MINIMAL REALIZATION AND L_2-SENSITIVITY ANALYSIS FOR 3-D SEPARABLE-DENOMINATOR DIGITAL FILTERS

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ABSTRACT

The problems of minimal state-space realization for a three-dimensional (3-D) separable-denominator digital filter and analysis of l_2-sensitivity for the realized 3-D state-space model are investigated. First, a 3-D transfer function with separable denominator is expressed as a cascade connection of three one-dimensional (1-D) transfer functions by applying a minimal decomposition technique. Next, each 1-D transfer function is realized by a state-space model of minimal order. The l_2-sensitivity of a 3-D separable-denominator transfer function with respect to the state-space paramters is then analyzed based on a pure l_2 norm. Finally, a numerical example is presented to evaluate the l_2-sensitivity of a 3-D state-space model realized from a given 3-D separable-denominator digital filter.

1. INTRODUCTION

In the case when implementing a state-space model for a given recursive digital filter with a finite binary representation, the truncation or rounding of the coefficients is required to meet the finite word length (FWL) constraints. This procedure may cause unacceptable degradation of the characteristics of a recursive digital filter, and may change a stable recursive digital filter to an unstable one. Such undesirable FWL effects can be reduced considerably by choosing the state-space model structure appropriately. Several techniques have been proposed to synthesize the 1-D state-space model structures that minimize the coefficient sensitivity [1]-[9]. The problem of minimizing the coefficient sensitivity for two-dimensional (2-D) state-space digital filters has also been explored extensively [10]-[15]. Among them, some evaluate the sensitivity by using a mixture of l_1/l_2 norms [10]-[13], while others rely on the use of a pure l_2 norm [14],[15]. In [15], the weighted-sensitivity minimization of 2-D state-space digital filters has been considered in both cases of a mixture of l_1/l_2 norms and a pure l_2 norm. It should be noted that the l_2-sensitivity minimization is more natural and reasonable than the conventional l_1/l_2 mixed sensitivity minimization, but it is more challenging. More recently, techniques have been presented for synthesizing 3-D separable-denominator state-space digital filters with low l_2-sensitivity [16],[17].

This paper considers the problem of realizing a minimal state-space model from 3-D separable-denominator digital filters, and analyzes the overall l_2-sensitivity of a minimal state-space model realized. A given 3-D transfer function with separable denominator is decomposed into three one-dimensional (1-D) transfer functions with a cascade connection. Each 1-D transfer function is then realized by a state-space model. It is noted that, unlike the analysis in [16], here we take into account 0 and 1 elements contained in the realized 3-D state-space model to evaluate the l_2-sensitivity more precisely. Moreover, unlike [17] where the sensitivity analysis is carried out only for the middle section of an MIMO 1-D system, this paper presents an l_2-sensitivity analysis for the entire 3-D system. As a result, for the first time an appropriate l_2-sensitivity measure for 3-D separable-denominator digital filters is defined, quantified, and numerically evaluated.

2. MINIMAL REALIZATION

Consider a stable 3-D separable in denominator digital filter described by

$$H(z_1, z_2, z_3) = \frac{N(z_1, z_2, z_3)}{D_1(z_1)D_2(z_2)D_3(z_3)} \tag{1}$$

where

$$N(z_1, z_2, z_3) = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\sum_{k=0}^{N_3} a_{ijk} z_1^{-i} z_2^{-j} z_3^{-k}$$

$$D_l(z_l) = 1 + b_{l1} z_l^{-1} + \cdots + b_{lN_l} z_l^{-N_l}, \quad l = 1, 2, 3$$

Suppose that there are no common factors between the numerator and denominator in (1). Since the 3-D digital filter in (1) is separable in denominator, the transfer function can be written as

$$H(z_1, z_2, z_3) = \frac{\boldsymbol{Z}_1^T}{D_1(z_1)} \boldsymbol{H}_2(z_2) \frac{\boldsymbol{Z}_3}{D_3(z_3)} \tag{2}$$

where

$$\boldsymbol{Z}_1 = [1, z_1^{-1}, \cdots, z_1^{-N_1}]^T, \quad \boldsymbol{Z}_3 = [1, z_3^{-1}, \cdots, z_3^{-N_3}]^T$$

$$\boldsymbol{H}_2(z_2) = \frac{\boldsymbol{\Delta}_0 + \boldsymbol{\Delta}_1 z_2^{-1} + \cdots + \boldsymbol{\Delta}_{N_2} z_2^{-N_2}}{D_2(z_2)}$$

$$\boldsymbol{\Delta}_m = \begin{bmatrix} a_{0m0} & a_{0m1} & \cdots & a_{0mN_3} \\ a_{1m0} & a_{1m1} & \cdots & a_{1mN_3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N_1m0} & a_{N_1m1} & \cdots & a_{N_1mN_3} \end{bmatrix}$$

$$m = 0, 1, \cdots, N_2$$

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The block diagram of a 3-D separable in denominator digital filter with order (N_1, p, N_3) is shown in Fig. 1.

