IMPROVED DESIGN OF FREQUENCY-RESPONSE-MASKING FILTERS USING ENHANCED SEQUENTIAL QUADRATIC PROGRAMMING

Wu-Sheng Lu

Dept. of Elec. and Comp. Engineering
University of Victoria
Victoria, BC, Canada V8W 3P6

Takao Hinamoto

Graduate School of Engineering
Hiroshima University
Higashi-Hiroshima 739-8527, Japan

ABSTRACT

Sequential quadratic programming (SQP) algorithms are widely recognized to be among the most successful algorithms for nonconvex optimization. This paper attempts to develop an SQP-based method for frequency-response-masking (FRM) filters. We explain how the complementarity conditions in the SQP algorithm help reduce the amount of computation required to update the Lagrange multipliers in a significant manner. Simulation results are presented to demonstrate the algorithm's performance that compares favorably with several existing design methods.

1. INTRODUCTION

The frequency-response-masking (FRM) technique originated in [1] has proved to be effective for the design of digital filters with narrow transition bands. Several design methods for linear-phase and low-group-delay, FIR and IIR, basic and multistage FRM filters have been investigated in the past, see [1]–[13] and the references cited there. Among others, available design methods include joint optimization of all subfilters using semidefinite programming (SDP) [11] and second-order cone programming (SOCP) [12][13]. Although these methods work well in general, a problem with them is the large number of constraints that inevitably effects design efficiency and, in the case of high-order FRM filters, may cause numerical difficulties.

In this paper, the joint optimization of subfilters is approached in a rather different way, namely, via an enhanced sequential quadratic programming (SQP) technique. Although, to the best knowledge of the authors, it appears to be the first attempt to use SQP for the design of FRM filters, SQP algorithms are widely recognized to be among the most successful algorithms for nonconvex constrained optimization problems [14]. Since the minimax design of an FRM filter can be formulated as a nonconvex constrained minimization problem, SQP is a natural candidate tool for the design. However, our primary reason to develop an SQP-based design methodology is that the complementarity conditions in an SQP formulation are found to be effective in reducing the number of constraints that actually participate in the optimization. Our design method is rather general in the sense that it is applicable to both basic and multistage FRM filters with linear phase response or low group delay. Because of space limitation and for illustration clarity, however, our attention here is focused on the class of basic, linear phase FIR FRM filters. Technical details of the proposed method are given in Secs. 2 and 3. Design examples with performance comparisons are presented in Sec. 4.

2. PROBLEM FORMULATION

Following [1], the reader is referred to the structure in Fig. 1 where all filters are assumed to have linear-phase responses, and the lengths of the masking filters are either both even or both odd. The transfer functions of the prototype and masking filters are respectively denoted by

$$H_a(z) = \sum_{k=0}^{N-1} h_k z^{-k},\ H_{ma}(z) = \sum_{k=0}^{N_a-1} h_k^{(a)} z^{-k},$$

$$H_{mc}(z) = \sum_{k=0}^{N_c-1} h_k^{(c)} z^{-k}$$

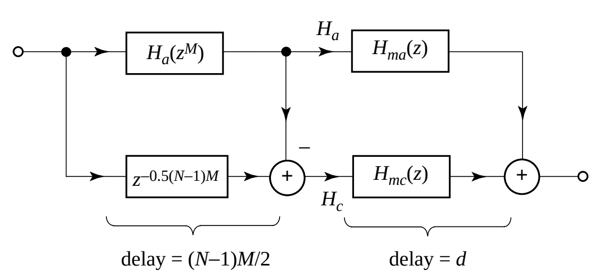

Fig. 1. A basic FRM filter structure.

