Transactions Letters

On the Capacity Formula for Multiple Input–Multiple Output Wireless Channels: A Geometric Interpretation

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***Abstract*— The capacity of multiple input, multiple output (MIMO) wireless channels is computed for Ricean channels. The novelty is a geometrical (ray-tracing) interpretation of the MIMO channel capacity formula to find array geometries which greatly enhance channel capacity compared to single input–single output (SISO) systems.**

***Index Terms*—Array signal processing, channel capacity, land mobile radio cellular systems.**

I. Introduction

A MULTIPLE input–multiple output (MIMO) wireless communications channel with a matrix transfer function of independent complex Gaussian random variables has an information-theoretic capacity which grows linearly with the number of antenna array elements n, for fixed power and bandwidth [1].

For line-of-sight (LOS) channels, we use ray tracing to construct a matrix transfer function (channel response) explicitly for some example environments and find array geometries which result in channel matrices with close to n nonnegligible eigenvalues, with corresponding high capacity. The LOS matrix channel response will change as the receiver is moved, so that a capacity distribution is obtained from the ensemble of sample matrix elements at different receiver locations. Also, a Rayleigh matrix may be added to the LOS channel matrix to form a matrix of Ricean scalars, from which a capacity distribution is obtained. In either case, we define an outage threshold x (say 0.01), and define C_x to be that capacity for which $\mathrm{Prob}\{C > C_x\} = 1 - x$.

In what follows, the MIMO channel capacity formula is used to compute the capacity for three example geometries, plus one Ricean example, with a discussion of the results.

II. Capacity Calculations and Array Geometry

The capacity in b/s/Hz of a MIMO wireless system with n_T transmit antennas and n_R receive antennas with an average received SNR ρ (independent of n_T) at each receive antenna

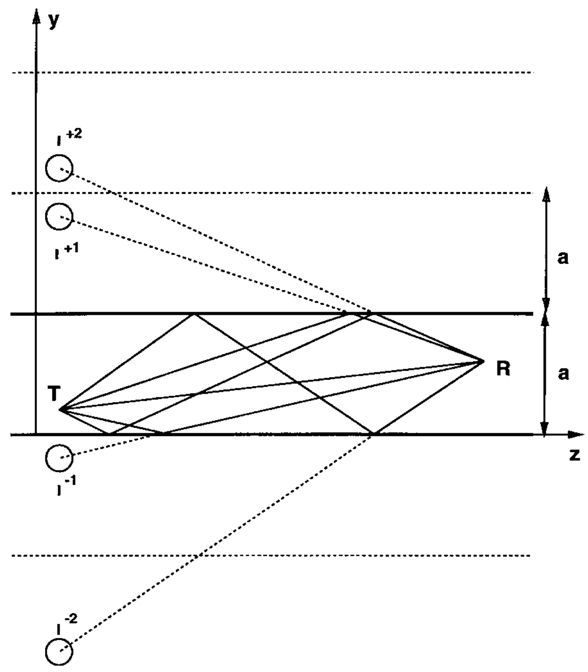

Fig. 1. Images in street canyon—top view $a = 25$ m, $\lambda = 1/3$ m, $T(x, y, z) = (1, 15, 0)$ m, $R(x, y, z) = (1, 5, 30)$ to $(1, 6, 40)$.

was obtained in [1] as $C = \log_2(\det[I_{n_R} + (\rho/n_T)HH^*])$, where the normalized channel matrix H contains complex scalars with unity average power loss, and H^* is the complex conjugate transpose of H. The capacity is expressed in b/s/Hz in the narrow-band limit with no frequency dependence. Normalization is achieved by dividing out the free space power loss and setting the parameter ρ to the desired SNR.[1] This result assumes that H is unknown to the receiver but n_R and ρ are known [1], [2]. The transmitted data has been demultiplexed into substreams which are separately independently coded and modulated on each antenna [6], so that the transmission from each antenna is different. The practicality of such MIMO wireless systems, using space-time coding with no bandwidth penalty, is illustrated in [3]–[5] and [7].