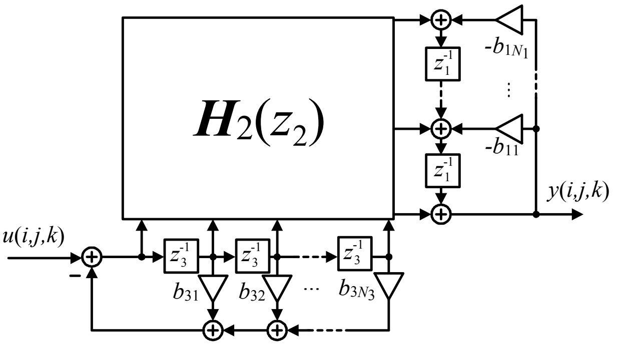

Fig. 1. Block diagram of a 3-D separable-denominator digital filter with order (N_1, p, N_3).

The 1-D transfer function $\boldsymbol{H}_2(z_2)$ with (N_3+1) inputs and (N_1+1) outputs can be realized by a minimal state-space model $(\boldsymbol{A}_2, \boldsymbol{B}_2, \boldsymbol{C}_2, \boldsymbol{\Delta}_0)_p$ as

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}_2\boldsymbol{x}(k) + \boldsymbol{B}_2\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= \boldsymbol{C}_2\boldsymbol{x}(k) + \boldsymbol{\Delta}_0\boldsymbol{u}(k) \end{aligned} \tag{3}$$

where $\boldsymbol{x}(k)$ is a $p \times 1$ state-variable vector, $\boldsymbol{u}(k)$ is an $(N_3+1)\times 1$ input vector, $\boldsymbol{y}(k)$ is an $(N_1+1)\times 1$ output vector, and $\boldsymbol{A}_2, \boldsymbol{B}_2, \boldsymbol{C}_2$ and $\boldsymbol{\Delta}_0$ are real constant matrices of appropriate dimensions. The transfer function of a 1-D system in (3) is given by

$$\boldsymbol{H}_2(z_2) = \boldsymbol{C}_2(z_2\boldsymbol{I}_p - \boldsymbol{A}_2)^{-1}\boldsymbol{B}_2 + \boldsymbol{\Delta}_0 \tag{4}$$

The 1-D trasfer functions $\boldsymbol{Z}_1^T/D_1(z_1)$ with (N_1+1) inputs and a scalar output and $\boldsymbol{Z}_3/D_3(z_3)$ with a scalar input and (N_3+1) outputs can be expressed in terms of the coefficient matrices of minimal state-space models $(\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{c}_1, \boldsymbol{d}_1)_{N_1}$ and $(\boldsymbol{A}_3, \boldsymbol{b}_3, \boldsymbol{C}_3, \boldsymbol{d}_3)_{N_3}$ as follows.

$$\begin{aligned} \frac{\boldsymbol{Z}_1^T}{D_1(z_1)} &= \boldsymbol{c}_1(z_1\boldsymbol{I}_{N_1} - \boldsymbol{A}_1)^{-1}\boldsymbol{B}_1 + \boldsymbol{d}_1 \\ \frac{\boldsymbol{Z}_3}{D_3(z_3)} &= \boldsymbol{C}_3(z_3\boldsymbol{I}_{N_3} - \boldsymbol{A}_3)^{-1}\boldsymbol{b}_3 + \boldsymbol{d}_3 \end{aligned} \tag{5}$$

where

$$\boldsymbol{A}_1 = \left[\begin{array}{c:c} \boldsymbol{0} & \\ \hdashline & \boldsymbol{a}_1 \\ \boldsymbol{I}_{N_1-1} & \end{array}\right], \quad \boldsymbol{B}_1 = \left[\begin{array}{cccc} & 0 & \cdots & 1 \\ \boldsymbol{b}_1 & \vdots & \iddots & \vdots \\ & 1 & \cdots & 0 \end{array}\right]$$

$$\boldsymbol{c}_1 = [\,0 \;\; \cdots \;\; 0 \;\; 1\,], \qquad \boldsymbol{d}_1 = [\,1 \;\; 0 \;\; \cdots \;\; 0\,]$$

$$\boldsymbol{a}_1 = \boldsymbol{b}_1 = [-b_{1N_1}, \cdots, -b_{12}, -b_{11}]^T$$

$$\boldsymbol{A}_3 = \left[\begin{array}{c:c} \boldsymbol{0} & \boldsymbol{I}_{N_3-1} \\ \hdashline \multicolumn{2}{c}{\boldsymbol{a}_3} \end{array}\right], \quad \boldsymbol{C}_3 = \left[\begin{array}{cccc} \multicolumn{4}{c}{\boldsymbol{c}_3} \\ \hdashline 0 & \cdots & 0 & 1 \\ \vdots & & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{array}\right]$$

