Direct Design of Orthogonal Filter Banks and Wavelets

Wu-Sheng Lu
Dept. of Electrical and Computer Engineering
University of Victoria
Victoria, BC, Canada V8W 3P6
Email: wslu@ece.uvic.ca

Takao Hinamoto
Graduate School of Engineering
Hiroshima University
Higashi-Hiroshima, 739-8527, Japan
Email: hinamoto@hiroshima-u.ac.jp

***Abstract*—This paper presents a new method for the design of two-channel conjugate quadrature (CQ) filter banks in which halfband filter and spectrum factorization are not required. Instead, a CQ filter is directly optimized subject to the perfect reconstruction and possibly other constraints (such as number of vanishing moments (VM)). We develop a design strategy in that the solution is approached sequentially with each update confined to within a small vicinity of the current iterate where the problem at hand behaves like a convex one, thus the update can be obtained as a solution of a *convex* problem. Four design scenarios are considered, namely the least squares designs with or without VM requirement, and equiripple designs with or without VM requirement. The simulation studies demonstrate that the proposed method is reliable to design high-order CQ filters with improved performance.**

I. Introduction

Two-channel conjugate quadrature (CQ) filter banks, also known as power-symmetric filter banks [1], are among the most popular building blocks for multirate systems and wavelet-based coding systems as they offer perfect reconstruction (PR) and other desirable properties. Several algorithms for the design of CQ filters have been proposed since 1980's, see e.g. references [1]-[6] and the relevant work cited therein. In addition, there are design techniques aimed at FIR compaction filters, that are also applicable to CQ filters [7], [8]. The dominating design technique in the literature is an *indirect* methodology in which the design is accomplished in two steps: construct a halfband filter (subject to certain nonnegativity constraint) followed by spectrum factorization of the halfband filter. In this paper, this design paradigm is altered with a *direct* method in which halfband filter and subsequent factorization are not required. Instead, a CQ filter is optimized directly subject to the PR and possibly other constraints (such as number of vanishing moments (VM)). The design so formulated turns out to be a *nonconvex* problem. We develop a design strategy in that the solution is approached sequentially with each update confined to within a small vicinity of the current iterate where the problem at hand behaves like a convex one, thus the update can be obtained as a solution of a *convex* problem. Four design scenarios are considered, namely the least squares designs with or without VM requirement, and equiripple designs with or without VM requirement. We show that these designs fit quite nicely into a convex QP framework, thus a single easy-to-access solver such as SeDuMi [9] can handle them all. The simulation studies demonstrate that the proposed method is reliable to design high-order CQ filters with improved performance.

II. Notation and Background

A two-channel causal FIR CQ filter bank is characterized by a pair of analysis filters H_0, H_1 and a pair of synthesis filters G_0 and G_1 as shown in Fig. 1, where the four filters are related by [2, p. 436]

$$\begin{aligned} H_1(z) &= -z^{-(N-1)}H_0(-z^{-1}) \\ G_0(z) &= H_1(-z) \\ G_1(z) &= -H_0(-z) \end{aligned} \qquad (1)$$

where $H_0(z) = \sum_{n=0}^{N-1} h_n z^{-n}$ is a lowpass FIR transfer function of length-N with N even. With (1), the aliasing is removed, and the PR is achieved if $H_0(z)$ satisfies

$$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2 \qquad (2)$$

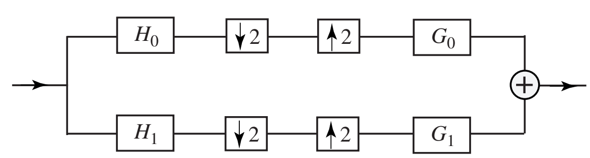

Fig. 1. A 2-channel CQ filter bank.

III. Least Squares Designs

By substituting $H_0(z) = \sum_{n=0}^{N-1} h_n z^{-n}$ into (2), we obtain a set of $N/2$ equality constraints in terms of impulse response $\{h_i,\ i = 0, 1, \ldots, N-1\}$ as

$$\sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta_m \quad \text{for } m = 0, 1, \ldots, (N-2)/2 \qquad (3)$$

where δ_m is the Dirac sequence with $\delta_0 = 1$ and $\delta_m = 0$ for nonzero m. Eq. (3) is known as the double shift orthogonality in the wavelet literature that explicitly characterizes the PR condition (2) in terms of the impulse response of $H_0(z)$. In addition to the PR conditions, CQ filters may be required to meet other constraints such as possessing certain number of VMs for constructing wavelets [3]-[6]. It is known that the number of VMs of a CQ filter bank is equal to the number of zeros of $H_0(z)$ at $\omega = \pi$. Because

$$\left.\frac{d^l H_0(e^{j\omega})}{d\omega^l}\right|_{\omega=\pi} = (-j)^l \sum_{n=0}^{N-1} (-1)^n \cdot n^l \cdot h_n$$

a CQ filter has L vanishing moments if

$$\sum_{n=0}^{N-1} (-1)^n \cdot n^l \cdot h_n = 0 \quad \text{for } l = 0, 1, \ldots L-1 \tag{4}$$

