

FREQUENCY-WEIGHTED L_2 -SENSITIVITY MINIMIZATION FOR 2-D STATE-SPACE DIGITAL FILTERS SUBJECT TO L_2 -SCALING CONSTRAINTS BY A QUASI-NEWTON METHOD

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ABSTRACT

This paper considers the problem of minimizing a frequency-weighted l_2 -sensitivity measure subject to l_2 -scaling constraints for 2-D state-space digital filters. First, the frequency-weighted l_2 -sensitivity is analyzed for 2-D state-space digital filters described by the Roesser local state-space model. Next, the minimization problem of the frequency-weighted l_2 -sensitivity subject to l_2 -scaling constraints is formulated. The constrained optimization problem is then converted into an unconstrained optimization formulation by using linear-algebraic techniques. An efficient quasi-Newton algorithm with closed-form formula for gradient evaluation is applied to solve the unconstrained optimization problem. The optimal state-space filter structure with minimum frequency-weighted l_2 -sensitivity and no overflow oscillations is constructed by applying the optimal coordinate transformation. Finally, a numerical example is presented to demonstrate the validity and effectiveness of the proposed technique.

1. INTRODUCTION

In many practical applications, it is desirable to realize a state-space model from a given transfer function so that the filter possesses minimum sensitivity with respect to the realization coefficients in a certain sense. So far, several techniques have been reported for synthesizing 2-D state-space filter structures with minimum coefficient sensitivity. These include the l_1/l_2 -mixed sensitivity minimization problem [1]-[6] and the l_2 -sensitivity minimization problem [6]-[10]. Some researchers have considered the minimization problem of the frequency-weighted sensitivity for 2-D state-space digital filters [4]-[7]. More recently, the minimization problem of l_2 -sensitivity subject to l_2 -scaling constraints has been explored for 2-D state-space digital filters [11],[12]. It is well known that the use of scaling constraints can be beneficial for suppressing overflow oscillations [13],[14]. However, frequency-weighted sensitivity measures have not yet been considered in [11],[12].

In this paper, we treat the problem of minimizing a frequency-weighted l_2 -sensitivity measure subject to l_2 -scaling constraints for 2-D state-space digital filters described by the Roesser local state-space (LSS)

model [15]. First, an expression for evaluating the l_2 -sensitivity is introduced, and the minimization problem of the l_2 -sensitivity subject to l_2 -scaling constraints is formulated. An iterative method is developed for solving the constrained optimization problem. This relies on the conversion of the constrained optimization problem into an unconstrained optimization formulation and utilizes an efficient quasi-Newton method with closed-form formula for gradient evaluation. The optimal filter structure with minimum l_2 -sensitivity and no overflow oscillations is constructed by applying the resulting coordinate transformation matrix. A numerical example is presented to demonstrate the validity and effectiveness of the proposed technique. The technique proposed here may be viewed as an extension of the l_2 -sensitivity minimization technique reported in [11] to the frequency-weighted l_2 -sensitivity minimization.

Throughout the paper, $\mathbf{I}_n$ stands for the identity matrix of dimension $n \times n$, $\oplus$ is used to denote the direct sum of matrices, the transpose (conjugate transpose) of a matrix $\mathbf{A}$ is indicated by $\mathbf{A}^T$ ($\mathbf{A}^*$), and the trace and i th diagonal element of a square matrix $\mathbf{A}$ are denoted by $\text{tr}[\mathbf{A}]$ and $(\mathbf{A})_{ii}$, respectively.

2. L_2 -SENSITIVITY ANALYSIS

Consider a LSS model $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)_{m,n}$ for 2-D IIR digital filters which is stable, separately locally controllable and separately locally observable [15],[16]

$$\begin{aligned} \mathbf{x}_{11}(i, j) &= \mathbf{A}\mathbf{x}(i, j) + \mathbf{b}u(i, j) \\ y(i, j) &= \mathbf{c}\mathbf{x}(i, j) + du(i, j) \end{aligned} \quad (1)$$

where

$$\begin{aligned} \mathbf{x}_{11}(i, j) &= \begin{bmatrix} \mathbf{x}^h(i+1, j) \\ \mathbf{x}^v(i, j+1) \end{bmatrix}, \quad \mathbf{x}(i, j) = \begin{bmatrix} \mathbf{x}^h(i, j) \\ \mathbf{x}^v(i, j) \end{bmatrix} \\ \mathbf{A} &= \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}, \quad \mathbf{c} = [\mathbf{c}_1 \quad \mathbf{c}_2] \end{aligned}$$