$$\boldsymbol{b}_3 = [\,0 \;\; \cdots \;\; 0 \;\; 1\,]^T, \qquad \boldsymbol{d}_3 = [\,1 \;\; 0 \;\; \cdots \;\; 0\,]^T$$

$$\boldsymbol{a}_3 = \boldsymbol{c}_3 = [-b_{3N_3}, \cdots, -b_{32}, -b_{31}]$$

3. L_2-SENSITIVITY ANALYSIS

The l_2-sensitivities of the 3-D digital filter in (2) with respect to coefficient matrices $\boldsymbol{A}_2$, $\boldsymbol{B}_2$, $\boldsymbol{C}_2$, $\boldsymbol{\Delta}_0$, $\boldsymbol{a}_1$, $\boldsymbol{b}_1$, $\boldsymbol{a}_3$ and $\boldsymbol{c}_3$ are defined as follows.

Definition 1: Let $\boldsymbol{X}$ be an $m \times n$ real matrix and let $f(\boldsymbol{X})$ be a scalar complex function of $\boldsymbol{X}$, differentiable with respect to all the entries of $\boldsymbol{X}$. The sensitivity function of $f(\boldsymbol{X})$ with respect to $\boldsymbol{X}$ is then defined as

$$\frac{\partial f(\boldsymbol{X})}{\partial \boldsymbol{X}} = \begin{bmatrix} \frac{\partial f(\boldsymbol{X})}{\partial x_{11}} & \frac{\partial f(\boldsymbol{X})}{\partial x_{12}} & \cdots & \frac{\partial f(\boldsymbol{X})}{\partial x_{1n}} \\ \frac{\partial f(\boldsymbol{X})}{\partial x_{21}} & \frac{\partial f(\boldsymbol{X})}{\partial x_{22}} & \cdots & \frac{\partial f(\boldsymbol{X})}{\partial x_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(\boldsymbol{X})}{\partial x_{m1}} & \frac{\partial f(\boldsymbol{X})}{\partial x_{m2}} & \cdots & \frac{\partial f(\boldsymbol{X})}{\partial x_{mn}} \end{bmatrix} \tag{6}$$

where x_{ij} denotes the (i,j)th entry of matrix $\boldsymbol{X}$.

Following Definition 1, the sensitivities of $H(z_1, z_2, z_3)$ with respect to matrices $\boldsymbol{A}_2$, $\boldsymbol{B}_2$, $\boldsymbol{C}_2$, $\boldsymbol{\Delta}_0$, $\boldsymbol{a}_1$, $\boldsymbol{b}_1$, $\boldsymbol{a}_3^T$ and $\boldsymbol{c}_3^T$ are evaluated as

$$\begin{aligned} \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{A}_2} &= [\boldsymbol{f}(z_2,z_3)\boldsymbol{g}(z_1,z_2)]^T \\ \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{B}_2} &= \left[\frac{\boldsymbol{Z}_3}{D_3(z_3)}\boldsymbol{g}(z_1,z_2)\right]^T \\ \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{C}_2} &= \left[\boldsymbol{f}(z_2,z_3)\frac{\boldsymbol{Z}_1^T}{D_1(z_1)}\right]^T \\ \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{\Delta}_0} &= \frac{\boldsymbol{Z}_1\boldsymbol{Z}_3^T}{D_1(z_1)D_3(z_3)} \\ \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{a}_1} &= [\,[\,0,\cdots,0,1\,]\,\boldsymbol{f}_1(z_1,z_2,z_3)\boldsymbol{g}_1(z_1)\,]^T \\ \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{b}_1} &= \left[[\,1,0,\cdots,0\,]\,\boldsymbol{H}_2(z_2)\frac{\boldsymbol{Z}_3}{D_3(z_3)}\boldsymbol{g}_1(z_1)\right]^T \\ \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{a}_3^T} &= \boldsymbol{f}_3(z_3)\boldsymbol{g}_3(z_1,z_2,z_3)\,[\,0,\cdots,0,1\,]^T \\ \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{c}_3^T} &= \left[\boldsymbol{f}_3(z_3)\frac{\boldsymbol{Z}_1^T}{D_1(z_1)}\boldsymbol{H}_2(z_2)\,[\,1,0,\cdots,0\,]^T\right]^T \end{aligned} \tag{7}$$