Thus a least squares (LS) design of CQ lowpass filter $H_0(z)$ having L VMs can be expressed as

$$\begin{array}{ll} \text{minimize} & \int_{\omega_a}^{\pi} \left| H_0(e^{j\omega}) \right|^2 d\omega \\ \text{subject to: constraints (3) and (4)} & \end{array} \tag{5}$$

where ω_a is the normalized stopband edge of $H_0(z)$. It can readily verified that the LS objective function in (5) is the quadratic form $h^T Q h$ where $h = \left[\begin{array}{cccc} h_0 & h_1 & \cdots & h_{N-1} \end{array} \right]^T$, and Q is a symmetric positive definite Toeplitz matrix characterized by its first row $\left[\begin{array}{cccc} \pi - \omega_a & -\sin\omega_a & \cdots & -\sin(N-1)\omega_a/(N-1) \end{array} \right]$, hence the objective function is globally strictly convex. Because of the constraints in (3), however, the feasible region is nonconvex and hence (5) is a nonconvex problem. To deal with (3), we develop a design strategy in that the solution is approached sequentially, where each update is limited to within a local region so that the problem at hand is approximately convex and the update can be obtained as a solution of a convex problem. Suppose we are in the kth iteration to update the coefficient vector from $h^{(k)}$ to $h^{(k+1)} = h^{(k)} + d$ to achieve two things: to reduce the filter's stopband energy and to better satisfy the constraints in (3). If we write (3) at $h^{(k+1)}$ as

$$\sum_n h_n^{(k)} d_{n+2m} + \sum_n d_n h_{n+2m}^{(k)} + \sum_n d_n d_{n+2m} = \delta_m \tag{6}$$

If the updating vector d is limited to be small in magnitude, then the last term on the right side of (6) is of second-order smallness and hence can be neglected. This yields

$$\sum_n h_n^{(k)} d_{n+2m} + \sum_n d_n h_{n+2m}^{(k)} = \delta_m - \sum_n h_n^{(k)} h_{n+2m}^{(k)} \equiv u_m^{(k)} \tag{7}$$

for m = 0, 1, … (N – 2)/2. The N/2 linear equations in (7) can be expressed as

$$C^{(k)} d = u^{(k)} \tag{8}$$

where $u^{(k)}$ is a vector whose mth component is $u_m^{(k)}$, and $C^{(k)}$ is an N/2 × N matrix defined by the right side of (7) in terms of $h^{(k)}$. Concerning the constraints on VMs at $h^{(k+1)}$, (4) gives

$$D h^{(k+1)} = D\left(h^{(k)} + d\right) = 0$$

i.e.,

$$D\, d = -D\, h^{(k)} \equiv v^{(k)} \tag{9}$$

where D is an $L \times N$ matrix whose lth row is generated as the (l – 1)th power of vector 0: 1: (N – 1), then with alternate sign change. The smallness of updating vector d is controlled by imposing a small upper bound β for the magnitude of each component, i.e.,$|d_i| \le \beta$ for i = 1, 2, …, N which can be put together as

$$A\, d \le b \tag{10}$$

where $A = \left[\begin{array}{c} I_N \\ -I_N \end{array} \right]$, $\quad b = \beta \cdot \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array} \right]^T$. With $h^{(k+1)} = h^{(k)} + d$ the objective function becomes a quadratic function of d and the kth iteration of LS problem (5) now assumes the form

$$\begin{array}{ll} \text{minimize} & d^T Q d + 2 d^T q^{(k)} + \kappa \\ \text{subject to:} & A\, d \le b \\ & \left[\begin{array}{c} C^{(k)} \\ D \end{array} \right] d = \left[\begin{array}{c} u^{(k)} \\ v^{(k)} \end{array} \right] \end{array} \tag{11}$$

with $q^{(k)} = Q h^{(k)}$ and κ a constant. Problem (11) has taken the VM requirement (5) into account. If VM is not a part of the design specifications, then (11) is modified by simply replacing the equality constraint with (8). The two design scenarios are unified by modifying (11) to

$$\begin{array}{ll} \text{minimize} & d^T Q d + 2 d^T q^{(k)} + \kappa \\ \text{subject to:} & A\, d \le b \\ & F^{(k)} d = w^{(k)} \end{array} \tag{12}$$

where

$$F^{(k)} = \begin{cases} \left[\begin{array}{c} C^{(k)} \\ D \end{array} \right] & \text{with VM} \\ C^{(k)} & \text{without VM} \end{cases} \tag{13a}$$

and

$$w^{(k)} = \begin{cases} \left[\begin{array}{c} u^{(k)} \\ v^{(k)} \end{array} \right] & \text{with VM} \\ u^{(k)} & \text{without VM} \end{cases} \tag{13b}$$

