Variable Fractional Delay FIR Filters with Sparse Coefficients

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***Abstract*—Implementing a variable fractional delay (VFD) filter in Farrow model is costly as each coefficient of a VFD filter is a polynomial rather than a numerical scalar as in a conventional digital filter. This paper presents a method for the design of VFD filters with sparse coefficients which admits efficient implementation. The design is accomplished in two phases with the first phase identifying locations in polynomial impulse response that are suitable to be set to zero and the second phase optimizing the remaining nonzero coefficients so as for the VFD filter to best approximate a desired frequency response. Performance evaluation and comparison of the proposed algorithm relative to an equivalent nonsparse counterpart are also presented.**

I. Introduction

Variable fractional delay (VFD) filters refer to digital filters capable of changing group delay continuously by tuning parameter(s) contained in filter coefficients. It is an important class of digital filters as it finds many applications in engineering and science, and algorithms for its design have been proposed, see [1]–[7] and the references cited therein. This paper addresses VFD FIR filters with sparse coefficients. Several authors have recently investigated sparse digital filters, these include general sparse FIR filters using convex minimization [8], [9] and sparse differentiators using orthogonal matching pursuit [10]. The work reported here may be considered as an extension of our effort made in [9] and is motivated by the fact that there are considerably more filter coefficients in a VFD filter relative to that in a conventional FIR filter because each coefficient of a VFD filter (in Farrow model [2]) is a polynomial rather than a numerical scalar. Consequently, implementing a VFD filter of even a moderate order requires a large amount of computation. In this regard, sparse VFD filters become attractive alternatives as they admit substantially more efficient implementations. We present a two-phase design method which is in spirit similar to that in [9], but the formulation as well as the solution method of the convex problems involved in the design method are quite different from [9]. Performance evaluation and comparison of the proposed algorithm relative to an equivalent nonsparse counterpart are presented through computer simulation studies.

II. VFD FIR Filters and Design Proplem

We adopt standard notation to denote the transfer function of a VFD FIR filter by

$$H(z,p)=\sum_{n=0}^{N} a_n(p) z^{-n} \tag{1}$$

where

$$a_n(p)=\sum_{k=0}^{K} a_{nk} p^k \tag{2}$$

for $n=0,\ 1,\ \ldots, N$ are polynomials of degree K with $p \in [0,\ 1]$ representing the fractional delay. If we let

$$\begin{aligned}\boldsymbol{\omega} &= [1\ \ e^{-j\omega}\ \cdots\ e^{-jN\omega}]^T\\ \boldsymbol{p} &= [1\ \ p\ \ \cdots\ \ p^K]^T\\ \boldsymbol{A} &= [a_{ij}] \in R^{(N+1)\times(K+1)}\end{aligned}$$

then the frequency response of the VFD filter can be expressed as

$$H(\omega,p)=\boldsymbol{\omega}^T \boldsymbol{A}\boldsymbol{p} \tag{3}$$

In designing a VFD filter, the desired variable frequency response is given as

$$H_d(\omega,p)=e^{-j\omega(D+p)} \tag{4}$$

where $0 \le p \le 1$ is the same parameter p encountered in (1)–(3) representing a fractional group delay in addition to an integer delay D which is typically chosen to be $(N-1)/2$ for an odd filter order N or $N/2$ for an even N. A weighted least-squares (WLS) design of a sparse VFD FIR filter of order N refers to finding a transfer function $H(z,p)$ of form (1) that minimizs WLS error

$$J(\boldsymbol{A}) = \frac{1}{2}\int_0^{\pi}\int_0^{1} W(\omega,p)|H(\omega,p)-H_d(\omega,p)|^2\, dp\, d\omega \tag{5a}$$

$$\text{subject to: sparsity}(\boldsymbol{A}) = N_z \tag{5b}$$

where N_z is an integer and sparsity$(\boldsymbol{A})$ denotes the number of zero entries in $\boldsymbol{A}$.