where

$$\begin{aligned} \boldsymbol{f}(z_2,z_3) &= (z_2\boldsymbol{I}_p - \boldsymbol{A}_2)^{-1}\boldsymbol{B}_2\frac{\boldsymbol{Z}_3}{D_3(z_3)} \\ \boldsymbol{g}(z_1,z_2) &= \frac{\boldsymbol{Z}_1^T}{D_1(z_1)}\boldsymbol{C}_2(z_2\boldsymbol{I}_p - \mathbf{A}_2)^{-1} \\ \boldsymbol{f}_1(z_1,z_2,z_3) &= (z_1\boldsymbol{I}_{N_1} - \boldsymbol{A}_1)^{-1}\boldsymbol{B}_1\boldsymbol{H}_2(z_2)\frac{\boldsymbol{Z}_3}{D_3(z_3)} \\ \boldsymbol{g}_1(z_1) &= \boldsymbol{c}_1(z_1\boldsymbol{I}_{N_1} - \boldsymbol{A}_1)^{-1} \\ \boldsymbol{f}_3(z_3) &= (z_3\boldsymbol{I}_{N_3} - \boldsymbol{A}_3)^{-1}\boldsymbol{b}_3 \\ \boldsymbol{g}_3(z_1,z_2,z_3) &= \frac{\boldsymbol{Z}_1^T}{D_1(z_1)}\boldsymbol{H}_2(z_2)\boldsymbol{C}_3(z_3\boldsymbol{I}_{N_3} - \boldsymbol{A}_3)^{-1} \end{aligned}$$

Definition 2: Let $\boldsymbol{X}(z_1, z_2, z_3)$ be an $m \times n$ complex matrix-valued function of complex variables z_1, z_2 and z_3. Let $x_{pq}(z_1, z_2, z_3)$ be the (p, q)th entry of $\boldsymbol{X}(z_1, z_2, z_3)$. The l_2-norm of $\boldsymbol{X}(z_1, z_2, z_3)$ is then defined as

$$\|\boldsymbol{X}(z_1,z_2,z_3)\|_2 = \left(\mathrm{tr}\left[\frac{1}{(2\pi j)^3} \oint_{|z_1|=1} \oint_{|z_2|=1} \oint_{|z_3|=1} \cdot \boldsymbol{X}(z_1,z_2,z_3)\boldsymbol{X}^*(z_1,z_2,z_3) \frac{dz_1}{z_1}\frac{dz_2}{z_2}\frac{dz_3}{z_3} \right] \right)^{\frac{1}{2}} \tag{8}$$

From (2), (4), (5) and Definitions 1 and 2, the overall l_2-sensitivity measure for the 3-D digital filter in (2) is defined as

$$\begin{aligned} S = & \left\| \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{A}_2} \right\|_2^2 + \left\| \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{B}_2} \right\|_2^2 \\ & + \left\| \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{C}_2} \right\|_2^2 + \left\| \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{\Delta}_0} \right\|_2^2 \\ & + \left\| \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{a}_1} \right\|_2^2 + \left\| \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{b}_1} \right\|_2^2 \\ & + \left\| \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{a}_3^T} \right\|_2^2 + \left\| \frac{\partial H(z_1,z_2,z_3)}{\partial \boldsymbol{c}_3^T} \right\|_2^2 \end{aligned} \tag{9}$$

The l_2-sensitivity measure in (9) can be expressed as

$$\begin{aligned} S = & \,\mathrm{tr}[\boldsymbol{M}_A(\boldsymbol{I}_p)] + \mathrm{tr}[\boldsymbol{W}_B] + \mathrm{tr}[\boldsymbol{K}_C] + \mathrm{tr}[\boldsymbol{N}_{\Delta_0}] \\ & + \mathrm{tr}[\boldsymbol{M}_1] + \mathrm{tr}[\boldsymbol{W}_1] + \mathrm{tr}[\boldsymbol{M}_3] + \mathrm{tr}[\boldsymbol{K}_3] \end{aligned} \tag{10}$$

where Gramians $\boldsymbol{M}_A(\boldsymbol{P})$, $\boldsymbol{W}_B$, $\boldsymbol{K}_C$, $\boldsymbol{N}_{\Delta_0}$, $\boldsymbol{M}_1$, $\boldsymbol{W}_1$, $\boldsymbol{M}_3$ and $\boldsymbol{K}_3$ can be computed using