Depending on whether the VM requirement is included or not, there are N/2 + L or N/2 linear equality constraints in (12). These constraints can be eliminated from the QP problem by expressing all solutions of $F^{(k)} d = w^{(k)}$ via the singular value decomposition (SVD) of $F^{(k)} = U^{(k)} \Sigma^{(k)} V^{(k)T}$ as [10]

$$d = V_r^{(k)} x + d_s \tag{14}$$

where x is an (N – r)-dimensional free vector with r being the rank of $F^{(k)}$ so r = N/2 + L or N/2 depending on whether VM is required, d_s is any solution of $F^{(k)} d = w^{(k)}$ (e.g. $d_s = \left(F^{(k)}\right)^{\dagger} w^{(k)}$, with † denoting pseudo-inverse), and $V_r^{(k)}$ is the matrix consisting of the last N – r columns of $V^{(k)}$. With (14), problem (12) is reduced to

$$\begin{array}{ll} \text{minimize} & x^T Q^{(k)} x + 2 x^T p^{(k)} \\ \text{subject to:} & A^{(k)} x \le \hat{b} \end{array} \tag{15}$$

where

$$\begin{array}{ll} Q^{(k)} = V_r^{(k)T} Q V_r^{(k)}, & p^{(k)} = V_r^{(k)T} \left(q^{(k)} + Q\, d_s \right) \\ A^{(k)} = A V_r^{(k)}, & \hat{b} = b - A d_s \end{array}$$

We note that (15) can be solved more efficiently relative to (12) because the number of design variables is now reduced to N/2 – L and also (15) involves less number of constraints. Note that the reduction in problem size is achieved at the cost of performing the SVD of matrix $F^{(k)}$ in each iteration. Problem (15) is a convex QP problem for which many efficient algorithms [10] and reliable software [9], [11] exist. Because $Q^{(k)}$ is positive definite, the solution, $x^{(k)}$, of (15) is a unique and globally optimal. By constructing $d^{(k)} = V_r^{(k)} x^{(k)} + d_s$ (see (14)) and then $h^{(k+1)} = h^{(k)} + d^{(k)}$, the kth iteration is complete. The algorithm continues until the magnitude of the updating vector $d^{(k)}$ is less than a prescribed tolerance and $h^{(k+1)}$ in this case is claimed to be the impulse response of $H_0(z)$. Filters $H_1(z)$, $G_0(z)$, and $G_1(z)$ are then constructed using (1).

IV. Minimax Designs

This section addresses the design of equiripple CQ filters. To this end we consider the minimization of the instantaneous power of lowpass filter $H_0(z)$ over its stopband subject to PR and VM constraints, namely,

$$\begin{array}{c} \underset{h}{\text{minimize}} \ \underset{\omega_a \le \omega \le \pi}{\text{maximize}} \ \left|H_0(e^{j\omega})\right| \\ \text{subject to: constraints (3) and (4)} \end{array} \quad (16)$$

By defining $c(\omega) = \left[\begin{array}{cccc} 1 & \cos\omega & \cdots & \cos(N-1)\omega \end{array}\right]^T$ and $s(\omega) = \left[\begin{array}{cccc} 0 & \sin\omega & \cdots & \sin(N-1)\omega \end{array}\right]^T$, we can write

$$\begin{aligned} \left|H_0(e^{j\omega})\right| &= \sqrt{\left(h^T c(\omega)\right)^2 + \left(h^T s(\omega)\right)^2} = \left\| \left[\begin{array}{c} c(\omega)^T \\ s(\omega)^T \end{array}\right] \cdot h \right\| \\ &\equiv \|T(\omega) \cdot h\| \end{aligned}$$

which in conjunction with the analysis in Sec. 3 leads (16) to

$$\begin{array}{c} \underset{x}{\text{minimize}} \ \underset{\omega_a \le \omega \le \pi}{\text{maximize}} \ \left\| T(\omega) \left(h^{(k)} + V_r^{(k)} x + d_s\right) \right\| \\ \text{subject to: } A^{(k)} x \le \hat{b} \end{array} \quad (17)$$

