_j(m) \quad (6a)$$

$$[\mathbf{T}]_i = Y_1 \tilde{E}_i(1) \quad (6b)$$

where, for convenience, the following notation was introduced:

$$\tilde{E}_i(n) = \frac{2}{a}\int_d^a E_i(x) \sin\left[m\pi\frac{x}{a}\right] dx \quad (7)$$

## 3 Surface current formulation (SCF)

An alternative formulation consists in deriving an integral equation for the induced current density on the iris instead of the tangential electric field at the gap. The tangential electric and magnetic fields, at the plane of the iris, are still given by eqns. 1 and 2.

From the discontinuity of $H_x$ and the continuity of $E_y$ along with the vanishing of $E_y$ over the iris, we get the following integral equation for the surface current density $J_y(x)$ [6]:

$$\sum_{m=1}^{\infty} \frac{1}{Y_m}\tilde{J}_y(m) \sin\left[m\pi\frac{x}{a}\right] = 2\sin\left[\pi\frac{x}{a}\right], \quad 0 \le x \le d \quad (8)$$

where the transform $\tilde{J}$ is now given by

$$\tilde{J}_y(n) = \frac{2}{a}\int_0^d J_y(x) \sin\left[n\pi\frac{x}{a}\right] dx \quad (9)$$

If the current density is now expanded in a series of basis functions $J_i(x)$ with expansion coefficients $b_i$ and Galerkin's method is applied to eqn. 8, we get a set of linear equations

$$[\mathbf{W}][b] = [\mathbf{S}] \quad (10)$$

The square and symmetric matrix **[W]** and the column vector **[S]** are given by

$$[\mathbf{W}]_{ij} = \sum_{m=1}^{\infty} \frac{\tilde{J}_i(m)\tilde{J}_j(m)}{Y_m} \quad (11)$$

and

$$[\mathbf{S}]_i = 2\tilde{J}_i(1) \quad (12)$$

## 4 Basis functions

To guarantee numerical efficiency, the basis functions should approximate the quantity under consideration reasonably well. In particular, they should contain information about the edge conditions. The tangential electric field $E_{gap}$ is known to approach zero at $x = d$ as $x^{1/2}$ and vanishes at $x = a$ as $x^1$ [10]. We also include the mirror image in the wall located at $x = a$. A set of basis functions which satisfy these requirements is given by

$$E_k(x) = \frac{\sin\left[k\pi\frac{x-d}{a-d}\right]}{[(x-d)(2a-d-x)]^{1/2}} \quad k = 1, 2, \ldots \quad (13)$$

The integrals involved in eqn. 3 can be expressed in terms of the Bessel function $J_0$ [11]. Similar arguments lead to the following set of basis functions for the surface current density:

$$J_k(x) = \frac{\sin\left[(2k-1)\pi\frac{x}{2d}\right]}{[(x+d)(d-x)]^{1/2}} \quad k = 1, 2, \ldots \quad (14)$$

The integrals in eqn. 9 can also be expressed in terms of the Bessel function $J_0$ [11].

## 5 Results

The two formulations are used to determined the reflection properties of an infinitely thin inductive iris.

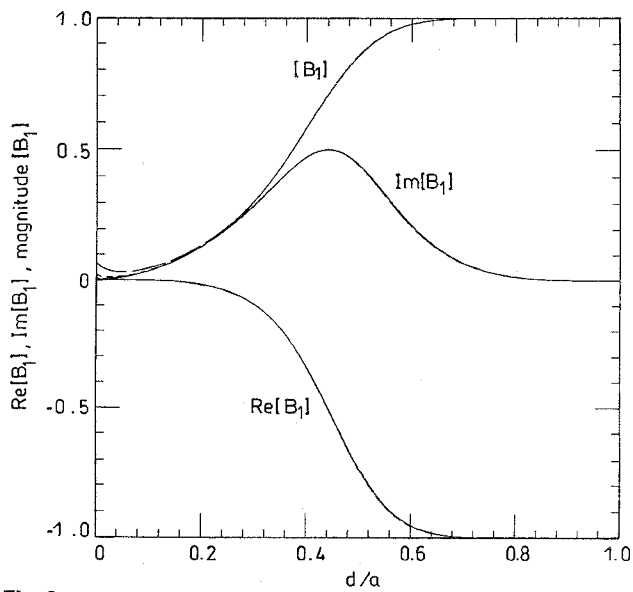


**Fig. 2** *Real part, imaginary part and magnitude of the reflection coefficient $B_1$ as a function of d/a as obtained from TEFF*
$a/\lambda = 0.8$ for M = 1, 2, 3, 4 and 5 basis functions

Fig. 2 shows the real part, the imaginary part and magnitude of the reflection coefficient $B_1$ as a function of $d/a$ for $M$ = 1, 2, 3, 4 and 5 basis functions as obtained from the tangential electric field formulation. The convergence of the numerical solution is evident. Except for the imaginary part when $d/a < 0.1$, one basis function is sufficient.