with an $m \times 1$ horizontal state vector $\mathbf{x}^h(i, j)$, an $n \times 1$ vertical state vector $\mathbf{x}^v(i, j)$, a scalar input $u(i, j)$, a scalar output $y(i, j)$, and real constant matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{b}_1, \mathbf{b}_2, \mathbf{c}_1, \mathbf{c}_2$ and d of appropriate dimensions. The transfer function of the LSS model in (1) is given

by

$$H(z_1, z_2) = \mathbf{c}(\mathbf{Z} - \mathbf{A})^{-1}\mathbf{b} + d \quad (2)$$

where $\mathbf{Z} = z_1\mathbf{I}_m \oplus z_2\mathbf{I}_n$.

To define the frequency-weighted l_2 -sensitivity of the LSS model in (1), we need the following definitions.

Definition 1: Let $\mathbf{X}$ be an $m \times n$ real matrix and let $f(\mathbf{X})$ be a scalar complex function of $\mathbf{X}$, differentiable with respect to all the entries of $\mathbf{X}$. The sensitivity function of $f(\mathbf{X})$ with respect to $\mathbf{X}$ is then defined as

$$\mathbf{S}_{\mathbf{X}} = \frac{\partial f(\mathbf{X})}{\partial \mathbf{X}}, \quad (\mathbf{S}_{\mathbf{X}})_{ij} = \frac{\partial f(\mathbf{X})}{\partial x_{ij}} \quad (3)$$

where x_{ij} denotes the (i, j) th entry of matrix $\mathbf{X}$.

Definition 2: In order to take into account the sensitivity behavior of the transfer function in a specified frequency band, or even at some discrete frequency points, the weighted sensitivity functions are defined as

$$\begin{aligned} \frac{\delta H(z_1, z_2)}{\delta \mathbf{A}} &= W_A(z_1, z_2) \frac{\partial H(z_1, z_2)}{\partial \mathbf{A}} \\ \frac{\delta H(z_1, z_2)}{\delta \mathbf{b}} &= W_B(z_1, z_2) \frac{\partial H(z_1, z_2)}{\partial \mathbf{b}} \\ \frac{\delta H(z_1, z_2)}{\delta \mathbf{c}^T} &= W_C(z_1, z_2) \frac{\partial H(z_1, z_2)}{\partial \mathbf{c}^T} \end{aligned} \quad (4)$$

where $W_A(z_1, z_2)$, $W_B(z_1, z_2)$, and $W_C(z_1, z_2)$ are scalar, stable, causal functions of the complex variables z_1 and z_2 .

Notice that δ in (4) is not meant to be a derivative operator, but rather a notation for defining the weighted parameter sensitivity.

Definition 3: Let $\mathbf{X}(z_1, z_2)$ be an $m \times n$ complex matrix valued function of the complex variables z_1 and z_2 . The l_2 norm of $\mathbf{X}(z_1, z_2)$ is then defined by

$$\begin{aligned} &\|\mathbf{X}(z_1, z_2)\|_2 \\ &= \left(\text{tr} \left[\frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \mathbf{X}(z_1, z_2) \mathbf{X}^*(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2} \right] \right)^{\frac{1}{2}} \end{aligned} \quad (5)$$

where $\Gamma_i = \{z_i : |z_i| = 1\}$ for $i = 1, 2$.

From Definitions 1-3, the overall frequency-weighted l_2 -sensitivity measure for the LSS model in (1) can be evaluated by

$$\begin{aligned} S &= \left\| \frac{\delta H(z_1, z_2)}{\delta \mathbf{A}} \right\|_2^2 + \left\| \frac{\delta H(z_1, z_2)}{\delta \mathbf{b}} \right\|_2^2 + \left\| \frac{\delta H(z_1, z_2)}{\delta \mathbf{c}^T} \right\|_2^2 \\ &= \left\| W_A(z_1, z_2) [\mathbf{F}(z_1, z_2) \mathbf{G}(z_1, z_2)]^T \right\|_2^2 \\ &\quad + \left\| W_B(z_1, z_2) \mathbf{G}^T(z_1, z_2) \right\|_2^2 + \left\| W_C(z_1, z_2) \mathbf{F}(z_1, z_2) \right\|_2^2 \end{aligned} \quad (6)$$

where

$$\mathbf{F}(z_1, z_2) = (\mathbf{Z} - \mathbf{A})^{-1}\mathbf{b}, \quad \mathbf{G}(z_1, z_2) = \mathbf{c}(\mathbf{Z} - \mathbf{A})^{-1}.$$

