Recovery of Sparse Signals from Noisy Measurements Using an ℓ_p-Regularized Least-Squares Algorithm

Jeevan K. Pant, Wu-Sheng Lu, and Andreas Antoniou
Department of Electrical and Computer Engineering, University of Victoria
P.O. Box 3055 STN CSC, Victoria, B.C. V8W 3P6, CANADA
{jkpant, wslu}@ece.uvic.ca, aantoniou@ieee.org

Abstract

A new algorithm for the reconstruction of sparse signals from noise corrupted compressed measurements is presented. The algorithm is based on minimizing an $\ell_{p,\epsilon}$-norm regularized ℓ_2 error. The minimization is carried out by iteratively taking descent steps along the basis vectors of the null space of the measurement matrix and its complement space. The step size is computed using a line search based on Banach's fixed-point theorem. Simulation results are presented, which demonstrate that the proposed algorithm yields improved reconstruction performance and requires a reduced amount of computation relative to several known algorithms.

1 Introduction

Compressive sensing (CS) ([1]-[3]) deals with the acquisition of *sparse signals* using a small number of measurements and the reconstruction of the signal from these measurements. One of the most successful algorithms for CS is *basis pursuit* (BP) which is based on ℓ_1-norm minimization [4]. Improved algorithms include the iteratively reweighted (IR) algorithm which is based on ℓ_p-norm minimization with $p < 1$ [5], the *smoothed ℓ_0-norm* (SL0) minimization algorithm [6], and the *unconstrained regularized ℓ_p-norm* (URLP) minimization algorithm [7]. A variant of BP known as *basis pursuit denoising* (BPDN) [4] was found to be particularly effective for signal reconstruction in the case of noise-corrupted measurements.

In this paper, we propose a new algorithm for the reconstruction of sparse signals in the CS framework where the measurements are corrupted by additive noise. The algorithm is based on minimizing an $\ell_{p,\epsilon}$-norm regularized ℓ_2 error with $p < 1$. By working in the null space of the measurement matrix and its orthogonal complement, descent directions for the $\ell_{p,\epsilon}$-norm regularized objective function can be efficiently computed. These principles along with Banach's fixed-point theorem [8] can be used to construct an efficient line search for the proposed algorithm. Simulation results are presented which demonstrate that the proposed algorithm yields improved reconstruction performance and requires a reduced amount of computation relative to several known algorithms.

2 Preliminaries

Let a real-valued discrete-time signal $\boldsymbol{x}$ of length N which has K nonzero components with $K \ll N$. Such a signal is said to be K-sparse. The measurement process in the CS framework can be modelled as

$$\boldsymbol{y} = \boldsymbol{\Phi x} + \boldsymbol{w} \tag{1}$$

where $\boldsymbol{y}$ is a measurement vector of length M, $\boldsymbol{\Phi}$ is a measurement matrix of size $M \times N$ with $M < N$, and $\boldsymbol{w}$ is a Gaussian noise vector with zero mean and variance σ^2. A tractable approach to recover $\boldsymbol{x}$ from $\boldsymbol{y}$ is to use BPDN [4] which entails solving the convex problem

$$\hat{\boldsymbol{x}} = \underset{\boldsymbol{x}}{\text{minimize}} \quad \frac{1}{2}\,||\boldsymbol{\Phi x} - \boldsymbol{y}||_2^2 + \lambda ||\boldsymbol{x}||_1 \tag{2}$$

Parameter λ is an appropriate regularization parameter and $||\boldsymbol{x}||_1$ is the ℓ_1 norm of $\boldsymbol{x}$.

