Symmetric Extension for Two-Channel Quincunx Filter Banks

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Abstract

In the case of one-dimensional filter banks, symmetric extension is a commonly used technique for constructing nonexpansive transforms of finite-length sequences. In this paper, we show how symmetric extension can be extended to the case of two-dimensional filter banks based on quincunx sampling. In particular, we show how, for filter banks of this type, one can construct nonexpansive transforms for input sequences defined on arbitrary rectangular regions.

1. Introduction

Fig. 1 shows a two-dimensional two-channel filter bank. Often, such a filter bank is defined so as to operate on sequences of infinite extent. In practice, however, we always deal with sequences of finite extent. Therefore, we usually require some means for adapting filter banks to such sequences. This leads to the boundary filtering problem. Furthermore, in many applications, such as image compression, it is desirable to employ a transform that is **nonexpansive** (i.e., maps a sequence of N samples to a new sequence of no more than N samples). Consequently, we seek a solution to the boundary problem that yields nonexpansive transforms.

In the 1-D case, symmetric extension [1, 2] is a commonly used technique for constructing nonexpansive transforms of finite-extent sequences. In this paper, we explain how this technique can be extended to the quincunx case.

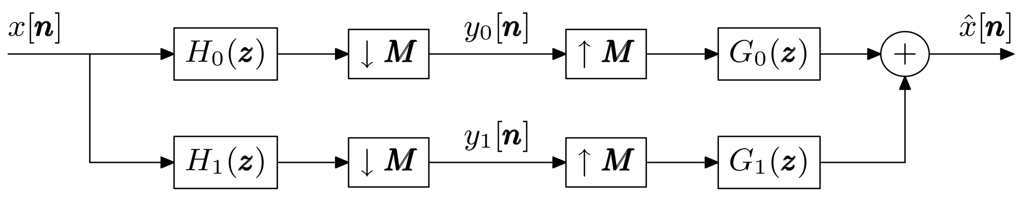

Figure 1: *Two-dimensional two-channel filter bank*

With the proposed symmetric extension algorithm, we use a structure for the forward transform as shown in Fig. 2(a). The input sequence $\tilde{x}$ is converted an infinite-extent periodic symmetric sequence x. By carefully constraining the choice of the analysis filters H_0 and H_1, the subband sequences y_0 and y_1 can always be symmetric and periodic. Then, we use these properties to extract the independent samples from y_0 and y_1. With some care, it is possible for the resulting transform to be nonexpansive. The structure for the inverse transform is shown in Fig. 2(b).

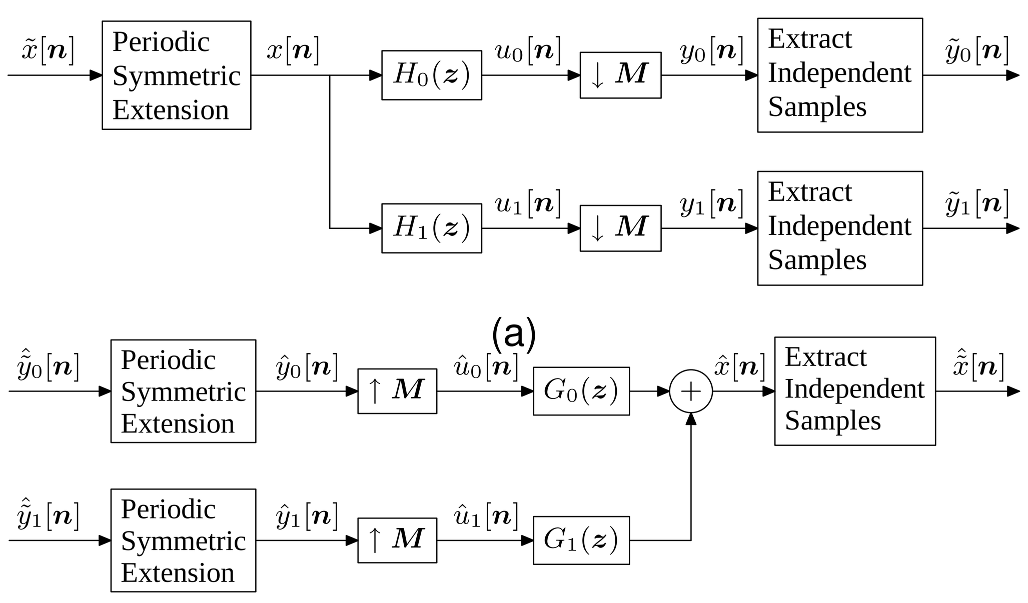

Figure 2: *Structure of the symmetric extension scheme: (a) analysis side, and (b) synthesis side.*

2. Symmetric Extension Preliminaries

The quincunx filter banks are two-dimensional two-channel filter banks based on the nonseparable quincunx lattice.