III. Design Method

This section presents a two-phase design method for sparse VFD filters. To begin with, we recall a closed-form expression [5] of $J(\boldsymbol{A})$ given by

$$J(\boldsymbol{A})=\frac{1}{2}\text{tr}(\boldsymbol{P}\boldsymbol{A}^T\boldsymbol{\Omega}\boldsymbol{A})-\text{tr}(\boldsymbol{S}\boldsymbol{A})+\text{const.} \tag{6}$$

where matrices $\boldsymbol{P}$, $\boldsymbol{\Omega}$ and $\boldsymbol{S}$ are related to weighting function $W(\omega,p)$. Suppose $W(\omega,p)$ is separable, namely $W(\omega,p) =$

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$W_1(\omega)W_2(p)$, then

$$P = \int_0^1 W_2(p)\boldsymbol{p}\boldsymbol{p}^T\,dp \quad (7a)$$

$$\boldsymbol{\Omega} = \text{Re}\left[\int_0^\pi W_1(\omega)\bar{\boldsymbol{\omega}}\boldsymbol{\omega}^T\,d\omega\right] \quad (7b)$$

$$\boldsymbol{S} = \int_0^1 W_2(p)\boldsymbol{p}\boldsymbol{\omega}_p^T\,dp \quad (7c)$$

with

$$\boldsymbol{\omega}_p^T = \text{Re}\left[\int_0^\pi W_1(\omega)\boldsymbol{\omega}^T e^{j\omega(D+p)}d\omega\right] \quad (7d)$$

As the condition numbers of $\boldsymbol{P}$ and $\boldsymbol{\Omega}$ can be very large even for a VFD filter of moderate order, Cholesky decompositions $\boldsymbol{P} = \boldsymbol{P}_1^T\boldsymbol{P}_1$ and $\boldsymbol{\Omega} = \boldsymbol{\Omega}_1^T\boldsymbol{\Omega}_1$ are used in [5] in the minimization of $J(\boldsymbol{A})$. Note that the Cholesky decompositions lead (6) (up to a constant) to

$$J(\boldsymbol{A}) = \frac{1}{2}\,\text{tr}\left(\boldsymbol{P}_1\boldsymbol{A}^T\boldsymbol{\Omega}_1^T\boldsymbol{\Omega}_1\boldsymbol{A}\boldsymbol{P}_1^T\right) - \text{tr}(\boldsymbol{S}\boldsymbol{A}) \quad (8)$$

If we denote by $\boldsymbol{a}$ and $\boldsymbol{s}$ the vectors generated by concatenating the columns of $\boldsymbol{A}$ and $\boldsymbol{S}$, respectively, via straightforward linear algebraic manipulations it can be shown that up to a constant $J(\boldsymbol{A})$ in (8) can be written as

$$J(\boldsymbol{a}) = \frac{1}{2}\|\boldsymbol{\Gamma}\boldsymbol{a} - \boldsymbol{y}\|_2^2,\ \boldsymbol{\Gamma} = \boldsymbol{P}_1 \otimes \boldsymbol{\Omega}_1,\ \boldsymbol{y} = \boldsymbol{\Gamma}^{-T}\boldsymbol{s} \quad (9)$$

where $\otimes$ denotes Kronecker product. We are now in a position to present algorithmic details of the design method.

A. Design Phase One

Given order (N, K), sparsity N_z and desired frequency response $H_d(\omega, p)$, phase 1 of the design identifies an index set of most adequate locations in coefficient matrix $\boldsymbol{A}$ to be set to zero that satisfies the target sparsity. Under the circumstances, it is natural to formulate the problem at hand as

$$\text{minimize}\ \ \mu\|\boldsymbol{a}\|_1 + J(\boldsymbol{A}) \quad (10)$$

where $\|\boldsymbol{a}\|_1 = \sum_{i=1}^{(K+1)(N+1)} |a_i|$ is the l_1-norm of $\boldsymbol{a} = \boldsymbol{A}(:)$ and $\mu > 0$ is a weight that controls the regularization level. By performing (10), one obtains a coefficient matrix $\hat{\boldsymbol{A}}$ that is sparser than that obtained by a conventional WLS design because of the presence of term $\mu\|\boldsymbol{a}\|_1$ which is known to promote signal sparsity [11] and, because of the presence of term $J(\boldsymbol{A})$, $\hat{\boldsymbol{A}}$ as the minimizer of (10) represents a VFD filter with a small WLS error. From (9), a formulation equivalent to (10) follows:

$$\underset{\boldsymbol{a}}{\text{minimize}}\ \ \mu\|\boldsymbol{a}\|_1 + \frac{1}{2}\|\boldsymbol{\Gamma}\boldsymbol{a} - \boldsymbol{y}\|_2^2 \quad (11)$$

The $l_1 - l_2$ optimization problem in (11) is nonsmooth but convex, for which efficient algorithms and software are available [12]. The particular algorithm that solves (11) for phase 1 combines a proximal-point technique with a fast iteration scheme known as FISTA [13]. For the sake of reader's convenience. We sketch the algorithm as follows.

