Design of Optimal Quincunx Filter Banks for Image Coding

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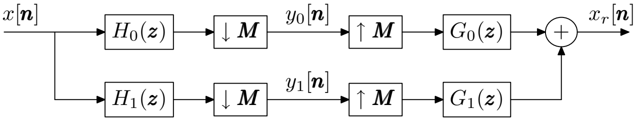
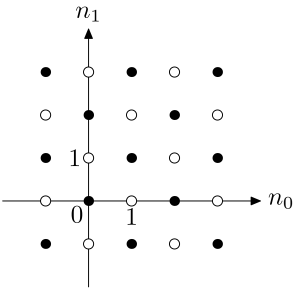
Outline

Introduction of Quincunx Filter Banks

- Two-dimensional two-channel nonseparable filter banks



- Quincunx lattice



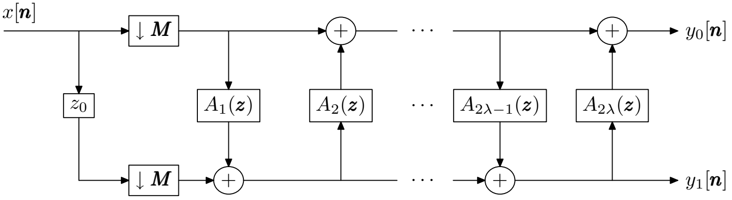
$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Motivation

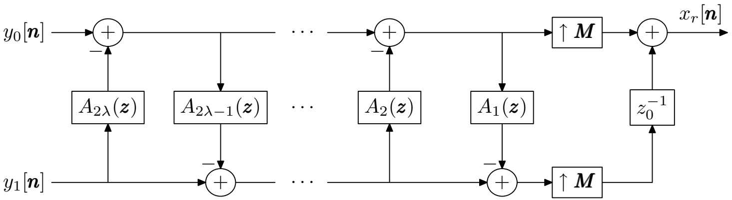
- Desirable properties for image coding
  - Perfect reconstruction (PR)
  - Linear phase
  - High coding gain
  - Vanishing moments
  - Good frequency selectivity
- Existing design methods
  - Transformation of variables
  - Direct optimization
  - Two-step lifting structure

Lifting Realization - Structure

- Analysis Side



- Synthesis Side



Lifting Realization - Transfer Functions

- Analysis filter transfer functions $H_0(\boldsymbol{z})$ and $H_1(\boldsymbol{z})$

$$H_k(\boldsymbol{z}) = H_{k,0}\left(\boldsymbol{z}^{\boldsymbol{M}}\right) + z_0 H_{k,1}\left(\boldsymbol{z}^{\boldsymbol{M}}\right),$$

$$\begin{bmatrix} H_{0,0}(\boldsymbol{z}) & H_{0,1}(\boldsymbol{z}) \\ H_{1,0}(\boldsymbol{z}) & H_{1,1}(\boldsymbol{z}) \end{bmatrix} = \prod_{k=1}^{\lambda} \left(\begin{bmatrix} 1 & A_{2k}(\boldsymbol{z}) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ A_{2k-1}(\boldsymbol{z}) & 1 \end{bmatrix} \right)$$

- Synthesis filter transfer functions $G_0(\boldsymbol{z})$ and $G_1(\boldsymbol{z})$

$$G_k(\boldsymbol{z}) = (-1)^{1-k} z_0^{-1} H_{1-k}(-\boldsymbol{z})$$

Lifting Realization - Advantages

1. PR is satisfied automatically.
2. Linear phase property can be imposed structurally.

Theorem

If each lifting filter A_k is symmetric with its group delay $\boldsymbol{c}_k$ satisfying

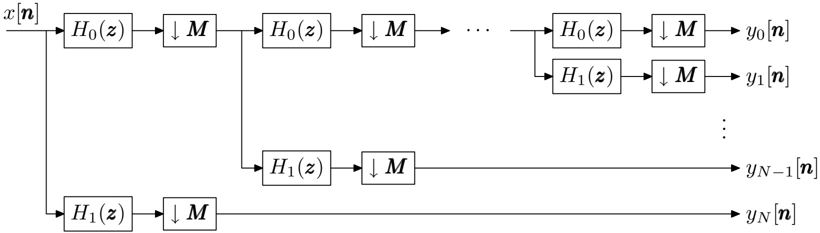
$$\boldsymbol{c}_k = (-1)^k \left[\tfrac{1}{2} \; \tfrac{1}{2} \right]^T,$$

then the analysis filters H_0 and H_1 are symmetric with group delays $[0 \;\; 0]^T$ and $[-1 \;\; 0]^T$, respectively.

