Unconstrained Regularized ℓ_p-Norm Based Algorithm for the Reconstruction of Sparse Signals

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***Abstract*—A new algorithm for signal reconstruction in a compressive sensing framework is presented. The algorithm is based on minimizing an unconstrained regularized ℓ_p norm with $p < 1$ in the null space of the measurement matrix. The unconstrained optimization involved is performed by using a quasi-Newton algorithm in which a new line search based on Banach's fixed-point theorem is used. Simulation results are presented, which demonstrate that the proposed algorithm yields improved reconstruction performance and requires a reduced amount of computation relative to several known algorithms.**

I. Introduction

One of the most successful algorithms used to recover sparse signals in *compressive sensing* (CS) is the ℓ_1-norm minimization based algorithm known as *basis pursuit* (BP) [1] - [4]. Alternative optimization based methods proposed recently in [5] and [6] are shown to offer improved performance relative to BP. In these contributions, it is demonstrated that more accurate signal reconstruction can be achieved by solving an ℓ_p-norm based minimization problem with $p < 1$. A computationally efficient algorithm based on the minimization of an approximate smoothed ℓ_0 norm, known as *smoothed* ℓ_0 (SL0) algorithm, is investigated in [7]. In [8], signal reconstruction is carried out by minimizing a reweighted approximate ℓ_0 norm in the null space of the sensing matrix.

In this paper, we propose a new algorithm for the reconstruction of sparse signals in the CS framework by minimizing an unconstrained regularized ℓ_p norm with $p < 1$ in the null space of the measurement matrix. By introducing a regularization parameter in the standard ℓ_p norm, the objective function involved becomes differentiable and its convex region is shown to be controllable by the regularization parameter. Moveover, by working in the null space of the measurement matrix, the optimization problem at hand becomes unconstrained and hence it can be solved by using efficient quasi-Newton methods. In addition, we propose a new line search for the quasi-Newton method based on Banach's fixed-point theorem. Simulation results are presented, which demonstrate that the proposed algorithm yields improved reconstruction performance and requires a reduced amount of computation relative to several known algorithms.

II. Preliminaries

Let $\boldsymbol{x}$ be a real-valued discrete-time signal of length N. Signal $\boldsymbol{x}$ is said to be K-sparse if the number of nonzero components of $\boldsymbol{x}$ is K with $K \ll N$. In CS, signal $\boldsymbol{x}$ is measured in terms of M random projections with M considerably less than N, namely,

$$\boldsymbol{y} = \boldsymbol{\Phi x} \tag{1}$$

where $\boldsymbol{y}$ is a measurement vector of length M and $\boldsymbol{\Phi}$ is a measurement matrix of size $M \times N$. Since $M < N$, recovering signal $\boldsymbol{x}$ from measurements $\boldsymbol{y}$ by solving (1) is ill-posed because (1) has infinitely many solutions [9]. In principle, the sparsest solution of (1) can be determined by minimizing the ℓ_0 norm of $\boldsymbol{x}$, i.e., $||\boldsymbol{x}||_0 = \sum_{i=1}^{N} |x|^0$, subject to the constraint in (1). Unfortunately, finding a solution with minimum ℓ_0 norm requires a combinatorial search among all the solutions of (1) for which the computation required grows exponentially as N increases. A tractable approach is to use BP which entails solving the convex problem

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad ||\boldsymbol{x}||_1 \quad \text{subject to:} \quad \boldsymbol{\Phi x} = \boldsymbol{y} \tag{2}$$

where $||\boldsymbol{x}||_1$ is the ℓ_1 norm of $\boldsymbol{x}$ defined as $||\boldsymbol{x}||_1 = \sum_{i=1}^{N} |x_i|$. A key result in CS is that if signal $\boldsymbol{x}$ is K-sparse, the elements of $\boldsymbol{\Phi}$ are drawn from a normal distribution and the number of measurements M satisfies the condition

$$M \geq cK \log(N/K) \tag{3}$$

with c a small constant, then $\boldsymbol{x}$ can be recovered by solving the problem in (2) [1] - [3]. Recently, it has been shown that improved performance for the reconstruction of sparse signals can be achieved by solving the ℓ_p-norm minimization problem

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad ||\boldsymbol{x}||_p^p \quad \text{subject to} \quad \boldsymbol{\Phi x} = \boldsymbol{y} \tag{4}$$

where $p < 1$ [5], [6].