The same quantities as obtained from the formulation based on the induced current density are shown in Fig. 3 . The results obtained with only one basis function are accurate only for the range $d/a < 0.5$. Two or more basis functions are, however, sufficient over the entire range of $d/a$.

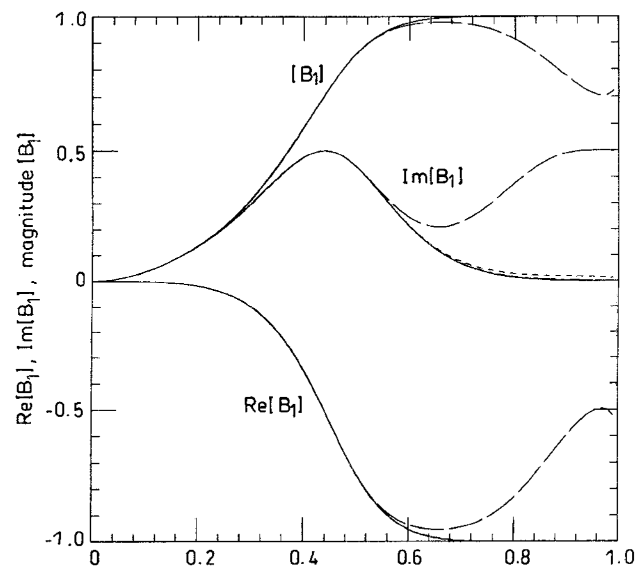


**Fig. 3** *Real part, imaginary part and magnitude of the reflection coefficient $B_1$ as a function of d/a as obtained from SCF*
$a/\lambda = 0.8$ for M = 1, 2, 3, 4 and 5 basis functions

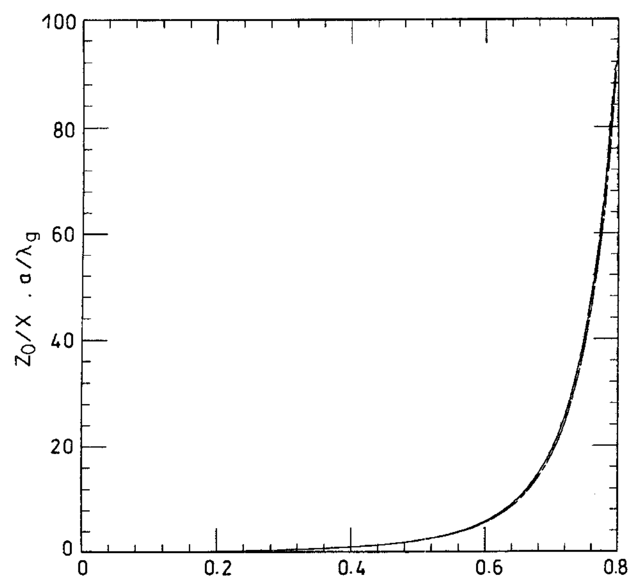


**Fig. 4** *$Z_0/X.a/\lambda_g$ as a function of d/a when $a/\lambda = 0.8$*
results from TEFF

Fig. 4 is a plot of the quantity $Z_0/X.a/\lambda_g$ as a function of $d/a$ when $a/\lambda_g = 0.8$ as obtained from the tangential electric field formulation. The solid line is the analytic expression given by Marcuvitz [8]. Again, it can easily be seen that the agreement between the numerical results and the analytical solution is excellent.

Fig. 5 shows $Z_0/X.a/\lambda_g$ obtained from the surface current formulation. Two or more basis functions are sufficient to guarantee convergence over the entire range of $d/a$.