The frequency-weighted l_2 -sensitivity measure in (6) is then written as

$$S = \text{tr}[\mathbf{M}_A] + \text{tr}[\mathbf{W}_B] + \text{tr}[\mathbf{K}_C] \quad (7)$$

where $\mathbf{M}_A$, $\mathbf{W}_B$, and $\mathbf{K}_C$ are obtained by the following general expression:

$$\mathbf{X} = \frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \mathbf{Y}(z_1, z_2) \mathbf{Y}^*(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}$$

with $\mathbf{Y}(z_1, z_2) = W_A(z_1, z_2) [\mathbf{F}(z_1, z_2) \mathbf{G}(z_1, z_2)]^T$ for $\mathbf{X} = \mathbf{M}_A$, $\mathbf{Y}(z_1, z_2) = W_B^*(z_1, z_2) \mathbf{G}^*(z_1, z_2)$ for $\mathbf{X} = \mathbf{W}_B$, and $\mathbf{Y}(z_1, z_2) = W_C(z_1, z_2) \mathbf{F}(z_1, z_2)$ for $\mathbf{X} = \mathbf{K}_C$. The matrices $\mathbf{K}_C$, $\mathbf{W}_B$, and $\mathbf{M}_A$ can be computed using

$$\begin{aligned} \mathbf{K}_C &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathbf{f}_C(i, j) \mathbf{f}_C^T(i, j) \\ \mathbf{W}_B &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathbf{g}_B^T(i, j) \mathbf{g}_B(i, j) \\ \mathbf{M}_A &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathbf{H}_A^T(i, j) \mathbf{H}_A(i, j) \end{aligned} \quad (8)$$

where

$$\begin{aligned} \mathbf{A}^{(1,0)} &= \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{A}^{(0,1)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix} \\ \mathbf{A}^{(0,0)} &= \mathbf{I}_{m+n}, \quad \mathbf{A}^{(-i,j)} = \mathbf{0} \ (i \geq 1), \quad \mathbf{A}^{(i,-j)} = \mathbf{0} \ (j \geq 1) \\ \mathbf{A}^{(i,j)} &= \mathbf{A}^{(1,0)} \mathbf{A}^{(i-1,j)} + \mathbf{A}^{(0,1)} \mathbf{A}^{(i,j-1)} \\ &= \mathbf{A}^{(i-1,j)} \mathbf{A}^{(1,0)} + \mathbf{A}^{(i,j-1)} \mathbf{A}^{(0,1)}, \quad (i, j) > (0, 0) \end{aligned}$$

$$\begin{aligned} \mathbf{f}(i, j) &= \mathbf{A}^{(i-1,j)} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{0} \end{bmatrix} + \mathbf{A}^{(i,j-1)} \begin{bmatrix} \mathbf{0} \\ \mathbf{b}_2 \end{bmatrix} \\ \mathbf{g}(i, j) &= \mathbf{c} \mathbf{A}^{(i-1,j)} \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathbf{c} \mathbf{A}^{(i,j-1)} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix} \end{aligned}$$

$$\mathbf{H}(i, j) = \sum_{(0,0) \leq (k,r) < (i,j)} \mathbf{f}(k, r) \mathbf{g}(i-k, j-r)$$

$$\mathbf{f}_C(i, j) = \sum_{(0,0) \leq (k,r) < (i,j)} w_C(k, r) \mathbf{f}(i-k, j-r)$$

$$\mathbf{g}_B(i, j) = \sum_{(0,0) \leq (k,r) < (i,j)} w_B(k, r) \mathbf{g}(i-k, j-r)$$

$$\mathbf{H}_A(i, j) = \sum_{(0,0) \leq (k,r) < (i,j)} w_A(k, r) \mathbf{H}(i-k, j-r)$$

with partial ordering for integer pairs (i, j) as described in [15, p. 2], and $w_A(k, r)$, $w_B(k, r)$, and $w_C(k, r)$ denoting the unit-sample responses of frequency-weighting functions $W_A(z_1, z_2)$, $W_B(z_1, z_2)$, and $W_C(z_1, z_2)$, respectively.