In what follows, H_a, H_{ma} and H_{mc} are referred to as *subfilters*. Without loss of generality, the FRM filter can be treated as a zero-phase FIR filter and all subfilters are assumed to be of odd length. The frequency response of the FRM filter is then given by

$$H(\omega, \boldsymbol{h}) = [\boldsymbol{a}^T \boldsymbol{c}(\omega)][\boldsymbol{a}_a^T \boldsymbol{c}_a(\omega) - \boldsymbol{a}_c^T \boldsymbol{c}_c(\omega)] + \boldsymbol{a}_c^T \boldsymbol{c}_c(\omega) \quad (1)$$

where

$$\begin{aligned}
\boldsymbol{a} &= [h_{(N-1)/2} \;\; 0.5h_{(N+1)/2} \;\; \cdots \;\; 0.5h_{N-1}]^T \\
\boldsymbol{c}(\omega) &= [1 \;\; \cos M\omega \;\; \cdots \;\; \cos[(N-1)M\omega/2]]^T \\
\boldsymbol{a}_a &= [h^{(a)}_{(N_a-1)/2} \;\; 0.5h^{(a)}_{(N_a+1)/2} \;\; \cdots \;\; 0.5h^{(a)}_{N_a-1}]^T \\
\boldsymbol{c}_a(\omega) &= [1 \;\; \cos\omega \;\; \cdots \;\; \cos[(N_a-1)\omega/2]]^T \\
\boldsymbol{a}_c &= [h^{(c)}_{(N_c-1)/2} \;\; 0.5h^{(c)}_{(N_c+1)/2} \;\; \cdots \;\; 0.5h^{(c)}_{N_c-1}]^T \\
\boldsymbol{c}_c(\omega) &= [1 \;\; \cos\omega \;\; \cdots \;\; \cos[(N_c-1)\omega/2]]^T
\end{aligned}$$

and $\boldsymbol{h} = [\boldsymbol{a}^T \;\; \boldsymbol{a}_a^T \;\; \boldsymbol{a}_c^T]^T$. The minimax design of the FRM filter amounts to finding a vector $\boldsymbol{h}$ that solves the minimax optimization problem

$$\underset{\boldsymbol{h}}{\text{minimize}}\{\underset{\omega\in\Omega}{\text{maximize}}\, W(\omega)|H(\omega,\boldsymbol{h}) - H_d(\omega)|\} \quad (2)$$

where $H_d(\omega)$ is a real-valued desired frequency response, $W(\omega) \geq 0$ is a weighting function, and $\Omega = \{\omega : 0 \leq \omega \leq \pi\}$.

Let β be an upper bound of $W((\omega)|H(\omega,\boldsymbol{h}) - H_d(\omega)|$ on Ω. As the first step of the optimization we convert the problem in (2) into a constrained minimization problem

$$\text{minimize} \quad \beta \quad (3a)$$

$$\text{subject to: } W(\omega)|H(\omega,\boldsymbol{h}) - H_d(\omega)| \leq \beta \text{ for } \omega \in \Omega \quad (3b)$$

For practical exercise of optimization techniques, the constraint in (3b) is imposed on a dense grid of frequencies $\Omega_d = \{0 \leq \omega_1 \leq \cdots \leq \omega_K \leq \pi\}$ and the problem in (3) becomes

$$\text{minimize} \quad \beta \quad (4a)$$

$$\text{subject to: } W(\omega_i)|H(\omega_i,\boldsymbol{h}) - H_d(\omega_i)| \leq \beta \text{ for } \omega_i \in \Omega_d \quad (4b)$$

3. DESIGN METHOD

3.1. An SQP-based algorithm

The constraints in (4b) can be made more specific as

$$\begin{aligned}
&p_i(\boldsymbol{h},\beta) = \\
&\beta + W(\omega_i)[H(\omega_i,\boldsymbol{h}) - H_d(\omega_i)] \geq 0, \;\; 1 \leq i \leq K \quad (5a) \\
&p_{K+i}(\boldsymbol{h},\beta) = \\
&\beta - W(\omega_i)[H(\omega_i,\boldsymbol{h}) - H_d(\omega_i)] \geq 0, \;\; 1 \leq i \leq K \quad (5b)
\end{aligned}$$

Defining $\boldsymbol{x} = [\beta \;\; \boldsymbol{h}^T]^T$ and $\boldsymbol{e} = [1 \;\; 0 \;\; \cdots \;\; 0]^T$, (4) can be expressed as

$$\text{minimize} \quad \boldsymbol{e}^T\boldsymbol{x} \quad (6a)$$

$$\text{subject to: } p_i(\boldsymbol{x}) \geq 0 \qquad 1 \leq i \leq 2K \quad (6b)$$

Since $-p_i(\boldsymbol{x})$ are not convex functions, (6) is a nonconvex problem.