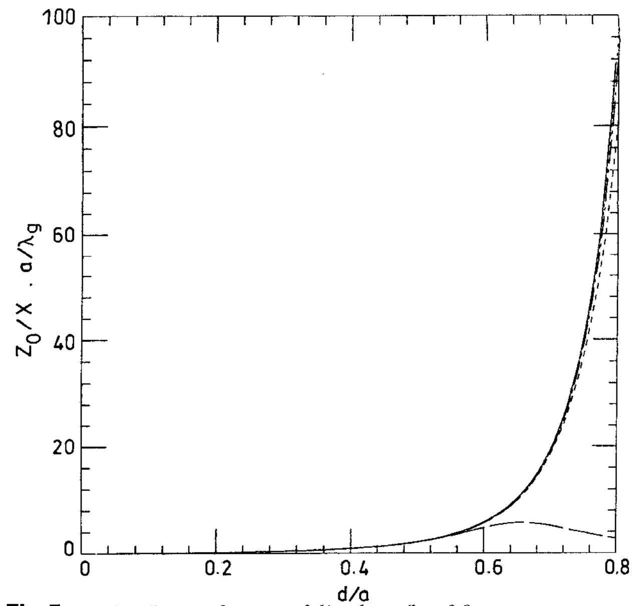


**Fig. 5** *$Z_0/X.a/\lambda_g$ as a function of d/a when $a/\lambda = 0.8$*
results from SCF

Although accurate results over the entire range of $d/a$ can be obtained with a single basis function when the two formulations are combined, these basis functions do not coincide with the asymptotic solutions of Maxwell's equations. Starting from the fact that the asymptotic solution of Maxwell's equations, at the edge of an infinitely thin perfect conductor, can be expressed in terms of the Bessel function $J_{1/2}$ [10], simple approximate solutions can be constructed. The tangential electric field $E_y$ approaches $J_{1/2}(\alpha x)$ as $x \rightarrow d^+$ and $J_1(\alpha x)$ as $x \rightarrow a$. Taking the mirror image in the electric wall at $x = a$, a simple function which satisfies these asymptotic properties is given by

$$R_E(\alpha, x) = J_{1/2}[\alpha(x-d)]J_{1/2}[\alpha(2a-d-x)]J_1[\alpha(a-x)] \quad (15)$$

It is found numerically that the function $R_E(\alpha = 0.95, x)$ approximates the real part of the electric field very accurately (Fig. 6).

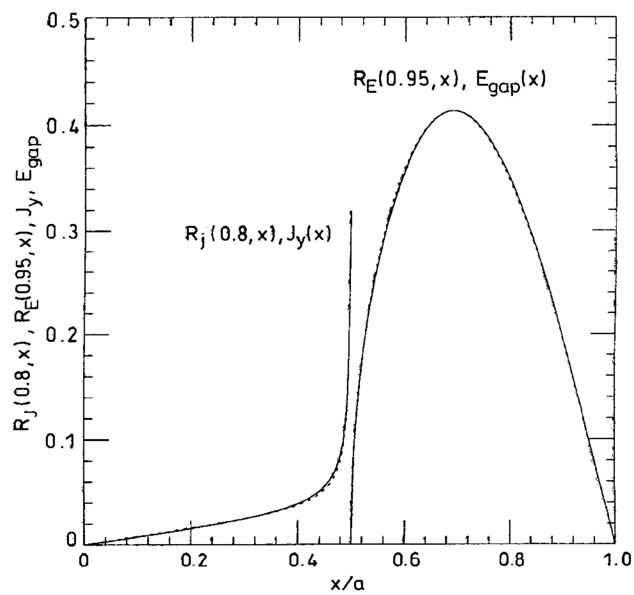


**Fig. 6** *Plots of $E_{gap}(x)$, $J_y(x)$ and the functions $R_E(0.95, x)$ and $R_J(0.8, x)$*
$a/\lambda = 0.8$ and d/a = 0.5. Numerical results (dashed lines) were obtained with 5 basis functions

Similarly, the surface current density is approximated by a function of the form

$$R_J(\gamma, x) = J'_{1/2}(\gamma(d-x)] + J'_{1/2}[\gamma(d+x)] \quad (16)$$

It was found that by choosing $\gamma = 0.8$, the real part of the surface current density can be approximated very accurately by this function (Fig. 6).

## 6 Conclusions

A detailed comparative study of two formulations of the inductive infinitely thin iris problem was presented along with two new sets of basis functions which contain the edge conditions and are mirror-imaged in the appropriate metallic wall of the rectangular waveguide. The formulation based on an integral equation for the tangential electric field at the gap converges to the analytic solution of the susceptance of the iris with only one basis function in the range $d/a > 0.1$, while the formulation based on the surface current density on the iris converges in the range $d/a < 0.5$ under the same conditions. Both formulations converge to the exact solutions when two or more basis functions are used over the entire range of $d/a$.

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