Two-Dimensional State-Space Digital Filters with Minimum Frequency-Weighted l_2-Sensitivity under l_2-Scaling Constraints

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***Abstract*— The minimization problem of frequency-weighted l_2-sensitivity subject to l_2-scaling constraints is formulated for two-dimensional (2-D) state-space digital filters described by the Roesser model. It is shown that the Fornasini-Marchesini second model can be readily imbedded in the Roesser model. An iterative method is developed to solve the constrained optimization problem. This method converts the problem into an unconstrained optimization formulation by using linear-algebraic techniques and solves it by applying an efficient quasi-Newton algorithm. A case study is presented to illustrate the utility of the proposed technique.**

I. INTRODUCTION

For 2-D state-space digital filters, the l_1/l_2-mixed sensitivity minimization problem [1]-[6] and l_2-sensitivity minimization problem [6]-[10] have been investigated. In [9], it has been argued that the sensitivity measure based on a pure l_2-norm is more natural and reasonable relative to the l_1/l_2-mixed sensitivity minimization. It should be realized that solutions for *frequency-weighted* sensitivity minimization would be of practical use as these solutions allow to emphasize or de-emphasize the filter's sensitivity in certain frequency regions of interest. Synthesis procedures of the optimal (finite word-length) FWL 2-D filter structures that minimize the frequency-weighted sensitivity measure have been considered [4]-[7]. However, the minimization methods proposed in the above work do not impose constraints on the scaling of the design variables. As a result, elimination of overflow cannot be ensured. More recently, the minimization problem of l_2-sensitivity subject to l_2-scaling constraints has been explored for a class of 2-D state-space digital filters [11]. However, frequency-weighted sensitivity measure has not yet been considered in [11].

This paper investigates the minimization problem of frequency-weighted l_2-sensitivity subject to l_2-scaling constraints for 2-D state-space digital filters described by the Roesser local state-space (LSS) model [12]. Moreover, it is shown that the Roesser LSS model is more general than either the Fornasini-Marchesini (FM) second LSS model [13] or its transposed-structure model [11],[14].

II. PROBLEM FORMULATION

Consider a stable, separately locally controllable and separately locally observable LSS model for 2-D recursive digital filters which was originally proposed by Roesser [12],[15]

$$\begin{bmatrix} \boldsymbol{x}^h(i+1,j) \\ \boldsymbol{x}^v(i,j+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix} u(i,j)$$
$$y(i,j) = \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} + d\,u(i,j) \tag{1}$$

where $\boldsymbol{x}^h(i,j)$ is an $m \times 1$ horizontal state vector, $\boldsymbol{x}^v(i,j)$ is an $n \times 1$ vertical state vector, $u(i,j)$ is a scalar input, $y(i,j)$ is a scalar output, and $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{A}_3$, $\boldsymbol{A}_4$, $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, and d are real constant matrices of appropriate dimensions. The transfer function of the LSS model in (1) is given by

$$H(z_1,z_2) = \boldsymbol{c}(\boldsymbol{Z}-\boldsymbol{A})^{-1}\boldsymbol{b} + d \tag{2}$$

where $\boldsymbol{Z} = z_1\boldsymbol{I}_m \oplus z_2\boldsymbol{I}_n$ and

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 \end{bmatrix}.$$

For the sake of simplicity, the LSS model in (1) is represented hereafter by $(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c},d)_{m,n}$.