3. l_2 -SENSITIVITY MINIMIZATION

3.1 Problem Formulation

Using a 2-D coordinate transformation defined by

$$\bar{\mathbf{x}}(i, j) = \mathbf{T}^{-1} \mathbf{x}(i, j) \quad (9)$$

where $\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_4$ is a block-diagonal nonsingular matrix with an $m \times m$ submatrix $\mathbf{T}_1$ and an $n \times n$ submatrix $\mathbf{T}_4$, we obtain a new realization $(\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}, d)_{m,n}$ characterized by

$$\bar{\mathbf{A}} = \mathbf{T}^{-1} \mathbf{A} \mathbf{T}, \quad \bar{\mathbf{b}} = \mathbf{T}^{-1} \mathbf{b}, \quad \bar{\mathbf{c}} = \mathbf{c} \mathbf{T}. \quad (10)$$

Applying the coordinate transformation in (9) to the LSS model in (1), the weighted l_2 -sensitivity measure in (7) is changed to

$$S(\mathbf{T}) = \text{tr}[\mathbf{T}^T \mathbf{M}_A(\mathbf{T}) \mathbf{T}] + \text{tr}[\mathbf{T}^T \mathbf{W}_B \mathbf{T}] + \text{tr}[\mathbf{T}^{-1} \mathbf{K}_C \mathbf{T}^{-T}] \quad (11)$$

where

$$\mathbf{M}_A(\mathbf{T}) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathbf{H}_A^T(i, j) \mathbf{T}^{-T} \mathbf{T}^{-1} \mathbf{H}_A(i, j).$$

It is noted that the local controllability Gramian $\mathbf{K}$ for the LSS model in (1) is given by

$$\begin{aligned} \mathbf{K} &= \frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \mathbf{F}(z_1, z_2) \mathbf{F}^*(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathbf{f}(i, j) \mathbf{f}^T(i, j) \end{aligned} \quad (12)$$

which is related to the local controllability Gramian $\bar{\mathbf{K}}$ for the new realization $(\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}, d)_{m,n}$ in (10) by

$$\bar{\mathbf{K}} = \mathbf{T}^{-1} \mathbf{K} \mathbf{T}^{-T}. \quad (13)$$

If l_2 -scaling constraints are imposed on the new local state vector $\bar{\mathbf{x}}(i, j)$, then it is required that

$$\begin{aligned} (\bar{\mathbf{K}}_1)_{ii} &= (\mathbf{T}_1^{-1} \mathbf{K}_1 \mathbf{T}_1^{-T})_{ii} = 1 \quad \text{for } i = 1, 2, \dots, m \\ (\bar{\mathbf{K}}_4)_{jj} &= (\mathbf{T}_4^{-1} \mathbf{K}_4 \mathbf{T}_4^{-T})_{jj} = 1 \quad \text{for } j = 1, 2, \dots, n \end{aligned} \quad (14)$$

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_3 & \mathbf{K}_4 \end{bmatrix}$$

with an $m \times m$ submatrix $\mathbf{K}_1$ and an $n \times n$ submatrix $\mathbf{K}_4$ along its diagonal.

The minimization problem of the frequency-weighted l_2 -sensitivity subject to l_2 -scaling constraints is now formulated as follows: *Given matrices $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}$, find an $(m+n) \times (m+n)$ block-diagonal nonsingular matrix $\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_4$ which minimizes $S(\mathbf{T})$ in (11) subject to l_2 -scaling constraints in (14).*

3.2 Problem Solution

Because the LSS model in (1) is assumed to be stable and separately locally controllable, submatrices $\mathbf{K}_1$ and $\mathbf{K}_4$ are symmetric and positive-definite [16]. Thus $\mathbf{K}_1^{1/2}$ and $\mathbf{K}_4^{1/2}$ satisfying $\mathbf{K}_i = \mathbf{K}_i^{1/2} \mathbf{K}_i^{1/2}$ for $i = 1, 4$ are also symmetric and positive-definite. By defining

$$\begin{aligned} \hat{\mathbf{T}} &= \hat{\mathbf{T}}_1 \oplus \hat{\mathbf{T}}_4 \\ &= (\mathbf{T}_1 \oplus \mathbf{T}_4)^T (\mathbf{K}_1 \oplus \mathbf{K}_4)^{-\frac{1}{2}}, \end{aligned} \quad (15)$$