By introducing an upper bound η (as an auxiliary variable) for the objective function in (17) over frequency grids $\Omega = \{\omega_a = \omega_1, \omega_2, \cdots, \omega_K = \pi\}$ in the stopband, (17) can be reformulated as

$$\begin{array}{ll} \text{minimize} & \eta \\ \text{subject to:} & \left\| T^{(k)}(\omega) x + e^{(k)}(\omega) \right\| \le \eta \quad \text{for } \omega \in \Omega \\ & A^{(k)} x \le \hat{b} \end{array} \quad (18)$$

where $T^{(k)}(\omega) = T(\omega) V_r^{(k)}$, $e^{(k)}(\omega) = T(\omega)\left(h^{(k)} + d_s\right)$. With regard to variable $\{\eta, x\}$, (18) is a second-order cone programming problem (SOCP) [10] – a class of convex quadratic problems for which efficient solvers such as SeDuMi [9] exist. Note that the number of variables in (18) remains as low as $N/2 - L + 1$, but there are now K second-order cone constraints in addition to $2N$ linear constraints. With properly defined $\{F^{(k)}, w^{(k)}\}$ using (13), $V_r^{(k)}$ and d_s in (14) are computed and the unique and global solution of problem (18), $x^{(k)}$, can be obtained. By constructing $d^{(k)} = V_r^{(k)} x^{(k)} + d_s$ and then $h^{(k+1)} = h^{(k)} + d^{(k)}$, and the kth iteration is complete. Like in the LS designs, the algorithm continues until the magnitude of the updating vector $d^{(k)}$ is less than a prescribed tolerance and $h^{(k+1)}$ is claimed to be the impulse response of $H_0(z)$.

V. Design Examples

As the first set of examples, the algorithm in Sec. 4 was utilized for the refinement of the three CQ filters of lengths 8, 16, 32 reported in [2]. Each of these filters was used as initial point for the algorithm. The number of frequency grids K and convergence tolerance ε were set to $\{K = 25,\ \varepsilon = 10^{-14}\}$, $\{K = 28,\ \varepsilon = 10^{-14}\}$, and $\{K = 50,\ \varepsilon = 10^{-17}\}$, respectively, and the value of η in (18) was set to $\eta = 2\sqrt{N} \times 10^{-3}$ where N is the filter length. It took the algorithm 7, 12, and 23 iterations to converge, respectively. The coefficients of $H_0(z)$ with $N = 32$ are shown in Table 1, where for comparison purposes the filter coefficients reported in [2] are also listed. The stopband edge as well as minimum attenuation (set to 40 dB) of each refined CQ filter were identical to that of the filter of same length from [2], and the performance of the filters are evaluated in terms of minimum stopband attenuation and satisfaction of the PR condition. The satisfaction of the PR condition was measured by the largest magnitude error among all $N/2$ equations in (3). Because all filters involved satisfy 40 dB minimum stopband attenuation, the evaluation results in Table 2 only list the largest equation errors of the three pairs of designs. Figure 2 depicts the magnitude responses of the CQ filters H_0 and H_1 of length 32 from [2] (dashed lines) and their refined versions (solid lines). It is observed that the magnitude responses are practically identical, but the refined filter offers considerably improved satisfaction of the PR condition (see Table 2).

As the second set of examples, the algorithms in Secs. 3 and 4 were applied to accomplish six LS designs and six minimax designs with length $N = 96$, normalized stopband edge $\omega_a = 0.56\pi$, and VM number $L = 0, 1, \ldots, 5$. The design results in terms of stopband energy (for LS designs), peak instantaneous power (for minimax designs), and largest equation error for PS conditions in (3) for various VMs are shown in Tables 3 and 4.

As a representative design, Figure 3 depicts the magnitude response of the equiripple CQ filters H_0 and H_1 of length $N = 96$ and $L = 3$.

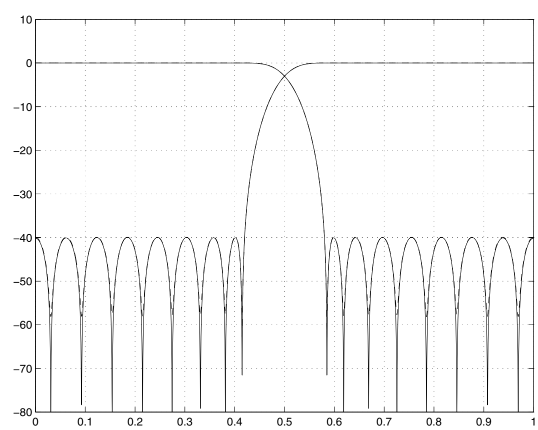

Fig. 2. Magnitude responses of $H_0(z)$ and $H_1(z)$ of [2] (dashed lines) and this paper (solid lines), $N = 32$.