Alternatively, an LSS model for a class of 2-D recursive digital filters can be described by [11],[14]

$$\begin{bmatrix} \boldsymbol{x}(i+1,j+1) \\ y(i,j) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1' & \boldsymbol{A}_2' \\ \boldsymbol{c}_1' & \boldsymbol{c}_2' \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(i,j+1) \\ \boldsymbol{x}(i+1,j) \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}' \\ d \end{bmatrix} u(i,j) \tag{3}$$

where $\boldsymbol{x}(i,j)$ is an $N \times 1$ local state vector, $u(i,j)$ is a scalar input, $y(i,j)$ is a scalar output, and $\boldsymbol{A}_1', \boldsymbol{A}_2', \boldsymbol{b}', \boldsymbol{c}_1', \boldsymbol{c}_2'$ and d are real constant matrices of appropriate dimensions. The transfer function of the LSS model in (3) is given by

$$D(z_1,z_2) = (z_1^{-1}\boldsymbol{c}_1' + z_2^{-1}\boldsymbol{c}_2') \cdot \left(\boldsymbol{I}_n - z_1^{-1}\boldsymbol{A}_1' - z_2^{-1}\boldsymbol{A}_2'\right)^{-1}\boldsymbol{b}' + d. \tag{4}$$

If we define

$$\boldsymbol{x}^h(i,j) = \boldsymbol{x}(i,j+1), \quad \boldsymbol{x}^v(i,j) = \boldsymbol{x}(i+1,j), \tag{5}$$

the LSS model in (3) can then be imbedded in that of (1) as a special case as follows:

$$\begin{bmatrix} \boldsymbol{x}^h(i+1,j) \\ \boldsymbol{x}^v(i,j+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1' & \boldsymbol{A}_2' \\ \boldsymbol{A}_1' & \boldsymbol{A}_2' \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}' \\ \boldsymbol{b}' \end{bmatrix} u(i,j)$$
$$y(i,j) = \begin{bmatrix} \boldsymbol{c}_1' & \boldsymbol{c}_2' \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} + d\,u(i,j) \tag{6}$$

where $m = n = N$. It is noted that $D(z_1, z_2)^T$ can be viewed as a transfer function of the FM second LSS model [13]. This reveals that the LSS model of $D(z_1, z_2)^T$ can be realized by a transposed structure of that in (6). Therefore, we conclude that the LSS model in (1) is more general than either the LSS model in (3) or the FM second LSS model [13].

Definition 1: Let $\boldsymbol{X}$ be an $m \times n$ real matrix and let $f(\boldsymbol{X})$ be a scalar complex function of $\boldsymbol{X}$, differentiable with respect to all the entries of $\boldsymbol{X}$. The sensitivity function of $f(\boldsymbol{X})$ with respect to $\boldsymbol{X}$ is defined as

$$\boldsymbol{S}_{\boldsymbol{X}} = \frac{\partial f(\boldsymbol{X})}{\partial \boldsymbol{X}}, \qquad (\boldsymbol{S}_{\boldsymbol{X}})_{ij} = \frac{\partial f(\boldsymbol{X})}{\partial x_{ij}} \tag{7}$$

where x_{ij} denotes the (i,j)th entry of matrix $\boldsymbol{X}$.

Definition 2: In order to take into account the sensitivity behavior of the transfer function in a specified frequency band, or even at some discrete frequency points, the weighted sensitivity functions are defined as

$$\begin{aligned} \frac{\delta H(z_1,z_2)}{\delta \boldsymbol{A}} &= W_A(z_1,z_2)\frac{\partial H(z_1,z_2)}{\partial \boldsymbol{A}} \\ \frac{\delta H(z_1,z_2)}{\delta \boldsymbol{b}} &= W_B(z_1,z_2)\frac{\partial H(z_1,z_2)}{\partial \boldsymbol{b}} \\ \frac{\delta H(z_1,z_2)}{\delta \boldsymbol{c}^T} &= W_C(z_1,z_2)\frac{\partial H(z_1,z_2)}{\partial \boldsymbol{c}^T} \end{aligned} \tag{8}$$

where $W_A(z_1, z_2)$, $W_B(z_1, z_2)$, and $W_C(z_1, z_2)$ are scalar, stable, causal functions of the complex variables z_1 and z_2.