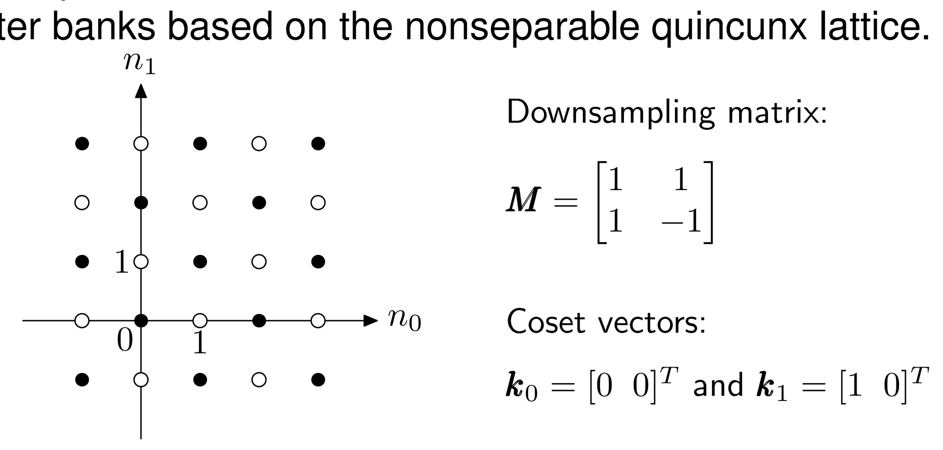

Figure 3: *The quincunx lattice.*

In this paper, several types of 2-D symmetry are of fundamental importance.

Definition 1 (Centrosymmetry)**.** A sequence x defined on $\mathbb{Z}^2$ is said to be **centrosymmetric** about $\boldsymbol{c}$ if, for some $\boldsymbol{c} \in \frac{1}{2}\mathbb{Z}^2$ and $S \in \{-1, 1\}$,

$$x[\boldsymbol{n}] = Sx[2\boldsymbol{c} - \boldsymbol{n}] \quad \text{for all } \boldsymbol{n} \in \mathbb{Z}^2.$$

Definition 2 (Quadrantal centrosymmetry)**.** A sequence x defined on $\mathbb{Z}^2$ is said to be **quadrantally centrosymmetric** about $\boldsymbol{c}$ if for some $S, T \in \{-1, 1\}$ and $\boldsymbol{c} = \begin{bmatrix} c_0 & c_1 \end{bmatrix}^T \in \frac{1}{2}\mathbb{Z}^2$,

$$\begin{aligned} x[n_0, n_1] &= STx[2c_0 - n_0, 2c_1 - n_1] \\ &= Sx[2c_0 - n_0, n_1] = Tx[n_0, 2c_1 - n_1] \end{aligned}$$

for all $n_0, n_1 \in \mathbb{Z}$.

In terms of S and T, four types of quadrantal centrosymmetry are possible [3] as shown in Fig. 4.

$$\begin{array}{cc|cc} d & b & b & d \\ c & a & a & c \\ \hline c & a & a & c \\ d & b & b & d \end{array} \qquad \begin{array}{cc|cc} -d & -b & b & d \\ -c & -a & a & c \\ \hline c & a & -a & -c \\ d & b & -b & -d \end{array} \qquad \begin{array}{cc|cc} d & b & -b & -d \\ c & a & -a & -c \\ \hline c & a & -a & -c \\ d & b & -b & -d \end{array} \qquad \begin{array}{cc|cc} d & b & b & d \\ c & a & a & c \\ \hline -c & -a & -a & -c \\ -d & -b & -b & -d \end{array}$$

(a) (b) (c) (d)

Figure 4: *Four types of quadrantal centrosymmetry: (a) even-even, (b) odd-odd, (c) odd-even, and (d) even-odd.*