Several algorithms for signal reconstruction from both noise-free and noisy measurements have been developed. One such algorithm is the ℓ_p-norm minimization based IR algorithm studied in [5] which solves the problem

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad ||\boldsymbol{x}||_p^p \quad \text{subject to} \quad \boldsymbol{\Phi x} = \boldsymbol{y} \tag{3}$$

with $p < 1$ where $||\boldsymbol{x}||_p$ is the ℓ_p norm of $\boldsymbol{x}$. In the URLP algorithm introduced in [7], a sparse signal is obtained as

$$\boldsymbol{x}^* = \boldsymbol{x}_s + \boldsymbol{V}_n \boldsymbol{\xi}^* \tag{4}$$

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where $\boldsymbol{x}_s$ is a solution of $\boldsymbol{\Phi x} = \boldsymbol{y}$, $\boldsymbol{V}_n$ is an $N \times (N-M)$ matrix composed of an orthonormal basis of the null space of $\boldsymbol{\Phi}$, and $\boldsymbol{\xi}^*$ is a vector obtained as

$$\boldsymbol{\xi}^* = \arg\min_{\boldsymbol{\xi}} \quad \sum_{i=1}^{N} \left[\left(x_{si} + \boldsymbol{v}_i^T \boldsymbol{\xi} \right)^2 + \epsilon^2 \right]^{p/2} \qquad (5)$$

where x_{si} is the ith component of $\boldsymbol{x}_s$, $\boldsymbol{v}_i^T$ is the ith row of matrix $\boldsymbol{V}_n$, and $p < 1$.

3 Minimization of the ℓ_p Regularized ℓ_2 Error

3.1 Problem formulation

We proposed to reconstruct a sparse signal from noisy measurements by solving the optimization problem

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad \frac{1}{2} ||\boldsymbol{\Phi x} - \boldsymbol{y}||_2^2 + \lambda ||\boldsymbol{x}||_{p,\epsilon}^p \qquad (6)$$

where $\lambda > 0$ is a fixed regularization parameter and $||\boldsymbol{x}||_{p,\epsilon}$ is a regularized ℓ_p norm of $\boldsymbol{x}$ defined as

$$||\boldsymbol{x}||_{p,\epsilon}^p = \sum_{i=1}^{N} (x_i^2 + \epsilon^2)^{p/2} \qquad (7)$$

With $\epsilon > 0$, function $||\boldsymbol{x}||_{p,\epsilon}$ is differentiable. Also, parameter ϵ is helpful in dealing with the local minima of the problem in (6).

Let $\boldsymbol{\Phi} = \boldsymbol{U} [\boldsymbol{\Sigma}\ \boldsymbol{0}] \boldsymbol{V}^T$ be the singular-value decomposition (SVD) of $\boldsymbol{\Phi}$. Matrix $\boldsymbol{V}$ can be expressed as $\boldsymbol{V} = [\boldsymbol{V}_r\ \boldsymbol{V}_n]$ where $\boldsymbol{V}_n$ is composed of the last $N-M$ columns of $\boldsymbol{V}$, which span the null space of $\boldsymbol{\Phi}$, while $\boldsymbol{V}_r$ is composed of the first M columns of $\boldsymbol{V}$, which span the orthogonal complement of the null space. This orthogonal complement is also known as the row space of $\boldsymbol{\Phi}$ in the literature.

By using the columns of $\boldsymbol{V} = [\boldsymbol{V}_r\ \boldsymbol{V}_n]$ as a set of orthonormal basis vectors, we can express a signal $\boldsymbol{x}$ of length N as

$$\boldsymbol{x} = \boldsymbol{V}_r \boldsymbol{\phi} + \boldsymbol{V}_n \boldsymbol{\xi} \qquad (8)$$

where $\boldsymbol{\phi}$ and $\boldsymbol{\xi}$ are vectors of length M and $N-M$, respectively. When measurement vector $\boldsymbol{y}$ is not corrupted by noise, vector $\boldsymbol{\phi}$ can be evaluated as

$$\boldsymbol{\phi} = \boldsymbol{\Sigma}^{-1} \boldsymbol{U}^T \boldsymbol{y} \qquad (9)$$

If measurement $\boldsymbol{y}$ is corrupted by noise, then vector $\boldsymbol{\phi}$ obtained from (9) is not optimal in general and we shall consider both $\boldsymbol{\phi}$ and $\boldsymbol{\xi}$ as independent variables.