$$\begin{aligned}
\boldsymbol{M}_A(\boldsymbol{P}) = & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} [\boldsymbol{0} \quad \boldsymbol{I}_p] \begin{bmatrix} \boldsymbol{A}_2^T & \boldsymbol{0} \\ \boldsymbol{C}_2^T \boldsymbol{R}_{ij}^T \boldsymbol{B}_2^T & \boldsymbol{A}_2^T \end{bmatrix}^k \\
& \cdot \begin{bmatrix} \boldsymbol{P}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_2 & \boldsymbol{B}_2 \boldsymbol{R}_{ij} \boldsymbol{C}_2 \\ \boldsymbol{0} & \boldsymbol{A}_2 \end{bmatrix}^k \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_p \end{bmatrix} \\
\boldsymbol{W}_B = & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (\boldsymbol{A}_2^T)^k \boldsymbol{C}_2^T \boldsymbol{R}_{ij}^T \boldsymbol{R}_{ij} \boldsymbol{C}_2 \boldsymbol{A}_2^k \\
\boldsymbol{K}_C = & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \boldsymbol{A}_2^k \boldsymbol{B}_2 \boldsymbol{R}_{ij} \boldsymbol{R}_{ij}^T \boldsymbol{B}_2^T (\boldsymbol{A}_2^T)^k \\
\boldsymbol{N}_{\Delta_0} = & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \boldsymbol{R}_{ij}^T \boldsymbol{R}_{ij} \\
\boldsymbol{M}_1 = & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} [\boldsymbol{0} \quad \boldsymbol{I}_{N_1}] \begin{bmatrix} \boldsymbol{A}_1^T & \boldsymbol{0} \\ \boldsymbol{c}_1^T \boldsymbol{r}_{ij}^T \boldsymbol{B}_1^T & \boldsymbol{A}_1^T \end{bmatrix}^k \\
& \cdot \begin{bmatrix} \boldsymbol{e}_{N_1} \boldsymbol{e}_{N_1}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1 \boldsymbol{r}_{ij} \boldsymbol{c}_1 \\ \boldsymbol{0} & \boldsymbol{A}_1 \end{bmatrix}^k \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_{N_1} \end{bmatrix} \\
\boldsymbol{W}_1 = & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (\boldsymbol{A}_1^T)^k \boldsymbol{c}_1^T \boldsymbol{r}_{ij}^T \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \boldsymbol{r}_{ij} \boldsymbol{c}_1 \boldsymbol{A}_1^k \\
\boldsymbol{M}_3 = & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} [\boldsymbol{I}_{N_3} \quad \boldsymbol{0}] \begin{bmatrix} \boldsymbol{A}_3 & \boldsymbol{b}_3 \hat{\boldsymbol{r}}_{ij} \boldsymbol{C}_3 \\ \boldsymbol{0} & \boldsymbol{A}_3 \end{bmatrix}^k \\
& \cdot \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_3^T & \boldsymbol{0} \\ \boldsymbol{C}_3^T \hat{\boldsymbol{r}}_{ij}^T \boldsymbol{b}_3^T & \boldsymbol{A}_3^T \end{bmatrix}^k \begin{bmatrix} \boldsymbol{I}_{N_3} \\ \boldsymbol{0} \end{bmatrix} \\
\boldsymbol{K}_3 = & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \boldsymbol{A}_3^k \boldsymbol{b}_3 \hat{\boldsymbol{r}}_{ij} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \hat{\boldsymbol{r}}_{ij}^T \boldsymbol{b}_3^T (\boldsymbol{A}_3^T)^k
\end{aligned}$$

with $\quad \boldsymbol{e}_{N_1} = (0, \cdots, 0, 1)^T$

$$\begin{aligned}
\frac{\boldsymbol{Z}_3 \boldsymbol{Z}_1^T}{D_3(z_3) D_1(z_1)} &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \boldsymbol{R}_{ij} z_1^{-i} z_3^{-j} \\
\boldsymbol{H}_2(z_2) \frac{\boldsymbol{Z}_3}{D_3(z_3)} &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \boldsymbol{r}_{ij} z_2^{-i} z_3^{-j} \\
\frac{\boldsymbol{Z}_1^T}{D_1(z_1)} \boldsymbol{H}_2(z_2) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \hat{\boldsymbol{r}}_{ij} z_1^{-i} z_2^{-j}
\end{aligned}$$