Let $f(\boldsymbol{a}) = \frac{1}{2}\|\boldsymbol{\Gamma}\boldsymbol{a} - \boldsymbol{y}\|_2^2$, the mth iterate $\boldsymbol{a}_m$ is updated to $\boldsymbol{a}_{m+1}$ by minimizing the proximal-point objective function

$$F(\boldsymbol{a}) = f(\boldsymbol{a}_m) + (\boldsymbol{a} - \boldsymbol{a}_m)^T\nabla f(\boldsymbol{a}_m) + \frac{L}{2}\|\boldsymbol{a} - \boldsymbol{a}_m\|_2^2 + \mu\|\boldsymbol{a}\|_1 \quad (12)$$

where L is the smallest Lipschitz constant of $\nabla f(\boldsymbol{a})$ that is equal to the largest eigenvalue of $\boldsymbol{\Gamma}\boldsymbol{\Gamma}^T$. It can readily be verified that minimizing $F(\boldsymbol{a})$ in (12) is equivalent to

$$\text{minimize}\left(\mu\|\boldsymbol{a}\|_1 + \frac{L}{2}\|\boldsymbol{a} - \boldsymbol{d}_m\|_2^2\right) \quad (13a)$$

where

$$\boldsymbol{d}_m = \boldsymbol{a}_m - \frac{1}{L}\boldsymbol{\Gamma}^T(\boldsymbol{\Gamma}\boldsymbol{a}_m - \boldsymbol{y}) \quad (13b)$$

It is well known [12] that the global solution of (13) is given by a soft shrinkage of $\boldsymbol{d}_m$, namely

$$\boldsymbol{a}_{m+1} = \mathcal{S}_{\mu/L}(\boldsymbol{d}_m) \quad (14a)$$

where the shrinkage operator is defined as

$$\mathcal{S}_\alpha(\boldsymbol{u}) = \text{sgn}(\boldsymbol{u}) \cdot \max\{|\boldsymbol{u}| - \alpha,\ \boldsymbol{0}\} \quad (14b)$$

The above algorithm can be considerably accelerated by incorporating FISIA type of iterations [13]. In doing so, the algorithm follows the steps given below.

Input: Data $\boldsymbol{\Gamma}$ and $\boldsymbol{y}$, parameter μ and iteration number M.

Step 1. Compute $\boldsymbol{A}_0 = \boldsymbol{\Gamma}^{-1}\boldsymbol{y},\ \boldsymbol{a}_0 = \boldsymbol{A}_0(:)$. Set $\boldsymbol{b}_1 = \boldsymbol{a}_0,\ t_1 = 1$, and $m = 1$.

Step 2. Compute $\boldsymbol{a}_m = \mathcal{S}_{\mu/L}\{\boldsymbol{b}_m - \frac{1}{L}\boldsymbol{\Gamma}^T(\boldsymbol{\Gamma}\boldsymbol{b}_m - \boldsymbol{y})\}$

Step 3. Compute $t_{m+1} = \dfrac{1 + \sqrt{1 + 4t_m^2}}{2}$

Step 4. Update $\boldsymbol{b}_{m+1} = \boldsymbol{a}_m + \left(\dfrac{t_m - 1}{t_{m+1}}\right)(\boldsymbol{a}_m - \boldsymbol{a}_{m-1})$

Step 5. If $m < M$, set $m = m + 1$ and repeat Step 2; otherwise stop and output $\boldsymbol{a}_m$ as solution $\hat{\boldsymbol{a}}$.

Hard thresholding is applied to vector $\hat{\boldsymbol{a}}$ that generates an index set $\mathcal{I} = \{i : |\hat{a}(i)| < \varepsilon\}$. An appropriate value of threshold ε^* can easily be identified so that the length of the associated index set $\mathcal{I}^*$ is equal to target sparsity N_z. The index set $\mathcal{I}^*$ is a key ingredient of design phase 2.