Using the SVD of $\boldsymbol{\Phi}$, we simplify the ℓ_2 term in (6) as

$$\begin{aligned} \frac{1}{2} ||\boldsymbol{\Phi x} - \boldsymbol{y}||_2^2 &= \frac{1}{2} ||\boldsymbol{\Sigma \phi} - \tilde{\boldsymbol{y}}||_2^2 \\ &= \frac{1}{2} \sum_{i=1}^{M} (\sigma_i \phi_i - \tilde{y}_i)^2 \end{aligned} \qquad (10)$$

where σ_i is the ith singular value of $\boldsymbol{\Phi}$, ϕ_i is the ith component of vector $\boldsymbol{\phi}$, and $\tilde{y}_i$ is the ith component of vector $\tilde{\boldsymbol{y}} = \boldsymbol{U}^T \boldsymbol{y}$.

Using (8) and (10), we recast the optimization problem in (6) as

$$\underset{\boldsymbol{\phi}, \boldsymbol{\xi}}{\text{minimize}} \quad F_{p,\epsilon}(\boldsymbol{\phi}, \boldsymbol{\xi}) \qquad (11)$$

where

$$F_{p,\epsilon}(\boldsymbol{\phi}, \boldsymbol{\xi}) = \frac{1}{2} \sum_{i=1}^{M} (\sigma_i \phi_i - \tilde{y}_i)^2 + \lambda ||\boldsymbol{x}||_{p,\epsilon}^p \qquad (12)$$

with $\boldsymbol{x}$ given in (8).

Below, we propose an algorithm for the solution of the optimization problem in (11).

3.2 Computation of descent direction

In the kth iteration of the proposed algorithm, signal $\boldsymbol{x}^{(k)}$ is updated to

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \alpha \boldsymbol{d}^{(k)} \qquad (13)$$

where

$$\boldsymbol{x}^{(k)} = \boldsymbol{V}_r \boldsymbol{\phi}^{(k)} + \boldsymbol{V}_n \boldsymbol{\xi}^{(k)} \qquad (14a)$$

$$\boldsymbol{d}^{(k)} = \boldsymbol{V}_r \boldsymbol{d}_r^{(k)} + \boldsymbol{V}_n \boldsymbol{d}_n^{(k)} \qquad (14b)$$

and $\alpha > 0$. The scalar α is determined using a line search (see Sec. 3.3), and the updating vectors $\boldsymbol{d}_r$ and $\boldsymbol{d}_n$ assume the forms

$$\boldsymbol{d}_r^{(k)} = \left[\delta_{r,1}^{(k)}\ \delta_{r,2}^{(k)}\ \cdots\ \delta_{r,M}^{(k)} \right]^T \qquad (15a)$$

and

$$\boldsymbol{d}_n^{(k)} = \left[\delta_{n,1}^{(k)}\ \delta_{n,2}^{(k)}\ \cdots\ \delta_{n,N-M}^{(k)} \right]^T \qquad (15b)$$

Vectors $\boldsymbol{d}_r^{(k)}$ and $\boldsymbol{d}_n^{(k)}$ in (15) are determined by minimizing the objective function $F_{p,\epsilon}(\boldsymbol{\phi}, \boldsymbol{\xi})$ along each of the directions defined by the column vectors of $[\boldsymbol{V}_r\ \boldsymbol{V}_n]$. In doing so, $\boldsymbol{d}_r^{(k)}$ and $\boldsymbol{d}_n^{(k)}$ become descent directions of $F_{p,\epsilon}$ and their components are found to be

$$\delta_{r,i}^{(k)} = -\frac{-\sigma_i u_i + \lambda p s_i}{\sigma_i^2 + \lambda p \beta_i} \qquad (16a)$$

and

$$\delta_{n,i}^{(k)} = -\frac{s_i}{\beta_i} \qquad (16b)$$

where $u_i = \tilde{y}_i - \sigma_i \phi_i$,

$$s_i = \sum_{j=1}^{N} x_j^{(k)} v_{ij} \gamma_j(\epsilon) \qquad (17a)$$

$$\beta_i = \sum_{j=1}^{N} v_{ij}^2 \gamma_j(\epsilon) \qquad (17b)$$

In (17a) and (17b), $x_j^{(k)}$ is the jth component of vector $\boldsymbol{x}^{(k)}$ and v_{ij} is the jth component of vector $\boldsymbol{v}_i$ where $\boldsymbol{v}_i$ is the ith column of matrix $\boldsymbol{V}_r$ if s_i and β_i are computed for Eq. (16a) and the ith column of matrix $\boldsymbol{V}_n$ if s_i and β_i are computed for Eq. (16b), and

$$\gamma_j(\epsilon) = \left[\left(x_j^{(k)}\right)^2 + \epsilon^2\right]^{p/2-1} \tag{18}$$