it follows that

$$\bar{\mathbf{K}} = \hat{\mathbf{T}}^{-T} \begin{bmatrix} \mathbf{I}_m & \mathbf{K}_1^{-\frac{1}{2}} \mathbf{K}_2 \mathbf{K}_4^{-\frac{1}{2}} \\ \mathbf{K}_4^{-\frac{1}{2}} \mathbf{K}_3 \mathbf{K}_1^{-\frac{1}{2}} & \mathbf{I}_n \end{bmatrix} \hat{\mathbf{T}}^{-1}. \quad (16)$$

Thus, the l_2 -scaling constraints in (14) can be written as

$$\begin{aligned} (\hat{\mathbf{T}}_1^{-T} \hat{\mathbf{T}}_1^{-1})_{ii} &= 1 \quad \text{for } i = 1, 2, \dots, m \\ (\hat{\mathbf{T}}_4^{-T} \hat{\mathbf{T}}_4^{-1})_{jj} &= 1 \quad \text{for } j = 1, 2, \dots, n. \end{aligned} \quad (17)$$

It is noted that the conditions in (17) are always satisfied by choosing $\hat{\mathbf{T}}_1^{-1}$ and $\hat{\mathbf{T}}_4^{-1}$ as

$$\begin{aligned} \hat{\mathbf{T}}_1^{-1} &= \left[\frac{\mathbf{t}_{11}}{\|\mathbf{t}_{11}\|}, \frac{\mathbf{t}_{12}}{\|\mathbf{t}_{12}\|}, \dots, \frac{\mathbf{t}_{1m}}{\|\mathbf{t}_{1m}\|} \right] \\ \hat{\mathbf{T}}_4^{-1} &= \left[\frac{\mathbf{t}_{41}}{\|\mathbf{t}_{41}\|}, \frac{\mathbf{t}_{42}}{\|\mathbf{t}_{42}\|}, \dots, \frac{\mathbf{t}_{4n}}{\|\mathbf{t}_{4n}\|} \right]. \end{aligned} \quad (18)$$

From (15), it follows that (11) can be written as

$$J_o(\hat{\mathbf{T}}) = \text{tr}[\hat{\mathbf{T}} \hat{\mathbf{M}}_A(\hat{\mathbf{T}}) \hat{\mathbf{T}}^T] + \text{tr}[\hat{\mathbf{T}} \hat{\mathbf{W}}_B \hat{\mathbf{T}}^T] + \text{tr}[\hat{\mathbf{T}}^{-T} \hat{\mathbf{K}}_C \hat{\mathbf{T}}^{-1}] \quad (19)$$

where

$$\hat{\mathbf{M}}_A(\hat{\mathbf{T}}) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \hat{\mathbf{H}}_A^T(i, j) \hat{\mathbf{T}}^{-1} \hat{\mathbf{T}}^{-T} \hat{\mathbf{H}}_A(i, j)$$

with

$$\begin{aligned} \hat{\mathbf{H}}_A(i, j) &= (\mathbf{K}_1 \oplus \mathbf{K}_4)^{-\frac{1}{2}} \mathbf{H}_A(i, j) (\mathbf{K}_1 \oplus \mathbf{K}_4)^{\frac{1}{2}} \\ \hat{\mathbf{W}}_B &= (\mathbf{K}_1 \oplus \mathbf{K}_4)^{\frac{1}{2}} \mathbf{W}_B (\mathbf{K}_1 \oplus \mathbf{K}_4)^{\frac{1}{2}} \\ \hat{\mathbf{K}}_C &= (\mathbf{K}_1 \oplus \mathbf{K}_4)^{-\frac{1}{2}} \mathbf{K}_C (\mathbf{K}_1 \oplus \mathbf{K}_4)^{-\frac{1}{2}}. \end{aligned}$$

Following the above arguments, the problem of finding an $(m+n) \times (m+n)$ block-diagonal nonsingular matrix $\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_4$ which minimizes $S(\mathbf{T})$ in (11) subject to l_2 -scaling constraints in (14) is converted into an unconstrained optimization problem of finding an $(m+n) \times (m+n)$ block-diagonal nonsingular matrix $\hat{\mathbf{T}} = \hat{\mathbf{T}}_1 \oplus \hat{\mathbf{T}}_4$ given by (15) which minimizes $J_o(\hat{\mathbf{T}})$ in (19).