A. Line-of-Sight Channels

We first consider an environment with only free space nonfading LOS propagation and T and R arrays of $n_T =$

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[1] This avoids the need to compute absolute propagation loss and then select the transmitted power to obtain the desired SNR.

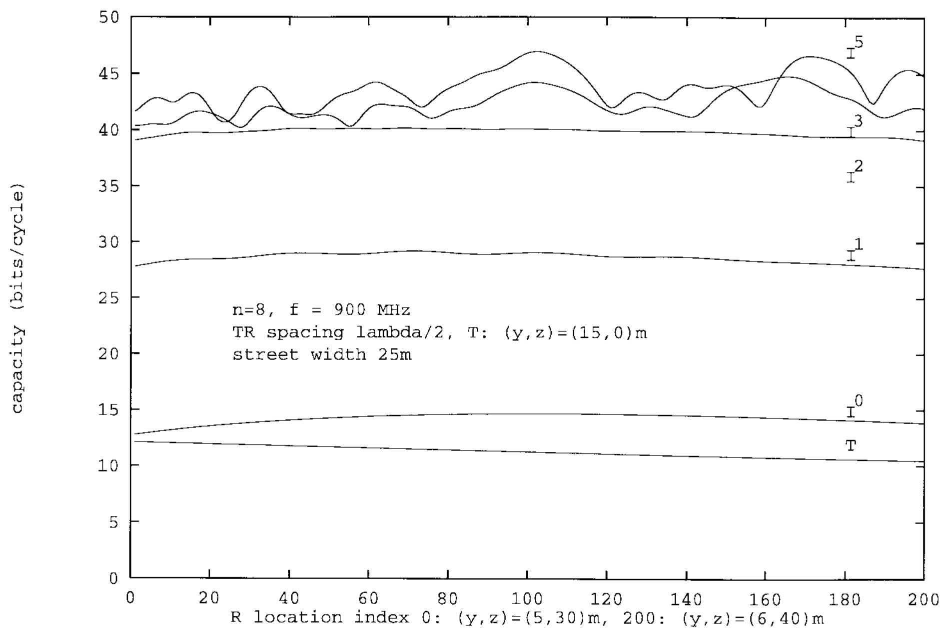

Fig. 2. Capacity versus R location in street canyon T and R height $x = 1$ m, $\rho = 20$ dB, $\lambda = 1/3$ m.