The solutions of (1) can be characterized as

$$\boldsymbol{x} = \boldsymbol{x}_s + \boldsymbol{V}_r \boldsymbol{\xi} \tag{5}$$

where $\boldsymbol{x}_s$ is a solution of (1), $\boldsymbol{V}_r$ is a $N \times (N - M)$ matrix whose columns constitute an orthonormal basis of the null space of $\boldsymbol{\Phi}$, and $\boldsymbol{\xi}$ is a parameter vector of dimension $N - M$.

In [8], the signal is reconstructed using (5) where vector $\boldsymbol{\xi}$ minimizes an approximate ℓ_0 norm of $\boldsymbol{x}$, namely,

$$F_\sigma(\xi) = \sum_{i=1}^{N} w_i \left\{ 1 - e^{-[x_{si} + \boldsymbol{v}_i^T \boldsymbol{\xi}]^2 / 2\sigma^2} \right\} \tag{6}$$

where x_{si} is the ith component of $\boldsymbol{x}_s$, $\boldsymbol{v}_i^T$ is the ith row of matrix $\boldsymbol{V}_r$, and w_i for $i = 1, 2, \ldots, N$ are weights that are used to enhance the performance of the recovery process.

III. Unconstrained Regularized ℓ_p-Norm Based Method for Sparse Signal Reconstruction

A. Regularized ℓ_p norm and problem conversion

We consider the regularized ℓ_p norm of $\boldsymbol{x}$

$$||\boldsymbol{x}||_{p,\epsilon}^p = \sum_{i=1}^{N} \left(x_i^2 + \epsilon^2\right)^{p/2} \quad (7)$$

where ϵ is a small positive scalar and $p < 1$. Note that $||\boldsymbol{x}||_{p,\epsilon}^p \rightarrow ||\boldsymbol{x}||_p^p$ as $\epsilon \rightarrow 0$. We propose to reconstruct $\boldsymbol{x}$ by minimizing $||\boldsymbol{x}||_{p,\epsilon}^p$ in (7) for a sufficiently small ϵ subject to the constraint in (1). Using (5), the constraint in (1) is eliminated and the problem at hand is converted into the unconstrained problem

$$\underset{\boldsymbol{\xi}}{\text{minimize}} \quad F_{p,\epsilon}(\boldsymbol{\xi}) = \sum_{i=1}^{N} \left[\left(x_{si} + \boldsymbol{v}_i^T \boldsymbol{\xi}\right)^2 + \epsilon^2\right]^{p/2} \quad (8)$$

Parameter ϵ plays two important roles in the proposed algorithm. First, the objective function $F_{p,\epsilon}(\boldsymbol{\xi})$ remains differentiable as long as ϵ is kept positive. In effect, for $\epsilon > 0$ the gradient of $F_{p,\epsilon}(\boldsymbol{\xi})$ is given by

$$\nabla F_{p,\epsilon}(\boldsymbol{\xi}) = p \cdot \boldsymbol{V}_r^T \cdot \boldsymbol{g} \quad (9)$$

where $\boldsymbol{g} = [g_1 \; g_2 \; \cdots \; g_N]^T$ and

$$g_i = \left[\left(x_{si} + \boldsymbol{v}_i^T \boldsymbol{\xi}\right)^2 + \epsilon^2\right]^{p/2-1} \left(x_{si} + \boldsymbol{v}_i^T \boldsymbol{\xi}\right) \quad (10)$$

Second, the region where $F_{p,\epsilon}(\boldsymbol{\xi})$ is convex is controlled by ϵ: the greater the ϵ, the larger the convex region. To see this, note that the Hessian of $||\boldsymbol{x}||_{p,\epsilon}^p$ is a diagonal matrix given by

$$\nabla^2 ||\boldsymbol{x}||_{p,\epsilon}^p = \text{diag}\{h_{11}, \; h_{22}, \; \cdots, \; h_{NN}\} \quad (11)$$

where

$$h_{ii} = p(x_i^2 + \epsilon^2)^{p/2-1} \left[(p-1)x_i^2 + \epsilon^2\right] \quad (12)$$