The above Gramians can also be computed by

$$\begin{aligned}
\boldsymbol{M}_A(\boldsymbol{P}) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} [\boldsymbol{0} \quad \boldsymbol{I}_p] \boldsymbol{M}_{ij}^A \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_p \end{bmatrix} \\
\boldsymbol{M}_{ij}^A &= \begin{bmatrix} \boldsymbol{A}_2 & \boldsymbol{B}_2 \boldsymbol{R}_{ij} \boldsymbol{C}_2 \\ \boldsymbol{0} & \boldsymbol{A}_2 \end{bmatrix}^T \boldsymbol{M}_{ij}^A \begin{bmatrix} \boldsymbol{A}_2 & \boldsymbol{B}_2 \boldsymbol{R}_{ij} \boldsymbol{C}_2 \\ \boldsymbol{0} & \boldsymbol{A}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{P}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \\
\boldsymbol{W}_B &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \boldsymbol{W}_{ij}^B \\
\boldsymbol{W}_{ij}^B &= \boldsymbol{A}_2^T \boldsymbol{W}_{ij}^B \boldsymbol{A}_2 + \boldsymbol{C}_2^T \boldsymbol{R}_{ij}^T \boldsymbol{R}_{ij} \boldsymbol{C}_2 \\
\boldsymbol{K}_C &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \boldsymbol{K}_{ij}^C \\
\boldsymbol{K}_{ij}^C &= \boldsymbol{A}_2 \boldsymbol{K}_{ij}^C \boldsymbol{A}_2^T + \boldsymbol{B}_2 \boldsymbol{R}_{ij} \boldsymbol{R}_{ij}^T \boldsymbol{B}_2^T \\
\boldsymbol{N}_{\Delta_0} &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \boldsymbol{R}_{ij}^T \boldsymbol{R}_{ij} \\
\boldsymbol{M}_1 &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} [\boldsymbol{0} \quad \boldsymbol{I}_{N_1}] \boldsymbol{M}_{ij}^1 \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_{N_1} \end{bmatrix} \\
\boldsymbol{M}_{ij}^1 &= \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1 \boldsymbol{r}_{ij} \boldsymbol{c}_1 \\ \boldsymbol{0} & \boldsymbol{A}_1 \end{bmatrix}^T \boldsymbol{M}_{ij}^1 \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1 \boldsymbol{r}_{ij} \boldsymbol{c}_1 \\ \boldsymbol{0} & \boldsymbol{A}_1 \end{bmatrix} + \begin{bmatrix} \boldsymbol{e}_{N_1} \boldsymbol{e}_{N_1}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \\
\boldsymbol{W}_1 &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \boldsymbol{W}_{ij}^1 \\
\boldsymbol{W}_{ij}^1 &= \boldsymbol{A}_1^T \boldsymbol{W}_{ij}^1 \boldsymbol{A}_1 + \boldsymbol{c}_1^T \boldsymbol{r}_{ij}^T \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \boldsymbol{r}_{ij} \boldsymbol{c}_1
\end{aligned}$$

$$\boldsymbol{M}_3 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} [\boldsymbol{I}_{N_3} \quad \boldsymbol{0}]\, \boldsymbol{M}_{ij}^3 \begin{bmatrix} \boldsymbol{I}_{N_3} \\ \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{M}_{ij}^3 = \begin{bmatrix} \boldsymbol{A}_3 & \boldsymbol{b}_3\hat{\boldsymbol{r}}_{ij}\boldsymbol{C}_3 \\ \boldsymbol{0} & \boldsymbol{A}_3 \end{bmatrix} \boldsymbol{M}_{ij}^3 \begin{bmatrix} \boldsymbol{A}_3 & \boldsymbol{b}_3\hat{\boldsymbol{r}}_{ij}\boldsymbol{C}_3 \\ \boldsymbol{0} & \boldsymbol{A}_3 \end{bmatrix}^T + \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{K}_3 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \boldsymbol{K}_{ij}^3$$

$$\boldsymbol{K}_{ij}^3 = \boldsymbol{A}_3\boldsymbol{K}_{ij}^3\boldsymbol{A}_3^T + \boldsymbol{b}_3\hat{\boldsymbol{r}}_{ij} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \hat{\boldsymbol{r}}_{ij}^T\boldsymbol{b}_3^T \tag{11}$$

By applying a coordinate transformation defined by $\overline{\boldsymbol{x}}(k) = \boldsymbol{T}^{-1}\boldsymbol{x}(k)$ to the 1-D system $(\boldsymbol{A}_2, \boldsymbol{B}_2, \boldsymbol{C}_2, \boldsymbol{\Delta}_0)_p$ in (3), we obtain a new realization $(\overline{\boldsymbol{A}}_2, \overline{\boldsymbol{B}}_2, \overline{\boldsymbol{C}}_2, \boldsymbol{\Delta}_0)_p$ characterized by

$$\overline{\boldsymbol{A}}_2 = \boldsymbol{T}^{-1}\boldsymbol{A}_2\boldsymbol{T}, \quad \overline{\boldsymbol{B}}_2 = \boldsymbol{T}^{-1}\boldsymbol{B}_2, \quad \overline{\boldsymbol{C}}_2 = \boldsymbol{C}_2\boldsymbol{T} \tag{12}$$

where $\boldsymbol{T}$ is a $p \times p$ nonsingular matrix. For the new realization, the l_2-sensitivity measure in (10) becomes

$$\begin{aligned} S(\boldsymbol{T}) = {} & J(\boldsymbol{T}) + \mathrm{tr}[\boldsymbol{N}_{\Delta_0}] + \mathrm{tr}[\boldsymbol{M}_1] + \mathrm{tr}[\boldsymbol{W}_1] \\ & + \mathrm{tr}[\boldsymbol{M}_3] + \mathrm{tr}[\boldsymbol{K}_3] \end{aligned} \tag{13}$$

where with $\boldsymbol{P} = \boldsymbol{T}\boldsymbol{T}^T$

$$J(\boldsymbol{T}) = \mathrm{tr}[\boldsymbol{T}^T\boldsymbol{M}_A(\boldsymbol{P})\boldsymbol{T}] + \mathrm{tr}[\boldsymbol{T}^T\boldsymbol{W}_B\boldsymbol{T}] + \mathrm{tr}[\boldsymbol{T}^{-1}\boldsymbol{K}_C\boldsymbol{T}^{-T}]$$