Eqs. (16)–(18) are derived in the Appendix.

3.3 Line search

By using a line search based on Banach's fixed-point theorem [8], the step size α can be computed as

$$\alpha = -\frac{q_1 + \lambda p q_2}{q_3 + \lambda p q_4} \tag{19}$$

where

$$q_1 = \sum_{j=1}^{M} \left(\sigma_j \phi_j^{(k)} - \tilde{y}_j\right) \sigma_j d_{rj}^{(k)} \tag{20a}$$

$$q_2 = \sum_{j=1}^{N} x_j^{(k)} d_{vj}^{(k)} \gamma_j(\alpha, \epsilon) \tag{20b}$$

$$q_3 = \sum_{j=1}^{M} \left(\sigma_j d_{rj}^{(k)}\right)^2 \tag{20c}$$

$$q_4 = \sum_{j=1}^{N} \left(d_{vj}^{(k)}\right)^2 \gamma_j(\alpha, \epsilon) \tag{20d}$$

In (20), $\phi_j^{(k)}$, $d_{rj}^{(k)}$, $x_j^{(k)}$, and $d_{vj}^{(k)}$ are the jth components of $\boldsymbol{\phi}^{(k)}$, $\boldsymbol{d}_r^{(k)}$, $\boldsymbol{x}^{(k)}$, and $\boldsymbol{d}_v^{(k)}$, respectively, and

$$\gamma_j(\alpha, \epsilon) = \left[\left(x_j^{(k)} + \alpha d_{vj}^{(k)}\right)^2 + \epsilon^2\right]^{p/2-1}$$

Step size α can be obtained through a finite number of iterations of the recursive formula in (19). Details of the line search algorithm can be found in [7].

3.4 Optimization

From (11) and (7), we note that the objective function in (11) is dependent on parameter ϵ. It turns out that the area of the region where the objective function is convex is proportional to the value of ϵ, namely, the larger the ϵ, the larger the convex region. Thus if a sufficiently large value of ϵ is used, the proposed algorithm will locate the global solution of the current objective function. On the other hand, if a very small value of ϵ is used, the objective function in (11) will approach the true value of the ℓ_2, ℓ_p objective function but it will become nonconvex and, consequently, it will have many suboptimal solutions. A good optimal solution can be obtained by using a sequential optimization whereby a series of objective functions are minimized starting with a large value of ϵ and gradually decreasing ϵ to a very small value. The detailed steps of such a sequential optimization are as follows:

- First, set ϵ to a large value, say, ϵ_1, typically $0.5 \leq \epsilon_1 \leq 1$, and initialize $\boldsymbol{\phi}$ and $\boldsymbol{\xi}$ to zero vectors.
- Solve the optimization problem in (11) by i) computing descent directions $\boldsymbol{d}_v$ and $\boldsymbol{d}_r$, ii) computing the step size α; and iii) updating solution $\boldsymbol{x}$ and coefficient vector $\boldsymbol{\phi}$.
- Reduce ϵ to a smaller value and again solve the problem in (11).
- Repeat this procedure until a small target value, say, ϵ_J, is reached.
- Output $\boldsymbol{x}$ as the solution.