Hence $||\boldsymbol{x}||_{p,\epsilon}^p$ is convex if and only if

$$|x_i| \leq \frac{\epsilon}{\sqrt{1-p}} \quad \text{for } 1 \leq i \leq N \quad (13)$$

Eq. (13) defines an N-dimensional hypercube whose volume is $\left(\frac{2\epsilon}{\sqrt{1-p}}\right)^N$. Therefore, for a fixed $p < 1$, the volume of the convex region in the $\boldsymbol{x}$ space is proportional to ϵ. Using (8), the Hessian of $F_{p,\epsilon}(\boldsymbol{\xi})$ is found to be

$$\nabla^2 F_{p,\epsilon}(\boldsymbol{\xi}) = \boldsymbol{V}_r^T \cdot \nabla^2 ||\boldsymbol{x}||_{p,\epsilon}^p \cdot \boldsymbol{V}_r$$

Hence $\nabla^2 F_{p,\epsilon}(\boldsymbol{\xi})$ is positive definite if $\nabla^2 ||\boldsymbol{x}||_{p,\epsilon}^p$ is positive definite. Consequently, we can use (13) and (5) to show that $F_{p,\epsilon}(\boldsymbol{\xi})$ is convex if

$$|\boldsymbol{v}_i^T \boldsymbol{\xi} + x_{si}| \leq \frac{\epsilon}{\sqrt{1-p}} \quad \text{for } 1 \leq i \leq N \quad (14)$$

From (14), we see that the size of the convex region of $F_{p,\epsilon}(\boldsymbol{\xi})$ is also proportional to the value of ϵ.

Based on the above analysis, we propose the following optimization technique for the problem in (8):

- First, we obtain the minimum ℓ_2-norm solution of (1) and use it as the special solution $\boldsymbol{x}_s$ in (5). An initial value of ϵ is selected to satisfy the inequality

  $$\epsilon \geq \sqrt{1-p} \cdot \underset{1 \leq i \leq N}{\text{maximum}} \, |x_{si}| \quad (15)$$

  which ensures that

  $$|x_{si}| \leq \frac{\epsilon}{\sqrt{1-p}} \quad \text{for } 1 \leq i \leq N \quad (16)$$

  It follows from (13) that such an $\boldsymbol{x}_s$ is in the convex region of $||\boldsymbol{x}||_{p,\epsilon}^p$. Now, Eq. (16) in conjunction with (14) implies that $\boldsymbol{\xi} = 0$ is in the convex region of $F_{p,\epsilon}(\boldsymbol{\xi})$. This justifies the choice of ϵ according to (15) and the use of $\boldsymbol{\xi} = 0$ as an initial point. A quasi-Newton algorithm such as the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm [10] is applied to minimize $F_{p,\epsilon}(\boldsymbol{\xi})$. The minimizer obtained is denoted as $\boldsymbol{\xi}_1^*$.
- Next, the value of ϵ is reduced by a certain amount and the BFGS algorithm is applied again to minimize $F_{p,\epsilon}(\boldsymbol{\xi})$ with $\boldsymbol{\xi}_1^*$ as initial point. The minimizer obtained is denoted as $\boldsymbol{\xi}_2^*$; it will be used as initial point in the next iteration.
- This procedure is repeated until the value of ϵ is reduced to a prescribed target value ϵ_J. The minimizer associated with $\epsilon = \epsilon_J$ is denoted by $\boldsymbol{\xi}_J^*$ and is used in (5) to reconstruct the sparse signal as $\boldsymbol{x}^* = \boldsymbol{x}_s + \boldsymbol{V}_r \boldsymbol{\xi}_J^*$.

B. Line search based on Banach's fixed-point theorem

Given parameters p and ϵ, the kth iteration of the BFGS algorithm involves a step to solve an one-dimensional optimization problem

$$\underset{\alpha}{\text{minimize}} \quad f(\alpha) \quad (17)$$

where

$$f(\alpha) = F_{p,\epsilon}(\boldsymbol{\xi}_k + \alpha \boldsymbol{d}_k)$$

Iterate $\boldsymbol{\xi}_k$ and search direction $\boldsymbol{d}_k$ are determined by using the BFGS algorithm. This is essentially a line search and it is known to have a great effect on the algorithm's performance in terms of computational efficiency and solution accuracy. Below we propose a new line search based on Banach's fixed-point theorem [11], which turns out to work very well for the optimization problem in (8). In the rest of this section, an α is said to be a *fixed point* for function $G(\alpha)$ if $\alpha = G(\alpha)$.