TABLE I
COEFFICIENTS OF $H_0(z)$ OF LENGTH 32

$H_0(z)$ of [2]	Refined $H_0(z)$
0.008494372478233	0.008521792387998
−0.000096178168735	−0.000175252185442
−0.008795047132403	−0.008799050141345
0.000708779549085	0.000743291070308
0.012204201560354	0.012239949917038
−0.001762639314795	−0.001916935471508
−0.015584559035738	−0.015574202133295
0.004082855675060	−0.004194284395171
0.017652220240893	0.017704339430360
−0.008385219782885	−0.008661200507376
−0.016747613884737	−0.016624150546330
0.018239062108698	0.018417032134584
0.005781735813341	0.005677864855465
−0.046926740909077	−0.047412526063976
0.057250054450732	0.058375341961644
0.354522945953839	0.355480526629108
0.504811839124518	0.504561109614265
0.264955363281817	0.263895772634852
−0.083290951611401	−0.084492709540863
−0.139108747584926	−0.138381320831234
0.033140360806592	0.033768695425686
0.090359384220331	0.090064413400439
−0.014687917291347	−0.015251239715143
−0.061033358867071	−0.060707173407903
0.006606122638754	0.006877980568124
0.040515550880357	0.040394285042527
−0.002631418173169	−0.002876919771416
−0.025925804761497	−0.025738116843614
0.000931953235019	0.001007773784971
0.015356389599162	0.015343739172949
−0.000119683269333	−0.000215732373560
−0.010570322584724	−0.010490177307673

TABLE II
LARGEST EQUATION ERROR FOR PR

Filter $H_0(z)$		Largest Eq. Error
$N=8$	$H_0(z)$ of [2]	8.3168×10^{-8}
	Refined $H_0(z)$	$< 10^{-15}$
$N=16$	$H_0(z)$ of [2]	2.6356×10^{-6}
	Refined $H_0(z)$	$< 10^{-15}$
$N=32$	$H_0(z)$ of [2]	2.1623×10^{-6}
	Refomed $H_0(z)$	$< 10^{-15}$

VI. CONCLUDING REMARKS

The simulation results in Sec. 5 indicate that the proposed design technique is able to produce highly satisfied CQ filters and is reliable for high-order designs. Our design idea is quite general and is in fact also applicable to M-channel filter banks, biorthogonal and cosine-modulated filter banks, and filter banks with additional requirements for phase response. These and other related design issues will be addressed in detail elsewhere.

TABLE III
LEAST SQUARES WITH N = 96, $\omega_a = 0.56\pi$

L	Energy in Stopband	Largest Equation Error
0	5.6213×10^{-10}	$< 10^{-15}$
1	5.6660×10^{-10}	$< 10^{-15}$
2	5.6660×10^{-10}	$< 10^{-15}$
3	5.8954×10^{-10}	$< 10^{-15}$
4	5.8954×10^{-10}	$< 10^{-15}$
5	6.2901×10^{-10}	7.6190×10^{-10}

TABLE IV
MINIMAX WITH N = 96, $\omega_a = 0.56\pi$

L	Instantaneous Power in Stopband	Largest Equation Error
0	2.8649×10^{-9}	$< 10^{-15}$
1	3.0323×10^{-9}	8.8247×10^{-7}
2	3.0654×10^{-9}	2.5128×10^{-5}
3	3.4075×10^{-9}	1.0654×10^{-6}
4	3.5281×10^{-9}	4.0553×10^{-7}
5	3.7121×10^{-9}	1.0982×10^{-5}

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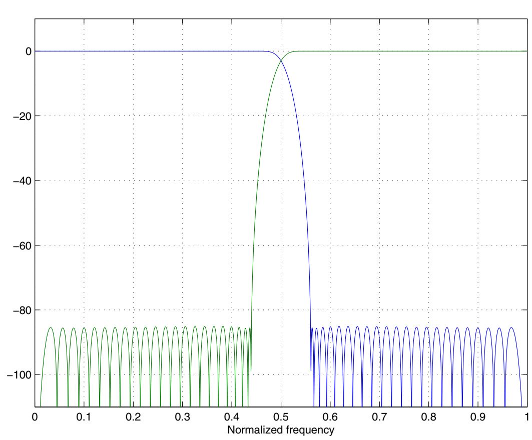

Fig. 3. Magnitude response of $H_0(z)$ and $H_1(z)$ with $N=96,\ L=3$.