In order to apply a quasi-Newton algorithm for the minimization of $J_o(\hat{\mathbf{T}})$ in (19) with respect to matrix $\hat{\mathbf{T}} = \hat{\mathbf{T}}_1 \oplus \hat{\mathbf{T}}_4$ given by (15), we define an $(m^2 + n^2) \times 1$ vector $\mathbf{x} = (\mathbf{t}_{11}^T, \mathbf{t}_{12}^T, \dots, \mathbf{t}_{1m}^T, \mathbf{t}_{41}^T, \mathbf{t}_{42}^T, \dots, \mathbf{t}_{4n}^T)^T$. In this way, $J_o(\hat{\mathbf{T}})$ is a function of $\mathbf{x}$, denoted by $J_o(\mathbf{x})$. The algorithm starts with a trivial initial point $\mathbf{x}_0$ obtained from an initial assignment $\hat{\mathbf{T}} = \mathbf{I}_{m+n}$. Then, in the k th iteration a quasi-Newton algorithm updates the most recent point $\mathbf{x}_k$ to point $\mathbf{x}_{k+1}$ as [17]

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k \quad (20)$$

where

$$\begin{aligned} \mathbf{d}_k &= -\mathbf{S}_k \nabla J_o(\mathbf{x}_k) \\ \alpha_k &= \arg \min_{\alpha} J_o(\mathbf{x}_k + \alpha \mathbf{d}_k) \\ \mathbf{S}_{k+1} &= \mathbf{S}_k + \left(1 + \frac{\gamma_k^T \mathbf{S}_k \gamma_k}{\gamma_k^T \delta_k}\right) \frac{\delta_k \delta_k^T}{\gamma_k^T \delta_k} \\ &\quad - \frac{\delta_k \gamma_k^T \mathbf{S}_k + \mathbf{S}_k \gamma_k \delta_k^T}{\gamma_k^T \delta_k} \\ \mathbf{S}_0 &= \mathbf{I}_{m^2+n^2}, \quad \delta_k = \mathbf{x}_{k+1} - \mathbf{x}_k \\ \gamma_k &= \nabla J_o(\mathbf{x}_{k+1}) - \nabla J_o(\mathbf{x}_k). \end{aligned}$$

Here, $\nabla J_o(\mathbf{x})$ is the gradient of $J_o(\mathbf{x})$ with respect to $\mathbf{x}$, and $\mathbf{S}_k$ is a positive-definite approximation of the inverse Hessian matrix of $J_o(\mathbf{x})$. This iteration process continues until

$$|J_o(\mathbf{x}_{k+1}) - J_o(\mathbf{x}_k)| < \varepsilon \quad (21)$$

where $\varepsilon > 0$ is a prescribed tolerance. If the iteration is terminated at step k , then $\mathbf{x}_k$ is viewed as a solution point.

The implementation of (20) requires the gradient of $J_o(\mathbf{x})$. Closed-form expressions for $\nabla J_o(\mathbf{x})$ are given below.

$$\begin{aligned} \frac{\partial J_o(\hat{\mathbf{T}})}{\partial t_{ij}} &= \lim_{\Delta \rightarrow 0} \frac{J_o(\hat{\mathbf{T}}_{ij}) - J_o(\hat{\mathbf{T}})}{\Delta} \\ &= 2(\beta_1 - \beta_2 + \beta_3 - \beta_4) \end{aligned} \quad (22)$$

where $\hat{\mathbf{T}}_{ij}$ is the matrix obtained from $\hat{\mathbf{T}} = \hat{\mathbf{T}}_1 \oplus \hat{\mathbf{T}}_4$ with a perturbed (i, j) th component, which is given by [18, p.655]