If $f(\alpha)$ in (17) is convex over a region of interest, R_a, and if it has a stationary point α^* in R_a, then α^* is a local minimizer of the problem in (17). The minimizer α^* can be obtained by solving the equation

$$f'(\alpha) = 0 \quad (18)$$

where

$$\begin{aligned} f'(\alpha) &= \frac{dF_{p,\epsilon}(\boldsymbol{\xi}_k + \alpha \boldsymbol{d}_k)}{d\alpha} \\ &= p \sum_{i=1}^{N} \left[\gamma_i(\alpha, \epsilon)^{p/2-1} \cdot (x_i + \alpha v_i) \cdot v_i\right] \quad (19) \end{aligned}$$

and

$$\gamma_i(\alpha, \epsilon) = (x_i + \alpha v_i)^2 + \epsilon^2 \tag{20}$$

with $x_i = x_{si} + \boldsymbol{v}_i^T \boldsymbol{\xi}_k$ and $v_i = \boldsymbol{v}_i^T \boldsymbol{d}_k$.

Using (19), (18) can be written as

$$\alpha = G(\alpha) \tag{21}$$

where

$$G_\epsilon(\alpha) = -\frac{\sum_{i=1}^{N} x_i \cdot v_i \cdot \gamma_i(\alpha, \epsilon)^{p/2-1}}{\sum_{i=1}^{N} v_i^2 \cdot \gamma_i(\alpha, \epsilon)^{p/2-1}} \tag{22}$$

Therefore, finding a minimizer of $f(\alpha)$ amounts to finding a fixed point of function $G_\epsilon(\alpha)$. The well-known Banach fixed-point theorem [11] states that if $G_\epsilon(\alpha)$ is a contraction mapping, i.e.,

$$|G_\epsilon(\alpha_1) - G_\epsilon(\alpha_2)| \le \eta |\alpha_1 - \alpha_2| \quad \text{for any } \alpha_1, \alpha_2 \tag{23}$$

with $\eta < 1$, then there exists a fixed point α^* for function $G_\epsilon(\alpha)$ and α^* can be obtained as the limiting point of sequence $\{\alpha_l, l = 1, 2, \ldots\}$ generated using the recursive relation

$$\alpha_{l+1} = G_\epsilon(\alpha_l) \quad \text{for } l = 1, 2, \ldots \tag{24}$$

Therefore, an approximate solution of (18) can be found by using a sufficient number of recursions of (24). It can be shown that $G_\epsilon(\alpha)$ satisfies the condition in (23) if the magnitude of $G'_\epsilon(\alpha)$ is bounded from above strictly by unity, i.e., $|G'_\epsilon(\alpha)| \le \eta < 1$ [11]. For function $G_\epsilon(\alpha)$ in (22), theoretical verification of the condition imposed on $G'_\epsilon(\alpha)$ turns out to be difficult. Nevertheless, as far as the problem in (8) is concerned, the condition was found to be satisfied in our experiments.

C. Algorithm

The proposed unconstrained regularized ℓ_p-norm (URLP) based algorithm is described in Table I. The input data for the algorithm include a value of $p < 1$, an initial value ϵ_1 satisfying (15), a target value ϵ_J, and the number of iterations J. The algorithm requires the minimum ℓ_2-norm solution $\boldsymbol{x}_s$ and matrix $\boldsymbol{V}_r$ (see (8)) which can be computed using the QR decomposition of $\boldsymbol{\Phi}^T$ as [10]

$$\begin{aligned} \boldsymbol{\Phi}^T &= \boldsymbol{Q} \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{Q} = [\ \boldsymbol{Q}_1 \quad \boldsymbol{Q}_2\ ] \\ \boldsymbol{x}_s &= \boldsymbol{Q}_1 \boldsymbol{R}^{-T} \boldsymbol{y}, \quad \boldsymbol{V}_r = \boldsymbol{Q}_2 \end{aligned} \tag{25}$$

The $J-2$ values of ϵ for which the optimization formulated in (8) is carried out are set between the initial value ϵ_0 and target value ϵ_J as

$$\epsilon_i = e^{-\beta i} \quad \text{for } i = 2, 3, \ldots, J-1 \tag{26}$$

where $\beta = \log(\epsilon_1/\epsilon_J)/(J-1)$.