3.5 Algorithm

The proposed $\ell_{p,\epsilon}$-norm regularized least-squares (LPeLS) algorithm for reconstructing sparse signals from compressed measurements is summarized in Table 1. The regularization parameter λ, number of iterations J, initial value ϵ_1, final value ϵ_J, and parameter p are supplied in Step 1. The algorithm uses the SVD to compute the singular values $\sigma_1, \sigma_2, \ldots, \sigma_M$ of $\boldsymbol{\Phi}$ and matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ whose columns are, respectively, the left and right singular vectors of $\boldsymbol{\Phi}$. The evaluation of the SVD is computationally demanding for measurement matrices of larger sizes. However, the computation can be performed offline and the resulting matrices can be stored and reused while reconstructing the signal.

The $J-2$ values of ϵ lying between the initial value ϵ_1 and final value ϵ_J are computed as

$$\epsilon_i = e^{-\beta i} \quad \text{for} \quad i = 1, 2, \ldots, J-1 \tag{21}$$

where $\beta = \log(\epsilon_1/\epsilon_J)/(J-1)$.

The computation of the step size using (19) in Step 4 requires vector $\boldsymbol{\phi}^{(k)}$ which is computed as

$$\boldsymbol{\phi}^{(k)} = \boldsymbol{V}_r^T \boldsymbol{x}^{(k)} \tag{22}$$

4 Experimental Results

Two experiments were performed to investigate the performance of the proposed algorithm. In the first experiment,

Table 1. LPeLS Algorithm

Step 1
Input: λ, p, ϵ_1, ϵ_J, J, L, $\mathbf{\Phi}$ and $\boldsymbol{y}$.
Set $\boldsymbol{x}^{(1)} = \mathbf{0}$ and $k = 1$.
Step 2
Compute ϵ_j for $j = 2, 3, \ldots, J-1$ using (21).
Step 3
Compute the SVD of $\mathbf{\Phi}$ to obtain $\boldsymbol{U}$, $\mathbf{\Sigma}$, $\boldsymbol{V}_r$ and $\boldsymbol{V}_n$.
Step 4
Repeat for $j = 1, 2, \ldots, J$
- i) Set $\epsilon = \epsilon_j$.
- ii) Repeat for $l = 1, 2, \ldots, L$
  - a) Use $\boldsymbol{x}^{(k)}$ as an initial value and compute $\boldsymbol{\phi}^{(k)}$ using Eq. (22).
  - b) Compute $\boldsymbol{d}^{(k)}$ using Eq. (14b).
  - c) Compute α using the line search based on Banach's fixed-point theorem using Eq. (19).
  - e) Compute $\boldsymbol{x}^{(k+1)}$ using Eq. (13).
  - f) $k = k + 1$.

Step 5
Set $\boldsymbol{x} = \boldsymbol{x}^{(k)}$ and stop.

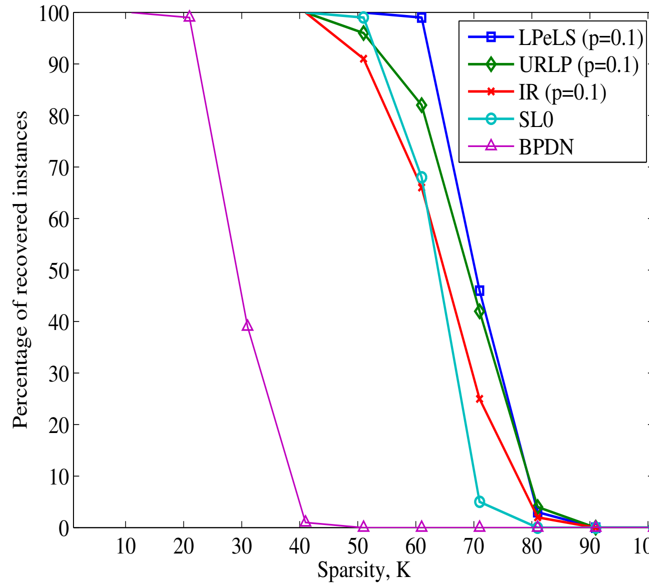

Figure 1. Percentage of recovered instances for the LPeLS, URLP, IR, SL0, and BPDN algorithms over 100 **runs with** $N = 1024$**,** $M = 200$**.**