$$\begin{aligned} \hat{\mathbf{T}}_{ij} &= \hat{\mathbf{T}} + \frac{\Delta \hat{\mathbf{T}} \mathbf{g}_{ij} \mathbf{e}_j^T \hat{\mathbf{T}}}{1 - \Delta \mathbf{e}_j^T \hat{\mathbf{T}} \mathbf{g}_{ij}}, \quad \hat{\mathbf{T}}_{ij}^{-1} = \hat{\mathbf{T}}^{-1} - \Delta \mathbf{g}_{ij} \mathbf{e}_j^T \\ \mathbf{g}_{ij} &= \partial \left\{ \frac{\mathbf{t}_j}{\|\mathbf{t}_j\|} \right\} / \partial t_{ij} = \frac{1}{\|\mathbf{t}_j\|^3} (t_{ij} \mathbf{t}_j - \|\mathbf{t}_j\|^2 \mathbf{e}_i) \\ \beta_1 &= \mathbf{e}_j^T \hat{\mathbf{M}}_A(\hat{\mathbf{T}}) \hat{\mathbf{T}} \mathbf{g}_{ij} \\ \beta_2 &= \mathbf{e}_j^T \hat{\mathbf{T}}^{-T} \left[\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \hat{\mathbf{H}}_A(p, q) \hat{\mathbf{T}}^T \hat{\mathbf{T}} \hat{\mathbf{H}}_A^T(p, q) \right] \mathbf{g}_{ij} \\ \beta_3 &= \mathbf{e}_j^T \hat{\mathbf{T}} \hat{\mathbf{W}}_B \hat{\mathbf{T}}^T \hat{\mathbf{T}} \mathbf{g}_{ij}, \quad \beta_4 = \mathbf{e}_j^T \hat{\mathbf{T}}^{-T} \hat{\mathbf{K}}_C \mathbf{g}_{ij}. \end{aligned}$$

4. NUMERICAL EXAMPLE

Consider a 2-D stable recursive digital filter specified by $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)_{2,2}$ where

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 1.888990 & -0.912190 & -0.114079 & 0.000000 \\ 1.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.242902 & -0.226159 & 1.888990 & 0.926336 \\ -0.244143 & 0.230044 & -0.984729 & 0.000000 \end{bmatrix} \\ \mathbf{b} &= [0.023466 \quad 0.000000 \quad -0.027123 \quad 0.092453]^T \\ \mathbf{c} &= [0.269725 \quad -0.851676 \quad -0.233354 \quad 0.000000] \\ d &= 0.08900 \end{aligned}$$

Let the frequency weighted functions be given by a 2-D FIR low-pass filter with the following unit-sample response:

$$\begin{aligned} w_A(i, j) &= w_B(i, j) = w_C(i, j) \\ &= 0.256322 \exp[-0.103203\{(i-4)^2 + (j-4)^2\}] \end{aligned}$$

for $(0, 0) \leq (i, j) \leq (20, 20)$, and zero elsewhere.

Using truncated versions of (12) and (8) over $(0, 0) \leq (i, j) \leq (200, 200)$ to evaluate the Gramians $\mathbf{K}$, $\mathbf{K}_C$, $\mathbf{W}_B$ and $\mathbf{M}_A$, it was found that

$$\mathbf{K} = \begin{bmatrix} 1.000000 & 0.978030 & 0.164896 & -0.167073 \\ 0.978030 & 1.000000 & 0.132858 & -0.133867 \\ 0.164896 & 0.132858 & 1.000000 & -0.985382 \\ -0.167073 & -0.133867 & -0.985382 & 1.000000 \end{bmatrix}$$

$$\mathbf{K}_C = 10 \times$$

$$\begin{bmatrix} 3.294482 & 3.241498 & 0.217805 & -0.239120 \\ 3.241498 & 3.294482 & 0.273305 & -0.285263 \\ 0.217805 & 0.273305 & 0.434813 & -0.413683 \\ -0.239120 & -0.285263 & -0.413683 & 0.405666 \end{bmatrix}$$

$$\mathbf{W}_B = 10^3 \times$$

$$\begin{bmatrix} 0.430004 & -0.378971 & 0.215395 & 0.250372 \\ -0.378971 & 0.344251 & -0.219055 & -0.242076 \\ 0.215395 & -0.219055 & 3.258040 & 2.969501 \\ 0.250372 & -0.242076 & 2.969501 & 2.795718 \end{bmatrix}$$

$$\mathbf{M}_A = 10^5 \times$$

$$\begin{bmatrix} 0.602109 & -0.525988 & 0.717257 & 0.794037 \\ -0.525988 & 0.469122 & -0.644409 & -0.712423 \\ 0.717257 & -0.644409 & 6.220951 & 5.654101 \\ 0.794037 & -0.712423 & 5.654101 & 5.338146 \end{bmatrix}$$