The line search based on Banach's fixed-point theorem (see Sec. III-B) is used in Step 4 of the algorithm. Starting from an initial value $\alpha = 0$, α is iteratively computed using (24). The details of the line-search algorithm are summarized in Table II.

TABLE I
URLP ALGORITHM

Step 1
Input: p, ϵ_1, ϵ_J, and J.
Set $\boldsymbol{\xi}^{(1)} = \boldsymbol{0}$.
Step 2
Compute ϵ_i for $i = 2, 3, \ldots, J-1$ using (26).
Step 3
Use (25) to compute $\boldsymbol{x}_s$ and $\boldsymbol{V}_r$.
Step 4
Repeat for $k = 1, \ldots, J$
i) Set $\epsilon = \epsilon_k$ and use $\boldsymbol{\xi}^{(k)}$ as an initial point. Apply the BFGS algorithm to solve the problem in (8). Denote the solution as $\boldsymbol{\xi}^{(k+1)}$.
Step 5
Set $\boldsymbol{\xi}^* = \boldsymbol{\xi}^{(J+1)}$, $\boldsymbol{x} = \boldsymbol{x}_s + \boldsymbol{V}_r \boldsymbol{\xi}^*$, and terminate.

TABLE II
LINE SEARCH BASED ON BANACH'S FIXED-POINT THEOREM

Step 1
Input: $\boldsymbol{\xi}_k$, $\boldsymbol{x}_s$, $\boldsymbol{V}_r$, $\boldsymbol{d}_k$, δ_t, and ϵ_k.
Set $l = 1$, $\alpha_1 = 0$, and $\delta_\alpha = \delta_t + 1$.
Step 2
Repeat until $\delta_\alpha < \delta_t$
i) Compute $G_\epsilon(\alpha_l)$ using (22).
ii) Set $\alpha_{l+1} = G_\epsilon(\alpha_l)$.
iii) $\delta_\alpha = \alpha_{l+1} - \alpha_l$.
iv) $l = l + 1$.
Step 3
Set $\alpha = \alpha_l$ and terminate.

IV. EXPERIMENTAL RESULTS

In the first experiment, the signal length N and the number of measurements M were set to 256 and 100, respectively. A total of fifteen values of sparsity K were chosen from 1 to 71 with an increment of 5. A K-sparse signal was constructed by assigning K random values drawn from a normal distribution $\mathcal{N}(0, 1)$ to K randomly selected locations of a zero vector of length N. Measurement matrix $\boldsymbol{\Phi}$ of size $M \times N$ was constructed by drawing its elements from $\mathcal{N}(0, 1)$ followed by a normalization step where each column was normalized to the unit 2 norm. The measurement was obtained as $\boldsymbol{y} = \boldsymbol{\Phi x}$. With $p = 0.1$, $\epsilon_1 = \sqrt{1-p} \cdot \min_{1 \le i \le N} |x_{si}|$, $\epsilon_J = 10^{-5}$, and $J = 9$, the URLP algorithm was applied and compared with BP [4], iterative reweighted (IR) with $p = 0.1$ [6], smoothed ℓ_0 norm (SL0) [7], and the null space reweighted approximate ℓ_0 norm (NRAL0) [8] algorithms. For each algorithm, the signal reconstruction was deemed perfect if the largest magnitude of the components of the reconstruction error vector $\hat{\boldsymbol{x}} - \boldsymbol{x}$ was less than 10^{-5} where $\hat{\boldsymbol{x}}$ and $\boldsymbol{x}$ are the reconstructed and test signals, respectively. For each value of K, the perfect reconstructions were counted over 100 runs. The results are plotted in Figure 1. It is observed that the performance of the URLP algorithm is better than that of the other algorithms. In the second experiment, the average CPU time required by the algorithms to converge was measured over 100 runs for typical instances with $M = N/2$ and $K = \textit{round}(M/2.5)$ where N varied in the range between 128 to 512. The CPU time was