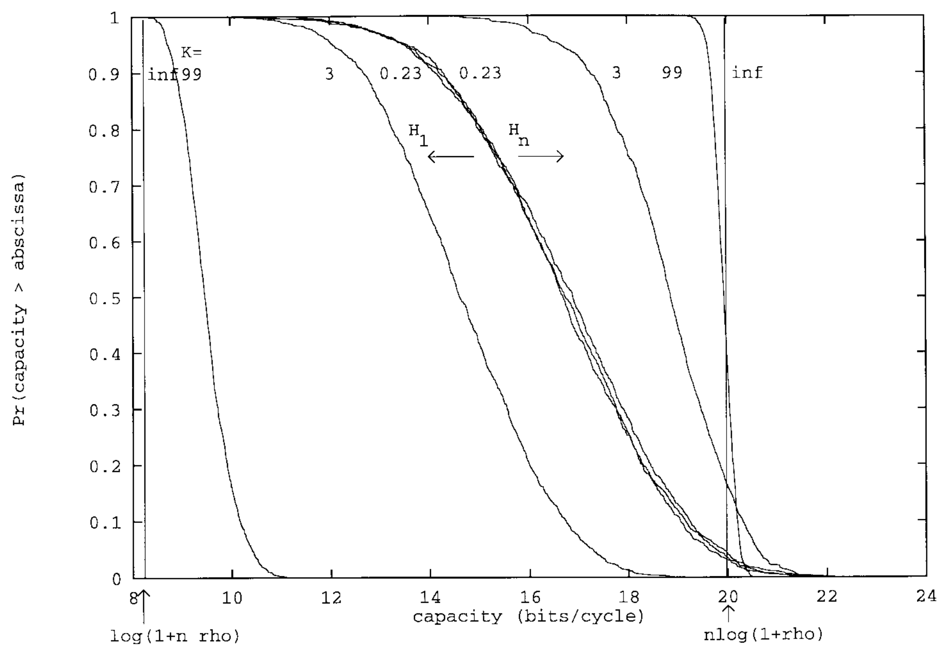

Fig. 3. Capacity on Ricean channels, $n = 3$.

$n_R = n$ antennas. The base and subscriber ends of the link are designated as T and R, respectively, but reciprocity applies. For LOS propagation, and a narrow-band channel at fixed carrier frequency $f_c = c/\lambda$, ray-tracing from T to R yields the channel transfer function matrix $H = H_{\text{LOS}}$ with complex scalar entries

$$H_{ik} = |T_1 - R_1| \frac{\exp(-j2\pi|T_i - R_k|/\lambda)}{|T_i - R_k|} \tag{1}$$

where T_i, R_i are coordinate vectors for the ith element of T, R. H_{ik} is normalized by the distance between the reference locations T_1, R_1, so that $H_{1,1} = 1$ and the absolute attenuation need not be calculated.

If the antennas are spaced less than $\lambda/2$ apart at both T and R, $H_{ik} = e^{j\theta_{ik}} \simeq e^{j\theta}$ for fixed θ for all i, k, $(HH^*)_{ik} \simeq n$, and $C = \log_2(1 + n\rho) = C_{\log}$, so that $C_{\log}$ increases logarithmically with n. For this case, $H = H_{\text{LOS}} = H_1$ is of rank 1, and the capacity gain is essentially due to the n-fold array gain in ρ.

For arrays of n more widely spaced antennas at T and/or R, the complex scalars H_{ik} all have magnitude near one but different phases θ_{ik}. For θ_{ik} so that $HH^* = nI_n$, $H =$