3. Reversible integer-to-integer transforms

Octave-Band Filter Banks

- N-level octave-band filter bank: analysis side



- Equivalent one-level analysis filters $\{\mathrm{H}'_i\}$

$$H'_i(\boldsymbol{z}) = \begin{cases} \prod_{k=0}^{N-1} H_0\left(\boldsymbol{z}^{\boldsymbol{M}^k}\right) & i = 0 \\ H_1\left(\boldsymbol{z}^{\boldsymbol{M}^{N-i}}\right) \prod_{k=0}^{N-i-1} H_0\left(\boldsymbol{z}^{\boldsymbol{M}^k}\right) & 1 \leq i \leq N-1 \\ H_1\left(\boldsymbol{z}\right) & i = N. \end{cases}$$

Coding Gain

- Measure of the energy compaction ability of a filter bank
- Coding gain G_{SBC} for an N-level octave-band filter bank

$$G_{SBC} = \prod_{k=0}^{N} (A_k B_k / \alpha_k)^{-\alpha_k},$$

$$A_k = \sum_{\boldsymbol{m} \in \mathbb{Z}^2} \sum_{\boldsymbol{n} \in \mathbb{Z}^2} h'_k[\boldsymbol{m}] h'_k[\boldsymbol{n}] r[\boldsymbol{m} - \boldsymbol{n}], B_k = \alpha_k \sum_{\boldsymbol{n} \in \mathbb{Z}^2} g_k'^2[\boldsymbol{n}],$$

$$\alpha_0 = 2^{-N}, \alpha_k = 2^{-(N+1-k)} \text{ for } k = 1, 2, \ldots, N,$$

- Autocorrelation r

$$r[n_0, n_1] = \begin{cases} \rho^{|n_0|+|n_1|} & \text{for separable model} \\ \rho^{\sqrt{n_0^2+n_1^2}} & \text{for isotropic model,} \end{cases}$$

where ρ is the correlation coefficient (typically, $0.90 \leq \rho \leq 0.95$).

Vanishing Moments

- $\tilde{N}$ dual vanishing moments $\Rightarrow$ $\tilde{N}$th order zero at $[0 \;\; 0]^T$ of $\hat{h}_1(\boldsymbol{\omega})$
- N primal vanishing moments $\Rightarrow$ Nth order zero at $[\pi \;\; \pi]^T$ of $\hat{h}_0(\boldsymbol{\omega})$
- Linear phase filter H with group delay $\boldsymbol{c} \in \mathbb{Z}^2$

$$\frac{\partial^{m_0+m_1}\hat{h}}{\partial\omega_0^{m_0}\partial\omega_1^{m_1}} = \begin{cases} \sum_{\boldsymbol{n}\in\mathbb{Z}^2} h[\boldsymbol{n}]\,(\boldsymbol{n}-\boldsymbol{c})^{\boldsymbol{m}}\cos\left(\boldsymbol{\omega}^T(\boldsymbol{n}-\boldsymbol{c})\right) & \text{for } |\boldsymbol{m}| \text{ even} \\ -\sum_{\boldsymbol{n}\in\mathbb{Z}^2} h[\boldsymbol{n}]\,(\boldsymbol{n}-\boldsymbol{c})^{\boldsymbol{m}}\sin\left(\boldsymbol{\omega}^T(\boldsymbol{n}-\boldsymbol{c})\right) & \text{otherwise,} \end{cases}$$

where $\boldsymbol{m} = [m_0 \;\; m_1]^T$ and $\boldsymbol{m} = m_0 + m_1$.

- $\tilde{N}$th order zero at $\boldsymbol{\omega} = [0 \;\; 0]^T$

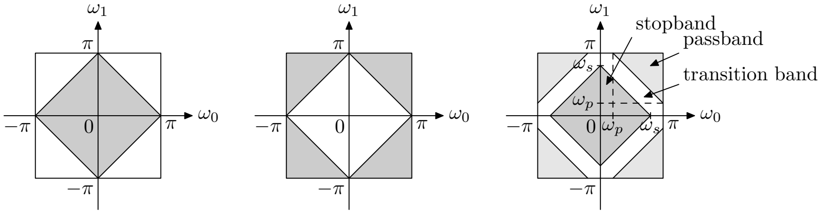
$$\sum_{\boldsymbol{n}\in\mathbb{Z}^2} h[\boldsymbol{n}]\,(\boldsymbol{n}-\boldsymbol{c})^{\boldsymbol{m}} = 0 \quad \text{for all even } |\boldsymbol{m}| \text{ such that } |\boldsymbol{m}| < \tilde{N}.$$