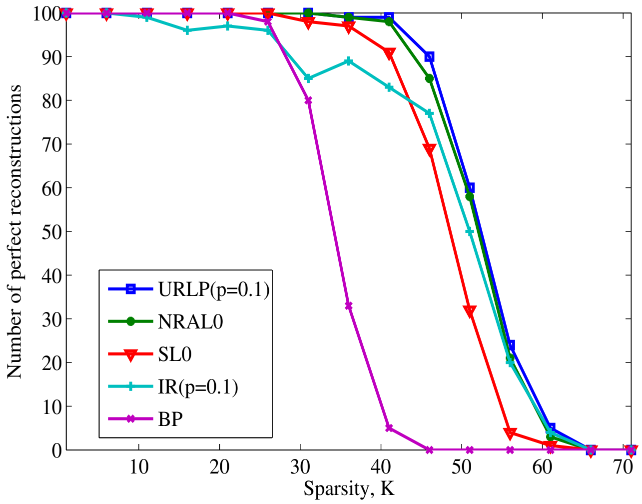

Fig. 1. Number of perfect reconstructions for URLP, NRAL0, SL0, IR, and BP algorithms over 100 runs with $N = 256$, $M = 100$.

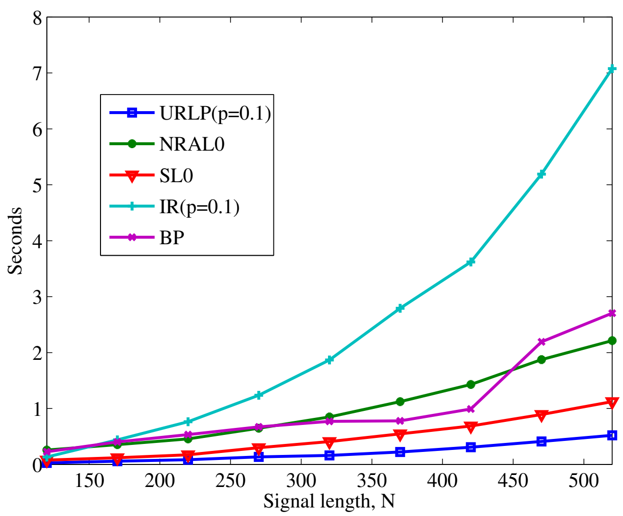

Fig. 2. Average CPU time required for URLP, NRAL0, SL0, IR, and BP algorithms over 100 runs with $M = N/2$, $K = M/2.5$.

measured using a PC desktop with Intel Core 2 CPU 6400 2.13 GHz processor using MATLAB command *cputime*. The results shown in Figure 2 indicate that the URLP algorithm requires the least amount of computation among the algorithms tested.

In the third experiment, the URLP algorithm was run for the same settings as for the first experiment with (a) the proposed line search based on Banach's fixed-point theorem and (b) Fletcher's inexact line search [10]. The reconstruction performance of the proposed algorithm for the two line searches was found to be the same as before. Next, the URLP algorithm was run for the same settings as the second experiment with both line searches. The CPU times required in the two cases for $N = 120$, 220, 320, 420, and 520 are given in Table III. It is noted that the amount of computation required by the URLP algorithm using the proposed line search is less than half of that required when the inexact line search is used. This demonstrates the crucial role of the proposed line search in reducing the numerical complexity of the URLP algorithm.

TABLE III
CPU Time Required by the URLP Algorithm with the Proposed Line Search and the Fletcher's Inexact Line Search, in Seconds

Signal length, N	120	220	320	420	520
Proposed line search	0.1218	0.2824	0.4394	0.6752	1.0678
Inexact line search	0.3844	0.5799	0.9009	1.4882	2.4240

V. Conclusion

We have proposed an algorithm for the reconstruction of sparse signals in the CS framework. The algorithm minimizes a regularized ℓ_p norm in the null space of the measurement matrix. The unconstrained optimization involved is solved using a quasi-Newton algorithm incorporating a new line search based on Banach's fixed-point thorem. Simulation results show that the proposed algorithm yields improved signal reconstruction performance and requires a reduced amount of computation relative to several competing algorithms. Also, the new line search is shown to be numerically more efficient than the conventional inexact line search.

Acknowledgement

The authors would like to thank the Natural Sciences and Engineering Research Council of Canada for supporting this research.

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