the signal length N and the number of measurements M were set to 1024 and 200, respectively. A total of eleven values of sparsity K were chosen from 1 to 101 with an increment of 10. A K-sparse signal $\boldsymbol{x}$ with energy value 100 was constructed as follows: i) a vector $\boldsymbol{x}$ of length N with all zero components was constructed, ii) a random vector of length K was constructed by drawing its components from a normal distribution $\mathcal{N}(0, 1)$ followed by a normalization step so that the ℓ_2 norm of the resulting vector is $\sqrt{100}$, and iii) the components of the resulting vector were set to randomly chosen K locations of vector $\boldsymbol{x}$. A measurement matrix $\mathbf{\Phi}$ of size $M \times N$ was constructed by drawing its elements from $\mathcal{N}(0, 1)$ followed by an orthonormalization step where the rows of $\mathbf{\Phi}$ were made orthonormal with each other. The measurement was obtained as $\boldsymbol{y} = \mathbf{\Phi}\boldsymbol{x} + \boldsymbol{w}$ where noise vector $\boldsymbol{w}$ was constructed by drawing its components from $\mathcal{N}(0, 0.01)$. The proposed LPeLS algorithm was used to reconstruct $\boldsymbol{x}$ from $\boldsymbol{y}$ with $p = 0.1$, $\lambda = 0.0008$, $\epsilon_1 = 0.8$, $\epsilon_J = 10^{-2}$, $J = 30$, and $L = 5$. The reconstruction performance of the LPeLS algorithm was compared with that of the BPDN [4], unconstrained regularized ℓ_p (URLP) [7] with $p = 0.1$, iterative reweighted (IR) with $p = 0.1$ [5], and smoothed ℓ_0 norm (SL0) [6] algorithms. For each algorithm, the signal was deemed reconstructed if the signal-to-noise ratio value, measured as $20\log_{10}(||\boldsymbol{x}||_2/||\boldsymbol{x} - \hat{\boldsymbol{x}}||_2)$, was greater than 27 dB where $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ are the initial and reconstructed signals, respectively. The results are shown in Figure 1. As can be seen, the performance of the LPeLS algorithm is better than that of the other algorithms.

In the second experiment, signal length N was varied in the range 128 to 512 where $M = N/2$ and $K = round(M/2.5)$. We constructed a measurement matrix $\mathbf{\Phi}$ and five K-sparse signals $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, $\boldsymbol{x}_4$, and $\boldsymbol{x}_5$ each with different values and locations of nonzero components. Five noisy measurements $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, $\boldsymbol{y}_3$, $\boldsymbol{y}_4$, and $\boldsymbol{y}_5$ were obtained by multiplying the sparse signals by $\mathbf{\Phi}$ and adding five different noise vectors constructed by drawing their components from $\mathcal{N}(0, 0.01)$. The LPeLS, URLP, IR, SL0, and BPDN algorithms were used to reconstruct signals from all five measurements and the CPU times required by the various algorithms to reconstruct the five signals were measured. For the proposed LPeLS algorithm, the SVD was performed once and the resulting matrices were reused while reconstructing five signals. For the URLP algorithm, the QR decomposition was performed once for all five reconstructions (see [7] for details). For the IR and SL0 algorithms, the pseudo-inverse of $\mathbf{\Phi}$ was computed only once as $\mathbf{\Phi}^T\left(\mathbf{\Phi}\mathbf{\Phi}^T\right)$ and reused for five signal reconstructions. The CPU times were measured using a PC desktop with Intel Core 2 CPU 6400 2.13 GHz processor using MATLAB command *cputime*. The results are shown in Figure 2. We observe that the LPeLS algorithm requires much less CPU time than that required by the URLP and IR algorithms, slightly less than that required by the BPDN algorithm, and slightly more than that required by the SL0 algorithm.