Definition 3 (Rotated quadrantal centrosymmetry)**.** A sequence x defined on $\mathbb{Z}^2$ is said to be **rotated quadrantally centrosymmetric** about $\boldsymbol{c}$ if, for some $S, T \in \{-1, 1\}$ and $\boldsymbol{c} = \begin{bmatrix} c_0 & c_1 \end{bmatrix}^T \in \frac{1}{2}\mathbb{Z}^2$ satisfying $c_0 + c_1 \in \mathbb{Z}$,

$$\begin{aligned} x[n_0, n_1] &= STx[2c_0 - n_0, 2c_1 - n_1] \\ &= Sx[c_0 + c_1 - n_1, c_0 + c_1 - n_0] \\ &= Tx[c_0 - c_1 + n_1, c_1 - c_0 + n_0] \end{aligned}$$

for all $n_0, n_1 \in \mathbb{Z}$.

Two examples of this kind of symmetry are shown in Fig. 5.

(a) (b)

Figure 5: *Rotated quadrantal centrosymmetry: (a) $\boldsymbol{c} \in \mathbb{Z}^2$, and (b) $2\boldsymbol{c} \in \mathrm{LAT}(\boldsymbol{M})$, $\boldsymbol{c} \notin \mathbb{Z}^2$.*

The symmetric extension scheme for mapping a finite-extent sequence to an infinite-extent symmetric and periodic sequence is given as follows.

Definition 4 (Symmetric extension of sequence)**.** Let $\tilde{x}$ be a (2-D) sequence defined on the rectangular region $\{0, 1, \ldots, L_0 - 1\} \times \{0, 1, \ldots, L_1 - 1\}$. Then, the symmetric extension x of $\tilde{x}$ is defined as

$$x[n_0, n_1] = \tilde{x}[f_w[n_0, L_0], f_w[n_1, L_1]], \qquad (1)$$

where the function f_w is given by

$$f_w[n, L] = \min\{\mathrm{mod}(n, 2L - 2), 2L - 2 - \mathrm{mod}(n, 2L - 2)\}.$$

The 2-D symmetrically extended sequence has quadrantal symmetry and periodicity properties as shown by Lemma 1.

Lemma 1. *Let $\tilde{x}$ be a sequence defined on the rectangular region $\{0, 1, \ldots, L_0 - 1\} \times \{0, 1, \ldots, L_1 - 1\}$. Let x denote the symmetric extension of $\tilde{x}$ as defined by (1). Then, x is $\boldsymbol{P}$-periodic with $\boldsymbol{M}^{-1}\boldsymbol{P}$ being an integer matrix, and is even-even quadrantally centrosymmetric about* $\boldsymbol{0}$.

An example of symmetric extension of a finite-extent sequence is shown in Fig. 6. The original sequence $\tilde{x}$ has four samples a, b, c, and d on a 2×2 square region.

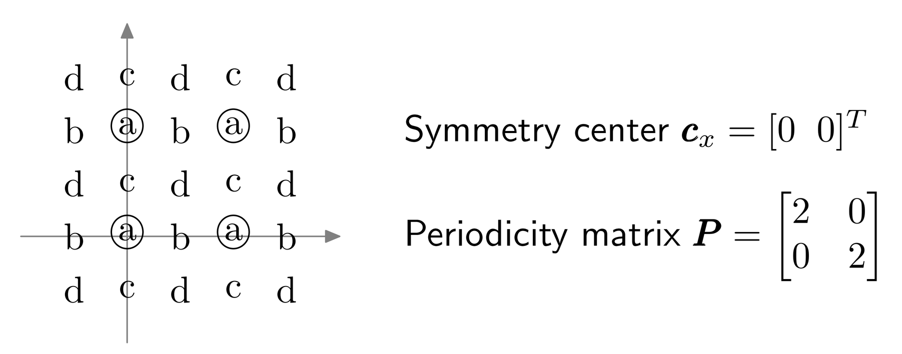

Figure 6: *The extended sequence x.*

Lemmas 2 to 5 show how the convolution and downsampling operations of a filter bank affect these properties.