Definition 3: Let $\boldsymbol{X}(z_1, z_2)$ be an $m \times n$ complex matrix valued function of the complex variables z_1 and z_2. The l_2 norm of $\boldsymbol{X}(z_1, z_2)$ is defined as

$$\begin{aligned} &||\boldsymbol{X}(z_1,z_2)||_2 \\ &= \left(\mathrm{tr}\left[\frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \boldsymbol{X}(z_1,z_2)\boldsymbol{X}^*(z_1,z_2)\frac{dz_1 dz_2}{z_1 z_2} \right] \right)^{\frac{1}{2}} \end{aligned} \tag{9}$$

where $j = \sqrt{-1}$ and $\Gamma_i = \{z_i : |z_i| = 1\}$ for $i = 1, 2$.

From (2) and Definitions 1-3, the overall frequency-weighted l_2-sensitivity measure for the LSS model in (1) can be evaluated by

$$\begin{aligned} S &= \left\| \frac{\delta H(z_1,z_2)}{\delta \boldsymbol{A}} \right\|_2^2 + \left\| \frac{\delta H(z_1,z_2)}{\delta \boldsymbol{b}} \right\|_2^2 + \left\| \frac{\delta H(z_1,z_2)}{\delta \boldsymbol{c}^T} \right\|_2^2 \\ &= \left\| W_A(z_1,z_2)[\boldsymbol{F}(z_1,z_2)\boldsymbol{G}(z_1,z_2)]^T \right\|_2^2 \\ &\quad + \left\| W_B(z_1,z_2)\boldsymbol{G}^T(z_1,z_2) \right\|_2^2 + \| W_C(z_1,z_2)\boldsymbol{F}(z_1,z_2)\|_2^2 \end{aligned} \tag{10}$$

where

$$\boldsymbol{F}(z_1,z_2) = (\boldsymbol{Z}-\boldsymbol{A})^{-1}\boldsymbol{b}, \quad \boldsymbol{G}(z_1,z_2) = \boldsymbol{c}\,(\boldsymbol{Z}-\boldsymbol{A})^{-1}.$$

The frequency-weighted l_2-sensitivity measure in (10) can be written as

$$S = \mathrm{tr}[\boldsymbol{M}_A] + \mathrm{tr}[\boldsymbol{W}_B] + \mathrm{tr}[\boldsymbol{K}_C] \tag{11}$$

where $\boldsymbol{M}_A$, $\boldsymbol{W}_B$, and $\boldsymbol{K}_C$ are obtained by the following general expression:

$$\boldsymbol{X} = \frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \boldsymbol{Y}(z_1,z_2)\boldsymbol{Y}^*(z_1,z_2)\frac{dz_1 dz_2}{z_1 z_2}$$

with $\boldsymbol{Y}(z_1, z_2) = W_A(z_1, z_2)[\boldsymbol{F}(z_1, z_2)\boldsymbol{G}(z_1, z_2)]^T$ for $\boldsymbol{X} = \boldsymbol{M}_A$, $\boldsymbol{Y}(z_1, z_2) = W_B^*(z_1, z_2)\boldsymbol{G}^*(z_1, z_2)$ for $\boldsymbol{X} = \boldsymbol{W}_B$, and $\boldsymbol{Y}(z_1, z_2) = W_C(z_1, z_2)\boldsymbol{F}(z_1, z_2)$ for $\boldsymbol{X} = \boldsymbol{K}_C$.