We should point out that the use of the SVD to com-

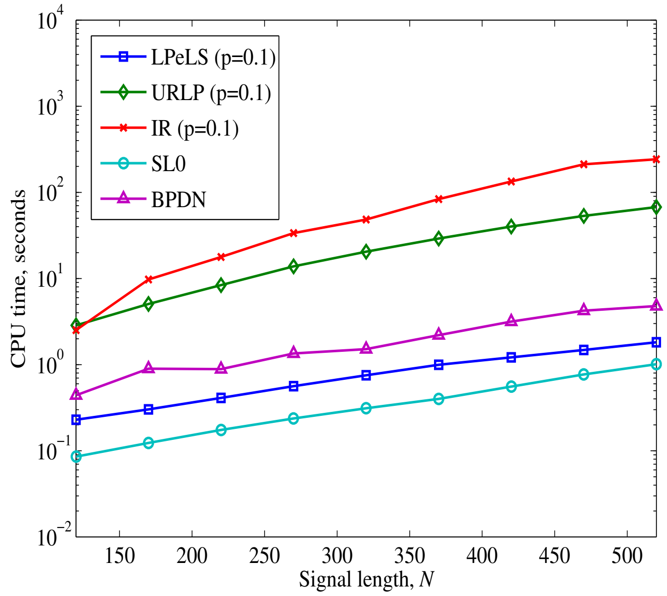

Figure 2. Average CPU time required by the LPeLS, URLP, IR, SL0, and BPDN algorithms over 100 **runs with** $M = N/2$**,** $K = M/2.5$**.**

pute matrices $\boldsymbol{U}$, $\boldsymbol{\Sigma}$, $\boldsymbol{V}_r$, and $\boldsymbol{V}_n$ in Step 3 of the algorithm is computationally expensive for data of moderate to large sizes. Below we discuss three cases where the computational burden can be reduced or eliminated.

- In CS, a measurement matrix is usually reused for both sensing and reconstructing. In such applications, the SVD can be computed offline, and vector $\tilde{y}$, matrices $\boldsymbol{V}_r$ and $\boldsymbol{V}_n$, and singular values $\sigma_1, \sigma_2, \ldots, \sigma_M$ can be stored and reused for the reconstruction process.

- In some applications, the measurement matrix $\boldsymbol{\Phi}$ is constructed by selecting a number of rows of a random orthonormal matrix $\boldsymbol{R}$. In these applications, we can use $\boldsymbol{V}_r = \boldsymbol{\Phi}^T$ and $\boldsymbol{V}_n = \boldsymbol{\Psi}^T$ where $\boldsymbol{\Psi}$ is formed by using the remaining rows of $\boldsymbol{R}$; on the other hand, matrix $\boldsymbol{U}$ is the identity matrix and the singular values $\sigma_1, \sigma_2, \ldots, \sigma_M$ are all equal to unity.

- When measurements are taken as a set of samples of a standard transform of the signal such as the Fourier, DCT, or orthogonal wavelet transform, $\boldsymbol{W}$ can be taken to be the orthogonal transfrom matrix. In such cases, the measurement matrix $\boldsymbol{\Phi}$ is composed of a number of rows of $\boldsymbol{W}$. Consequently, we can assign $\boldsymbol{V}_r = \boldsymbol{\Phi}^T$ and $\boldsymbol{V}_n = \boldsymbol{\Psi}^T$ where $\boldsymbol{\Psi}$ is formed by using the remaining rows of $\boldsymbol{W}$. In these applications matrix $\boldsymbol{U}$ is the identity matrix and the singular values are all equal to unity.

5 Conclusion

We have proposed an algorithm for the reconstruction of sparse signals from noise-corrupted compressed measurements. The algorithm minimizes an $\ell_{p,\epsilon}$-norm regularized ℓ_2 error by iteratively taking steps along the basis vectors of the null space of the measurement matrix and its complement space. The step size is determined using a line sarch based on Banach's fixed-point theorem. Simulation results show that the proposed algoirthm yields improved signal reconstruction performance and requires a reduced amount of computation relative to several known algorithms.

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Appendix

Eqs. (16)–(18) can be derived as detailed below.