Lemma 2 (Preservation of symmetry under convolution)**.** *Let x and h be sequences defined on $\mathbb{Z}^2$, and define $y = x * h$. If x and h are quadrantally centrosymmetric about $\boldsymbol{c}_x$ and $\boldsymbol{c}_h$, respectively, then y is quadrantally centrosymmetric about $\boldsymbol{c}_y = \boldsymbol{c}_x + \boldsymbol{c}_h$.*

Lemma 3 (Preservation of periodicity under convolution)**.** *Let x and h be sequences defined on $\mathbb{Z}^2$, with x being $\boldsymbol{P}$-periodic. Then, $y = x * h$ is $\boldsymbol{P}$-periodic.*

Lemma 4 (Downsampling of periodic sequence)**.** *Let $\boldsymbol{M}$ be an arbitrary sampling matrix. Let x be $\boldsymbol{P}$-periodic such that $\boldsymbol{M}^{-1}\boldsymbol{P}$ is an integer matrix. Then, $(\downarrow \boldsymbol{M})x$ is $(\boldsymbol{M}^{-1}\boldsymbol{P})$-periodic.*

Lemma 5 (Downsampling of quadrantally centrosymmetric sequence)**.** *Let x be a quadrantally centrosymmetric sequence with symmetry center $\boldsymbol{c}_x \in \mathbb{Z}^2$, and $\boldsymbol{M} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. Define $y = (\downarrow \boldsymbol{M})x$. Then, y is rotated quadrantally centrosymmetric about $\boldsymbol{M}^{-1}\boldsymbol{c}_x$.*

3. Symmetric Extension Algorithm

For a perfect reconstruction quincunx filter bank, if both analysis filters have quadrantal centrosymmetry with group delays $\boldsymbol{d}_0, \boldsymbol{d}_1 \in \mathbb{Z}^2$, then $\boldsymbol{d}_0$ and $\boldsymbol{d}_1$ must be in different cosets of the quincunx lattice, and H_0 and H_1 must have the even-even type symmetry. Such a PR filter bank is compatible with symmetric extension.

Theorem 1 (Symmetric extension algorithm)**.** *Consider the filter bank shown in Fig. 2, where $\tilde{x}$ is defined on the rectangular region $\{0, 1, \ldots, L_0 - 1\} \times \{0, 1, \ldots, L_1 - 1\}$ and x is the symmetric extension of $\tilde{x}$ as given by (1). If H_0 and H_1 are quadrantally centrosymmetric with group delays $\boldsymbol{d}_0 = \begin{bmatrix} d_{0,0} & d_{0,1} \end{bmatrix}^T \in \mathbb{Z}^2$ and $\boldsymbol{d}_1 = \begin{bmatrix} d_{1,0} & d_{1,1} \end{bmatrix}^T \in \mathbb{Z}^2$, respectively, then the subband y_0 can be completely characterized by N_0 samples with indices $\boldsymbol{n} = \begin{bmatrix} n_0 & n_1 \end{bmatrix}^T$ given by*

$$\begin{aligned} \lceil (d_{0,0} + d_{0,1})/2 \rceil &\le n_0 \le \lfloor (d_{0,0} + d_{0,1} + L_0 + L_1)/2 \rfloor - 1, \\ \text{and } \max\{d_{0,0} - n_0, n_0 - d_{0,1} - L_1 + 1\} &\le n_1 \\ &\le \min\{d_{0,0} + L_0 - 1 - n_0, n_0 - d_{0,1}\}; \end{aligned}$$

y_1 can be completely characterized by N_1 samples with indices $\boldsymbol{n} = \begin{bmatrix} n_0 & n_1 \end{bmatrix}^T$ given by

$$\begin{aligned} \lceil (d_{1,0} + d_{1,1})/2 \rceil &\le n_0 \le \lfloor (d_{1,0} + d_{1,1} + L_0 + L_1)/2 \rfloor - 1, \\ \text{and } \max\{d_{1,0} - n_0, n_0 - d_{1,1} - L_1 + 1\} &\le n_1 \\ &\le \min\{d_{1,0} + L_0 - 1 - n_0, n_0 - d_{1,1}\}; \end{aligned}$$

and $N_0 + N_1 = L_0 L_1$ (i.e., the transform is nonexpansive).