The Lagrangian of (6) is defined by

$$L(\boldsymbol{x},\boldsymbol{\mu}) = \boldsymbol{e}^T\boldsymbol{x} - \sum_{i=1}^{2K} \mu_i p_i(\boldsymbol{x})$$

where μ_i for $1 \leq i \leq 2K$ are the Lagrange multipliers. The solution of problem (6) must satisfy the Karush-Kuhn-Tucker (KKT) conditions [15]

$$\begin{aligned}
\nabla L(\boldsymbol{x},\boldsymbol{\mu}) &= \boldsymbol{0} && (7a) \\
p_i(\boldsymbol{x}) &\geq 0 \qquad 1 \leq i \leq 2K && (7b) \\
\mu_i &\geq 0 \qquad 1 \leq i \leq 2K && (7c) \\
\mu_i p_i(\boldsymbol{x}) &= 0 \qquad 1 \leq i \leq 2K && (7d)
\end{aligned}$$

It is the KKT conditions that form the basis of our design algorithm and a subsequent analysis of the algorithm. Suppose one starts with a reasonable initial point $\boldsymbol{x}_0$ (which may be produced using the method in [1]) and an initial $\boldsymbol{\mu}_0 = \boldsymbol{0}$. In the kth iteration, $\{\boldsymbol{x}_k, \boldsymbol{\mu}_k\}$ is updated to $\{\boldsymbol{x}_{k+1}, \boldsymbol{\mu}_{k+1}\} = \{\boldsymbol{x}_k, \boldsymbol{\mu}_k\} + \{\boldsymbol{\delta}_x, \boldsymbol{\delta}_\mu\}$ such that (7a), (7b), and (7c) are approximately satisfied up to the first order, and (7c) is precisely satisfied. This first-order approximation leads to

$$\begin{aligned}
\boldsymbol{Y}_k\boldsymbol{\delta}_x + \boldsymbol{e} - \boldsymbol{A}_k^T\boldsymbol{\mu}_{k+1} &= \boldsymbol{0} && (8a) \\
\boldsymbol{A}_k\boldsymbol{\delta}_x &\geq -\boldsymbol{c}_k && (8b) \\
\boldsymbol{\mu}_{k+1} &\geq \boldsymbol{0} && (8c) \\
(\boldsymbol{\mu}_{k+i})_i(\boldsymbol{A}_k\boldsymbol{\delta}_k + \boldsymbol{c}_k)_i &= 0 \quad 1 \leq i \leq 2K && (8d)
\end{aligned}$$

where $\boldsymbol{Y}_k = \nabla^2 L(\boldsymbol{x}_k,\boldsymbol{\mu}_k)$, $\boldsymbol{c}_k = [p_1(\boldsymbol{x}_k) \;\; \cdots \;\; p_{2K}(\boldsymbol{x}_k)]^T$, and

$$\boldsymbol{A}_k = \begin{bmatrix} \nabla^T p_1(\boldsymbol{x}_k) \\ \vdots \\ \nabla^T p_{2K}(\boldsymbol{x}_k) \end{bmatrix} \quad (9)$$

Equations (8a)–(8d) turn out to be the *exact* KKT conditions for the quadratic programming (QP) problem

$$\text{minimize} \;\; \frac{1}{2}\boldsymbol{\delta}^T\boldsymbol{Y}_k\boldsymbol{\delta} + \boldsymbol{\delta}^T\boldsymbol{e} \quad (10a)$$

$$\text{subject to: } \boldsymbol{A}_k\boldsymbol{\delta} \geq -\boldsymbol{c}_k \quad (10b)$$