Since $\boldsymbol{f}(z_2, z_3)$ is the transfer function from the filter input to the state-variable vector $\boldsymbol{x}(k)$, a controllability Gramian $\boldsymbol{K}$ can be derived from

$$\begin{aligned} \boldsymbol{K} &= \frac{1}{(2\pi j)^2} \oint_{|z_2|=1} \oint_{|z_3|=1} \boldsymbol{f}(z_2, z_3)\boldsymbol{f}^*(z_2, z_3) \frac{dz_2}{z_2}\frac{dz_3}{z_3} \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \boldsymbol{A}_2^k\boldsymbol{B}_2\boldsymbol{r}_j\boldsymbol{r}_j^T\boldsymbol{B}_2^T(\boldsymbol{A}_2^T)^k \end{aligned} \tag{14}$$

or equivalently,

$$\boldsymbol{K} = \sum_{j=0}^{\infty} \boldsymbol{K}_j, \quad \boldsymbol{K}_j = \boldsymbol{A}_2\boldsymbol{K}_j\boldsymbol{A}_2^T + \boldsymbol{B}_2\boldsymbol{r}_j\boldsymbol{r}_j^T\boldsymbol{B}_2^T \tag{15}$$

where

$$\frac{\boldsymbol{Z}_3}{D_3(z_3)} = \sum_{j=0}^{\infty} \boldsymbol{r}_j z_3^{-j}$$

In this case, l_2-scaling constraints are given by

$$(\overline{\boldsymbol{K}})_{ii} = (\boldsymbol{T}^{-1}\boldsymbol{K}\boldsymbol{T}^{-T})_{ii} = 1 \quad \text{for} \quad i = 1, 2, \cdots, p. \tag{16}$$

The problem for obtaining the coordinate transformation matrix $\boldsymbol{T}$ that minimizes $J(\boldsymbol{T})$ in (13) subject to the l_2-scaling constraints in (16) has been addressed in [17]. We note however that the l_2-sensitivity of the entire 3-D system as formulated in (13) is not available in the literature. It is also noted that, unlike the analysis in [16], the l_2-sensitivity evaluated here has also taken into account all the 0 and 1 entries in the realized 3-D state-space model. As a result, the l_2-sensitivity formulated in (13) is more accurate.

4. A NUMERICAL EXAMPLE

Consider a stable 3-D separable in denominator digital filter specified by

$$\boldsymbol{\Delta}_0 = 10^{-2}\begin{bmatrix} 0.00730 & 0.34297 & -0.09594 & 0.20541 \\ 3.33408 & -5.73707 & 3.94939 & -1.61598 \\ -1.46081 & 2.66051 & -1.68094 & 0.68022 \\ 1.12651 & -1.62192 & 1.24735 & -0.55781 \end{bmatrix}$$

$$\boldsymbol{\Delta}_1 = 10^{-2}\begin{bmatrix} 2.81318 & -5.00467 & 3.46926 & -0.84798 \\ -5.29980 & 9.24831 & -6.29206 & 2.80791 \\ 4.95232 & -8.39641 & 5.73329 & -1.62170 \\ 0.72029 & -1.34272 & 0.95941 & 0.54827 \end{bmatrix}$$

$$\boldsymbol{\Delta}_2 = 10^{-2}\begin{bmatrix} -0.69409 & 1.54874 & -0.94779 & 0.39116 \\ 3.93785 & -6.79910 & 4.66564 & -1.96344 \\ -2.37995 & 4.20737 & -2.75482 & 0.95329 \\ 0.70545 & -0.90615 & 0.73168 & -0.55633 \end{bmatrix}$$

$$\boldsymbol{\Delta}_3 = 10^{-2}\begin{bmatrix} 1.67681 & -2.69078 & 1.98218 & -0.33567 \\ -0.59937 & 1.11289 & -0.71981 & 0.43504 \\ 1.82472 & -2.93685 & 2.11591 & -0.43417 \\ 1.28875 & -2.01749 & 1.51782 & -0.09016 \end{bmatrix}$$

$$\begin{aligned} [b_{11} \quad b_{12} \quad b_{13}] &= [b_{31} \quad b_{32} \quad b_{33}] \\ &= [-1.81600 \quad 1.23756 \quad -0.31382] \\ [b_{21} \quad b_{22} \quad b_{23}] &= [-1.81611 \quad 1.23775 \quad -0.31391] \end{aligned}$$