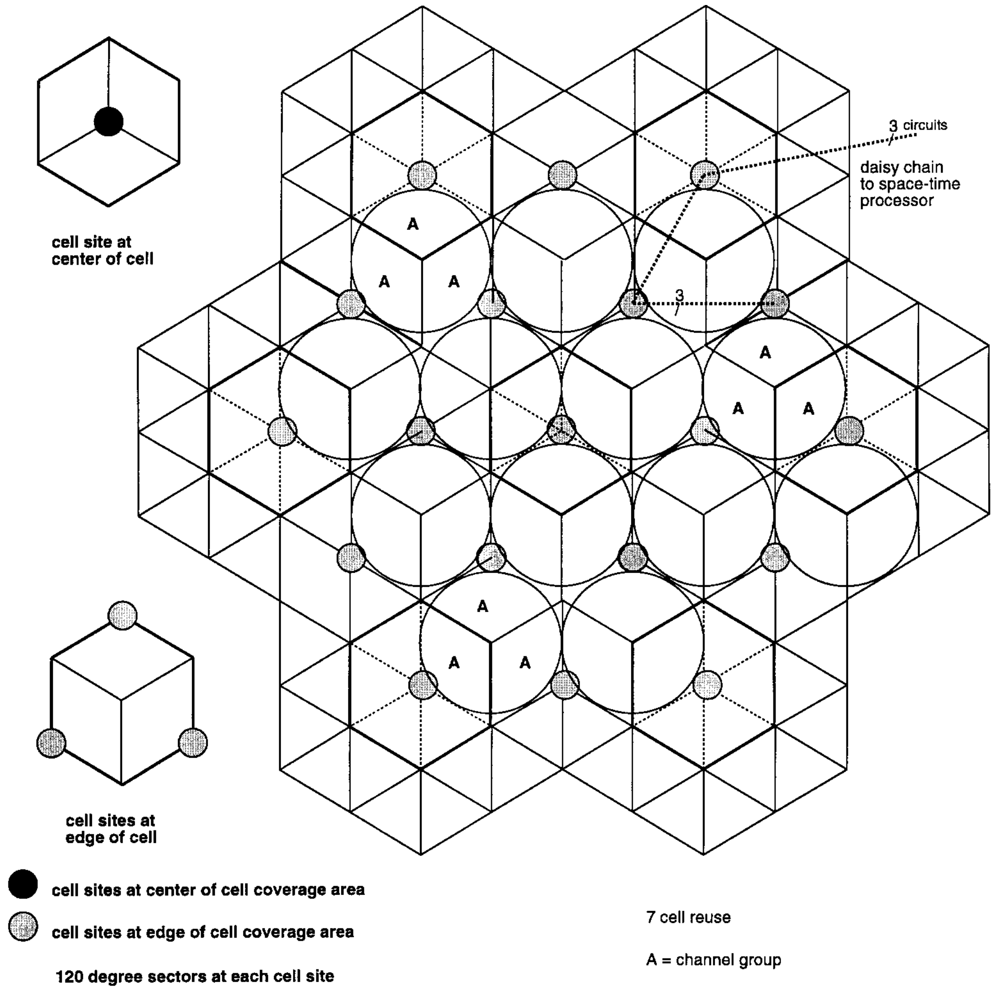

Fig. 4. Cell sites at center and edge of coverage area.

$H_{\rm LOS} = H_n$ is of rank n, and $C = n\log_2(1+\rho) = C_{\rm lin}$, so that $C_{\rm lin}$ increases linearly with n. An example is $\theta_{ik} = \frac{\pi}{n}[(i-i_0)-(k-k_0)]^2$ where for $n = 2$ and $i_0 = k_0 = 0, H_{\max} = \begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix}$, and the corresponding array geometry is two linear arrays broadside to each other. In what follows, we show three more examples of geometric arrangements for which $H_{\rm LOS} \simeq H_n$.

One such arrangement is an n-element T array spread along an arc at angles ϕ_{k-1} for $k = 1, \cdots, n$, and a linear R array oriented broadside to the center of the arc. From (1), for arc radius $D \gg \lambda$, and interelement spacing z_r, $H_{ik} = \exp(j\frac{2\pi z_r}{\lambda}[(i-1) - \frac{n-1}{2}]\sin\phi_{k-1})$ which corresponds to the autocorrelation $R_{xx}(i-k) = \frac{1}{2\triangle}\,{\rm Re}\int_{-\triangle}^{\triangle} e^{j\frac{2\pi z_r}{\lambda}(i-k)\sin\beta}\,d\beta$ from [8, eq. (A-13)] specialized for the n discrete angles of arrival ϕ_{k-1}. $C \simeq C_{\rm lin}$ when the angle subtended by the arc $2\triangle = \phi_{n-1} - \phi_0$ is consistent with the beamwidth $2\triangle = \lambda/z_r$ at which $R_{xx}(i-k) = 0$. The radiation pattern of the R array with all elements in phase $E(\phi) = \frac{\sin(n\gamma/2)}{n\sin(\gamma/2)}$, where $\gamma = 2\pi z_r \sin\phi/\lambda$. For $z_r = \lambda/2$, and the n array elements of T placed along the arc at ϕ_{k-1} exactly in the nulls of $R, H_{ik} = \exp(j\frac{2\pi}{n}(i-i_0)(k-k_0))$ where $i_0 = k_0 = \frac{n+1}{2}$, for which $HH^* = nI_n$ so that $H = H_n$ and $C = C_{\rm lin}$.