B. Design Phase Two

The objective of phase 2 is to find a coefficient matrix $\boldsymbol{A}$ that minimizes the WLS error $J(\boldsymbol{A})$ in (5a) subject to sparsity constraint (5b). Following (9), the problem at hand can be formulated as

$$\underset{\boldsymbol{a}}{\text{minimize}}\quad J(\boldsymbol{a}) = \frac{1}{2}\|\boldsymbol{\Gamma}\boldsymbol{a} - \boldsymbol{y}\|_2^2 \quad (15a)$$

$$\text{subject to:}\quad a(i) = 0 \quad \text{for } i \in \mathcal{I}^* \quad (15b)$$

which is a convex quadratic programming problem. Note the length of $\boldsymbol{a}$ is $(K+1)(N+1)$. Denote index sets $\mathcal{I}_0 = \{i : i = 1,\ 2,\ \ldots,\ (K+1)(N+1)\}$ and $\bar{\mathcal{I}}^* = \{i : i \in \mathcal{I}_0$ and $i \notin \mathcal{I}^*\}$. For a vector $\boldsymbol{x}$ of length $(K+1)(N+1)$, denote by $\boldsymbol{x}(\mathcal{S})$ the vector composed of the entries of $\boldsymbol{x}$ whose indices are in set $\mathcal{S}$. Problem (15) can be solved by simply substituting

(15b) into the objective function in (15a) so that (15) becomes an unconstrained convex quadratic problem. In this way, the unique global solution $\boldsymbol{a}^*$ of the problem can be specified as

$$\boldsymbol{a}^*(\bar{\mathcal{I}}^*) = \boldsymbol{\Gamma}_s^{-1}\boldsymbol{y} \quad \text{and} \quad \boldsymbol{a}^*(\mathcal{I}^*) = \boldsymbol{0} \tag{16a}$$

where $\boldsymbol{\Gamma}_s$ is composed of those columns of $\boldsymbol{\Gamma}$ whose indices are in set $\bar{\mathcal{I}}^*$. The optimal coefficient matrix $\boldsymbol{A}^*$ is obtained by converting vector $\boldsymbol{a}^*$ into a $K+1$ by $N+1$ matrix.

IV. Performance Evaluation

We present a design case to serve the purpose of evaluating the proposed design algorithm. The algorithm was applied to design a VFD FIR filter of order $N = 65$, hence $D = 32$. The order of the polynomials $a_{n,k}(p)$ was set to $K = 7$, and the cutoff frequency $\omega_c = 0.9\pi$. The performance of VFD filters were evaluated in terms of maximum error

$$e_{max} = \max\{e(\omega, p),\ 0 \le \omega \le 0.9\pi,\ 0 \le p \le 1\}$$

with

$$e(\omega, p) = 20\log_{10}|H(\omega, p) - H_d(\omega, p)|$$

and L_2-error

$$e_2 = \left[\int_0^{0.9\pi}\int_0^1 |H(\omega, p) - H_d(\omega, p)|^2 dp\, d\omega\right]^{1/2}$$

The weighting function was set to $W(\omega, p) = W_1(\omega)W_2(p)$ with $W_2(p) \equiv 1$ for $0 \le p \le 1$ and

$$W_1(\omega) = \begin{cases} 1 & \text{for } \omega \in [0,\ 0.88\pi) \\ 3 & \text{for } \omega \in [0.88\pi,\ 0.8994\pi) \\ 0 & \text{for } \omega \in [0.8994\pi,\ \pi] \end{cases}$$

With $(N, K) = (65, 7)$, $\mu = 1 \times 10^{-5}$, and $M = 80$, phase 1 of the proposed algorithm produced an coefficient vector $\hat{\boldsymbol{a}}$. In our design example, the target sparsity was set to $N_z = 198$ which means that 37.5% of the entries in $\hat{\boldsymbol{A}}$ were set to zero. With $\varepsilon^* = 10^{-3}$, hard thresholding of $\hat{\boldsymbol{a}}$ yielded an index set $\mathcal{I}^*$ of length 198. This index set was used for the algorithm to proceed with phase 2 which produced an optimal filter coefficient $\boldsymbol{A}^*$ with sparsity($\boldsymbol{A}^*$) = 198. The maximum and L_2 errors of the sparse VFD filter obtained were found to be $e_{max} = 0.0021$ and $e_2 = -75.25$dB, respectively. The frequency response error and fractional delay of the VFD filter over $[0,\ 0.9\pi]$ and $0 \le p \le 1$ are depicted in Fig. 1(a) and Fig. 2(a), respectively. For comparison, an equivalent nonsparse VFD FIR filter with $(N, K) = (65, 4)$ was designed to minimize the WLS error in (5a) without imposing coefficient sparsity. Note that with $(N, K) = (65, 4)$ the total number of nonzero coefficients in the nonsparse VFD filter is 330 — exactly the same as in the sparse VFD filter designed above. Also note that the nonsparse VFD filter possesses the same order ($N = 65$) so that the integer delay D was unaltered. The maximum and L_2 errors of the nonsparse VFD filter were found to be 0.0609 and $e_2 = -45.28$dB, respectively. Its frequency response error and fractional delay are shown in Fig. 1(b) and Fig. 2(b), respectively.