Frequency Selectivity

- Error function of a linear phase filter H

$$e_h = \int_{[-\pi,\,\pi)^2} W(\boldsymbol{\omega}) \left| \hat{h}_a(\boldsymbol{\omega}) - D\hat{h}_d(\boldsymbol{\omega}) \right|^2 \mathrm{d}\boldsymbol{\omega}$$

- Ideal frequency responses and weighting function



Lowpass Highpass Weighting function

- Frequency response constraint: $e_h \leq \delta_h$

Design problems

- Lifting parameterization of linear-phase filter banks
- Maximize coding gain subject to vanishing moments and frequency response constraints
- Iterative second-order cone programming

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{b}^T \boldsymbol{x} \\ \text{subject to:} \quad & \|\boldsymbol{A}_i^T \boldsymbol{x} + \boldsymbol{c}_i\| \leq \boldsymbol{b}_i^T \boldsymbol{x} + d_i \quad \text{for } i = 1, \ldots, q. \end{aligned}$$

- Linear/quadratic approximations

Two Lifting Steps - Problem Formulation (1)

- Lifting filter coefficients $\boldsymbol{x}$
- Vanishing moments
  - Constraint: an underdetermined linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$
  - Solutions: $\boldsymbol{x} = \boldsymbol{x}_s + \boldsymbol{V}_r\phi$
- Coding gain
  - Define $G = -10\log_{10} G_{SBC}$
  - For a given ϕ, seek a small perturbation $\boldsymbol{\delta}_{\boldsymbol{\phi}}$ such that $G(\phi + \boldsymbol{\delta}_{\boldsymbol{\phi}})$ is reduced relative to $G(\phi)$
  - $\|\boldsymbol{\delta}_{\boldsymbol{\phi}}\|$ is small $\Rightarrow G(\phi + \boldsymbol{\delta}_{\boldsymbol{\phi}}) \approx G(\phi) + \boldsymbol{g}^T\boldsymbol{\delta}_{\boldsymbol{\phi}}$
  - Iteratively minimize $\boldsymbol{g}^T\boldsymbol{\delta}_{\boldsymbol{\phi}}$, update ϕ until $|G(\phi + \boldsymbol{\delta}_{\boldsymbol{\phi}}) - G(\phi)| < \varepsilon$

Two Lifting Steps - Problem Formulation (2)

- Frequency selectivity
  - Analysis highpass filter frequency response

$$\hat{h}_1(\boldsymbol{\omega}) = \hat{a}_1(\boldsymbol{M}^T\boldsymbol{\omega}) + e^{j\omega_0}$$

    where $\hat{a}_1(\boldsymbol{M}^T\boldsymbol{\omega})$ is linear in $\boldsymbol{\phi}$
  - Error function

$$e_{h_1} = \boldsymbol{\phi}^T\boldsymbol{H}_{\boldsymbol{\phi}}\boldsymbol{\phi} + \boldsymbol{\phi}^T\boldsymbol{s}_{\boldsymbol{\phi}} + C_{\boldsymbol{\phi}}$$

  - Frequency response constraint is a second-order cone

$$\left\|\tilde{\boldsymbol{H}}_k\boldsymbol{\delta}_{\boldsymbol{\phi}} + \tilde{\boldsymbol{s}}_k\right\| \leq \delta'_{h_1}$$

Two Lifting Steps - Design Algorithm

Algorithm 1

1. Select an initial point $\boldsymbol{\phi}_0$
2. For the kth iteration, solve

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{g}^T \boldsymbol{\delta}_{\boldsymbol{\phi}} \\ \text{subject to:} \quad & \left\| \tilde{\boldsymbol{H}}_k \boldsymbol{\delta}_{\boldsymbol{\phi}} + \tilde{\boldsymbol{s}}_k \right\| \leq \delta'_{h_1} \\ & \left\| \boldsymbol{\delta}_{\boldsymbol{\phi}} \right\| \leq \beta, \end{aligned}$$

   update $\boldsymbol{\phi}$ by $\boldsymbol{\phi}_{k+1} = \boldsymbol{\phi}_k + \boldsymbol{\delta}_{\boldsymbol{\phi}}$
3. When $|G(\boldsymbol{\phi}_{k+1}) - G(\boldsymbol{\phi}_k)| < \varepsilon$, output and stop