Let the solution of (10) be denoted by $\boldsymbol{\delta}_x$, the Lagrange multiplier $\boldsymbol{\mu}_{k+1}$ can then be determined by (8a) and (8d) as follows. First, the $2K$ components of $\boldsymbol{A}_k\boldsymbol{\delta}_x + \boldsymbol{c}_k$ are examined. For the component indices with $(\boldsymbol{A}_k\boldsymbol{\delta}_x + \boldsymbol{c}_k)_i > 0$, the complementarity conditions in (8d) imply that $(\boldsymbol{\mu}_{k+1})_i = 0$. Since $\boldsymbol{\delta}_x$ satisfies (10b), the rest of indices are those where $(\boldsymbol{A}_k\boldsymbol{\delta}_x + \boldsymbol{c}_k)_i = 0$ and the complementarity conditions are satisfied regardless of the values of $(\boldsymbol{\mu}_{k+1})_i$. These possibly nonzero Lagrange multipliers can be determined using (8a) as

$$\hat{\boldsymbol{\mu}}_{k+1} = (\boldsymbol{A}_{ak}\boldsymbol{A}_{ak}^T)^{-1}\boldsymbol{A}_{ak}(\boldsymbol{Y}_k\boldsymbol{\delta}_x + \boldsymbol{e}) \qquad (11)$$

where the rows of $\boldsymbol{A}_{ak}$ are those rows of $\boldsymbol{A}_k$ satisfying $(\boldsymbol{A}_k\boldsymbol{\delta}_x + \boldsymbol{c}_k)_i = 0$ and $\hat{\boldsymbol{\mu}}_{k+1}$ denotes the associated Lagrange multiplier. Having computed $\hat{\boldsymbol{\mu}}_{k+1}$, vector $\boldsymbol{\mu}_{k+1}$ can be obtained by inserting zeros wherever necessary in $\hat{\boldsymbol{\mu}}_{k+1}$. It should be stressed that typically the number of nonzero Lagrange multiplies, say $\hat{K}$, is much smaller than the number of constraints imposed in (4b), K (usually $\hat{K} < 0.1K$). Consequently, computing $\hat{\boldsymbol{\mu}}_{k+1}$ using (11) which involves inversion of an $\hat{K} \times \hat{K}$ matrix does not impose a computational burden. Moreover, since in the $2K$ linear constraints in (10b) only a small fraction of them are *active*, solving the QP problem in (10) can be carried out efficiently when an active-set type algorithm [15] is utilized.

Having obtained $\boldsymbol{\delta}_x$ and $\boldsymbol{\mu}_{k+1}$, point $\boldsymbol{x}_k$ is then updated to $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{\delta}_x$, and $\boldsymbol{Y}_k$, $\boldsymbol{c}_k$, and $\boldsymbol{A}_k$ are updated to $\boldsymbol{Y}_{k+1}$, $\boldsymbol{c}_{k+1}$, and $\boldsymbol{A}_{k+1}$, respectively. The iteration continues until a convergence criterion in terms of the progress made, i.e., $\|\boldsymbol{\delta}_x\|_2$, or the total number of iterations is met. The coefficients of the optimized subfilters can be found in the solution vector $\boldsymbol{x}^*$ as $\boldsymbol{h}^* = \boldsymbol{x}^*(2 : \text{end})$.

3.2. Convex relaxation of problem (10)

The Hessian matrix of the Lagrangian, $\boldsymbol{Y}_k$, is in general not positive definite, hence problem (10) is not a convex QP problem. A convex relaxation of problem (10) can be made by replacing the Hessian matrix $\boldsymbol{Y}_k$ in (10a) with a positive definite matrix, still denoted by $\boldsymbol{Y}_k$, with $\boldsymbol{Y}_0 = \boldsymbol{I}$ using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) recusion [15] that updates $\boldsymbol{Y}_k$ to

$$\boldsymbol{Y}_{k+1} = \boldsymbol{Y}_k + \frac{\boldsymbol{\eta}_k\boldsymbol{\eta}_k^T}{\boldsymbol{\delta}_x^T\boldsymbol{\eta}_k} - \frac{\boldsymbol{v}_k\boldsymbol{v}_k^T}{\boldsymbol{\delta}_x^T\boldsymbol{v}_k} \qquad (12a)$$