In the rest of the section, the proposed algorithm is further illustrated by examining the significance of each design phase.

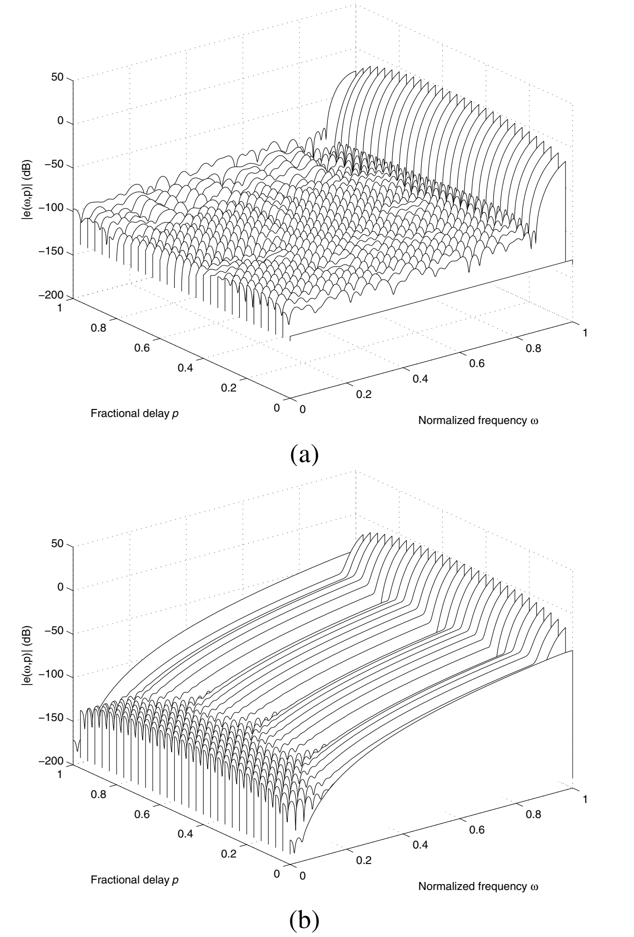


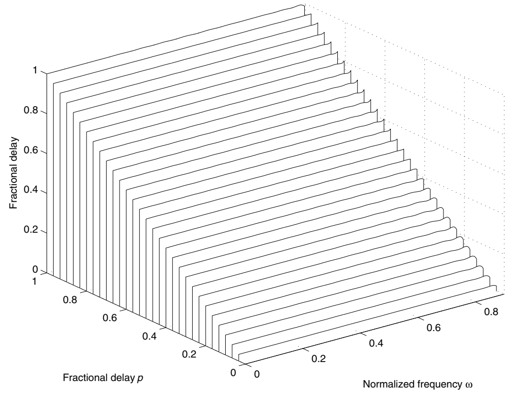
Fig. 1. Profile of frequency response error $|e(\omega, p)|$ over $0 \le \omega \le 0.9\pi$ and $0 \le p \le 1$ of (a) sparse VFD filter with $(N, K) = (65, 7)$ and $N_z = 198$; (b) equivalent nonsparse VFD filter with $(N, K) = (65, 4)$.

