QUIZ #3

Que. a) Find DFT of $f(k) = [1 \;\; 0 \;\; 1]$

b) Find Inverse DFT to get back to $[1 \;\; 0 \;\; 1]$

Solution: (a) DFT of $f(k)$, $F(r) = \sum_{k=0}^{N_0-1} f(k)\,(W_{N_0})^{rk} = \sum_{k=0}^{N_0-1} f(k)\left(e^{-j2\pi k/N_0}\right)^{r}$

$$\Rightarrow \quad F(r) = \sum_{k=0}^{2} f(k)\left(e^{-j2\pi k/N_0}\right)^{r} \qquad \Big| \because N_0 = 3$$

$\therefore$

$$r=0 \Rightarrow \quad F(0) = \sum_{k=0}^{2} f(k)\left(e^{-j2\pi k/3}\right)^{0} = f(0) + f(1) + f(2) = 1 + 0 + 1 = 2$$

$$r=1 \Rightarrow \quad F(1) = \sum_{k=0}^{2} f(k)\left(e^{-j2\pi k/3}\right)^{1} = f(0) + f(1)e^{-j2\pi/3} + f(2)e^{-j4\pi/3} = 1 + 0 \cdot e^{-j2\pi/3} + 1 \cdot e^{-j4\pi/3}$$

$$= 1 + e^{-j4\pi/3} = 1 - \frac{1}{2} + j\frac{\sqrt{3}}{2} = \frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$r=2 \Rightarrow \quad F(2) = \sum_{k=0}^{2} f(k)\left(e^{-j2\pi k/3}\right)^{2} = f(0) + f(1)e^{-j4\pi/3} + f(2)e^{-j8\pi/3} = 1 + 0 + e^{-j8\pi/3}$$

$$= 1 + e^{-j8\pi/3} = 1 - \frac{1}{2} - j\frac{\sqrt{3}}{2} = \frac{1}{2} - j\frac{\sqrt{3}}{2}$$

$$\therefore \text{ DFT of } f(k),\; F(r) = \left[2 \quad \frac{1}{2} + j\frac{\sqrt{3}}{2} \quad \frac{1}{2} - j\frac{\sqrt{3}}{2}\right] \leftarrow$$

(b) IDFT of $F(r)$, $f(k) = \frac{1}{N_0}\sum_{r=0}^{N_0-1} F(r)\,(W_{N_0})^{-rk} = \frac{1}{N_0}\sum_{r=0}^{N_0-1} F(r)\left(e^{j2\pi r/N_0}\right)^{k}$

$$\Rightarrow \quad f(k) = \frac{1}{3}\sum_{r=0}^{2} F(r)\left(e^{j2\pi r/3}\right)^{k}$$

$\therefore$

$$k=0 \Rightarrow \quad f(0) = \frac{1}{3}\sum_{r=0}^{2} F(r)\left(e^{j2\pi r/3}\right)^{0} = \frac{1}{3}\left[F(0) + F(1) + F(2)\right]$$

$$= \frac{1}{3}\left[2 + \frac{1}{2} + \cancel{j\frac{\sqrt{3}}{2}} + \frac{1}{2} - \cancel{j\frac{\sqrt{3}}{2}}\right] = \frac{1}{3} \times 3 = 1$$

$$k=1 \Rightarrow \quad f(1) = \frac{1}{3}\sum_{r=0}^{2} F(r)\left(e^{j2\pi r/3}\right)^{1} = \frac{1}{3}\left[F(0) \cdot e^{j2\pi \times 0/3} + F(1)e^{j2\pi/3} + F(2)e^{j4\pi/3}\right]$$

$$= \frac{1}{3}\left[2 \cdot 1 + (1 + e^{-j4\pi/3})e^{j2\pi/3} + (1 + e^{-j8\pi/3})e^{j4\pi/3}\right]$$

$$= \frac{1}{3}\left[2 + e^{j2\pi/3} + e^{-j2\pi/3} + e^{j4\pi/3} + e^{-j4\pi/3}\right] = \frac{1}{3}\left[2 + 2\cos(2\pi/3) + 2\cos(4\pi/3)\right]$$