The 3-D separable-denominator digital filter with specified coefficients is realized by the state-space model in (3) as

$$\boldsymbol{A}_2 = \begin{bmatrix} 0.00000 & -0.19089 & 0.29060 \\ 0.74393 & -86.40470 & 133.71075 \\ -0.27211 & -57.01643 & 88.22081 \end{bmatrix}$$

$$\boldsymbol{B}_2 = 10^3\begin{bmatrix} 0.00602 & -0.00921 & 0.00699 & -0.00095 \\ -1.10247 & 1.68622 & -1.27902 & 0.17267 \\ -0.71455 & 1.09291 & -0.82977 & 0.11192 \end{bmatrix}$$

$$\boldsymbol{C}_2 = \begin{bmatrix} 0.07236 & 0.06711 & -0.10298 \\ 0.01930 & 0.01789 & -0.02745 \\ 0.05887 & 0.05460 & -0.08378 \\ 0.07079 & 0.06565 & -0.10073 \end{bmatrix}.$$

Using (11) with truncation $(0,0) \le (i,j) \le (100,100)$ to evaluate the Gramians $\boldsymbol{M}_A(\boldsymbol{I}_3)$, $\boldsymbol{W}_B$, $\boldsymbol{K}_C$, $\boldsymbol{N}_{\Delta_0}$, $\boldsymbol{M}_1$, $\boldsymbol{W}_1$, $\boldsymbol{M}_3$ and $\boldsymbol{K}_3$, we arrived at

$$\boldsymbol{M}_A(\boldsymbol{I}_3) = 10^7\begin{bmatrix} 6.713807 & 4.577834 & -7.015937 \\ 4.577834 & 3.166229 & -4.852755 \\ -7.015935 & -4.852755 & 7.437629 \end{bmatrix}$$

$$\boldsymbol{W}_B = 10^2\begin{bmatrix} 1.195455 & 0.863340 & -1.323327 \\ 0.863340 & 0.652270 & -0.999976 \\ -1.323327 & -0.999976 & 1.533035 \end{bmatrix}$$

$$\boldsymbol{K}_C = 10^8\begin{bmatrix} 0.000066 & -0.013350 & -0.008678 \\ -0.013350 & 3.814232 & 2.483684 \\ -0.008678 & 2.483684 & 1.617300 \end{bmatrix}$$

$$\boldsymbol{N}_{\Delta_0} = 10^2 \begin{bmatrix} 8.134287 & 7.400517 & 5.695102 & 3.736423 \\ 7.400517 & 8.134287 & 7.400517 & 5.695102 \\ 5.695102 & 7.400517 & 8.134287 & 7.400517 \\ 3.736423 & 5.695102 & 7.400517 & 8.134287 \end{bmatrix}$$

$$\boldsymbol{M}_1 = 10^3 \begin{bmatrix} 2.572596 & 2.480170 & 2.221844 \\ 2.480170 & 2.572596 & 2.480170 \\ 2.221844 & 2.480170 & 2.572596 \end{bmatrix}$$

$$\boldsymbol{W}_1 = \begin{bmatrix} 5.502192 & 5.005856 & 3.852279 \\ 5.005856 & 5.502192 & 5.005856 \\ 3.852279 & 5.005856 & 5.502192 \end{bmatrix}$$

$$\boldsymbol{M}_3 = 10^3 \begin{bmatrix} 2.340179 & 2.244464 & 1.979670 \\ 2.244464 & 2.340179 & 2.244464 \\ 1.979670 & 2.244464 & 2.340179 \end{bmatrix}$$

$$\boldsymbol{K}_3 = 10 \begin{bmatrix} 2.713421 & 2.468651 & 1.899762 \\ 2.468651 & 2.713421 & 2.468651 \\ 1.899762 & 2.468651 & 2.713421 \end{bmatrix}.$$

The l_2-sensitivity in (13) was then computed as

$$S(\boldsymbol{I}_3) = 7.163549 \times 10^8$$

where $J(\boldsymbol{I}_3) = 7.163398 \times 10^8$, $\mathrm{tr}[\boldsymbol{N}_{\Delta_0}] = 3.253715 \times 10^3$ $\mathrm{tr}[\boldsymbol{M}_1] + \mathrm{tr}[\boldsymbol{W}_1] = 7.734294 \times 10^3$ and $\mathrm{tr}[\boldsymbol{M}_3] + \mathrm{tr}[\boldsymbol{K}_3] = 7.101941 \times 10^3$. It is noted that only the value of $J(\boldsymbol{T})$ for a scaling matrix $\boldsymbol{T}$ was provided in the numerical example given in [17].

5. CONCLUSION

A technique for the minimal realization of a 3-D separable-denominator digital filter has been presented, and the l_2-sensitivity of a minimal state-space model realized from a given 3-D separable-denominator digital filter has been analyzed. The practical application of 3-D filters can be found in geophysical signal processing such as 3-D seismic projection/migration. It is noted that a 3-D FIR digital filter can be approximated by a 3-D separable-denominator IIR digital filter [18].

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