The second example arrangement is an n-element T array with elements spread evenly around a circle of radius $D \gg \lambda$, and a similar R array on a circle of radius $D_r \leq \lambda$ at the center of the T array. From (1), $H_{ik} = \exp(j\frac{2\pi D_r}{\lambda}\cos[(i-k)\frac{2\pi}{n}])$. For $D_r = \lambda/2$, the T elements are not in the nulls of the R array, but the elements of HH^* for which $i-k$ is odd are zero, and the off-diagonal even elements are approximately $0.3n$. Nonetheless, the capacity approaches $C_{\rm lin}$, consistent with the observation [8] that small correlation (<0.3) has negligible effect on performance. Further calculations confirm that the capacity is robust in the presence of rotation or lateral movement of R or perturbations in the placement of the T elements.

The third example is an urban street geometry with two parallel reflectors representing the building walls separated by the street width a (Fig. 1). $I^{\pm m}$ represents an image due to m specular reflections from the walls, and I^0 is the "ground reflection" image not visible in the figure. For this street geometry, the elements of H may be written

$$\frac{H_{ik}}{|T_1 - R_1|} = \frac{\exp(-j2\pi|T_i - R_k|/\lambda)}{|T_i - R_k|} + \sum_{k=-m}^{m} \Gamma^k \frac{\exp\left(-j2\pi\left|I_i^k - R_k\right|/\lambda\right)}{\left|I_i^k - R_k\right|} \quad (2)$$

where Γ is the amplitude reflection coefficient,[2] m is the maximum number of reflections considered, and we have assumed isotropic array elements.

For eight-element linear arrays with $\lambda/2$ spacing oriented perpendicular to the street, Fig. 2 shows how the capacity increases as more images are added (and the angular spread of rays increases) and approaches $C_{\rm lin}$ with seven images $(|m| \leq 3)$ plus T. Furthermore, the received signal envelope looks increasingly Rayleigh-like as more images are added.

B. Ricean Channels

Next we consider the capacity for Ricean channels, where the deterministic component $H_{\rm LOS}$ is fixed as either H_1 or H_n. We follow the simulation methods of [1] using the normalized Ricean channel matrix $H = (aH_{\rm LOS} + bH_{\rm Rayleigh})$ with $a^2 + b^2 = 1$ and Ricean K-factor $K = a^2/b^2$. The results for $n = 3$ with $\rho = 100$ (20 dB) (Fig. 3) quantify how for closely spaced $(\leq\lambda)$ array elements at both T and R, and no reflectors or scatterers such that $H_{\rm LOS} \simeq H_1$, the capacity decreases with increasing K toward $C = C_{\rm log}$. However, for array geometries such as the above examples, where $H_{\rm LOS} \simeq H_n$, the capacity increases with increasing K toward $C = C_{\rm lin}$. For $K = 0, C$ corresponds to that obtained in [1].

III. Discussion

Capacities approaching $C_{\rm lin} = n\log_2(1+\rho)$ can be achieved for MIMO channels in an LOS (non-Rayleigh) environment by spreading out the elements of T either explicitly (by placing one element of T at each of n sites), or implicitly (by adding reflectors which create images of T). The results of the second example suggest that three sectors in a conventional cellular system can be combined to form one "edge-excited" (inward-facing) cell (Fig. 4) to enhance capacity for R not close to a base station. This is reminiscent of soft handoff in CDMA systems where multiple base stations serve one mobile, except that in this case, each base station carries a different substream of the transmitted data. The results of the third example suggest that in the absence of reflectors, we may use n antennas at each of n sites, thus replicating the effect of images of the n-element T array. The ray-tracing channel model for $H = H_{\rm LOS} + H_{\rm Rayleigh}$ described here may be useful for the performance evaluation of MIMO wireless systems which use spatial diversity through space-time coding to exploit the available capacity with no bandwidth penalty (e.g. [2], [5]–[7]).

[2] In general, the Γ are different for each image, since they depend on the angles of incidence and reflection, and the surface characteristics. Here we assume Γ has the same constant value 0.6 for all reflections, except the ground reflection Γ^0 which is set to -1. This approximation is sufficient to illustrate the capacity gain.

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