Define a state-space coordinate transformation by [12],[15]

$$\begin{bmatrix} \overline{\boldsymbol{x}}^h(i,j) \\ \overline{\boldsymbol{x}}^v(i,j) \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_1^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}_4^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} \tag{12}$$

where $\boldsymbol{T}_1$ and $\boldsymbol{T}_4$ are $m \times m$ and $n \times n$ nonsingular matrices, respectively. New realizations can then be characterized as $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{b}}, \overline{\boldsymbol{c}}, d)_{m,n}$ with

$$\overline{\boldsymbol{A}} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}, \quad \overline{\boldsymbol{b}} = \boldsymbol{T}^{-1}\boldsymbol{b}, \quad \overline{\boldsymbol{c}} = \boldsymbol{c}\boldsymbol{T} \tag{13}$$

where $\boldsymbol{T} = \boldsymbol{T}_1 \oplus \boldsymbol{T}_4$. For the new realizations, the frequency-weighted l_2-sensitivity measure in (11) is changed to

$$\begin{aligned} S(\boldsymbol{P}) &= \mathrm{tr}[\boldsymbol{M}_A(\boldsymbol{P})\boldsymbol{P}] + \mathrm{tr}[\boldsymbol{W}_B\boldsymbol{P}] + \mathrm{tr}[\boldsymbol{K}_C\boldsymbol{P}^{-1}] \\ &= \mathrm{tr}[\boldsymbol{N}_A(\boldsymbol{P})\boldsymbol{P}^{-1}] + \mathrm{tr}[\boldsymbol{W}_B\boldsymbol{P}] + \mathrm{tr}[\boldsymbol{K}_C\boldsymbol{P}^{-1}] \end{aligned} \tag{14}$$

where $\boldsymbol{P} = \boldsymbol{T}\boldsymbol{T}^T = \boldsymbol{P}_1 \oplus \boldsymbol{P}_4$ and

$$\boldsymbol{M}_A(\boldsymbol{P}) = \frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \boldsymbol{Y}(z_1,z_2)\boldsymbol{P}^{-1}\boldsymbol{Y}^*(z_1,z_2)\frac{dz_1 dz_2}{z_1 z_2}$$
$$\boldsymbol{N}_A(\boldsymbol{P}) = \frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \boldsymbol{Y}^*(z_1,z_2)\boldsymbol{P}\boldsymbol{Y}(z_1,z_2)\frac{dz_1 dz_2}{z_1 z_2}$$

with $\boldsymbol{Y}(z_1, z_2) = W_A(z_1, z_2)[\boldsymbol{F}(z_1, z_2)\boldsymbol{G}(z_1, z_2)]^T$.

If l_2-scaling constraints are imposed on the horizontal and vertical state vectors $\overline{\boldsymbol{x}}^h(i,j)$ and $\overline{\boldsymbol{x}}^v(i,j)$, we require that [16]

$$\begin{aligned} (\overline{\boldsymbol{K}}_1)_{\xi\xi} &= (\boldsymbol{T}_1^{-1}\boldsymbol{K}_1\boldsymbol{T}_1^{-T})_{\xi\xi} = 1 \quad \text{for} \quad \xi = 1, 2, \cdots, m \\ (\overline{\boldsymbol{K}}_4)_{\zeta\zeta} &= (\boldsymbol{T}_4^{-1}\boldsymbol{K}_4\boldsymbol{T}_4^{-T})_{\zeta\zeta} = 1 \quad \text{for} \quad \zeta = 1, 2, \cdots, n \end{aligned} \tag{15}$$

where

$$\begin{aligned} \boldsymbol{K} &= \frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \boldsymbol{F}(z_1,z_2)\boldsymbol{F}^*(z_1,z_2)\frac{dz_1 dz_2}{z_1 z_2} \\ &= \begin{bmatrix} \boldsymbol{K}_1 & \boldsymbol{K}_2 \\ \boldsymbol{K}_3 & \boldsymbol{K}_4 \end{bmatrix} \end{aligned}$$

is the local controllability Gramian for the LSS model in (1) with an $m \times m$ submatrix $\boldsymbol{K}_1$ and an $n \times n$ submatrix $\boldsymbol{K}_4$ along its diagonal [15].