Two Lifting Steps - Comments

- β: upper bound of $\left\|\boldsymbol{\delta}_{\boldsymbol{\phi}}\right\|$
  - Too large: $\boldsymbol{g}^T\boldsymbol{\delta}_{\boldsymbol{\phi}}$ cannot correctly reflect the actual reduction in G
  - Too small: the solution to the SOCP subproblem is restricted to an unnecessarily small region around $\boldsymbol{\phi}_k$
  - Should be chosen such that

$$\boldsymbol{g}^T\boldsymbol{\delta} \approx G(\boldsymbol{\phi}+\boldsymbol{\delta}) - G(\boldsymbol{\phi}) \quad \text{for} \quad \|\boldsymbol{\delta}\| = \beta$$

- δ_{h_1}: upper bound of the error function e_{h_1}
  - Too small: feasible region may be empty
  - Chosen to be a scaled version of e_{h_1} evaluated at $\boldsymbol{\phi}_k$

$$\delta_{h_1} = d\left(\boldsymbol{\phi}_k^T\boldsymbol{H}_{\boldsymbol{\phi}}\boldsymbol{\phi}_k + \boldsymbol{\phi}_k^T\boldsymbol{s}_{\boldsymbol{\phi}} + c_{\boldsymbol{\phi}}\right) \quad \text{for some} \quad 0 < d \leq 1$$

  - Error e_{h_1} is reduced after each iteration.

More Than Two Lifting Steps - Problem Formulation

- Lifting filter coefficients $\boldsymbol{x}$
- Coding gain: linear approximation

$$G(\boldsymbol{x}+\boldsymbol{\delta_x}) = G(\boldsymbol{x}) + \boldsymbol{g}^T\boldsymbol{\delta_x}$$

- Vanishing moments
  - Polynomial equations in $\boldsymbol{x}$
  - Approximated by

$$\boldsymbol{A}_k\boldsymbol{\delta_x} = \boldsymbol{b}_k$$

  - Moments are nearly vanishing
- Frequency selectivity
  - Frequency response: polynomial in $\boldsymbol{x}$
  - Error function e_{h_1}: approximated by $\boldsymbol{\delta_x}^T\boldsymbol{H}_k\boldsymbol{\delta_x} + \boldsymbol{\delta_x}^T\boldsymbol{s}_k + C_k$
  - Constraint: approximated by the second-order cone

$$\left\|\tilde{\boldsymbol{H}}_k\boldsymbol{\delta_x} + \tilde{\boldsymbol{s}}_k\right\| \le \delta'_{h_1}$$

More Than Two Lifting Steps - Design Algorithm

Algorithm 2

1. Select an initial point $\boldsymbol{x}_0$
2. For the kth iteration, solve

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{g}^T \boldsymbol{\delta}_{\boldsymbol{x}} \\ \text{subject to:} \quad & \boldsymbol{A}_k \boldsymbol{\delta}_{\boldsymbol{x}} = \boldsymbol{b}_k \\ & \left\| \tilde{\boldsymbol{H}}_k \boldsymbol{\delta}_{\boldsymbol{x}} + \tilde{\boldsymbol{s}}_k \right\| \leq \delta'_{h_1} \\ & \|\boldsymbol{\delta}_{\boldsymbol{x}}\| \leq \beta, \end{aligned}$$

   update $\boldsymbol{x}$ by $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{\delta}_{\boldsymbol{x}}$
3. When $|G(\boldsymbol{x}_{k+1}) - G(\boldsymbol{x}_k)| < \varepsilon$, output and stop

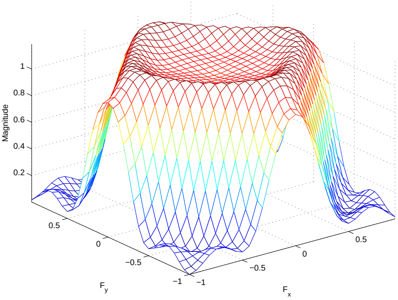
Design Examples

- Isotropic image model with $\rho = 0.95$ for six levels of decomposition
- CAL1: two 6×6 lifting filters
- CAL2: three 4×4 lifting filters

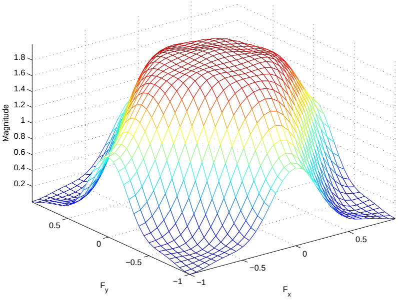
Comparison with existing filter banks

Filter banks	Support of analysis filters	Coding gain(dB)	Vanishing moments		
			$\tilde{N}$	N	Max. order
CAL1	13×13, 7×7	12.06	2	2	0
CAL2	9×9, 13×13	12.23	2	2	10^{-12}
KS	13×13, 7×7	11.95	6	6	0
9/7	9×9, 7×9, 9×7, 7×7	12.09	4	4	0