Let $\boldsymbol{e}_i$ be a vector of length M whose ith component is unity and the rest of its components are zero. Provided that $\boldsymbol{\phi}$ and $\boldsymbol{\xi}$ are given, a descent direction along the ith component of vector $\boldsymbol{\phi}$ in (12) can be obtained by solving the one-dimensional optimization problem

$$\underset{\delta}{\text{minimize}} \quad F(\delta) \tag{23}$$

where

$$\begin{aligned} F(\delta) &= \frac{1}{2}\,||\boldsymbol{\Sigma}\,(\boldsymbol{\phi} + \delta \boldsymbol{e}_i) - \tilde{\boldsymbol{y}}||_2^2 + \lambda\,||\boldsymbol{x} + \delta \boldsymbol{v}_i||_{p,\epsilon}^p \\ &= \frac{1}{2}\,[\sigma_i(\phi_i + \delta) - \tilde{y}_i]^2 \\ &\quad + \lambda \sum_{j=1}^{N} \left[(x_j + \delta v_{ij})^2 + \epsilon^2\right]^{p/2} \end{aligned} \tag{24}$$

In Eq. (24) $\boldsymbol{x} = \boldsymbol{V}_r\boldsymbol{\phi} + \boldsymbol{V}_n\boldsymbol{\xi}$, $\boldsymbol{v}_i$ is the ith column of vector $\boldsymbol{V}_r$, and x_j and v_{ij} are the jth components of vectors $\boldsymbol{x}$ and $\boldsymbol{v}_i$, respectively. By equating the first-order derivative of $F(\delta)$ to zero, we obtain

$$\delta = -\frac{-\sigma_i \tilde{y}_i + \sigma_i^2 \phi_i + \lambda p \sum_{j=1}^{N} \gamma_j(\epsilon) x_j v_{ij}}{\sigma_i^2 + \lambda p \sum_{j=1}^{N} \gamma_j(\epsilon) v_{ij}^2} \tag{25}$$

where

$$\gamma_j(\epsilon) = \left[(x_j + \delta v_{ij})^2 + \epsilon^2\right]^{p/2-1} \tag{26}$$

Now Eq. (25) is a fixed-point equation which can be used to determine δ iteratively. A descent step can, however, be determined using only the first iteration of (25), which can be done by using $\delta = 0$ in the right-hand side of (26) to compute δ. With $\delta = 0$, Eq. (26) simplifies to (18) and Eq. (16a) follows from (25).

For the descent directions along the components of vector $\boldsymbol{\xi}$ in (12) and (8), the fidelity term becomes a constant. As a result, $-\sigma_l \tilde{y}_l$ and $\sigma_l^2 \phi_l$ in the numenerator and σ_i^2 in the denominator of (25) can be deleted. Consequently, from (25) Eq. (16b) can be obtained.

References

[1] E. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information," *IEEE Trans. Inf. Theory*, vol. 52, pp. 489-509, Feb. 2006.

[2] D. L. Donoho, "Compressed sensing," *IEEE Trans. Inf. Theory*, vol. 52, pp. 1289-1306, Apr. 2006.

[3] E. Candès and T. Tao, "Near-optimal signal recovery from random projections: universal encoding strategies," *IEEE Trans. Inf. Theory*, vol. 52, pp. 5406-5425, Dec. 2006.

[4] S. S. Chen, D. L. Donoho, and M. A. Saunders, "Atomic decomposition by basis pursuit," *SIAM J. Scientif. Comput.*, vol. 20, no. 1, pp. 33-61, 1999.

[5] R. Chartrand and W. Yin, "Iteratively reweighted algorithms for compressive sensing," in *Proc. IEEE Inter. Conf. Acoustics, Speech, Signal Process.*, pp. 3869-3872, 2008.

[6] H. Mohimani, M. Babie-Zadeh, and C. Jutten, "A fast approach for overcomplete sparse decomposition based on smoothed ℓ_0-norm," *IEEE Trans. Signal Process.*, vol. 57, pp. 289-301, Jan. 2009.

[7] J. K. Pant, W.-S. Lu, and A. Antoniou, "Unconstrained regularized ℓ_p-norm based algorithm for the reconstruction of sparse signals," in *Proc. IEEE Inter. Symp. Circuits and Syst.* 2011.

[8] R. P. Agrawal, M. Meehan, and D. O'Regan, *Fixed Point Theory and Applications,* Cambridge University Press, 2001.