Thus, the l_2-scaling constrained frequency-weighted l_2-sensitivity minimization problem can be formulated as follows: *Given matrices $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, obtain a block-diagonal* *nonsingular matrix $\boldsymbol{T} = \boldsymbol{T}_1 \oplus \boldsymbol{T}_4$ which minimizes $S(\boldsymbol{P})$ in (14) subject to l_2-scaling constraints in (15).*

III. PROBLEM SOLUTION

By defining

$$\hat{\boldsymbol{T}} = \hat{\boldsymbol{T}}_1 \oplus \hat{\boldsymbol{T}}_4 = (\boldsymbol{T}_1 \oplus \boldsymbol{T}_4)^T (\boldsymbol{K}_1 \oplus \boldsymbol{K}_4)^{-\frac{1}{2}}, \qquad (16)$$

it follows that

$$\overline{\boldsymbol{K}} = \hat{\boldsymbol{T}}^{-T} \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{K}_1^{-\frac{1}{2}} \boldsymbol{K}_2 \boldsymbol{K}_4^{-\frac{1}{2}} \\ \boldsymbol{K}_4^{-\frac{1}{2}} \boldsymbol{K}_3 \boldsymbol{K}_1^{-\frac{1}{2}} & \boldsymbol{I}_n \end{bmatrix} \hat{\boldsymbol{T}}^{-1}. \quad (17)$$

Thus, the l_2-scaling constraints in (15) can be written as

$$\begin{aligned} (\hat{\boldsymbol{T}}_1^{-T} \hat{\boldsymbol{T}}_1^{-1})_{\xi\xi} &= 1, \quad \xi = 1, 2, \cdots, m \\ (\hat{\boldsymbol{T}}_4^{-T} \hat{\boldsymbol{T}}_4^{-1})_{\zeta\zeta} &= 1, \quad \zeta = 1, 2, \cdots, n. \end{aligned} \qquad (18)$$

It is obvious that the conditions in (18) are always satisfied by choosing $\hat{\boldsymbol{T}}_1^{-1}$ and $\hat{\boldsymbol{T}}_4^{-1}$ as

$$\begin{aligned} \hat{\boldsymbol{T}}_1^{-1} &= \left[\frac{\boldsymbol{t}_{11}}{||\boldsymbol{t}_{11}||}, \frac{\boldsymbol{t}_{12}}{||\boldsymbol{t}_{12}||}, \cdots, \frac{\boldsymbol{t}_{1m}}{||\boldsymbol{t}_{1m}||} \right] \\ \hat{\boldsymbol{T}}_4^{-1} &= \left[\frac{\boldsymbol{t}_{41}}{||\boldsymbol{t}_{41}||}, \frac{\boldsymbol{t}_{42}}{||\boldsymbol{t}_{42}||}, \cdots, \frac{\boldsymbol{t}_{4n}}{||\boldsymbol{t}_{4n}||} \right]. \end{aligned} \qquad (19)$$

Substituting matrix $\boldsymbol{T} = \boldsymbol{T}_1 \oplus \boldsymbol{T}_4$ which satisfies (16) into $S(\boldsymbol{P})$ in (14), the frequency-weighted l_2-sensitivity measure can be expressed as

$$J_o(\boldsymbol{x}) = \mathrm{tr}[\hat{\boldsymbol{T}} \hat{\boldsymbol{M}}_A(\hat{\boldsymbol{P}}) \hat{\boldsymbol{T}}^T] + \mathrm{tr}[\hat{\boldsymbol{T}} \hat{\boldsymbol{W}}_B \hat{\boldsymbol{T}}^T] + \mathrm{tr}[\hat{\boldsymbol{T}}^{-T} \hat{\boldsymbol{K}}_C \hat{\boldsymbol{T}}^{-1}] \qquad (20)$$

where $\hat{\boldsymbol{P}} = \hat{\boldsymbol{T}}^T \hat{\boldsymbol{T}}$ and

$$\boldsymbol{x} = (\boldsymbol{t}_{11}^T, \boldsymbol{t}_{12}^T, \cdots, \boldsymbol{t}_{1m}^T, \boldsymbol{t}_{41}^T, \boldsymbol{t}_{42}^T, \cdots, \boldsymbol{t}_{4n}^T)^T$$