A. Justification of Phase One

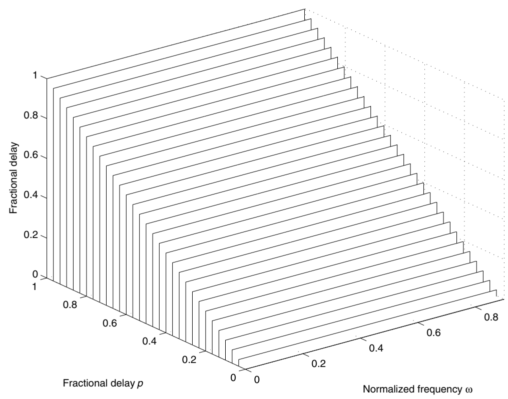
To this end, we started with a conventional (nonsparse) WLS optimal VFD FIR filter of order $(N, K) = (65, 7)$ whose coefficient matrix is denoted by $\boldsymbol{A}$. Without going to phase 1, a hard thresholding was applied to $\boldsymbol{A}$ with $\varepsilon = 3.8 \times 10^{-3}$ to generate 198 zero entries. Denote the index set associated with these zeros by $\mathcal{I}^{**}$. With the identified index set $\mathcal{I}^{**}$ (rather than $\mathcal{I}^*$), phase 2 was carried out to design a sparse VFD filter. The maximum and L_1 errors of the filter were found to be $e_{max} = 0.0025$ and $e_2 = -73.41$dB, respectively, see Fig. 3 for its frequency response error and fractional delay. On comparing with the maximum and L_2 errors $e_{max} = 0.0021$ and $e_2 = -75.25$dB achieved by the proposed algorithm, this justifies phase 1 of the design method that identifies a better index set to achieve a target sparsity.

B. Justification of Phase Two

Suppose one terminates the design process right after phase 1 is complete. The result is a coefficient vector $\hat{\boldsymbol{a}}$ and an index set $\mathcal{I}^*$ with $|\mathcal{I}^*| = 198$. A sparse $\tilde{\boldsymbol{a}}$ with sparsity($\tilde{\boldsymbol{a}}$) = 198 is generated by setting $\tilde{a}(i) = 0$ for $i \in \mathcal{I}^*$. A sparse coefficient matrix $\tilde{\boldsymbol{A}}$ can be constructed by converting $\tilde{\boldsymbol{a}}$ into a $(K +$

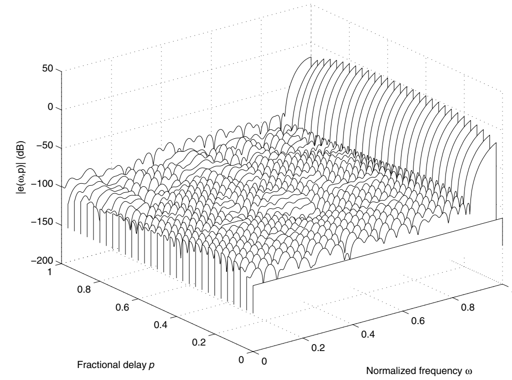


(a)

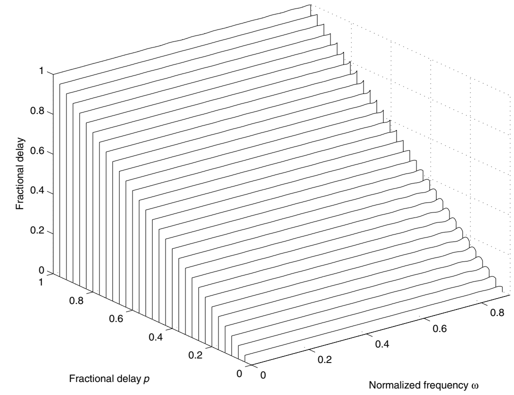


(b)

Fig. 2. Fractional delay over $0 \leq \omega \leq 0.9\pi$ and $0 \leq p \leq 1$ of (a) sparse VFD filter with $(N, K) = (65, 7)$ and $N_z = 198$; (b) equivalent nonsparse VFD filter with $(N, K) = (65, 4)$.



(a)



(b)

Fig. 3. Profile of (a) frequency response error $|e(\omega, p)|$ and (b) fractional delay of the VFD filter without phase 1: $e_{max} = 0.0025$, $e_2 = -73.41$dB.

$1) \times (N + 1)$ matrix with $(N, K) = (65, 7)$. The maximum and L_2 errors of the VFD filter associated with $\tilde{\boldsymbol{A}}$ were found to be $e_{max} = 0.1291$ and $e_2 = -43.23$dB, respectively. The severely degraded performance is attributed to the absence of phase 2 of the design.

V. Conclusion

A two-phase technique for the design of variable fractional delay FIR filters that are WLS optimal subject to a target coefficient sparsity has been proposed. The design algorithm is easy to implement and computationally efficient because it is based on $l_1 - l_2$ convex optimization. Computer simulations for performance evaluation and comparison of the proposed algorithm relative to an equivalent nonsparse counterpart have also been presented.

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