From (7), the frequency-weighted l_2 -sensitivity of filter $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)_{2,2}$ was found to be

$$S = 126.9935053243 \times 10^4.$$

Choosing $\hat{\mathbf{T}} = \mathbf{I}_2 \oplus \mathbf{I}_2$ as the initial estimate and a tolerance $\varepsilon = 10^{-8}$ in (21), it took the proposed quasi-Newton algorithm 21 iterations to converge to the solution

$$\hat{\mathbf{T}}^{opt} = \begin{bmatrix} 0.904809 & 0.697047 \\ -0.173105 & 1.128976 \end{bmatrix} \oplus \begin{bmatrix} 0.839016 & 0.582244 \\ -0.399854 & 0.939719 \end{bmatrix}$$

or equivalently,

$$\mathbf{T}^{opt} = \begin{bmatrix} 1.141842 & 0.575681 \\ 1.111047 & 0.768680 \end{bmatrix} \oplus \begin{bmatrix} 0.266819 & -0.911117 \\ -0.094981 & 0.976390 \end{bmatrix}$$

From (19), the minimized frequency-weighted l_2 -sensitivity measure was given by

$$J_o(\hat{\mathbf{T}}^{opt}) = 4.0943096873 \times 10^4.$$

The profile of the frequency-weighted l_2 -sensitivity measure $J_o(\mathbf{x})$ during the first 21 iterations is shown in Fig. 1, from which it is observed that with a tolerance $\varepsilon = 10^{-8}$ the algorithm converges with 21 iterations.

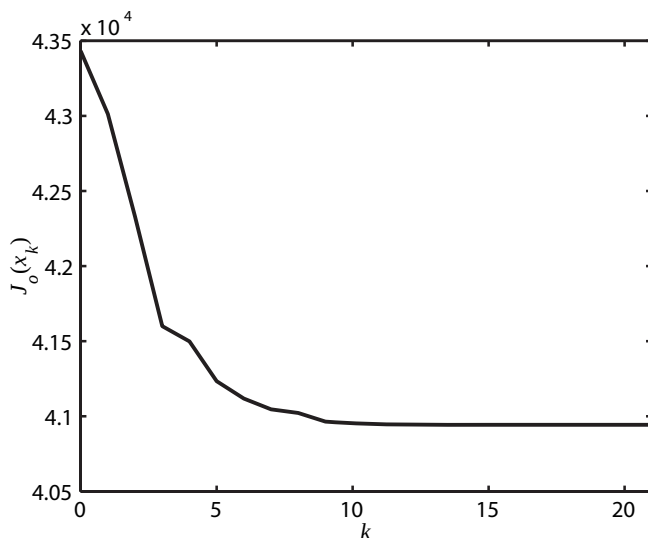


Fig. 1. Profile of $J_o(\mathbf{x})$ during the first 21 iterations.

The optimal filter structure $(\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}, d)_{2,2}$ (that minimizes (11) subject to the l_2 -scaling constraints in (14)) is synthesized by substituting matrix $\mathbf{T} = \mathbf{T}^{opt}$ into (10) as

$$\bar{\mathbf{A}} = \begin{bmatrix} 0.930710 & -0.144845 & -0.098266 & 0.335553 \\ 0.140214 & 0.958280 & 0.142034 & -0.485008 \\ 0.024963 & -0.000865 & 0.958846 & 0.115600 \\ -0.021316 & 0.037074 & -0.175824 & 0.930144 \end{bmatrix}$$

$$\bar{\mathbf{b}} = [0.075755 \quad -0.109496 \quad 0.331951 \quad 0.126980]^T$$

$$\bar{\mathbf{c}} = [-0.638268 \quad -0.499391 \quad -0.062263 \quad 0.212613].$$

5. CONCLUSION

In this paper, we have investigated the minimization problem of the frequency-weighted l_2 -sensitivity subject to l_2 -scaling constraints for 2-D state-space digital filters described by the Roesser LSS model. An efficient iterative technique has been presented to solve the problem. This technique relies on the conversion of the constrained optimization problem into an unconstrained optimization problem which is solved using an efficient quasi-Newton algorithm. The optimal state-space filter structure with minimum frequency-weighted l_2 -sensitivity and no overflow oscillations has been constructed by applying the resulting coordinate-transformation matrix. Our computer simulation results have shown the validity and effectiveness of the proposed technique.

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