$$\hat{\boldsymbol{M}}_A(\hat{\boldsymbol{P}}) = \frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} \hat{\boldsymbol{Y}}(z_1, z_2) \hat{\boldsymbol{P}}^{-1} \hat{\boldsymbol{Y}}^*(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}$$

with

$$\begin{aligned} \hat{\boldsymbol{Y}}(z_1, z_2) &= (\boldsymbol{K}_1 \oplus \boldsymbol{K}_4)^{\frac{1}{2}} \boldsymbol{Y}(z_1, z_2) (\boldsymbol{K}_1 \oplus \boldsymbol{K}_4)^{-\frac{1}{2}} \\ \boldsymbol{Y}(z_1, z_2) &= W_A(z_1, z_2) [\boldsymbol{F}(z_1, z_2) \boldsymbol{G}(z_1, z_2)]^T \\ \hat{\boldsymbol{W}}_B &= (\boldsymbol{K}_1 \oplus \boldsymbol{K}_4)^{\frac{1}{2}} \boldsymbol{W}_B (\boldsymbol{K}_1 \oplus \boldsymbol{K}_4)^{\frac{1}{2}} \\ \hat{\boldsymbol{K}}_C &= (\boldsymbol{K}_1 \oplus \boldsymbol{K}_4)^{-\frac{1}{2}} \boldsymbol{K}_C (\boldsymbol{K}_1 \oplus \boldsymbol{K}_4)^{-\frac{1}{2}}. \end{aligned}$$

This means that the problem of obtaining an $(m+n) \times (m+n)$ block-diagonal nonsingular matrix $\boldsymbol{T} = \boldsymbol{T}_1 \oplus \boldsymbol{T}_4$ which minimizes $S(\boldsymbol{P})$ in (14) subject to the l_2-scaling constraints in (15) can be converted into an unconstrained optimization problem of obtaining an $(m^2 + n^2) \times 1$ vector $\boldsymbol{x}$ which minimizes $J_o(\boldsymbol{x})$ in (20).

By applying a quasi-Newton algorithm to minimize $J_o(\boldsymbol{x})$ in (20), in the kth iteration the most recent point $\boldsymbol{x}_k$ is updated to point $\boldsymbol{x}_{k+1}$ as [17]

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha_k \boldsymbol{d}_k \qquad (21)$$

where

$$\begin{aligned} \boldsymbol{d}_k &= -\boldsymbol{S}_k \nabla J_o(\boldsymbol{x}_k), \qquad \alpha_k = arg \min_{\alpha} \; J_o(\boldsymbol{x}_k + \alpha \boldsymbol{d}_k) \\ \boldsymbol{S}_{k+1} &= \boldsymbol{S}_k + \left(1 + \frac{\boldsymbol{\gamma}_k^T \boldsymbol{S}_k \boldsymbol{\gamma}_k}{\boldsymbol{\gamma}_k^T \boldsymbol{\delta}_k}\right) \frac{\boldsymbol{\delta}_k \boldsymbol{\delta}_k^T}{\boldsymbol{\gamma}_k^T \boldsymbol{\delta}_k} - \frac{\boldsymbol{\delta}_k \boldsymbol{\gamma}_k^T \boldsymbol{S}_k + \boldsymbol{S}_k \boldsymbol{\gamma}_k \boldsymbol{\delta}_k^T}{\boldsymbol{\gamma}_k^T \boldsymbol{\delta}_k} \\ \boldsymbol{S}_0 &= \boldsymbol{I}_{m^2+n^2}, \qquad \boldsymbol{\delta}_k = \boldsymbol{x}_{k+1} - \boldsymbol{x}_k \\ \boldsymbol{\gamma}_k &= \nabla J_o(\boldsymbol{x}_{k+1}) - \nabla J_o(\boldsymbol{x}_k). \end{aligned}$$