Frequency Responses of CAL1

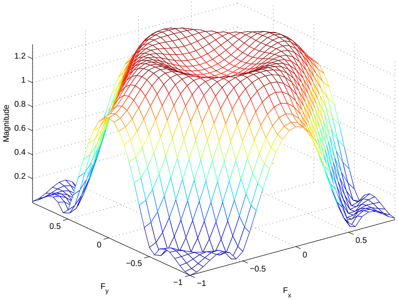


Analysis lowpass

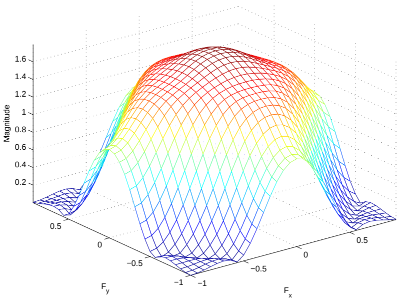


Synthesis lowpass

Frequency Responses of CAL2

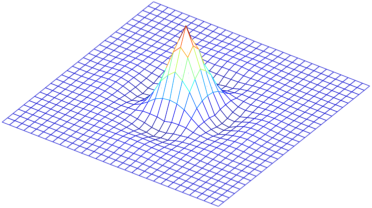


Analysis lowpass

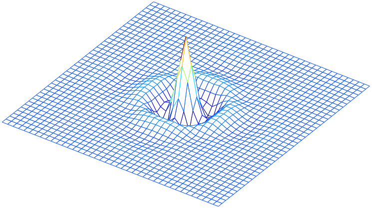


Synthesis lowpass

Scaling and Wavelet Functions for CAL1

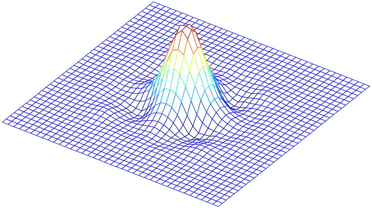


Primal scaling

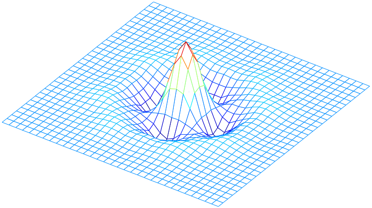


Primal wavelet

Scaling and Wavelet Functions for CAL2



Primal scaling



Primal wavelet

Image Coding Results

- Image coder: separable/nonseparable based on the lifting scheme
- Reversible integer-to-integer mappings
- Test images: grayscale images in the JPEG-2000 test set
- Coding
  - Lossy coding at various bit rates
  - Six/three levels of decomposition for quincunx/separable transforms
  - Difference measured in terms of PSNR
- Coding results: CAL1 and CAL2 outperform KS in 80% cases

Experimental Results for `finger`

Test image: `finger`

CR[†]	PSNR (dB)			
	CAL1	CAL2	KS	9/7
128	19.88	19.95	19.67	**19.98**
64	21.70	**21.75**	21.53	21.72
32	**24.52**	24.39	24.36	24.20
16	27.75	**27.83**	27.65	27.61

[†]compression ratio

- CAL1 and CAL2 outperform the KS filter bank.
- CAL1 and CAL2 outperform the 9/7 filter bank except at the lowest bit rate.

Conclusion

- New optimization-based design method is proposed.
- This method yields linear-phase PR quincunx filter banks with high coding gain, good analysis/synthesis filter frequency responses, and prescribed vanishing moments properties.
- Effectiveness is demonstrated by the experimental results.

Optimal Design for a Particular Image

- Optimize with the autocorrelation function of the `finger` image
- CAL1f: same filter support as CAL1
- CAL2f: same filter support as CAL2
- Coding gains for the `finger` image

CAL1f	CAL2f	CAL1	CAL2	KS	9/7
12.76	12.35	12.17	12.04	12.27	12.05

- Coding results

CR†	PSNR (dB)					
	CAL1f	CAL2f	CAL1	CAL2	KS	9/7
128	19.92	19.35	19.88	19.95	19.67	**19.98**
64	**21.82**	21.37	21.70	21.75	21.53	21.72
32	**24.53**	24.21	24.52	24.39	24.36	24.20
16	**27.84**	27.63	27.75	27.83	27.65	27.61

†compression ratio