The following is an example. The analysis filters are

$$\begin{aligned} H_0(\boldsymbol{z}) &= \tfrac{1}{32}\Big(28 - 2z_1^{-1}z_2^{-1} - 2z_1z_2^{-1} - 2z_1^{-1}z_2 - 2z_1z_2 \\ &\quad - z_1^{-2} - z_1^2 - z_2^{-2} - z_2^2 + 4z_1^{-1} + 4z_1 + 4z_2^{-1} + 4z_2\Big) \\ H_1(\boldsymbol{z}) &= z_1^{-1} + \left(1 + z_1^{-1}z_2^{-1} + z_1^{-1}z_2 + z_1^{-2}\right), \end{aligned}$$

and the synthesis filters satisfy that $G_0(\boldsymbol{z}) = H_1(-\boldsymbol{z})$ and $G_1(\boldsymbol{z}) = -H_0(-\boldsymbol{z})$. The group delays of the analysis filters are $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$, respectively. The frequency responses of the filters are shown in Fig. 7.

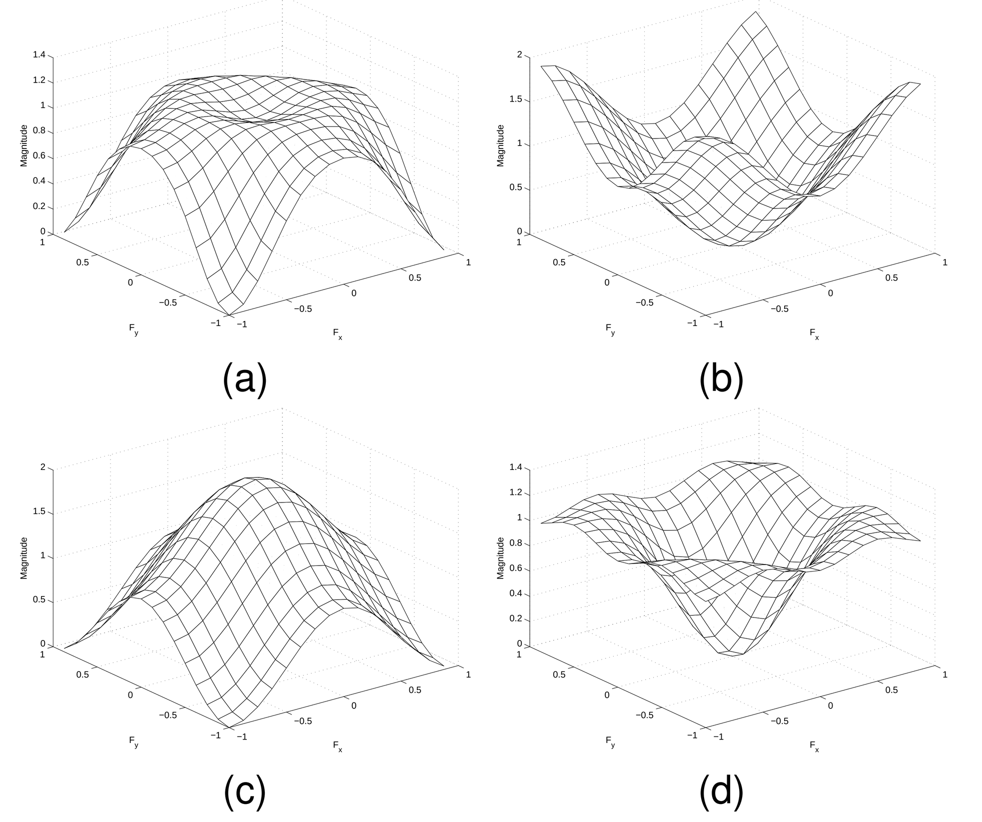

Figure 7: *Frequency responses of (a) analysis lowpass, (b) analysis highpass, (c) synthesis lowpass, and (d) synthesis highpass.*

There are also other types of PR filter banks that lead to nonexpansive transforms with slight variations on the above algorithm. The constraints on the symmetry types and group delays of the analysis filters are different depending on how the original input sequence $\tilde{x}$ is extended. For details, please refer to [4].

4. Conclusions

In this paper, we have investigated how to preserve symmetry and periodicity under the convolution and downsampling operations of a quincunx filter bank. This led us to propose a new symmetric extension algorithm which can be used to construct nonexpansive transforms associated with quincunx filter banks. This scheme is potentially useful in any application that processes finite-extent sequences using such filter banks.

References

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