Here, $\nabla J_o(\boldsymbol{x})$ is the gradient of $J_o(\boldsymbol{x})$ with respect to $\boldsymbol{x}$, and $\boldsymbol{S}_k$ is a positive-definite approximation of the inverse Hessian matrix of $J_o(\boldsymbol{x})$. The algorithm starts with a trivial initial point $\boldsymbol{x}_0$ obtained from an initial assignment $\hat{\boldsymbol{T}} = \boldsymbol{I}_{m+n}$, and this iteration process continues until

$$|J_o(\boldsymbol{x}_{k+1}) - J_o(\boldsymbol{x}_k)| < \varepsilon \qquad (22)$$

where $\varepsilon > 0$ is a prescribed tolerance.

IV. A NUMERICAL EXAMPLE

Consider a 2-D stable recursive digital filter realization $(\boldsymbol{A}^o, \boldsymbol{b}^o, \boldsymbol{c}^o, d)_{2,2}$ where

$$\boldsymbol{A}^o = \begin{bmatrix} \boldsymbol{A}_1^o & \boldsymbol{A}_2^o \\ \boldsymbol{A}_1^o & \boldsymbol{A}_2^o \end{bmatrix}, \quad \boldsymbol{b}^o = \begin{bmatrix} \boldsymbol{b}_1^o \\ \boldsymbol{b}_2^o \end{bmatrix}, \quad \boldsymbol{c}^o = \begin{bmatrix} \boldsymbol{c}_1^o & \boldsymbol{c}_2^o \end{bmatrix}$$

with

$$\boldsymbol{A}_1^o = \begin{bmatrix} 0.0 & 0.481228 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.510378 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.525287 \\ -0.031857 & 0.298663 & -0.808282 & 1.044600 \end{bmatrix}$$

$$\boldsymbol{A}_2^o = \begin{bmatrix} -0.226080 & 0.776837 & 0.024693 & -0.000933 \\ -0.843550 & 1.610400 & -0.309366 & 0.065898 \\ -1.260339 & 2.005100 & -0.453220 & 0.203118 \\ -1.121498 & 1.636435 & -0.590516 & 0.562890 \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{b}_1^o &= \boldsymbol{b}_2^o = [\, 0.0 \quad 0.0 \quad 0.0 \quad 0.198473]^T \\ \boldsymbol{c}_1^o &= [\, -0.567054 \quad 0.231913 \quad 0.197016 \quad 0.239932] \\ \boldsymbol{c}_2^o &= [\, 0.464344 \quad 0.441837 \quad -0.061100 \quad 0.105505] \\ d &= 0.009430. \end{aligned}$$

This 2-D filter was obtained by imbedding the LSS model of Example 2 in [11] into the Roesser LSS model. The frequency-weighted l_2-sensitivity of the LSS model in (1) is obtained by carrying out the l_2-scaling for the above realization with a diagonal coordinate matrix

$$\begin{aligned} \boldsymbol{T}^o = \mathrm{diag}\{&1.000001, 1.000002, 1.000003, 1.000003, \\ &1.000001, 1.000002, 1.000003, 1.000003\} \end{aligned}$$

and using frequency-weighted functions given by 2-D FIR digital low-pass filters with the unit-sample response [18]

$$\begin{aligned} w_A(i,j) &= w_B(i,j) = w_C(i,j) \\ &= 0.256322 \exp[-0.103203\{(i-4)^2 + (j-4)^2\}] \end{aligned}$$

for $(0,0) \leq (i,j) \leq (20,20)$, and zero elsewhere. The above frequency-weighted functions were selected to emphasize the filter's sensitivity in the passband and de-emphasize it in the stopband. The frequency-weighted l_2-sensitivity of the LSS model in (1) $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)_{2,2}$ was found to be