$$= \frac{1}{3}\left[2 + 2\left(-\frac{1}{2}\right) + 2\left(-\frac{1}{2}\right)\right] = \frac{1}{3}\left[2 - 1 - 1\right] = 0$$

$$k=2 \Rightarrow \quad f(2) = \frac{1}{3}\sum_{r=0}^{2} F(r)\left(e^{j2\pi r/3}\right)^{2} = \frac{1}{3}\left[F(0) + F(1)e^{j4\pi/3} + F(2)e^{j8\pi/3}\right]$$

$$= \frac{1}{3}\left[2 + (1 + e^{-j4\pi/3})e^{j4\pi/3} + (1 + e^{-j8\pi/3})e^{j8\pi/3}\right] = \frac{1}{3}\left[2 + e^{j4\pi/3} + 1 + e^{j8\pi/3} + 1\right]$$

$$= \frac{1}{3}\left[2 - \frac{1}{2} - j\sqrt{3}/2 + 1 - 1/2 + j\sqrt{3}/2 + 1\right] = 3 \times \frac{1}{3} = 1$$

$\therefore \quad f(k) = [\,1 \quad 0 \quad 1\,] \leftarrow$

<u>ALTERNATIVE SOLUTION:</u>

(using matrix)

ⓐ $$f = [\,1 \quad 0 \quad 1\,]$$

$$\therefore \quad \text{DFT of } f \Rightarrow \quad F = Wf = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3 & W_3^2 \\ 1 & W_3^2 & W_3^4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow F = \begin{bmatrix} 1.1 + 1.0 + 1.1 \\ 1.1 + W_3.0 + W_3^2.1 \\ 1.1 + W_3^2.0 + W_3^4.1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 + W_3^2 \\ 1 + W_3^4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 + e^{-j4\pi/3} \\ 1 + e^{-j8\pi/3} \end{bmatrix} \qquad [\because W_3 = e^{-j2\pi/3}]$$

$$\therefore \quad F = \begin{bmatrix} 2 \\ \frac{1}{2} + j\frac{\sqrt{3}}{2} \\ \frac{1}{2} - j\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\Rightarrow \quad F = \left[\, 2 \quad \left(\tfrac{1}{2} + j\tfrac{\sqrt{3}}{2}\right) \quad \left(\tfrac{1}{2} - j\tfrac{\sqrt{3}}{2}\right) \right]' \leftarrow$$

ⓑ $$\text{IDFT of } F \Rightarrow f = W^{-1}F = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-4} \end{bmatrix} \begin{bmatrix} 2 \\ 1 + e^{-j4\pi/3} \\ 1 + e^{-j8\pi/3} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{j2\pi/3} & e^{j4\pi/3} \\ 1 & e^{j4\pi/3} & e^{j8\pi/3} \end{bmatrix} \begin{bmatrix} 2 \\ 1 + e^{-j4\pi/3} \\ 1 + e^{-j8\pi/3} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 + 1 + e^{-j4\pi/3} + 1 + e^{-j8\pi/3} \\ 2 + e^{j2\pi/3} + e^{-j2\pi/3} + e^{j4\pi/3} + e^{-j4\pi/3} \\ 2 + e^{j4\pi/3} + 1 + e^{j8\pi/3} + 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 + 1 \cancel{- \frac{1}{2}} \cancel{- j\frac{\sqrt{3}}{2}} + \cancel{1} \cancel{- \frac{1}{2}} \cancel{+ j\frac{\sqrt{3}}{2}} \\ 2 + 2\cos(2\pi/3) + 2\cos(4\pi/3) \\ 2 \cancel{- \frac{1}{2}} \cancel{- j\frac{\sqrt{3}}{2}} + 1 - \cancel{\frac{1}{2}} \cancel{+ j\frac{\sqrt{3}}{2}} + \cancel{1} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 2 + 2(-\frac{1}{2}) + 2(-\frac{1}{2}) \\ 3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow f = [\,1 \quad 0 \quad 1\,]' \leftarrow$$