where

$$\boldsymbol{v}_k = \boldsymbol{Y}_k\boldsymbol{\delta}_x$$

$$\boldsymbol{\eta}_k = \theta\boldsymbol{\gamma}_k + (1-\theta)\boldsymbol{v}_k \qquad (12b)$$

$$\boldsymbol{\gamma}_k = -(\boldsymbol{A}_{k+1} - \boldsymbol{A}_k)^T\boldsymbol{\mu}_{k+1} \qquad (12c)$$

$$\theta = \begin{cases} 1 & \text{if } \boldsymbol{\delta}_x^T(\boldsymbol{\gamma}_k - 0.2\boldsymbol{v}_k) \geq 0 \\ \frac{0.8\boldsymbol{\delta}_x^T\boldsymbol{v}_k}{\boldsymbol{\delta}_x^T(\boldsymbol{v}_k - \boldsymbol{\gamma}_k)} & \text{otherwise} \end{cases} \qquad (12d)$$

In this way, (10) becomes a convex QP problem which possesses a unique global minimizer that can be obtained using an efficient algorithm such as an active-set algorithm. A desirable feature of the BFGS update is that if $\boldsymbol{Y}_k$ is positive definite, then $\boldsymbol{Y}_{k+1}$ is also positive definite. With $\boldsymbol{Y}_0 = \boldsymbol{I}$, therefore, the QP subproblems involved in the entire design process are all guaranteed to be convex QP problems.

3.3. Implementation

Initial subfilters can be obtained using the method proposed in [1]. For given $H_d(\omega)$, weighting function $W(\omega)$, a grid of frequencies Ω_d, and an initial $\boldsymbol{h}_0$, the value of β_0 can be calculated as

$$\beta_0 = \max_{\Omega_d} W(\omega_i)|H(\omega_i, \boldsymbol{h}_0) - H_d(\omega_i)|$$

The SQP-based algorithm starts with initial point $\boldsymbol{x}_0 = [\beta_0 \;\; \boldsymbol{h}_0^T]^T$ and $\boldsymbol{Y}_0 = \boldsymbol{I}$. For the design of basic FIR FRM filters, the matrix $\boldsymbol{A}_k$ in (9) is a $2K \times (N + N_a + N_c + 5)/2$ matrix whose ith and $(K+i)$th rows for $1 \leq i \leq K$ are given by $[1 \;\; W(\boldsymbol{a}_a^T\boldsymbol{c}_a - \boldsymbol{a}_c^T\boldsymbol{c}_c)\boldsymbol{c} \;\; W(\boldsymbol{a}^T\boldsymbol{c})\boldsymbol{c}_a \;\; W(1 - \boldsymbol{a}^T\boldsymbol{c})\boldsymbol{c}_c]$ and $-[-1 \;\; W(\boldsymbol{a}_a^T\boldsymbol{c}_a - \boldsymbol{a}_c^T\boldsymbol{c}_c)\boldsymbol{c} \;\; W(\boldsymbol{a}^T\boldsymbol{c})\boldsymbol{c}_a \;\; W(1 - \boldsymbol{a}^T\boldsymbol{c})\boldsymbol{c}_c]$ respectively, where the frequency-dependence for W, $\boldsymbol{c}$, $\boldsymbol{c}_a$, and $\boldsymbol{c}_c$ have been omitted. Reliable convex QP solvers are available, for example, in MATLAB Optimization Toolbox: `quadprog` uses an interior-point method while `qp` adopts an active-set method.

4. DESIGN EXAMPLES

The method described in Sections 2 and 3 was applied to design two one-stage linear-phase FRM filters that were addressed in the literature [1][6][11].