$$S = 394423.679690.$$

By choosing $\hat{\boldsymbol{T}} = \boldsymbol{I}_2 \oplus \boldsymbol{I}_2$ (therefore $\boldsymbol{T} = (\boldsymbol{K}_1 \oplus \boldsymbol{K}_4)^{1/2}$ in (16)) as an initial estimate and a tolerance $\varepsilon = 10^{-8}$ in (22), the quasi-Newton algorithm took 54 iterations to converge to

$$\hat{\boldsymbol{T}}^{opt} = \begin{bmatrix} 3.056671 & -2.673365 & 0.575882 & -0.429287 \\ -0.331629 & 2.142411 & -0.401503 & -0.192081 \\ -2.530651 & 0.932586 & 0.553002 & -0.136935 \\ 1.754363 & -0.312582 & 0.624509 & 0.515370 \end{bmatrix}$$
$$\oplus \begin{bmatrix} 1.307170 & -0.419919 & 0.045538 & -0.194118 \\ 0.762443 & 0.830435 & -0.297531 & 0.062104 \\ -0.405202 & 0.189220 & 0.976564 & -0.250656 \\ 1.071478 & -0.069804 & 0.315533 & 0.828727 \end{bmatrix}$$

or equivalently,

$$\boldsymbol{T}^{opt} = \begin{bmatrix} 0.690639 & 0.697890 & -0.986414 & 1.417930 \\ 0.205523 & 0.802226 & -0.589043 & 1.251725 \\ 0.091157 & 0.584768 & -0.335877 & 1.196490 \\ -0.023116 & 0.340604 & -0.305900 & 1.128518 \end{bmatrix}$$
$$\oplus \begin{bmatrix} 0.595272 & 0.843931 & 0.159783 & 1.068904 \\ 0.406418 & 0.767684 & 0.282059 & 0.990157 \\ 0.270942 & 0.580437 & 0.379363 & 1.000879 \\ 0.148987 & 0.449337 & 0.230545 & 1.074708 \end{bmatrix}.$$

The minimized frequency-weighted l_2-sensitivity was found to be

$$J_o(\hat{\mathbf{T}}^{opt}) = 4670.176797.$$

The profile of the l_2-sensitivity measure $J_o(\hat{\boldsymbol{T}})$ during the first 54 iterations is shown in Fig. 1, from which it is seen that with a tolerance $\varepsilon = 10^{-8}$ the algorithm converges with 54 iterations.

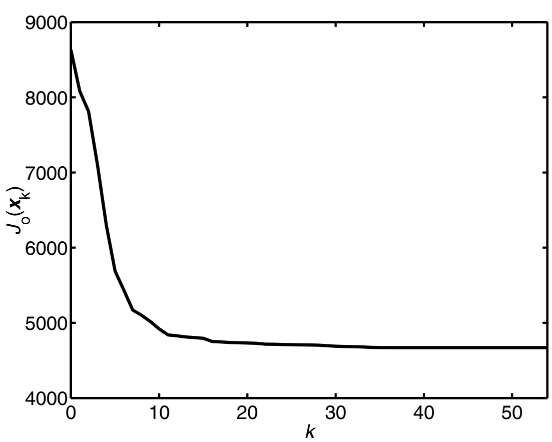

Fig. 1. Profile of $J_o(\hat{\boldsymbol{T}})$ during the first 54 iterations.

V. CONCLUSION

The minimization problem of the frequency-weighted l_2-sensitivity subject to l_2-scaling constraints for 2-D state-space digital filters described by the Roesser LSS model have been investigated. It has been shown that the FM second LSS model can be imbedded in the Roesser LSS model as a special case. An iterative algorithm has been developed to solve the problem. This algorithm relies on the conversion of the constrained optimization problem into an unconstrained optimization formulation and utilizes an efficient quasi-Newton algorithm. Our computer simulation results have demonstrated the validity and effectiveness of the proposed technique.

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