Example 1: The design parameters were $N = 45$, $N_a = 41$, $N_c = 33$, $M = 9$, $\omega_p = 0.6\pi$, $\omega_a = 0.61\pi$, $W(\omega) \equiv 1$ for $\omega \in [0, \;\omega_p] \bigcup [\omega_a, \;\pi]$ and $K = 950$. It took the algorithm 110 iterations to converge to a solution FRM filter whose amplitude response and passband ripple are shown in Fig. 2a and b, respectively. It is interesting to note that among the $2K = 1900$ inequality constraints (see (6b)), the average number of active constraints in the entire design process was only 27. In other words, the average size of the matrix $\boldsymbol{A}_{ak}\boldsymbol{A}_{ak}^T$ in (11) was 27×27. The maximum passband ripple and minimum stopband attenuation were 0.0667 dB and 42.38 dB, respectively, which compare favorably with the design of the same FRM filter in [1] (with passband ripple = 0.0896 dB and stopband attenuation = 40.96 dB) and in [11] (with passband ripple = 0.0674 dB and stopband attenuation = 42.25 dB).

Example 2: The design parameters were $N = 123$, $N_a = 56$, $N_c = 78$, $M = 21$, $\omega_p = 0.4\pi$, $\omega_a = 0.61\pi$, and $K = 1100$. The weighting function $W(\omega)$ was piecewise constant with $W(\omega) \equiv 1$ in the passband and $W(\omega) \equiv 12$ in the stopband. It took the proposed algorithm 150 iterations to converge to a solution FRM filter. The average number of active constraints was 67. The amplitude response and passband ripple of the FRM filter are depicted in Fig. 3(a) and (b), respectively. The maximum passband ripple and minimum stopband attenuation are 0.0898 dB and 61.66 dB. For comparison, the maximum ripple and minimum stopband

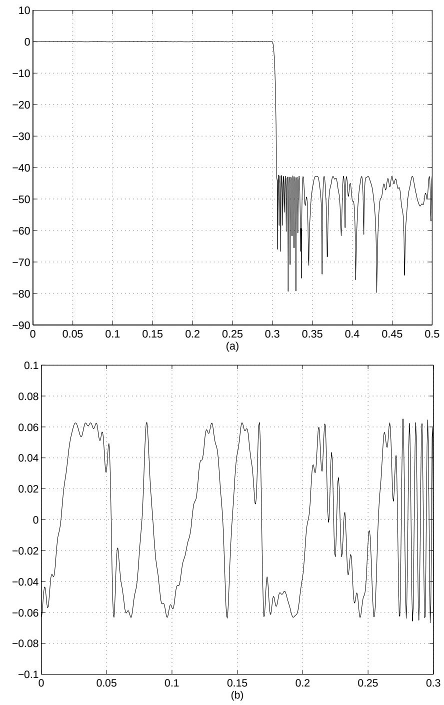

Fig. 2. (a) Amplitude response and (b) passband ripple of the FRM filter in Example 1, all in dB.

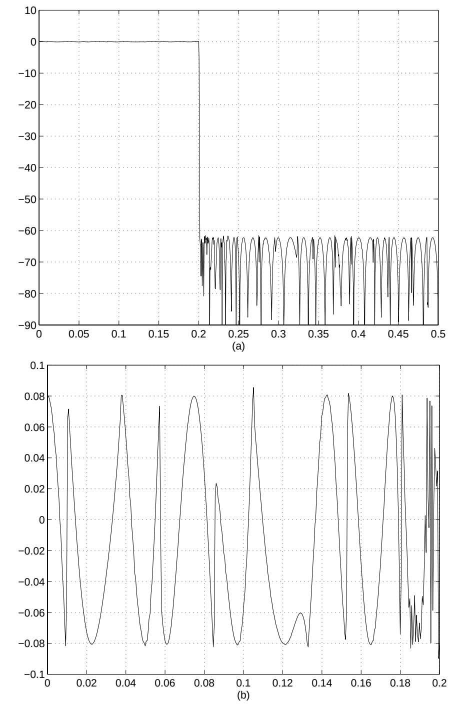

Fig. 3. (a) Amplitude response and (b) passband ripple of the FRM filter in Example 2, all in dB.

attenuation are 0.0864 dB and 60 dB in [6] and are 0.0855 dB and 60.93 dB in [11].

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